The Stokes system with boundary condition involving a pressure

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$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{F}, \quad \nabla \cdot \mathbf{u} = G \quad \text{in } \Omega,$$

$$\pi = h, \quad \mathbf{u} \cdot \tau = g \quad \text{on } \partial\Omega$$

$\Omega \subset R^2$ is a bounded simply connected domain. τ is the tangential vector on $\partial\Omega$.

Ch. Amrouche, P. Penel, N. Seloula, *Some remarks on the boundary conditions in the theory of Navier-Stokes equations*, Ann. Math. Blaise Pascal 20 (2013), 37–73.

in $W^{1,p}(\Omega, R^3) \times W^{1,p}(\Omega)$ for $\Omega \subset R^3$, $\partial\Omega \in C^{1,1}$

Theorem.

Let $\Omega \subset R^2$ *be a bounded domain with connected boundary of class* $\mathcal{C}^{k,1}$, $k \in N$, $1 < p, q < \infty$, $1/q < s < k+1$, $s - 1/q \notin N_0$, $1/p < t < k$, $t - 1/p \notin N_0$, *and* $t \leq s+1$, $p \leq q$. *If* $t = s+1$ *suppose moreover that* $p = q$. *If* $g \in W^{t-1/p,p}(\partial\Omega)$, $h \in W^{s-1/q,q}(\partial\Omega)$, $\mathbf{F} \in W^{s-1,q}(\Omega, R^2)$, $G \in W^{s,q}(\Omega)$, *then there exists a unique solution* $(\mathbf{u}, \pi) \in W^{t,p}(\Omega, R^2) \times W^{s,q}(\Omega)$ *of the problem*

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{F}, \quad \nabla \cdot \mathbf{u} = G \quad \textit{in } \Omega,$$

$$\mathbf{u} \cdot \tau = g, \quad \pi = h \quad \textit{on } \partial\Omega.$$

Theorem.

Let $\Omega \subset R^2$ *be a bounded domain with connected boundary of class* $\mathcal{C}^{k,1}$, $k \in N$, $1 < p, q, r, \beta < \infty$, $1/q < s < k+1$, $1/p < t < k$, *and* $t \leq s+1$, $p \leq q$. *If* $t = s+1$ *suppose moreover that* $p = q$ *and* $r \leq \beta$. *If* $g \in B^{p,\beta}_{t-1/p}(\partial\Omega)$, $h \in B^{q,r}_{s-1/q}(\partial\Omega)$, $\mathbf{F} \in B^{q,r}_{s-1}(\Omega, R^2)$, $G \in B^{q,r}_s(\Omega)$, *then there exists a unique solution* $(\mathbf{u}, \pi) \in B^{p,\beta}_t(\Omega, R^2) \times B^{q,r}_s(\Omega)$ *of the problem*

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{F}, \quad \nabla \cdot \mathbf{u} = G \quad \textit{in } \Omega,$$

$$\mathbf{u} \cdot \tau = g, \quad \pi = h \quad \textit{on } \partial\Omega.$$

Theorem.
Let $k \in N$, $0 < \gamma < 1$, $\Omega \subset R^2$ be a bounded domain with connected boundary of class $\mathcal{C}^{k+2,\gamma}$. Suppose that $\mathbf{F} \in \mathcal{C}^{k-1,\gamma}(\overline{\Omega}, R^2)$, $G \in \mathcal{C}^{k,\gamma}(\overline{\Omega})$, $h \in \mathcal{C}^{k,\gamma}(\partial\Omega)$, $g \in \mathcal{C}^{k+1,\gamma}(\partial\Omega)$. Then there exists a unique solution $(\mathbf{u}, \pi) \in \mathcal{C}^{k+1,\gamma}(\overline{\Omega}, R^2) \times \mathcal{C}^{k,\gamma}(\overline{\Omega})$ of the problem

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{F}, \quad \nabla \cdot \mathbf{u} = G \quad \text{in } \Omega,$$

$$\mathbf{u} \cdot \tau = g, \quad \pi = h \quad \text{on } \partial\Omega.$$

$a > 0$ *fixed*

The nontangential approach regions of opening a *at the point* $\mathbf{x} \in \partial\Omega$

$$\Gamma_a(\mathbf{x}) = \{\mathbf{y} \in \Omega; |\mathbf{x} - \mathbf{y}| < (1 + a) dist(\mathbf{y}, \partial\Omega)\}.$$

The nontangential maximal function of $\mathbf{v}$ *on* $\partial\Omega$

$$M_a(\mathbf{v})(\mathbf{x}) = \sup\{|\mathbf{v}(\mathbf{y})|; \mathbf{y} \in \Gamma_a(\mathbf{x})\}.$$

The nontangential limit of $\mathbf{v}$ *at* $\mathbf{x} \in \partial\Omega$

$$\mathbf{v}(\mathbf{x}) = \lim_{\Gamma(\mathbf{x}) \ni \mathbf{y} \to \mathbf{x}} \mathbf{v}(\mathbf{y})$$

Theorem.
Let $\Omega \subset R^2$ *be a bounded domain with connected Lipschitz boundary. Let* $1 < p \leq 2 \leq q < \infty$, $h \in L^q(\partial\Omega)$, $g \in L^p(\partial\Omega)$. *Then there exists a unique* $(\mathbf{u}, \pi)$ *such that*

$$-\Delta\mathbf{u} + \nabla\pi = 0, \quad \nabla\cdot\mathbf{u} = 0 \quad \textit{in } \Omega,$$

the nontangential maximal function of $\mathbf{u}$ *is in* $L^q(\partial\Omega)$ *(and therefore* $\mathbf{u} \in B^{q,q}_{1/q}(\Omega, R^2)$ *i.e* $\mathbf{u} \in W^{1/q,q}(\Omega, R^2)$*), the nontangential maximal function of* π *is in* $L^p(\partial\Omega)$ *(and therefore* $\pi \in B^{2,p}_{1/p}(\Omega)$*) and the boundary conditions*

$$\mathbf{u}\cdot\tau = g, \quad \pi = h \quad \textit{on } \partial\Omega$$

are satisfiesd in the sense of nontangential limits.