# Measure–valued solutions and their applications in numerical analysis

Eduard Feireisl
based on joint work with P. Gwiazda, A. Świerczewska-Gwiazda (Warsaw), M. Medviďová/Lukáčová (Mainz), E.Wiedemann (Hannover)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

**SIAM PDE conference, Baltimore, 9–12 December 2017**

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

# Measure–valued solutions revisited

**Motto: The larger the better**

*Measure valued solutions* $\equiv$ the largest class of objects in which the smooth (classical) solutions of a given problem are uniquely determined by the data

**Advantages**

- Limits of approximate problems (low viscosity, low Mach number limit)
- Limits of approximate solutions obtained via *numerical schemes*

# Compressible Navier-Stokes system

**Field equations**

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

**Isentropic EOS, Newton's rheological law**

$$p(\varrho) = a \varrho^\gamma$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

**No-slip boundary conditions**

$$\mathbf{u}|_{\partial \Omega} = 0$$

# Dissipative solutions

**Energy (entropy) inequality**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x \boxed{\leq} 0$$

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

**Known results**

- **Local strong solution for any data and global weak solutions for small data.** Matsumura and Nishida [1983], Valli and Zajaczkowski [1986], among others
- **Global-in-time weak solutions.** $p(\varrho) = \varrho^\gamma$, $\gamma \geq 9/5, N = 3$, $\gamma \geq 3/2, N = 2$ P.L. Lions [1998], $\gamma > 3/2$, $N = 3$, $\gamma > 1$, $N = 2$ EF, Novotný, Petzeltová [2000], $\gamma = 1$, $N = 2$ Plotnikov and Vaigant [2014]
- **Measure-valued solutions.** Neustupa [1993], related results Málek, Nečas, Rokyta, Růžička, Nečasová - Novotný

# Numerical method [T. Karper]

## FV framework

$$\text{regular tetrahedral mesh}, \ Q_h = \{ v \mid v = \text{piece-wise constant} \}$$

## FE framework - Crouzeix - Raviart

$$V_h = \left\{ v \ \middle| \ v = \text{piece-wise affine}, \ \tilde{v}_\Gamma \text{ continuous on face } \Gamma \right\}$$

$$\tilde{v}_\Gamma \equiv \frac{1}{|\Gamma|} \int_\Gamma v \ \mathrm{dS}_x$$

## Upwind discretization of convective terms

$$\langle h\mathbf{u}; \nabla_x \varphi \rangle_E \approx \sum_\Gamma \int_\Gamma \mathrm{Up}[h, \mathbf{u}][[\varphi]] \ \mathrm{dS}_x$$

# Dissipative upwind operator

## Upwind operator

$$\mathrm{Up}[r_h, \mathbf{u}_h] = \underbrace{\{r_h\} \langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}_{\text{convective part}} - \frac{1}{2} \underbrace{\max\{h^\alpha ; |\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma|\} [[r_h]]}_{\text{dissipative part}}$$

$$= \underbrace{r_h^{\mathrm{out}} [\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma]^- + r_h^{\mathrm{in}} [\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma]^+}_{\text{standard upwind}} - \frac{h^\alpha}{2} [[r_h]] \chi \left( \frac{\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right)$$

## Auxilliary function

$$\chi(z) = \begin{cases} 0 \text{ for } z < -1, \\ z + 1 \text{ if } -1 \leq z \leq 0 \\ 1 - z \text{ if } 0 < z \leq 1 \\ 0 \text{ for } z > 1 \end{cases}$$

# Numerical scheme

**Discrete time derivative - implicit scheme**

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

**Continuity method**

$$\int_{\Omega_h} D_t \varrho_h^k \phi \mathrm{d}x - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \, [[\phi]] \, \mathrm{dS}_x = 0$$

**Momentum method**

$$\int_{\Omega_h} D_t \left(\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle\right) \cdot \boldsymbol{\phi} \mathrm{d}x - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle, \mathbf{u}_h^k] \cdot [[\langle \boldsymbol{\phi} \rangle]] \, \mathrm{dS}_x$$

$$- \int_{\Omega_h} p(\varrho_h^k) \mathrm{div}_h \boldsymbol{\phi} \mathrm{d}x$$

$$+\mu \int_{\Omega_h} \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\phi} \mathrm{d}x + \left(\frac{\mu}{3} + \eta\right) \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\phi} \mathrm{d}x = 0$$

# Convergence results for Karper's scheme

**Convergence to weak solutions**

**Karper [2013]:** Convergence to a weak solution if $\boxed{\gamma > 3}$

**Error estimates**

**Gallouet, Herbin, Maltese, Novotný [2015]**

Convergence to smooth solutions + error estimates if $\boxed{\gamma > 3/2}$, $\Omega$ a polyhedral domain

# Convergence for general adiabatic coefficient

**EF, M. Lukáčová/Medviďová [2016]**

Let $\Omega \subset R^3$ be a smooth bounded domain. Let

$$1 < \gamma < 2, \ \Delta t \approx h, \ 0 < \alpha < 2(\gamma - 1).$$

Suppose that the initial data are smooth and that the compressible Navier-Stokes system admits a smooth solution in $[0, T]$ in the class

$$\varrho, \ \nabla_x \varrho, \ \mathbf{u}, \nabla_x \mathbf{u} \in C([0, T] \times \overline{\Omega})$$

$$\partial_t \mathbf{u} \in L^2(0, T; C(\overline{\Omega}; R^3)), \ \varrho > 0, \ \mathbf{u}|_{\partial\Omega} = 0.$$

Then

$$\varrho_h \to \varrho \text{ (strongly) in } L^\gamma((0, T) \times K)$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2((0, T) \times K; R^3)$$

for any compact $K \subset \Omega$.

# General strategy

**Basic properties of numerical scheme**

Show stability, consistency, discrete energy inequality

**Measure valued solutions**

Show convergence of the scheme to a

**dissipative measure – valued solution**

**Weak-strong uniqueness**

Use the weak-strong uniqueness principle in the class of measure-valued solutions. Strong and measure valued solutions emanating from the same initial data coincide as long as the latter exists

# Measure-valued solutions

## Parameterized (Young) measure

$$\nu_{t,x} \in L^{\infty}_{\rm weak}((0,T) \times \Omega; \mathcal{P}([0,\infty) \times R^N), \ [s, \mathbf{v}] \in [0,\infty) \times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s \rangle , \ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v} \rangle \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$$

## Field equations revisited

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s \rangle \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t = \langle R_1; \nabla_x \varphi \rangle$$

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle \cdot \nabla_x \varphi + \langle \nu_{t,x}; p(s) \rangle {\rm div}_x \varphi \ {\rm d}x \ {\rm d}t$$

$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ {\rm d}x \ {\rm d}t + \langle R_2; \nabla_x \varphi \rangle$$

# Dissipativity

**Energy inequality**

$$\int_\Omega \left\langle \nu_{\tau,x}; \left( \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t + \boxed{\mathcal{D}(\tau)}$$

$$\leq \int_\Omega \left\langle \nu_0; \left( \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x$$

**Compatibility**

$$\left| R_1[0,\tau] \times \overline{\Omega} \right| + \left| R_2[0,\tau] \times \overline{\Omega} \right| \leq \xi(\tau)\mathcal{D}(\tau), \ \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \right\rangle \, \mathrm{d}x \, \mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

# Truly measure-valued solutions

**Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)**

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded $L^p$ weak solutions to the Euler system.

# Do we need measure valued solutions?

**Limits of problems with higher order viscosities**

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u}, \ \nabla_x^2 \mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} \left( (-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x \mathbf{u} \ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

**Limits of numerical solutions**

Numerical solutions resulting from Karlsen-Karper and other schemes

**Sub-critical parameters**

$p(\varrho) = a\varrho^\gamma$, $\gamma < \gamma_{\mathrm{critical}}$

# Weak (mv) - strong uniqueness

**Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]**

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

# Relative energy (entropy)

**Relative energy functional**

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \, \mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s |\mathbf{v}|^2 + P(s) \right\rangle \, \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; s \rangle |\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s \rangle P'(r) \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

# Relative energy (entropy) inequality

**Relative energy inequality**

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

# Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} [\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \mathbf{U}] \, \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega [\langle \nu_{t,x}; s \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[ \left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \left\langle R_1; \frac{1}{2} \nabla_x \left( |\mathbf{U}|^2 - P'(r) \right) \right\rangle \ \mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \, \mathrm{d}t$$

# Regularity

**Theorem - EF, P.Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann**

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\ \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \ \middle| \ 0 \leq s \leq \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.
Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x),\mathbf{u}(t,x)}$$

where $\varrho$, $\mathbf{u}$ is a smooth solution.

# Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

# Corollary

**Convergence of numerical solutions**

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution