Institute of Mathematics
Czech Academy of Sciences

Combinatorial homotopy theory for operads

Jovana Obradović

Category Theory 2019

Edinburgh, Scotland, July 7-13, 2019

Operads

Algebras

Operads	*As*
Algebras	associative algebras

$$(ab)c = a(bc)$$

Operads, algebras, resolutions & polytopes

Operads	*As*	A_∞-operad
Algebras	associative algebras	A_∞-algebras

$$\mu_{a,b,c} : (ab)c \to a(bc)$$

Operads	*As*	A_∞-operad	$\mathcal{O}$
Algebras	associative algebras	A_∞-algebras	operads

$$(a \circ_i b) \circ_{j+i-1} c = a \circ_i (b \circ_j c) \qquad (a \circ_i b) \circ_{j+k-1} c = (a \circ_j c) \circ_i b$$

Operads, algebras, resolutions & polytopes

Operads	*As*	A_∞-operad	$\mathcal{O}$	$\mathcal{O}_\infty$-operad
Algebras	associative algebras	A_∞-algebras	operads	strongly homotopy operads

$$\beta^{i,j}_{a,b,c} : (a \circ_i b) \circ_{j+i-1} c \to a \circ_i (b \circ_j c) \qquad \theta^{i,j}_{a,b,c} : (a \circ_i b) \circ_{j+k-1} c \to (a \circ_j c) \circ_i b$$

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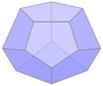

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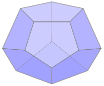

Goal: Provide a combinatorial description of the $\mathcal{O}_\infty$-operad.

Operadic polytopes $\subseteq$ Hypergraph polytopes a.k.a. Nestohedra

K. Došen, Z. Petrić
Hypergraph polytopes
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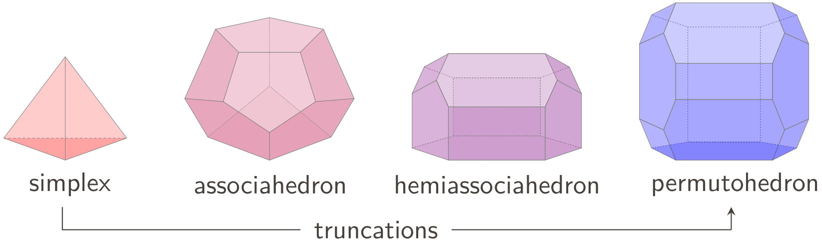

Hypergraph terminology

$\mathbf{H} = (H, \mathbf{H})$ H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \backslash \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

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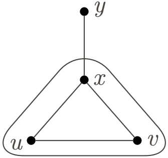

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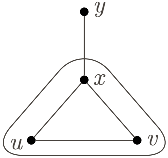

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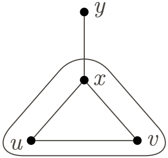

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$$Sat(\mathbf{H}) = \mathbf{H} \cup \{\{x, y, u\}, \{x, y, v\}, \{x, y, u, v\}\}$$

Hemiassociahedron

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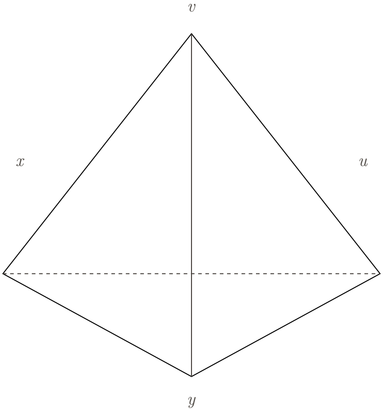

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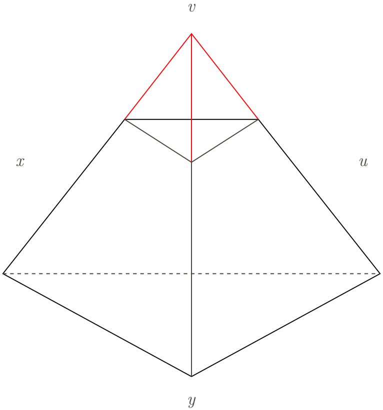

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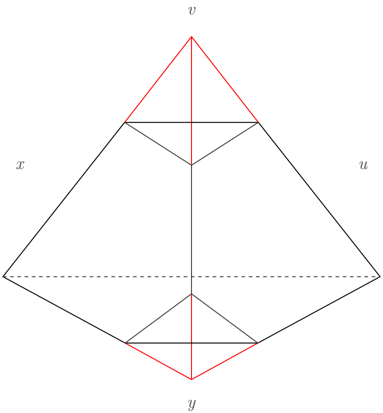

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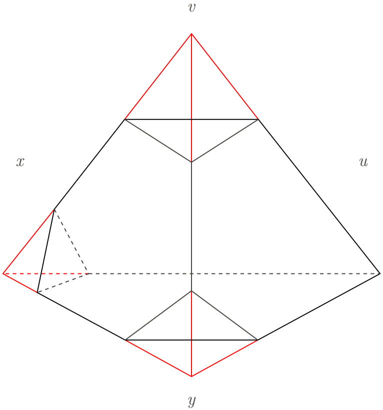

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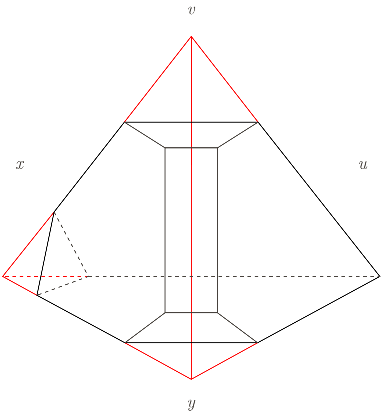

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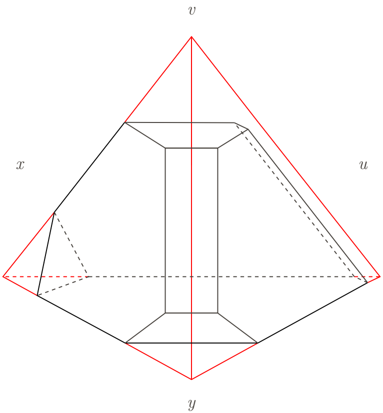

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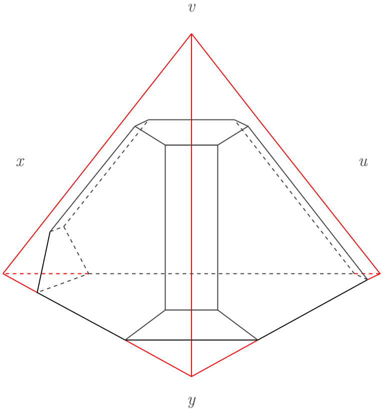

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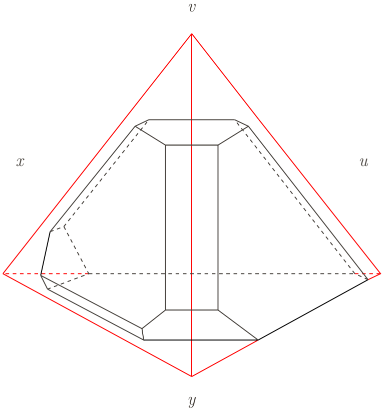

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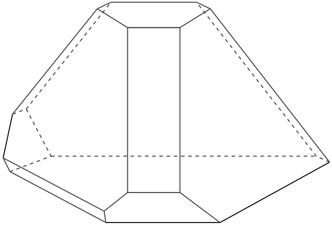

Combinatorial description of hypergraph polytopes

Poset $\mathcal{A}(\mathbf{H})$ of constructs of $\mathbf{H}$

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- Elements: constructs $C : \mathbf{H}$. Pick $\emptyset \neq Y \subseteq H$.
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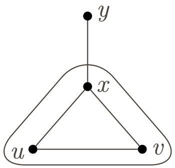

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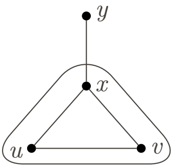

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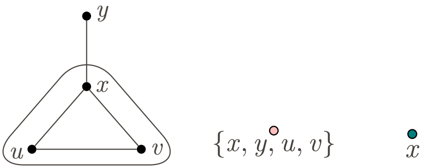

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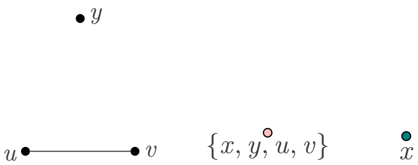

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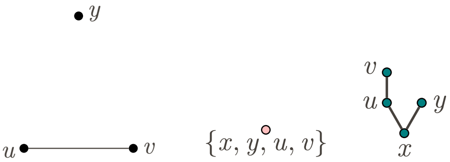

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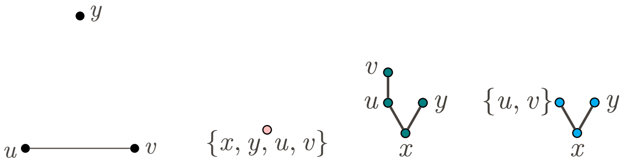

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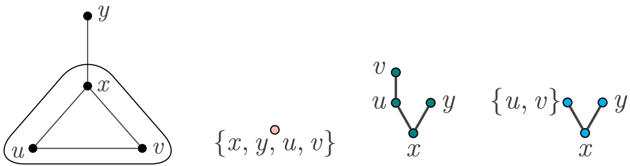

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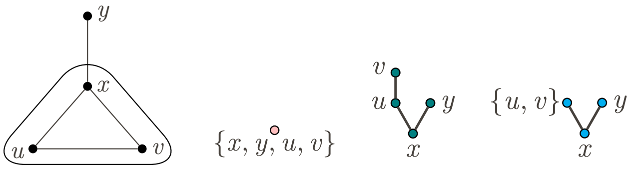
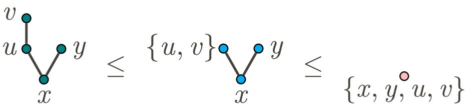


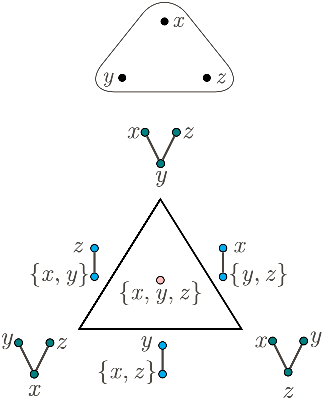
- Partial order: edge contraction



Hypergraph polytopes as truncated simplices

2-dim simplex

$$\mathbf{H} = \{\{x\},\{y\},\{z\},\{x,y,z\}\}$$



truncating $x\{y,z\}$

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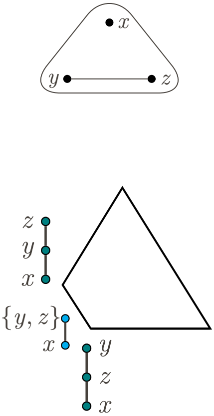
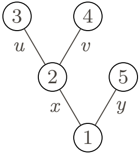


Operadic polytopes



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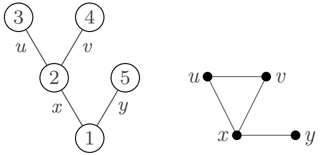

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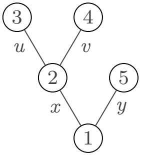
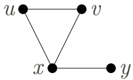
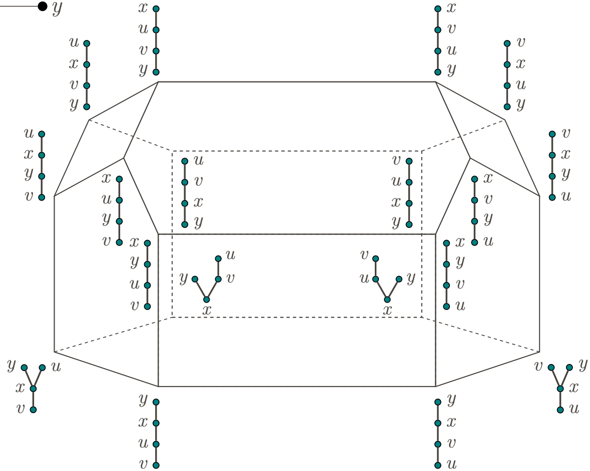


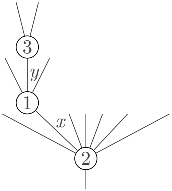


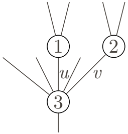


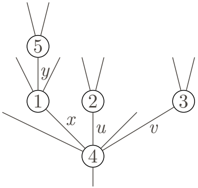
The $\mathcal{O}_\infty$-operad

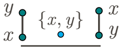
$$\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n) := \mathsf{Span}_{\Bbbk}\left(\bigoplus_{\mathcal{T} \in \mathcal{O}(n_1, \ldots, n_k; n)} \mathcal{A}(\mathbf{H}_{\mathcal{T}}) \right)$$

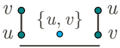
$$(\mathcal{T}_1, C_1) \circ_i (\mathcal{T}_2, C_2) = (\mathcal{T}_1 \bullet_i \mathcal{T}_2, C_1 \bullet_i C_2) \qquad d(\mathcal{T}, C) = \sum_V (\mathcal{T}, C[X\{Y\}/V])$$

$\mathcal{T}_1 =$ 

$\mathcal{T}_2 =$ 

$\mathcal{T}_1 \bullet_2 \mathcal{T}_2 =$ 



$\circ_2$


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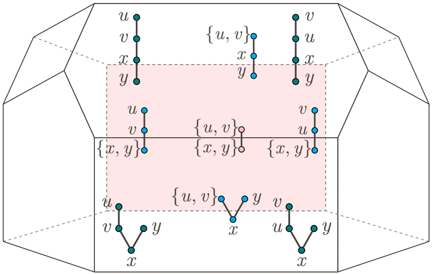

Minimal models for operads (Markl)

Theorem

The $\mathcal{O}_\infty$-operad is the minimal model of the $\mathbb{N}$-coloured operad $\mathcal{O}$, i.e.

- $\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}(E)$,
- there exists a quasi-isomorphism $\alpha_{\mathcal{O}} : \mathcal{O}_\infty \to \mathcal{O}$, and
- $d_{\mathcal{O}_\infty}(E) \subseteq \mathcal{T}_{\mathbb{N}}(E)^{(\geq 2)}$.

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Proof.

$$\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}\left(\bigoplus_{k\geq 2} \bigoplus_{n_1,\ldots,n_k\geq 1} \bigoplus_{\mathcal{T}\in\mathcal{O}(n_1,\ldots,n_k;n)} \mathbb{k} \right)$$

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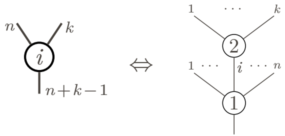

Thank you!

*This work has been supported by the Praemium Academiae of M. Markl and RVO:67985840.