Convergence of a new finite volume scheme for the Euler equations of gas dynamics

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based on joint work with
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Euler equations: *complete system*

- *multi-d compressible system* in $(0,T) \times \Omega$

$$
\begin{aligned}
\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) &= 0 \\
\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, \vartheta) &= 0 \\
\partial_t \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, \vartheta) \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, \vartheta) + p(\rho, \vartheta) \right) \mathbf{u} \right) &= 0
\end{aligned}
$$

- *perfect gas law*

$$
p = (\gamma - 1)\rho e = \rho \vartheta, \ \ \gamma > 1
$$

- *physical entropy production* (2^{nd} law of thermodynamics)

$$
\partial_t(\rho s) + \mathrm{div}_x \left(s \rho \mathbf{u} \right) \geq 0, \quad s = \log \left(\frac{\vartheta^{c_v}}{\rho} \right), \ c_v = \frac{1}{\gamma - 1}
$$

- *no–flux* boundary condition & initial condition

Euler equations: *complete system*

- *multi-d compressible system* in $(0,T)\times\Omega$ in conservative variables

$$\begin{aligned}\partial_t\rho+\mathrm{div}_x\mathbf{m}&=0\\ \partial_t\mathbf{m}+\mathrm{div}_x\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right)+\nabla_x p&=0\\ \partial_t E+\mathrm{div}_x\left((E+p)\frac{\mathbf{m}}{\rho}\right)&=0\end{aligned}$$

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$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{m} \\ E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \mathbf{m} \\ \frac{\mathbf{m}\otimes\mathbf{m}}{\rho} + p\mathbb{I} \\ (E+p)\,\frac{\mathbf{m}}{\rho} \end{pmatrix}$$

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Euler equations: *isentropic system*

- *multi-d compressible system* in $(0,T)\times\Omega$ in conservative variables

$$\partial_t \rho + \mathrm{div}_x \mathbf{m} = 0$$
$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho} \right) + \nabla_x p(\rho) = 0$$

- *isentropic* state equation; pressure potential

$$p(\rho) = a\rho^\gamma,\ a>0,\ \gamma>1; \quad P(\rho) = \frac{a}{\gamma-1}\rho^\gamma$$

- *total energy* inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + P(\rho) \right] \mathrm{d}x \le 0$$

- *space–periodic* boundary condition & initial condition

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Numerical analysis: *general strategy*

- suitable approximation scheme

  • *stability* • *consistency* • *convergence*

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Challenges in fluid mechanics:

- **lack of convergence** for particular initial data
  (Kevin-Helmholtz and Rayleigh-Taylor instabilities)
  - in multi-d compressible inviscid flows
    oscillations on fine scales persist
  - *lack of compactness* to show convergence in nonlinear terms

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- **link to non-uniqueness** of weak entropy solutions in multi-d
  [De Lellis, Székelyhidi 2009, 2010],
  ill–posedness for the isentropic and complete Euler equations
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- **hypothesis** that physical solutions are *not* in the class
  of integrable functions, i.e. weak solutions

New approach:

- take a concept of **Young measures** that can describe *limits of oscillating sequences*
- use a **dissipative measure–valued solution** to show *convergence of numerical solution* for equations of fluid dynamics
- use a concept of $\mathcal{K}$**(Komlós)-convergence** to show *convergence* of *arithmetic means* of numerical solutions to a **dissipative solution**

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Dissipative measure–valued solutions

$\equiv$

the most general concept of solution

(in which the classical solutions are uniquely determined by the data)

to obtain the convergence of a numerical scheme

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-> $\mathcal{V}_{t,x} : (t,x) \in (0,T) \times \Omega \mapsto \mathcal{P}(\mathcal{F})$ is weakly-(*) measurable

$\mathcal{F} = \left\{ \mathbf{U} = (\rho, \mathbf{m}, E) \;\middle|\; \rho \geq 0,\ \mathbf{m} \in \mathbb{R}^d,\ E \geq 0 \right\}$ $\mathcal{F} = \left\{ \mathbf{U} = (\rho, \mathbf{m}) \;\middle|\; \rho \geq 0,\ \mathbf{m} \in \mathbb{R}^d \right\}$

-> if $\|\mathbf{U}_h\|_{L^\infty((0,T)\times\Omega)} < \infty$ then

$G(\mathbf{U}_{h_k}) \to \{G(\mathbf{U})\}$ weakly-(*) in $L^\infty((0,T) \times \Omega)$, $G \in C_c(\mathcal{F})$ and

$$\{G(\mathbf{U})\} = \int_{\mathcal{F}} G(\mathbf{U}) \mathrm{d}\mathcal{V}_{t,x}(t,x) \equiv \langle \mathcal{V}_{t,x}; G(\mathbf{U}) \rangle \quad \text{a.a. } (t,x) \in (0,T) \times \Omega$$

[Ball 1989] FTYM for L^p-bounded sequences, $p \in (1, \infty)$

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- MVS assigns a probability distribution (likely value) for a.e. point in space-time
- **dissipative MVS** allows possible *concentration defect* $\mathbb{C}_d$ that is dominated by the *dissipation defect* $\mathcal{D} \geq 0$

$$\|\mathbb{C}_d\|_{\mathcal{M}} \leq \int_0^t \mathcal{D}(\tau), \qquad \text{for a.a. } t \in (0,T)$$

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► discrete entropy / energy inequality plays a crucial role!

3) *complete Euler*

employ **DMV-strong uniqueness principle**

• *strong convergence to strong solution*

isentropic Euler

employ **$\mathcal{K}$-convergence**

• *strong convergence of arithmetic means to dissipative solution*

► averaging for different mesh steps,
not averaging with respect to perturbed initial data[1]!

[1] Fjordholm, Käppeli, Mishra, Tadmor 2015–2016

complete Euler system

Dissipative measure–valued solutions

A family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega_h}$ is a DMVS to the *complete Euler system* if

i) *continuity equation*

$$\left[\int_{\Omega_h} \langle \mathcal{V}_{t,x}; \rho\rangle \varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_h} \left[\langle \mathcal{V}_{t,x}; \rho\rangle \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \nabla_x \varphi\right] \ \mathrm{d}x \, \mathrm{d}t,$$

for any $\varphi \in C^1([0,T] \times \overline{\Omega}_h)$;

ii) *momentum equation*

$$\left[\int_{\Omega_h} \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_{\Omega_h} \left[\langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p\rangle \mathrm{div}_x \boldsymbol{\varphi}\right] \ \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_{\overline{\Omega}_h} \nabla_x \varphi : \mathrm{d}\mathbb{C}_d,$$

for any $\boldsymbol{\varphi} \in C^1([0,T] \times \overline{\Omega}_h; \mathbb{R}^N)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\Omega_h} = 0$;

Dissipative measure–valued solutions

A family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega_h}$ is a DMVS to the *complete Euler system* if

iii) *energy inequality*

$$\int_{\Omega_h} \langle \mathcal{V}_{\tau,x}; E \rangle \ \mathrm{d}x \leq \int_{\Omega_h} E_0 \ \mathrm{d}x, \qquad \text{for a.a. } \tau \in [0,T];$$

iv) *(renormalized) entropy inequality*

$$\left[\int_{\Omega_h} \langle \mathcal{V}_{t,x}; \rho\chi(s) \rangle \, \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_{\Omega_h} \left[\langle \mathcal{V}_{t,x}; \rho\chi(s) \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \chi(s)\mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C^1([0,T] \times \overline{\Omega}_h)$, $\varphi \geq 0$, and any χ,

$\chi : \mathbb{R} \to \mathbb{R}$ a non–decreasing concave function, $\chi(s) \leq \overline{\chi}$ for all $s \in \mathbb{R}$;

v) *concentration defect vs. energy dissipation defect*

$$\int_0^\tau \int_{\overline{\Omega}_h} 1 \ \mathrm{d}|\mathbb{C}_d| \lesssim \int_{\Omega_h} E_0 \, \mathrm{d}x - \int_{\Omega_h} \langle \mathcal{V}_{\tau,x}; E \rangle \ \mathrm{d}x, \ \text{ for a.a. } \tau \in [0,T].$$

New finite volume scheme

Brenner's model: for *viscous heat conducting fluids*

two velocities approach: $\mathbf{v} = \mathbf{u} - K\nabla_x \log(\rho)$

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{v}) = 0$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{v}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\partial_t E + \mathrm{div}_x(E\mathbf{v}) + \mathrm{div}_x(p\mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathrm{div}_x\left(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u}\right), \; E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e,$$

where $\mathbb{S}(\nabla_x \mathbf{u}) = \eta_1 \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta_2 \mathrm{div}_x \mathbf{u}\mathbb{I}, \; \mathbf{q} = -\kappa \nabla_x \vartheta$

Brenner 2005–2006

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- [Guermond, Popov 2014]

Brenner's model is a **viscous regularization of the complete compressible Euler equations** that is **compatible with thermodynamics**:

- is entropy stable
- satisfies the minimum entropy principle
- preserves positivity of density and temperature

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For the ansatz

$$\mathbb{S}(\nabla_x\mathbf{u}) = h\lambda\rho\nabla_x\mathbf{u} + h^\alpha\nabla_x\mathbf{u},\ \kappa = c_v\rho K = c_v h\rho\lambda,\ \lambda \geq 0,\ h > 0,$$

Brenner's model rewrites in the conservative variables as

$$\partial_t\rho + \mathrm{div}_x(\mathbf{m}) = \lambda h\,\mathrm{div}_x(\nabla_x\rho)$$
$$\partial_t\mathbf{m} + \mathrm{div}_x(\mathbf{m}\otimes\mathbf{u}) + \nabla_x p = \lambda h\,\mathrm{div}_x(\nabla_x\mathbf{m}) + h^\alpha\Delta_x\mathbf{u}$$
$$\partial_t E + \mathrm{div}_x((E+p)\mathbf{u}) = \lambda h\,\mathrm{div}_x(\nabla_x E) + h^\alpha\mathrm{div}_x(\nabla_x\mathbf{u}\cdot\mathbf{u})$$

New finite volume scheme

$$\partial_t \rho + \mathrm{div}_x(\mathbf{m}) = \lambda h\, \mathrm{div}_x(\nabla_x \rho)$$

$$\partial_t \mathbf{m} + \mathrm{div}_x(\mathbf{m} \otimes \mathbf{u}) + \nabla_x p = \lambda h\, \mathrm{div}_x\, (\nabla_x \mathbf{m}) + h^\alpha \Delta_x \mathbf{u}$$

$$\partial_t E + \mathrm{div}_x((E+p)\mathbf{u}) = \lambda h\, \mathrm{div}_x(\nabla_x E) + h^\alpha \mathrm{div}_x(\nabla_x \mathbf{u} \cdot \mathbf{u})$$

motivates a finite volume scheme

$$\frac{d\rho_K(t)}{dt} + \frac{|\sigma|}{|K|} \sum_{\sigma \in \partial K} \mathcal{F}_h(\rho_h, \mathbf{u}_h) = 0$$

$$\frac{d\mathbf{m}_K(t)}{dt} + \frac{|\sigma|}{|K|} \sum_{\sigma \in \partial K} (\mathcal{F}_h(\mathbf{m}_h, \mathbf{u}_h) + \overline{p_h}\mathbf{n}) = h^{\alpha-1} \frac{|\sigma|}{|K|} \sum_{\sigma \in \partial K} [\![\mathbf{u}_h]\!]_\sigma$$

$$\frac{dE_K(t)}{dt} + \frac{|\sigma|}{|K|} \sum_{\sigma \in \partial K} \left(\mathcal{F}_h(E_h, \mathbf{u}_h) + \overline{p_h}\mathbf{u}_K \cdot \mathbf{n}_{\sigma,K} + \frac{p_K}{2} [\![\mathbf{u}_h]\!]_\sigma \cdot \mathbf{n}_{\sigma,K} \right)$$

$$= \frac{h^{\alpha-1}}{2} \frac{|\sigma|}{|K|} \sum_{\sigma \in \partial K} [\![\mathbf{u}_h^2]\!]_\sigma$$

$$\mathcal{F}_h(r, \mathbf{u}) = \mathrm{Up}[r, \mathbf{u}] - \mu_h [[r]] = \overline{r}\, \overline{u} \cdot \mathbf{n} - \frac{1}{2} |\overline{u} \cdot \mathbf{n}| [[r]] - \mu_h [[r]], \quad \mu_h \geq 0$$

Semi-discrete FV scheme: *complete Euler*

$$\frac{d\rho_K(t)}{dt} + \frac{|\sigma|}{|K|}\sum_{\sigma\in\partial K}\mathcal{F}_h(\rho_h,\mathbf{u}_h) = 0, \qquad K\in\mathcal{T}_h,\ t>0$$

$$\frac{d\mathbf{m}_K(t)}{dt} + \frac{|\sigma|}{|K|}\sum_{\sigma\in\partial K}\Big(\mathcal{F}_h(\mathbf{m}_h,\mathbf{u}_h) + \overline{p_h}\mathbf{n}\Big) = h^{\alpha-1}\frac{|\sigma|}{|K|}\sum_{\sigma\in\partial K}[\![\mathbf{u}_h]\!]_\sigma$$

$$\frac{dE_K(t)}{dt} + \frac{|\sigma|}{|K|}\sum_{\sigma\in\partial K}\left(\mathcal{F}_h(E_h,\mathbf{u}_h) + \overline{p_h}\mathbf{u}_K\cdot\mathbf{n}_{\sigma,K} + \frac{p_K}{2}[\![\mathbf{u}_h]\!]_\sigma\cdot\mathbf{n}_{\sigma,K}\right)$$

$$= \frac{h^{\alpha-1}}{2}\frac{|\sigma|}{|K|}\sum_{\sigma\in\partial K}[\![\mathbf{u}_h^2]\!]_\sigma$$

$$\mathbf{u}_h = \frac{\mathbf{m}_h}{\rho_h},\ p_h = (\gamma-1)\left(E_h - \frac{1}{2}\frac{|\mathbf{m}_h|^2}{\rho_h}\right) \quad \text{and} \quad \mathbf{U}_K(0) = (\Pi_h(\mathbf{U}^0))_K$$

Brenner-type numerical flux

$$\mathcal{F}_h(r,\mathbf{u}) = \mathrm{Up}[r,\mathbf{u}] - \mu_h[[r]] = \overline{r}\,\overline{u}\cdot\mathbf{n} - \frac{1}{2}|\overline{u}\cdot\mathbf{n}|[[r]] - \mu_h[[r]], \quad \mu_h\ge 0$$

Ω polyhedral domain in $\mathbb{R}^d$, $d=2,3$, $\mathcal{T}_h$ regular mesh, $\sigma = K|L$ cell-interface, K compact polygonal element, $\overline{r} = \frac{r_K+r_L}{2}$, $[[r]] = r_L - r_K$, no-flux boundary condition

Scheme properties: *complete Euler*

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- Entropy stability (discrete renormalized entropy inequality):

$$\frac{d}{dt}\int_{\Omega_h} \rho_h \chi(s_h)\Phi_h \ \mathrm{d}x - \sum_{\sigma\in\Sigma_{int}} \int_\sigma Up[\rho_h\chi(s_h),\mathbf{u}_h][[\Phi_h]]\,\mathrm{d}S_h = \sigma_h \geq 0$$

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- Minimum entropy principle: $s_h(t) \geq s_0,\ t \geq 0$

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- Minimum entropy principle: $s_h(t) \geq s_0, \ t \geq 0$
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$$h^{\alpha-1}\int_0^T \sum_{\sigma\in\Sigma_{int}} \int_\sigma [[\mathbf{u}_h]]^2 + \lambda_h[[\rho_h]]^2 + \lambda_h[[\vartheta_h]]^2\,\mathrm{d}S_h\,\mathrm{d}t \lesssim 1,$$

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- Existence of discrete solutions: unique global in time solution for $h > 0$

Convergence result: *complete Euler*

[Feireisl, Lukáčová, M. 2018b]

Assumptions:

- initial data: $\rho_{0,h} \geq \underline{\rho} > 0, \ E_{0,h} - \frac{1}{2}\frac{|\mathbf{m}_{0,h}|^2}{\rho_{0,h}} > 0$
- $\exists\, \underline{\rho}, \overline{\vartheta} > 0$ such that $0 < \underline{\rho} \leq \rho_h, \ \vartheta_h \leq \overline{\vartheta}$ uniformly for $h \to 0$
- scheme parameters satisfy

$$h^\beta \lesssim \mu_h \lesssim 1, \ 0 \leq \beta < 1, \quad 0 < \alpha < \frac{4}{3}$$

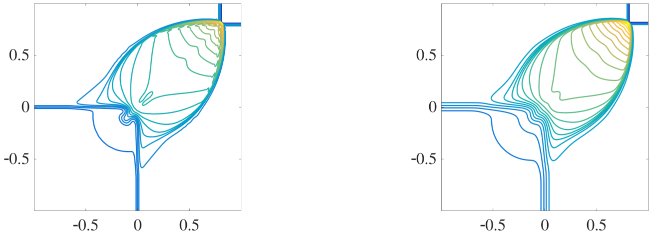
Result:

- *convergence to DMVS*
  the family of approximate solutions $\{\mathbf{U}_h\}_{h>0}$ up to a subsequence generates a Young measure representing a dissipative measure-valued solution of the complete Euler system
- *convergence to strong solution*[2]
  if the complete Euler possess L-continuous solution then

$$\mathbf{U}_h \to \mathbf{U} \text{ in } L^1((0,T)\times\Omega) \text{ as } h \to 0^+$$

[2]using *DMV-strong uniqueness* [Březina, Feireisl 2017]

- $\gamma = 1.4$, $\alpha = 1.5$: density ρ at $t = 0.25$



(a) $\beta = 0.5$ (b) $\beta = 0.2$

- $\gamma = 1.4$, $\alpha = 1.2$, $\beta = 0.2$

h	$\|\|e_\rho\|\|_{L^1}$	EOC	$\|\|e_p\|\|_{L^1}$	EOC	$\|\|e_m\|\|_{L^1}$	EOC	$\|\|e_E\|\|_{L^1}$	EOC
1/100	9.05e-03	–	8.59e-03	–	3.80e-02	–	8.61e-03	–
1/200	4.53e-03	1.00	4.29e-03	1.00	1.91e-02	0.99	4.30e-03	1.00
1/400	2.26e-03	1.00	2.14e-03	1.00	9.57e-03	1.00	2.15e-03	1.00
1/800	1.13e-03	1.00	1.07e-03	1.00	4.79e-03	1.00	1.07e-03	1.01
1/1600	5.66e-04	1.00	5.35e-04	1.00	2.39e-03	1.00	5.37e-04	0.99
1/3200	2.83e-04	1.00	2.68e-04	1.00	1.19e-03	1.01	2.69e-04	1.00

isentropic Euler system

Dissipative solutions and $\mathcal{K}$-convergence

- **dissipative solution** $\equiv$ barycenter (expected value) of a suitable DMVS to the isentropic Euler system with non-increasing total energy [Breit, Feireisl, Hofmanová 2019a]

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-> [Feireisl, Lukáčová, M. 2019]: extension to sequences of *numerical solutions* in the measure-valued framework

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{U}_{h_{n_k}} \text{ converges a.a. to a dissipative solution } \mathbf{U}$$

Implicit FV scheme: *isentropic Euler*

$$\mathrm{D}_{\Delta t}(\rho_K^k) + \frac{|\sigma|}{|K|}\sum_{\sigma\in\partial K}\mathcal{F}_h(\rho_h^k, \mathbf{u}_h^k) = 0, \qquad K \in \mathcal{T}_h,\ k = 1, \ldots, N_T$$

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and $\mathbf{U}_K^0 = (\Pi_h(\mathbf{U}^0))_K$

Brenner-type numerical flux

$$\mathcal{F}_h(r, \mathbf{u}) = \mathrm{Up}[r, \mathbf{u}] - \mu_h[[r]] = \overline{r}\,\overline{u}\cdot\mathbf{n} - \frac{1}{2}|\overline{u}\cdot\mathbf{n}|[[r]] - \mu_h[[r]], \quad \mu_h \geq 0$$

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- Existence of discrete solutions: [Hošek, She 2018]

I. Convergence result: *isentropic system*

[Feireisl, Lukáčová, M. 2019]

Assumptions:

- initial data: $0 < \rho_0 \in L^\gamma(\mathbb{T}^d)$, $E_0 = \int_{\mathbb{T}^d} \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + P(\rho_0)\,\mathrm{d}x < \infty$
- adiabatic coefficient is in the physical range $1 < \gamma < 2$
- scheme parameters satisfy

$$h^\beta \lesssim \mu_h \lesssim 1,\ 0 < \beta < 1, \quad 0 < \alpha < \left(2 - \frac{2}{\gamma}\right) - \frac{\beta}{\gamma}$$

Result:

- *strong convergence of arithmetic means to dissipative solution*

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{U}_{h_{n_k}} \to \mathbf{U} \text{ in } L^1((0,T)\times\Omega), \quad E_{h_{n_k}} \to E \text{ pointwise in } [0,T]$$

- *unconditional convergence*[3]

  if the complete Euler equations possess *weak solution* then

$$\mathbf{U}_{h_n} \to \mathbf{U} \text{ in } L^1((0,T)\times\Omega) \text{ as } n \to \infty$$

[3] using *DMV-strong uniqueness* [Feireisl, Ghoshal, Jana 2019]

* ≈ rarefaction waves emanating from Riemann data

II. Convergence result: *isentropic system*

[Feireisl, Lukáčová, M. 2019]

Assumptions:

- initial data[4]: $\rho_0 > 0$, $\rho_0, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d)$
- adiabatic coefficient is in the physical range $1 < \gamma < 2$
- scheme parameters satisfy

$$h^\beta \lesssim \mu_h \lesssim 1, \ 0 < \beta < 1, \quad 0 < \alpha < \left(2 - \frac{2}{\gamma}\right) \frac{\beta}{\gamma}$$

- additionally[5]

$$\sup_{\sigma \in \mathcal{E}} \frac{|[\![\rho_{h_n}]\!]_\sigma|}{h} + \sup_{\sigma \in \mathcal{E}} \frac{|[\![\mathbf{u}_{h_n}]\!]_\sigma|}{h} \leq L$$

Result:

- *strong convergence to strong solution*[6]

$$\mathbf{U}_{h_n} \to \mathbf{U} \text{ in } L^1((0,T) \times \mathbb{T}^d) \text{ as } n \to \infty$$

---

[4] [Benzoni, Gavage, Serre 2007], [Majda 1984]
[5] blow up criterion [Alinhac 1995]
[6] using *DMV-strong uniqueness* [Feireisl, Ghoshal, Jana 2019]

Negative result: *isentropic system*

[Feireisl, Lukáčová, M. 2019]

Assumptions:

- initial data: $0 < \rho_0 \in L^\gamma(\mathbb{T}^d)$, $E_0 = \int_{\mathbb{T}^d} \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\rho_0} + P(\rho_0)\, \mathrm{d}x < \infty$
- adiabatic coefficient is in the physical range $1 < \gamma < 2$
- scheme parameters satisfy

$$h^\beta \lesssim \mu_h \lesssim 1,\ 0 < \beta < 1, \quad 0 < \alpha < \left(2 - \frac{2}{\gamma}\right) - \frac{\beta}{\gamma}$$

Result:

- *absence of smooth solution*

  if there exists a function $g \in BC(\mathbb{R}^{d+1})$ such that

$$\frac{1}{N} \sum_{k=1}^{N} g(\mathbf{U}_{h_{n_k}}) \to G \neq g(\mathbf{U}) \text{ on a set of positive measure in } (0,T) \times \mathbb{T}^d$$

then the isentropic Euler *does not admit* a classical solution

Concluding remarks

- Cauchy problem for **compressible Euler equations** is in general ill-posed $\longrightarrow$ DMVS as the right concept of solution (?)

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complete Euler

  - convergence of FV scheme provided gas remains in its non-degenerate regime $\approx$ **no L^∞-bounds assumed!**

isentropic Euler

  - strong convergence of arithmetic means when limit solution possesses minimal regularity
  - possible evidence of singularities due to presence of oscillations in numerical solutions

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- FV scheme inspired by Brenner's two velocities approach is suitable for DMVS framework ⟶ **convergence of numerical solutions** (✓)

- DMVS framework substitutes linearity in the Lax equivalence theorem
  • *stability* • *consistency* • *convergence*

suitable numerical scheme → DMVS framework → convergence of numerical solutions

Thank you for your attention!