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# On solvability of inhomogeneous boundary-value problems in Sobolev spaces

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**We study**

the characteristics of solvability of systems of linear ordinary differential equations of arbitrary order on a finite interval with the most general (generic) inhomogeneous boundary conditions in Sobolev spaces. Boundary conditions can be both overdetermined and underdetermined.

Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m, n, r, l\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given.

**Linear boundary-value problem**

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c. \tag{2}$$

Here matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m \times m}$, vector-valued function $f(\cdot) \in (W_p^n)^m$, vector $c \in \mathbb{C}^l$, linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen; vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

If $l < r$, then the boundary conditions are underdetermined.
If $l > r$, then the boundary conditions are overdetermined.

The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (2) with operator (3) is generic condition for this equation.

It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) $y^{(k)}(\cdot)$ with $0 < k \leq n + r$.

Thus, boundary conditions can contain derivatives whose order is greater than the order of the equation.

In case $1 \leq p < \infty$, the linear continuous operator $B\colon (W_p^{n+r})^m \to \mathbb{C}^l$ admits the unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m. \tag{4}$$

Here, the matrices $\alpha_k \in \mathbb{C}^{l\times m}$, and the matrix-valued function $\Phi(\cdot) \in L_{p'}\left([a,b];\mathbb{C}^{l\times m}\right)$, $1/p + 1/p' = 1$.

For $p = \infty$ this formula also defines an operator $B\colon (W_\infty^{n+r})^m \to \mathbb{C}^l$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

## Complex Sobolev space $W_p^{n+r} := W_p^{n+r}\big([a,b];\mathbb{C}\big)$

$$W_p^{n+r}\big([a,b];\mathbb{C}\big) := \big\{y \in C^{n+r-1}[a,b] : y^{(n+r-1)} \in AC[a,b],\, y^{(n+r)} \in L_p[a,b]\big\}$$

This space is Banach one relative to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \|y^{(k)}\|_p + \|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is the norm in $L_p\big([a,b];\mathbb{C}\big)$.
By $\|\cdot\|_{n+r,p}$, we also denote the norms in Banach spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \quad \text{and} \quad (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They consist of the vector-valued functions and matrix-valued functions, respectively, all components of which belong to $W_p^{n+r}$.

With problem (1), (2), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{5}$$

A linear continuous operator $T\colon X \to Y$, where $X$ and $Y$ are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If this operator is Fredholm, then its range $T(X)$ is closed in $Y$ and the index is finite:

$$\operatorname{ind} T := \dim\ker T - \dim(Y/T(X)) \in \mathbb{Z}.$$

**Theorem 1.**

*The linear operator (5) is a bounded Fredholm operator with index $mr - l$.*

Family of matrix Cauchy problems with the initial conditions

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a,b),$$

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1, \dots, r\}.$$

By $[BY_k]$, we denote the numerical $m \times l$ matrix, in which $j$-th column is result of the action of $B$ on $j$-th column of $Y_k(\cdot)$.

**Definition 1.**

A block numerical matrix

$$M(L,B) := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{mr \times l} \qquad (6)$$

is characteristic matrix to problem (1), (2). It consists of $r$ rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m \times l}$.

**Theorem 2.**

*The dimensions of kernel and cokernel of the operator* (5) *are equal to the dimensions of kernel and cokernel of matrix* (6), *respectively:*

$$\dim\ker(L,B) = \dim\ker\big(M(L,B)\big),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}\big(M(L,B)\big).$$

**Corollary 1.**

*The operator* (5) *is invertible if and only if* $l = mr$ *and the square matrix* $M(L,B)$ *is nondegenerate.*

Consider problem (1), (2), where $r = 1$, putting $A(t) \equiv 0$ with the next boundary conditions:

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n)}(t) \mathrm{d}t, \quad y(\cdot) \in (W_p^n)^m.$$

Then we have

$$BY = \sum_{s=0}^{n-1} \alpha_s Y^{(s)}(a) + \int_a^b \Phi(t) Y^{(n)}(t) \mathrm{d}t, \quad Y(\cdot) = I_m,$$

$$M(L,B) = \alpha_0.$$

The numerical matrix $\alpha_0$ does not depend on $p$, $\alpha_1, \ldots, \alpha_{n-1}$, and $\Phi(\cdot)$. Thus, the statement of Theorem 2 holds:

$$\dim\ker(M(L,B)) = \dim\ker(\alpha_0),$$
$$\dim\operatorname{coker}(M(L,B)) = \dim\operatorname{coker}(\alpha_0).$$

**Boundary-value problems depending on the parameter $k \in \mathbb{N}$**

$$L(k)y(t,k) := y^{(r)}(t,k) + \sum_{j=1}^{r} A_{r-j}(t,k)y^{(r-j)}(t,k) = f(t,k), \quad t \in (a,b), \quad (7)$$

$$B(k)y(\cdot,k) = c(k), \quad k \in \mathbb{N}, \quad (8)$$

where $A_{r-j}(\cdot,k)$, $f(\cdot,k)$, $c(k)$, and linear continuous operator $B(k)$ satisfy the above conditions to problem (1), (2).

**The sequence of linear continuous operators**

$$(L(k),B(k))\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l,$$

**and characteristic matrices**

$$M\big(L(k),B(k)\big) := \big([B(k)Y_0(\cdot,k)],\ldots,[B(k)Y_{r-1}(\cdot,k)]\big) \subset \mathbb{C}^{mr\times l}.$$

Let's formulate a sufficient condition for convergence of the characteristic matrices.

**Theorem 3.**

If the sequence of operators $\big(L(k),B(k)\big)$ converges strongly to the operator $(L,B)$ then the sequence of characteristic matrices $M\big(L(k),B(k)\big)$ converges to the matrix $M(L,B)$ for $k \to \infty$.

From Theorem 3 follows sufficient conditions of semicontinuity from above the dimensions of the kernel and cokernel of the operator $(L, B)$.

**Corollary 2.**

*Under assumptions in Theorem 3, the following inequalities hold starting with sufficiently large $k$:*

$$\dim\ker(L(k), B(k)) \leq \dim\ker(L, B),$$
$$\dim\operatorname{coker}(L(k), B(k)) \leq \dim\operatorname{coker}(L, B).$$

The Corollary 2 implies the consequences of the stability of the invertibility of the sequence of operators $(L(k), B(k))$, the existence and uniqueness of the solution to problem (7), (8). In particular, for sufficiently large $k$, we have:

1) if $l = mr$ and operator $(L, B)$ is invertible, then the operators $(L(k), B(k))$ are also invertible;
2) if problem (1), (2) has a solution, then problems (7), (8) also have a solution;
3) if problem (1), (2) has a unique solution, then problems (7), (8) also have a unique solution [1, 3].

For each $k \to \infty$, we write the operator $B(k)$ in the form (4), where $\alpha_s = \alpha_s(k)$, $\Phi(t) = \Phi(t,k)$.
In the case of $1 \le p < \infty$, based on a unique analytic representation of the operator $B$ in (4), we formulate necessary and sufficient conditions that guarantees a strong convergence of the sequence of operators $(L(k), B(k))$ to the operator $(L, B)$.

**Theorem 4.**

Condition $(L(k), B(k)) \xrightarrow{s} (L, B)$ is equivalent to conditions:

1. $\|L(k) - L\| \to 0$;
2. $L(k)y \to Ly$ for each $y \in (W_p^{n+r})^m$;
3. $\alpha_s(k) \to \alpha_s$ in $\mathbb{C}^{l\times m}$ for each $s \in \{0, \ldots, n-1\}$;
4. $\|\Phi(\cdot,k)\|_q = O(1)$;
5. $\int_a^t \Phi(\tau,k)d\tau \to \int_a^t \Phi(\tau)d\tau$ в $\mathbb{C}^{l\times m}$ for each $t \in (a,b]$.

In the case of $1 \leq p < \infty$, we formulate necessary and sufficient conditions that guarantees the uniform convergence of the sequence of operators $(L(k), B(k))$ to the operator $(L, B)$.

**Theorem 5.**

Condition $\big\| (L(k), B(k)) - (L, B) \big\| \to 0$ is equivalent to conditions:

1. $\|L(k) - L\| \to 0$;
6. $\|\Phi(\cdot, k) - \Phi(\cdot)\|_q \to 0$.

The condition 6 is stronger than conditions 4 and 5.

$$L(k)y(t,k) := y'(t,k) + A(t,k)y(t,k) = f(t,k), \quad B(k)y(\cdot,k) = c(k). \qquad (9)$$

Denote by $Y(\cdot,k) \in (W_p^n)^{m\times m}$, respectively, the solution of the sequence of matrix differential equations

$$Y'(t,k) + A(t,k)Y(t,k) = 0, \quad t \in (a,b), \quad k \in \mathbb{N}, \quad Y(a,k) = I_m. \qquad (10)$$

Denote by $M(L(k),B(k)) := [B(k)Y(\cdot,k)] \in \mathbb{C}^{m\times l}$.
From (4), we have

$$B(k)Y = \sum_{s=0}^{n-1} \alpha_s(k)Y^{(s)}(a) + \int_a^b \Phi(t,k)Y^{(n)}(t)\mathrm{d}t. \qquad (11)$$

Suppose that for the problem (9) the conditions of the Theorem 4 are satisfied:

*a)* $\alpha_s(k) \to \alpha_s$ in $\mathbb{C}^{l\times m}$ for each $s \in \{0,\ldots,n-1\}$;

*b)* $\|\Phi(\cdot,k)\|_q = O(1)$;

*c)* $\int\limits_a^t \Phi(\tau,k)d\tau \to \int\limits_a^t \Phi(\tau)d\tau$ in $\mathbb{C}^{l\times m}$ for each $t \in (a,b]$.

Then we have a strong convergence of the sequence of operators $(L(k),B(k))$ to the operator $(L,B)$.
Then by the Theorem 3 we have the convergence of the sequence of characteristic matrices.
In (10), put $A(t,k)\to 0$, then $Y(t,k)\to I_m$. Substituting this value into equality (11), we have

$$M(L(k),B(k))\to \alpha_0.$$

Therefore, starting with some number $k$

$$\dim\ker\big(M(L(k),B(k))\big)\leq \dim\ker(\alpha_0),$$
$$\dim\operatorname{coker}\big(M(L(k),B(k))\big)\leq \dim\operatorname{coker}(\alpha_0).$$

In particular, if the numerical matrix $\alpha_0$ is square and nondegenerate, then starting from some number $k_0$ all boundary-value problems are well-posedness.

**This results can be used in:**

finding approximate solutions of complex boundary-value problems, reducing this problem to finding solutions of simpler multipoint boundary-value problems.

**Linear boundary-value problem**

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{12}$$

$$By = c, \tag{13}$$

where $B\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$.

**A sequence of multipoint boundary-value problems**

$$(L_k y_k)(t) := y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y_k^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{14}$$

$$B_k y_k := \sum_{j=0}^{N} \sum_{l=0}^{n+r-1} \beta_k^{(l,j)} y^{(l)}(t_{k,j}) = c. \tag{15}$$

**Theorem 6.**

In case $1 \leq p < \infty$, for the boundary-value problem (12), (13) there is a sequence of multipoint boundary-value problems of the form (14), (15) such that they are well-posedness for sufficiently large $k$ and the asymptotic property is fulfilled

$$y_k \to y \quad \text{in} \quad (W_p^{n+r})^m \quad \text{for} \quad k \to \infty.$$

The sequence can be chosen independently of $f$ and $c$, and constructed explicitly.

# Publication of results

1. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 10, 1526–1537. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-019-01588-w*
2. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems with parameter in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 11, 1677–1687. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-019-01599-7*
3. *Atlasiuk, O. M.; Mikhailets, V. A.* On solvability of inhomogeneous boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2019), no. 11, 3–7. *DOI: 10.15407/dopovidi2019.11.003*

4. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems in Sobolev spaces. J. Math. Sci. (Scopus, SJR Q3) 247 (2020), no. 2, 238–247. *DOI: 10.1007/s10958-020-04799-w*
5. *Atlasiuk, O. M.; Mikhailets, V. A.* On Fredholm parameter-dependent boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2020), no. 6, 3–6. *DOI: 10.15407/dopovidi2020.06.003*
6. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems with a parameter in Sobolev spaces. Ukrainian Math. J. 72 (2021), no. 8, 1175–1184. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-020-01859-x*


# Thank you for your attention!