Self-adjoint Sturm–Liouville operators with singular coefficients

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The main aim of the talk is

a presentation of some results on self-adjointness of semibounded symmetric in Hilbert space $L^2(\mathbb{R})$ Sturm–Liouville operators with strongly singular coefficients and their spectra.
The case of 1D-Schrödinger operators is studied in more detail.

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Content

1. Schrödinger operators.
2. Localization principals.
3. Operators with measure-valued potentials.
4. Hill–Schrödinger operators.
5. Sturm–Liouville operators.
6. References.

Let us consider a differential operator in Hilbert space $L^2(\mathbb{R})$

$$L_{00}\colon y(\cdot) \to -y''(x) + q(x)y(x),$$

where real-valued function $q \in L^2_{loc}(\mathbb{R})$, with a domain

$$Dom\,(L_{00}) = C^{\infty}_{comp}(\mathbb{R}).$$

It is densely defined and symmetric in $L^2(\mathbb{R})$.

The operator is important in quantum mechanics and mathematical physics problems.

At the same time, only those operators L_{00} that are essentially self-adjoint have a physical meaning. For this purpose, the function q must satisfy certain conditions. The simplest of them is given by

Theorem (H. Weyl, 1910)

If

$$q(x) \geq const, \quad x \in \mathbb{R},$$

then the operator L_{00} is semibounded below and essentially self-adjoint.

Carleman showed that this Theorem can be generalized to multidimensional Schrödinger operators

$$L_{00} = -\triangle + q(x), \quad x \in \mathbb{R}^n,$$

in Hilbert space $L^2(\mathbb{R}^n)$.

Povzner (1953) and later independently Wienholtz (1958) showed that if the function q is real-valued and *continuous*, and the operator L_{00} is semibounded, then it is essentially self-adjoint in space $L^2(\mathbb{R}^n)$.

Later, the conditions for the regularity of the potential q were significantly weakened.

Schrödinger operators with a *singular* potential q naturally arise in mathematical models of physical processes in strongly inhomogeneous media. In these models the potential is not a locally summable function, it is a Radon measure or a distribution.

In this connection, a question arises.

How to define the Schrödinger operator in this case?

Since the middle of 20th century, the study of this topic has been devoted to a large number of papers, mainly of a physical character, where various approaches are used.

In the one-dimensional case, the most natural and general approach is the interpretation of an *ordinary* differential expression with singular coefficients as quasi-differential in the Shin-Zettl sense.

Applied to the one-dimensional Schrödinger operator, it consists of the following.

Let a *formal* differential expression be given

$$ly = -y''(x) + Q'(x)y(x), \quad x \in \mathbb{R}, \tag{1}$$

where the function Q is real-valued and belongs to the class $L^2_{loc}(\mathbb{R})$, and the derivative is in the sense of the theory of distributions.

In particular, if the function Q has a locally bounded variation, then Q' is a Radon measure on $\mathbb{R}$.

We determine the quasi-derivatives of the function y by putting

$$D^{[0]}y := y,$$
$$D^{[1]}y := y' - Qy,$$
$$D^{[2]}y := -\left(D^{[1]}y\right)' - QD^{[1]}y - Q^2 y =: -ly.$$

If the function Q is smooth, then this definition is equivalent to the standard one. In general case

$$Dom(L) = \{y \in L^2(\mathbb{R}) |\, y, y^{[1]} \in AC_{loc}(\mathbb{R}), :\, D^{[2]}y \in L^2(\mathbb{R})\} \subset H^1_{loc}(\mathbb{R}).$$

We define the operator L_{00} as a restriction L to a set of functions with a compact support.

We can show that

- The set $Dom(L_{00})$ is dense in Hilbert space $L^2(\mathbb{R})$.
- The operator L_{00} is symmetric in $L^2(\mathbb{R})$.

We denote by L_0 the closure of the operator L_{00} in space $L^2(\mathbb{R})$.

Theorem 1.

If the symmetric operator L_0 is bounded below in space $L^2(\mathbb{R})$, then it is self-adjoint.

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Theorem 1.

If the symmetric operator L_0 is bounded below in space $L^2(\mathbb{R})$, then it is self-adjoint.

Let $t > 0$ is an arbitrary number. Put

$$\omega_n^t := (nt/2,\, nt/2 + t), \quad n \in \mathbb{Z}.$$

We denote by $\lambda(\omega_n^t)$ the smallest eigenvalue of the lower bounded self-adjoint operator in the Hilbert space $L^2(\omega_n^t)$ generated by the differential expression l and homogeneous Dirichlet boundary conditions.

The following two theorems generalize known results of Ismagilov (1961).

Theorem 2 (the first localization principal).

The minimal operator L_0 is bounded below and self-adjoint if and only if the sequence of numbers $(\lambda(\omega_n^t))_{n=-\infty}^{+\infty}$ is bounded below.

Theorem 3 (the second localization principal).

The operator L_0 is bounded below self-adjoint operator with discrete spectrum if and only if

$$(\star) \qquad \lambda(\omega_n^t) \to \infty \qquad \text{as} \qquad |n| \to \infty.$$

- It follows from Theorem 2 and 3 that if for a certain $t > 0$ the sequence $(\lambda(\omega_n^t))_{n=-\infty}^{n=\infty}$ is bounded below or satisfies $(\star)$, this sequence will have the same property for every $t > 0$.
- If Theorem 2 and 3 it is possible to replace all the intervals ω_n^l, where $n \in \mathbb{Z}$, with their shifts at an arbitrary chosen number $a \in \mathbb{R}$.
- Analogs of Theorem 2 and 3 are true in the case where the differential expression is given on a semiaxis.

From the point of view of physical applications, the case when the potential of Schrödinger operator is the Radon measure on $\mathbb{R}$ is of particular interest. For such operators, it is possible to give more constructive conditions of boundedness below and discreteness of the spectrum.

Let us consider Schrödinger operator

$$Lu = -u'' + Q'(x)u, \quad x \in \mathbb{R},$$

where the function $Q \in BV_{loc}(\mathbb{R})$.

We will additionally assume that the next assumption is fulfilled.

Assumption.

There exists a number $c > 0$ such that for every interval $\triangle \subset \mathbb{R}$ of length $|\triangle| \leq 1$

(B) $\int_{\triangle} dQ(x) \geq -c.$

Without reducing the generality, we will observe that $c \geq 2$.

Theorem 4.

Under the condition (B) operator L_0 is self-adjoint bounded below and

$$L_0 = L \geq -2c^2 I.$$

The following theorem gives necessary and sufficient conditions for the spectra of the minimal operators to be discrete.

Theorem 5.

Let the potential Q' satisfies the Condition (B). The spectrum of the operator L_0 is discrete if and only if the Molchanov condition is satisfied,

$$\lim_{|a|\to\infty} \int_a^{a+h} dQ(x) = +\infty,$$

for all $h > 0$.

This criterion generalizes Molchanov's, Brinc's, and the Albeverio–Kostenko–Malamud criteria.

Since the Schrödinger operator is bounded below when the condition (B) is fulfilled, the question arises about the description of the closed quadratic form t generated by this operator.
The answer to it is given by the following

Theorem 6.

Let the potential Q' satisfies the Condition (B). Then for closed quadratic form t of the operator L_0, we have

$$Dom(t) = \{u \in H^1(\mathbb{R}) | \exists \lim_{M,N\to\infty} \int_{-M}^{N} |u|^2 dQ \in \mathbb{R}\},$$

$$t[u] = \int_R |u'|^2 dx + \lim_{M,N\to\infty} \int_{-M}^{N} |u|^2 dQ,$$

$$u \in Dom(t).$$

Let's consider the Hill–Schrödinger differential expression

$$lu = -u'' + q(x)u, \quad x \in \mathbb{R},$$

where the potential is a real-valued 1-periodic generalized function $q \in H^{-1}_{loc}(\mathbb{R})$.
Its Fourier-Schwartz series has the form

$$q(x) = \sum_{k \in \mathbb{Z}} \hat{q}(k) \exp^{ik2\pi x},$$

where the coefficients of the series satisfy the conditions

$$\bar{\hat{q}}(k) = \hat{q}(-k), \quad k \in \mathbb{Z}, \quad \sum_{k \in \mathbb{Z}} \left(1 + 2|k|\right)^{-2} \left|\hat{q}(k)\right|^2 < \infty.$$

This differential expression is correctly defined as quasi-differential. In the Hilbert space $L^2(\mathbb{R})$, it defines a self-adjoint bounded below operator $L = L_0$, with

$$Dom(L) = \{u \in H^1(\mathbb{R}) \mid -u'' + q(x)u \in L^2(\mathbb{R})\},$$

where u'' and $q(x)u$ are understood in the sense of distributions.

The spectrum of the operator is purely absolutely continuous and has a band and gap structure. Its spectrum zones are interlaced with spectral gaps.

As in the case of the locally summable potential, the endpoints of the spectral gaps $\{\lambda_0^+(q),\, \lambda_n^\pm(q)\}_{n=1}^\infty$ satisfy the inequalities

$$-\infty < \lambda_0^+(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \ldots,$$

where $\{\lambda_0^+(q),\, \lambda_n^\pm(q)\}_{n=1}^\infty$, for even/odd n, are the eigenvalues of the corresponding periodic/antiperiodic boundary-value problems on the interval $[0,1]$.

By $\gamma_q(n) := \lambda_n^+(q) - \lambda_n^-(q)$, $n \in \mathbb{N}$, we denote the lengths of the spectral gaps of the operator L.
If $q \in L^2_{loc}(\mathbb{R})$, than as known

$$\gamma_q(n) \to 0, \quad n \to \infty,$$

and the rate of convergence increases along with the smoothness of the potential.

In the case of a singular potential, the situation changes qualitatively. The examples show that in this case the sequence $\{\gamma_q(n)\}_{n\in\mathbb{N}}$ can be unbounded.

In this connection, the question arises of finding the conditions under which the sequence $\{\gamma_q(n)\}_{n\in\mathbb{N}}$ belongs to the class l_∞ or c_0.

We denote by $s(q)$ the order of smoothness of the distribution in the Hilbert scale of Sobolev spaces on the circle $\mathbb{T}$:

$$s(q) := \sup\{s \in \mathbb{R} |\, q \in H^s(\mathbb{T})\} \geq -1.$$

As is known

$$q \in H^s(\mathbb{T}) \Leftrightarrow \sum_{k \in \mathbb{Z}} \left(1 + 2|k|\right)^{2s} |\hat{q}(k)|^2 < \infty.$$

Theorem 7.

Let the operator L have $s(q) \in [-1, -1/2)$. Then the sequence $\{\gamma_n(q)\}_{n \in \mathbb{N}}$ is unbounded.

Theorem 7 is supplemented by

Theorem 8.

Let the order of the singularity of the potential $s(q) \in (-1/2, 0]$. Then the sequence

$$\gamma_q(n) - 2|\hat{q}(n)| \in c_0.$$

In particular

(i) $\gamma_q(n) \in l_\infty \Leftrightarrow \{\hat{q}(n)\}_{n\in\mathbb{Z}} \in l_\infty$;

(ii) $\gamma_q(n) \in c_\infty \Leftrightarrow \{\hat{q}(n)\}_{n\in\mathbb{Z}} \in c_\infty$.

Corollary.

Let $\{n_k\}_{k\in\mathbb{N}}$ is an arbitrary sequence in $\mathbb{N}$ and $c \in [0, +\infty]$. Then if $s(q) \in (-1/2, 0]$, than

$$\gamma_q(n_k) \to 2c \Leftrightarrow |\hat{q}(n_k)| \to c, \quad k \to \infty.$$

In particular, there exist q, with $s(q) = 0$, and

$$\gamma_q(n) \notin l_\infty.$$

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The question of whether Theorem 7 is true when $s(q) = -\frac{1}{2}$ remains open.

This case is especially important for applications because it covers 1-periodic Radon measures on $\mathbb{R}$. It requires a separate study.

Theorem 9.

Let the potential q is Radon measure on $\mathbb{R}$. Then

$$\gamma_q(n) \in l_\infty.$$

Let's consider the operators that are generated by the differential expression

$$l[u] := -\left(pu'\right)' + qu + i((ru)' + ru'),$$

with real-valued coefficients $\{p, q, r\}$ given on $\mathbb{R}$.

If these coefficients are sufficiently regular, then the mapping

$$L_{00} : u \to l[u], \quad u \in C^{\infty}_{comp}(\mathbb{R})$$

defines a densely defined preminimal symmetric operator L_{00} in the Hilbert space $L^2(\mathbb{R})$.

Naturally, the question arises about the self-adjointness conditions of the closure of this operator L_0.

Hartman (1948) and Rellich (1951) showed that if the operator L_{00} is bounded below and the conditions are satisfied

$$r \equiv 0, \quad 0 < p \in C^2(\mathbb{R}),$$

q is piecewise continuous on $\mathbb{R}$ and function p satisfies the condition

$$\int_0^{\infty} p^{-1/2}(t)\mathrm{d}t = \int_{-\infty}^{0} p^{-1/2}(t)\mathrm{d}t = \infty, \tag{2}$$

then the minimal operator L_0 is self-adjoint.

As Stetkaer–Hansen (1966) showed, the conditions for the regularity of the coefficients can be weakened by replacing them with the following ones

$$r \equiv 0, \quad 0 < p \quad \text{is locally Lipshitz}, \quad q \in L^2_{loc}(\mathbb{R}).$$

Clark and Gesztesy (2003) obtained another sufficient condition for the self-adjointness of the operator L_0.
It has the form

$$\|p\|_{L^\infty(-\rho,-\rho/2)}, \quad \|p\|_{L^\infty(\rho/2,\,\rho)} = O(\rho^2), \quad \rho \to \infty. \qquad (3)$$

At the same time, it is assumed that

$$r \equiv 0, \quad 0 < p \in W^1_{2,loc}(\mathbb{R}).$$

The examples show that the conditions (2) and (3) are independent.

As Wienholtz (1958) showed, similar results are valid for multidimensional elliptic operators of the second order in space $L^2(\mathbb{R}^n)$.

The more general result on the self-adjointness of such operators with smooth complex-valued coefficients was obtained by Berezansky (1968).

He showed that such operator L_{00} given on $C^{\infty}_{comp}(\mathbb{R})$ is essentially self-adjoint in space $L^2(\mathbb{R}^n)$ under the condition of globally finite rate of propagation.
That is, if every solution of a hyperbolic differential equation

$$u_{tt} + Lu = 0,$$

which has a compact support for $t = 0$ has compact support for any $t > 0$. These results were further developed.

We introduce and investigate symmetric operators L_0 associated in the complex Hilbert space $L^2(\mathbb{R})$ with a formal differential expression

$$l[u] := -(pu')' + qu + i((ru)' + ru')$$

under minimal conditions on the regularity of the coefficients.

They are assumed to satisfy conditions

$$q = Q' + s, \quad \frac{1}{|\rho|^{1/2}}, \quad \frac{Q}{|\rho|^{1/2}}, \quad \frac{r}{|\rho|^{1/2}} \quad \in L^2_{loc}(\mathbb{R}), \tag{4}$$

$$s \in L^1_{loc}(\mathbb{R}), \quad 1/p \neq 0, \quad \text{a. e.},$$

where the derivative of function Q is understood in the sense of a theory of distributions, and all functions {p, Q, r, s} are real-valued.

In particular, the coefficients q and r' can be Radon measures on $\mathbb{R}$, while function p can be discontinuous.

These operators are defined using Shin–Zettl quasi-derivatives.

$$u^{[0]} := u, \quad u^{[1]} := pu' - (Q + ir)u, \tag{5}$$

$$u^{[2]} := \left(u^{[1]}\right)' + \frac{Q - it}{p} u^{[1]} + \left(\frac{Q^2 + r^2}{p} - s\right) u = -l[u].$$

$$Dom(l) := \{u \in L^2(\mathbb{R}) | \quad u, u^{[1]} \in AC_{loc}(\mathbb{R})\}.$$

This definition is motivated by the fact that

$$\langle -u^{[2]}, u\rangle \equiv \langle -(pu')' + qu + i\left((ru)' + ru'\right), u\rangle \quad u \in C^{\infty}_{comp}(\mathbb{R})$$

in the sense of a theory of distributions.

We define for the quasi-differential expression l the operators L and L_{00} as:

$$Lu := l[u], \quad Dom(L) := \{u \in L^2(\mathbb{R}) | \quad u, u^{[1]} \in AC_{loc}(\mathbb{R}), \quad l[u] \in L^2(\mathbb{R})\},$$

$$L_{00}u := Lu, \quad Dom(L_{00}) := \{u \in Dom(L) | \quad supp\, u \subset\subset \mathbb{R}\}.$$

Their definitions coincide with the classical ones if the coefficients l are sufficiently smooth.

One can prove that the operator L_{00} is densely defined in $L^2(\mathbb{R})$ and is symmetric.

Theorem 10.

Let the coefficients of the formal differential expression l satisfy the assumptions (4) and also

i) $p \in W^1_{2,loc}(\mathbb{R}), \quad p > 0,$

ii) $\int_{-\infty}^{0} p^{-1/2}(t)\mathrm{d}t = \int_0^{\infty} p^{-1/2}(t)\mathrm{d}t = \infty.$

Then, if operator L_{00} is bounder below, then it is essentially self-adjoint and

$$L_{00}^* = L.$$

In the next theorem, additional conditions on coefficient p are imposed not on the entire axis, but only on a sequence of finite interval. However, outside of these intervals the function p *can vanish and be discontinuous.*

Theorem 11.

Suppose the assumptions (4) satisfy and the operator L_{00} is bounded below. Suppose the sequence of intervals $\triangle_n := [a_n, b_n]$ exist such that

$$-\infty < a_n < b_n < \infty, \quad b_n \to -\infty, \quad n \to -\infty, \quad a_n \to \infty, \quad n \to \infty,$$

where the coefficients p satisfy the additional conditions

i) $p_n := p|\triangle_n \in W_2^1(\triangle_n), \quad p_n > 0,$

ii) $\exists c > 0 \colon p_n(x) \leq c|\triangle_n|^2, \quad n \in \mathbb{Z},$

where $|\triangle_n|$ is the length of interval $\triangle_n$.
Then semibounded operator L_{00} is essentially self-adjoint and

$$L_{00}^* = L.$$

1. Mikhailets, V.; Murach, A.; Novikov, V. Localizations principles for Schrödinger operator with a singular matrix potential. *Methods Funct. Anal. Topology.* 23 (2017), no. 4, 367–377.
2. Mikhailets, V.; Molyboga, V. Schrödinger operators with measure-valued potentials: semiboundness and spectrum. *Methods Funct. Anal. Topology.* 24 (2018), no. 3, 240–254.
3. Mikhailets, V.; Molyboga, V. Remarks on Schrödinger operators with singular matrix potentials. *Methods Funct. Anal. Appl.* 19 (2013), no. 2, 161–167.

4. Goriunov, A.; Mikhailets, V.; Molyboga, V. Povzner–Wienholtz–type theorems for Sturm–Liouville operators with singular coefficients. *arXiv.2110.11750* (2021).
5. Mikhailets, V.; Sobolev, A. Common eigenvalue problem and periodic Schrödinger operators. *J. Funct. Anal.* 165 (1999), no. 1, 150–172.
6. Mikhailets, V. A; Molyboga , V. M. On spectral gaps of the Hill—Schrödinger operator with singular potential. *Reports of the National Academy of Sciences of Ukraine* (2018), no. 10, 3–8.

Thank you for your attention!