INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

A proof of uniqueness of the Gurarii space

Wiesław Kubiś
Sławomir Solecki

Preprint No. 11-2011

PRAHA 2011

A proof of uniqueness of the Gurarii space *

Wiesław Kubiś † and Sławomir Solecki ‡

Abstract

We present a short and elementary proof of isometric uniqueness of the Gurarii space.

1 Introduction

A *Gurarii space*, constructed by Gurarii [3] in 1965, is a separable Banach space $\mathbb{G}$ satisfying the following condition: given finite-dimensional Banach spaces $X \subseteq Y$, given $\varepsilon > 0$, and given an isometric linear embedding $f\colon X \to \mathbb{G}$ there exists an injective linear operator $g\colon Y \to \mathbb{G}$ extending f and satisfying $\|g\| \cdot \|g^{-1}\| < 1+\varepsilon$. It is not hard to prove straight from this definition that such a space is unique up to isomorphism of norm arbitrarily close to one. The question whether the Gurarii space is unique up to isometry remained open for some time. It was answered affirmatively by Lusky [6] in 1976 using deep techniques developed by Lazar and Lindenstrauss [5]. Subsequently, another proof of uniqueness was given by Henson using model theoretic methods of continuous logic. (This proof remains unpublished.) The natural question whether there is an elementary proof of uniqueness occurred to several mathematicians. This question was made current by recent increased interest in universal, homogeneous structures and their automorphism groups; see, for example, [4] and [7]. The aim of this note is to provide just such a simple and elementary proof of isometric uniqueness of the Gurarii space. This proof is given in Section 2. In Section 3, we give an elementary argument showing isometric universality of the Gurarii space among separable Banach spaces.

In order to state the theorem precisely, we introduce some notions. Let X, Y be Banach spaces, $\varepsilon > 0$. A linear operator $f\colon X \to Y$ is an *ε-isometry* if

$$(1+\varepsilon)^{-1} \cdot \|x\| < \|f(x)\| < (1+\varepsilon) \cdot \|x\|.$$

* *2010 Mathematics Subject classification.* 46B04, 46B20. *Key words and phrases.* Gurarii space, isometry

†Research of Kubiś supported in part by the Grant IAA 100 190 901 and by the Institutional Research Plan of the Academy of Sciences of Czech Republic No. AVOZ 101 905 03.

‡Research of Solecki supported by NSF grant DMS-1001623.

holds for every $x \in X \setminus \{0\}$. We use strict inequalities for the sake of convenience. In particular, in the case of finite dimensional spaces, every ε-isometry is an ε'-isometry for some $0 < \varepsilon' < \varepsilon$. Note that the inverse of a bijective ε-isometry is again an ε-isometry. By an *isometry* we mean a linear operator $f \colon X \to Y$ that is an ε-isometry for every $\varepsilon > 0$, that is, $\|f(x)\| = \|x\|$ holds for every $x \in X$. (A word of caution about our terminology may be in place: in the literature, such functions are often called *isometric embeddings*, with the word "isometry" reserved for a *bijective* isometric embedding.)

We will give a proof of the following theorem.

Theorem 1.1. *Let E, F be separable Gurarii spaces, $0 < \varepsilon < 1$. Assume $X \subseteq E$ is a finite dimensional space and $f \colon X \to F$ is an ε-isometry. Then there exists a bijective isometry $h \colon E \to F$ such that $\|h \restriction X - f\| < 2\varepsilon$.*

By taking X to be the trivial space, we obtain the following corollary.

Corollary 1.2 (Lusky [6])**.** *The Gurarii space is unique up to a bijective isometry.*

2 Proof of uniqueness of the Gurarii space

Lemma 2.1. *Let X, Y be finite dimensional Banach spaces and let $f \colon X \to Y$ be an ε-isometry, where $0 < \varepsilon < 1$. Consider the algebraic sum $X \oplus Y$ and the canonical embeddings $i \colon X \to X \oplus Y$ and $j \colon Y \to X \oplus Y$. Then there exists a norm $\|\cdot\|$ on $X \oplus Y$ such that*

$$\|j \circ f - i\| < 2\varepsilon$$

and both i and j are isometries.

Proof. We will denote by $\|\cdot\|_X$, $\|\cdot\|_Y$ the norms of X and Y, respectively. Given $x^* \in S_X^*$, denote by $\overline{x}^*$ a fixed functional on Y satisfying $\overline{x}^* \circ f = x^*$ and

$$\|\overline{x}^*\|_Y^* = \|x^* f^{-1}\|_{f[X]}^*.$$

The existence of $\overline{x}^*$ is a direct consequence of Hahn-Banach's Theorem.

Now define

$$\varphi_X(x,y) = \sup_{x^* \in S_X^*} \left| x^*(x) + \frac{1}{\|\overline{x}^*\|_Y^*} \overline{x}^*(y) \right|.$$

It is clear that φ_X is a seminorm on $X \oplus Y$. Observe that $\varphi_X(x,0) = \|x\|_X$ and $\varphi_X(0,y) \leqslant \|y\|_Y$. Next, define

$$\varphi_Y(x,y) = \sup_{y^* \in S_Y^*} \left| \frac{1}{\|y^* f\|_X^*} y^* f(x) + y^*(y) \right|.$$

Again, φ_Y is a seminorm on $X \oplus Y$ such that $\varphi_Y(x,0) \leqslant \|x\|_X$ and $\varphi_Y(0,y) = \|y\|_Y$. Finally, define

$$\|(x,y)\| = \max\Big\{\varphi_X(x,y), \varphi_Y(x,y), \varepsilon\|x\|_X, \varepsilon\|y\|_Y\Big\}.$$

Now $\|\cdot\|$ is a norm on $X \oplus Y$ and, since $\varepsilon < 1$, we have that $\|(x,0)\| = \|x\|_X$ and $\|(0,y)\| = \|y\|_Y$. Hence, i and j are isometries with respect to $\|\cdot\|$. It remains to check that $\|jf(x) - i(x)\| < 2\varepsilon\|x\|_X$.

Fix $x \in S_X$ and let $u = jf(x) - i(x) = (-x, f(x)) \in X \oplus Y$. Note that, by compactness, in the definitions of φ_X, φ_Y the supremum can be replaced by the maximum. So fix $x^* \in S_X^*$ and $y^* \in S_Y^*$ such that

$$\varphi_X(u) = \left| x^*(-x) + \frac{1}{\|\overline{x}^*\|_Y^*}\overline{x}^* f(x)\right|$$

and

$$\varphi_Y(u) = \left|\frac{1}{\|y^* f\|_X^*} y^* f(-x) + y^* f(x)\right|.$$

Since $\overline{x}^* f(x) = x^*(x)$, we have

$$\varphi_X(u) = \left|\frac{1}{\|x^* f^{-1}\|_{f[X]}^*} - 1\right| \cdot |x^*(x)| = \left|\frac{1}{\|x^* f^{-1}\|_{f[X]}^*} - 1\right|.$$

Similarly,

$$\varphi_Y(u) = \left|1 - \frac{1}{\|y^* f\|_X^*}\right| \cdot |y^* f(x)| < (1+\varepsilon)\cdot\left|1 - \frac{1}{\|y^* f\|_X^*}\right|.$$

Now recall that both f and f^{-1} are ε-isometries and $\|x^*\|_X^* = 1 = \|y^*\|_Y^*$, therefore $(1+\varepsilon)^{-1} < \|x^* f^{-1}\|_{f[X]}^* < 1 + \varepsilon$ and $(1+\varepsilon)^{-1} < \|y^* f\|_X^* < 1+\varepsilon$. It follows that $\varphi_X(u) < \varepsilon$ and $\varphi_Y(u) < \varepsilon(1+\varepsilon) < 2\varepsilon$. Finally, since $\varepsilon\|x\|_X < \varepsilon$, we have that $\|u\| = \max\{\varphi_X(u), \varphi_Y(u)\} < 2\varepsilon$. This completes the proof. □

Lemma 2.2. *Let E be a Gurarii space and let $f\colon X \to Y$ be an ε-isometry, where X is a finite dimensional subspace of E and $0 < \varepsilon < 1$. Then for every $\delta > 0$ there exists a δ-isometry $g\colon Y \to E$ such that $\|gf(x) - x\| < 2\varepsilon\|x\|$ for every $x \in X$.*

Proof. Use Lemma 2.1 together with the definition of a Gurarii space. □

Proof of Theorem 1.1. Let $\{X_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional subspaces of E such that $X_0 = X$ and $\bigcup_{n\in\mathbb{N}} X_n$ is dense in E. Similarly, let $\{Y_n\}_{n\in\mathbb{N}}$ be a chain of finite dimensional subspaces of F such that $Y_0 = f[X]$ and $\bigcup_{n\in\mathbb{N}} Y_n$ is dense in F. Fix a strictly decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers. The precise conditions on $\{\varepsilon_n\}_{n\in\mathbb{N}}$ will be specified later. We define inductively two sequences of linear operators $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ so that the following conditions are satisfied.

(0) $X_0 = X$, $Y_0 = f[X]$, and $f_0 = f$;

(1) $f_n \colon X_{k_n} \to Y_{\ell_n}$ is an ε_{2n}-isometry and $k_n < \ell_n$;

(2) $g_n \colon Y_{\ell_n} \to X_{k_{n+1}}$ is an ε_{2n+1}-isometry and $\ell_n < k_{n+1}$;

(3) $\|g_n f_n(x) - x\| < 2\varepsilon_{2n}\|x\|$ for $x \in X_{k_n}$;

(4) $\|f_{n+1} g_n(y) - y\| < 2\varepsilon_{2n+1}\|y\|$ for $y \in Y_{\ell_n}$.

Condition (0) tells us how to start the inductive construction. Here we pick $\varepsilon_0 > 0$ so that (1) holds for $n = 0$ and $\varepsilon_0 < \varepsilon$. Suppose f_i, g_i have been constructed for $i < n$. We easily find f_n and g_n using Lemma 2.2. Thus, the construction can be carried out.
Fix $n \in \mathbb{N}$ and $x \in X_{k_n}$ with $\|x\| = 1$. Using (4), we get

$$\|f_{n+1} g_n f_n(x) - f_n(x)\| < 2\varepsilon_{2n+1}\|f_n(x)\| \leqslant 2\varepsilon_{2n+1}(1 + \varepsilon_{2n}) < 4\varepsilon_{2n+1}.$$

Using (3), we get

$$\|f_{n+1} g_n f_n(x) - f_{n+1}(x)\| \leqslant \|f_{n+1}\| \cdot \|g_n f_n(x) - x\| < (1 + \varepsilon_{2n+2}) \cdot 2\varepsilon_{2n} < 2(\varepsilon_{2n} + \varepsilon_{2n+2}).$$

These inequalities give

$$(\dagger) \qquad \|f_n(x) - f_{n+1}(x)\| < 2(\varepsilon_{2n} + 2\varepsilon_{2n+1} + \varepsilon_{2n+2}).$$

Now it is clear that if the series $\sum_{n\in\mathbb{N}} \varepsilon_n$ converges, then the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy. Let us make a stronger assumption, namely that

$$(\ddagger) \qquad 2(2\varepsilon_1 + \varepsilon_2) + \sum_{n=1}^{\infty} 2(\varepsilon_{2n} + 2\varepsilon_{2n+1} + \varepsilon_{2n+2}) < 2\varepsilon - 2\varepsilon_0.$$

Given $x \in \bigcup_{n\in\mathbb{N}} X_n$, define $h(x) = \lim_{n\geqslant m} f_n(x)$, where m is such that $x \in X_{k_m}$. Then h is an ε_n-isometry for every $n \in \mathbb{N}$, hence it is an isometry. Consequently, it uniquely extends to an isometry on E, which we denote also by h. Furthermore, ($\dagger$) and ($\ddagger$) give

$$\|f(x) - h(x)\| \leqslant \sum_{n=0}^{\infty} 2(\varepsilon_{2n} + 2\varepsilon_{2n+1} + \varepsilon_{2n+2}) < 2\varepsilon.$$

It remains to see that h is a bijection. To this end, we check as before that $\{g_n(y)\}_{n\geqslant m}$ is a Cauchy sequence for every $y \in Y_{\ell_m}$. Once this is done, we obtain an isometry g_∞ defined on F. Conditions (3) and (4) tell us that $g_\infty \circ h = \mathrm{id}_E$ and $h \circ g_\infty = \mathrm{id}_F$. This completes the proof. □

3 On universality of the Gurarii space

It is known that the Gurarii space is isometrically universal among separable Banach spaces. Indeed, as pointed out by Gevorkjan [2], universality follows from the results of Lazar and Lindenstrauss [5] and Michael and Pełczyński [8]: the dual of the Gurarii space is a non-separable L_1 space, therefore the Gurarii space contains an isometric copy of $C([0,1])$. The reader may also consult the recent paper [1] for another approach.

We conclude with applying our method to proving universality directly, without referring to the structure of the dual or to universality of other Banach spaces.

Lemma 3.1. *Let X_0, X_1, Y_0 be finite-dimensional Banach spaces such that $X_0 \subseteq X_1$ and let $f \colon X_0 \to Y_0$ be an ε-isometry, where $\varepsilon > 0$. Then there exist a finite-dimensional Banach space Y_1 containing Y_0 and an isometry $g \colon X_1 \to Y_1$ such that*

$$\|g \restriction X_0 - f\| < 2\varepsilon.$$

Proof. A standard and well known amalgamation property for Banach spaces says that there exist $W \supseteq Y_0$ and an ε-isometry $f' \colon X_1 \to W$ such that $f' \restriction X_0 = f$. More precisely, $W = (X_1 \oplus Y_0)/\Delta$, where $X_1 \oplus Y_0$ is endowed with the ℓ_1-norm and

$$\Delta = \{(z, -f(z)) \colon z \in X_0\}.$$

The space Y_0 is naturally identified with the subspace of W and $f'(x)$ is the equivalence class of $(x, 0)$ (where $x \in X_1$).

Finally, the desired isometry g is provided by Lemma 2.1. □

Theorem 3.2. *Every separable Banach space can be isometrically embedded into the Gurarii space.*

Proof. Let $\mathbb{G}$ denote the Gurarii space. Fix a separable Banach space X and let $\{X_n\}_{n \in \mathbb{N}}$ be a chain of finite-dimensional spaces such that $X_0 = \{0\}$ and $\bigcup_{n \in \mathbb{N}} X_n$ is dense in X. In case X is finite-dimensional, we set $X_n = X$ for $n > 0$. We inductively define $f_n \colon X_n \to \mathbb{G}$ so that

(i) f_n is a 2^{-n}-isometry,

(ii) $\|f_{n+1} \restriction X_n - f_n\| < 3 \cdot 2^{-n}$,

for every $n \in \mathbb{N}$. We set $f_0 = 0$. Suppose f_n has already been defined. Let $Y = f_n[X_n]$. Using Lemma 3.1, we find a finite-dimensional space $W \supseteq Y$ and an isometry $g \colon X_{n+1} \to W$ such that $\|g \restriction X_n - f_n\| < 2 \cdot 2^{-n}$. Using the property of the Gurarii space, we find a $2^{-(n+1)}$-isometry $h \colon W \to \mathbb{G}$ such that $h \restriction Y$ is the inclusion $Y \subseteq \mathbb{G}$. Now set $f_{n+1} := h \circ g$. Given $x \in X_n$ with $\|x\| = 1$, we have that $\|g(x) - f_n(x)\| < 2 \cdot 2^{-n}$ and hence

$$\|f_{n+1}(x) - f_n(x)\| = \|h(g(x)) - h(f_n(x))\| < (1 + 2^{-(n+1)}) \cdot 2 \cdot 2^{-n} \leqslant 3 \cdot 2^{-n}.$$

This shows (ii). Finally, we obtain a sequence $\{f_n\}_{n\in\mathbb{N}}$ that is pointwise Cauchy on each X_n. By (i) and (ii), $f_\infty(x) := \lim_{n\to\infty} f_n(x)$ is a well-defined linear isometry on $\bigcup_{n\in\mathbb{N}} X_n$. This isometry extends uniquely to an isometry $f\colon X \to \mathbb{G}$. □

References

[1] Avilés, A., Cabello Sánchez, F.C., Castillo, J.M.F., González, M., Moreno, Y., *Banach spaces of universal disposition*, J. Funct. Anal. **261** (2011), 2347–2361

[2] Gevorkjan, Ju.L., *Universality of the spaces of almost universal placement*, Funkcional. Anal. i Priložen **8** (1974), 72 (in Russian); Functional Anal. Appl. **8** (1974), 157 (in English)

[3] Gurariĭ, V.I., *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Sibirsk. Mat. Ž. **7** (1966), 1002–1013 (in Russian)

[4] Kechris, A. S., Pestov, V. G., Todorcevic, S., *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), 106–189

[5] Lazar, A.J., Lindenstrauss, J., *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. **126** (1971), 165–193

[6] Lusky, W., *The Gurarij spaces are unique*, Arch. Math. (Basel) **27** (1976), 627–635

[7] Melleray, J., *Some geometric and dynamical properties of the Urysohn space*, Topology Appl. **155** (2008), 1531–1560

[8] Michael, E., Pełczyński, A., *Separable Banach spaces which admit l_n^∞ approximations*, Israel J. Math. **4** (1966), 189–198

Kubiś's address:
Mathematical Institute, Academy of Sciences of the Czech Republic, Prague, Czech Republic
Institute of Mathematics, Jan Kochanowski University, Kielce, Poland
kubis@math.cas.cz, wkubis@pu.kielce.pl

Solecki's address:
Department of Mathematics, University of Illinois, Urbana, Illinois 61801, USA
Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland
ssolecki@math.uiuc.edu