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A NOTE ON MAXIMAL COMMUTATORS AND COMMUTATORS OF MAXIMAL FUNCTIONS

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Abstract. In this paper maximal commutators and commutators of maximal functions with functions of bounded mean oscillation are investigated. New pointwise estimates for them are proved.

1. Introduction

In 1976, Coifman, Rochberg and Weiss [5] studied the L_p boundedness of the commutator $[b, T]$ generated by the Calderón-Zygmund singular integral operator T and a function b, where $[b, T]$ is defined by

$$[b, T](f)(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)[b(x) - b(y)]f(y)dy, \tag{1.1}$$

with a Calderón-Zygmund kernel K and $\mathrm{BMO}(\mathbb{R}^n)$ function b. Using the L_p boundedness of commutator $[T, b]$, Coifman, Rochberg and Weiss successfully gave a decomposition of Hardy space $H^1(\mathbb{R}^n)$.

The commutator defined by (1.1) is called CRW-type commutator, and CRW-type commutator plays an important role in studying the regularity of solution of partial differential equations.

For the Hilbert transform H, and other classical singular integral operators, a well known and important result due to Coifman, Rochberg and Weiss (cf. [5]) states that a locally integrable function b in $\mathbb{R}^n$ is in BMO if and only if the commutator is bounded in $L_p(\mathbb{R}^n)$, for some (and for all) $p \in (1, \infty)$.

It was shown (see [7], [1], [11], for instance) that so-called maximal commutator

$$C_b(f)(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|dy.$$

plays an important role in the study of commutators of singular integral operators with BMO symbols. The operator C_b have been investigated in [15] and [16]. Garcia-Cuerva et al. [7] proved that C_b is bounded in $L_p(\mathbb{R}^n)$ for any $p \in (1, \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$,

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and Alphonse [1] proved that C_b enjoys weak-type $L(\log L)$ estimate. In same papers this operator named also maximal commutator. If b is non-negative, the operator C_b controls commutator of maximal function.

The commutator of maximal function was studied by Milman et al. in [12] and [2]. Using real interpolation techniques, in [12], Milman and Schonbek proved a commutator result that applies to the Hardy-Littlewood maximal operator M. In [2], Bastero, Milman and Ruiz proved that the commutator of the maximal operator with locally integrable function b is bounded in L_p if and only if b is in BMO with bounded negative part. As we know only these two papers are devoted to the problem of boundedness of the commutator of maximal function in Lebesgue spaces. The maximal operator C_b was studied intensively and exists plenty of results about it. We see that results for C_b and $[M,b]$ is not same.

In this paper maximal commutators and commutators of maximal functions with functions of bounded mean oscillation are investigated. New pointwise estimates for them are proved. By the way, new results for Lebesgue spaces are obtained. For example, we give weak type estimate for commutator of maximal function which is new, as we know.

The paper is organized as follows. We start with notations and give some preliminaries in Section 2. In Section 3 we present new pointwise estimate for maximal commutator. In Section 4 we give new proof of results by Garcia-Cuerva et al. from [7] and Alphonse from [1]. In latter case we also show necessity. Note that in [1] only sufficient part was proved. In Section 5 we obtain pointwise estimate of commutators of maximal operator by iterated maximal function. Using this estimate we obtain new proof of the result by M. Milman and T. Schonbek from [12]. Weak-type inequalities were also proved by means of this estimation.

2. Notations and Preliminaries

Let $0 \le \alpha < n$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. Denote by M_α fractional maximal operator:

$$M_\alpha f(x) := \sup_{x\in Q} |Q|^{\frac{\alpha-n}{n}} \int_Q |f|.$$

Definition 2.1. For the Hardy-Littlewood maximal operator M and a locally integrable function b, we define a commutator of maximal function by

$$[M,b]f(x) = M(bf)(x) - b(x)Mf(x).$$

For the sake of completeness we recall the definition of the spaces and some properties of the spaces we are going to use.

The non-increasing rearrangement (see, e.g., [4, p. 39]) of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x\in\mathbb{R}^n : |f(x)| > \lambda\}| \le t\} \quad (0 < t < \infty).$$

Let $p \in [1,\infty)$. The Lorentz space $L^{p,\infty}$ is defined as

$$L^{p,\infty}(\mathbb{R}^n) := \left\{ f : \quad f \text{ measurable on } \mathbb{R}^n; \quad \|f\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t) < \infty \right\}.$$

Definition 2.2. Let $g \in L_1^{\text{loc}}(\mathbb{R}^n)$. Then g is said to have bounded mean oscillation ($g \in \text{BMO}$) if the seminorm given by

$$\|g\|_* := \sup_B \frac{1}{B} \int_B |g(y) - g_B| dy \tag{2.1}$$

is finite. Here, B denotes any ball of $\mathbb{R}^n$ and g_B is the average of g over B.

The most important result regarding BMO is the following Theorem of F. John and L. Nirenberg (see [6], p.164, for instance).

Lemma 2.3 ([10]). *There exists constants C_1, C_2 depending only on the dimension n, such that for every $f \in \text{BMO} = \text{BMO}(\mathbb{R}^n)$, every cube Q and every $t > 0$:*

$$|\{x \in Q : |f(x) - f_Q| > t\}| \le C_1 |Q| \exp\left\{-\frac{C_2}{\|f\|_*} t\right\}. \tag{2.2}$$

Lemma 2.4 ([10] and [3]). *For $p \in (0, \infty)$, $\text{BMO}(p) = \text{BMO}$, with equivalent norms, where*

$$\|f\|_{\text{BMO}(p)} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^p dy\right)^{\frac{1}{p}}.$$

Lemma 2.5 ([6], p. 166). *Let $f \in \text{BMO}$ and $p \in (0, \infty)$. Then for every λ such that $0 < \lambda < C_2/\|f\|_*$, where C_2 is the same constant appearing in* (2.2), *we have*

$$\sup_Q \frac{1}{|Q|} \int_Q \exp\{\lambda |f(x) - f_Q|\} dx < \infty.$$

Denote by $b^+(x) = \max\{b(x), 0\}$ and $b^-(x) = -\min\{b(x), 0\}$, consequently $b = b^+ - b^-$ and $|b| = b^+ + b^-$.

The boundedness of the commutator for the Hardy-Littlewood maximal operator investigated in [12] and [2].

Theorem 2.6. ([12], Theorem 4.6) *If $b \ge 0$ is in* BMO, *then*

$$\|[M, b]f\|_{L_p(\mathbb{R}^n)} \le c\|b\|_* \|f\|_{L_p(\mathbb{R}^n)}, \quad 1 < p < \infty.$$

Theorem 2.7. ([2], Proposition 4) *Let $1 < p < \infty$ and b be a real valued, locally integrable function in $\mathbb{R}^n$. Then the commutator $[M, b]$ is bounded in $L_p(\mathbb{R}^n)$ if and only if b is a* $\text{BMO}(\mathbb{R}^n)$ *function such that its negative part b^- is bounded.*

Let Ω be a cube of $\mathbb{R}^n$. A continuously increasing function on $[0, \infty]$, say $\Psi : [0, \infty] \to [0, \infty]$ such that $\Psi(0) = 0$, $\Psi(1) = 1$ and $\Psi(\infty) = \infty$, will be referred to as an Orlicz function. If Ψ is a Orlicz function, then

$$\Phi(t) = \sup\{ts - \Psi(s); s \in [0, \infty]\}$$

is the complementary Orlicz function to Ψ.

The generalized Orlicz space denoted by $L^\Psi(\Omega)$ consists of all functions $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_\Omega \Psi\left(\frac{|g|}{\alpha}\right)(x) dx < \infty$$

for some $\alpha > 0$.

Let us define the Ψ-average of g over a cube Q contained in Ω by

$$\|g\|_{\Psi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Psi\left(\frac{|g(y)|}{\alpha}\right) dy \leq 1\right\}.$$

When A is a Young function, i.e. a convex Orlicz function, the quantity

$$\|f\|_{\Psi} = \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|f(y)|}{\alpha}\right) dy \leq 1\right\}$$

is well known Luxemburg norm in the space $L^{\Psi}(\Omega)$ (see [14]).

We define the weak $L(1+\log^+ L)$-average of g over a cube Q contained in Ω analogously by

$$\|g\|_{WL(1+\log^+ L),Q} = \inf\left\{\alpha > 0 : \sup_{t>0}\frac{1}{|Q|}\frac{|\{x \in Q : |g(x)| > \alpha t\}|}{\frac{1}{t}\left(1+\log^+\frac{1}{t}\right)} \leq 1\right\}.$$

If $f \in L^{\Psi}(\mathbb{R}^n)$, the maximal function of f with respect to Ψ is defined by setting

$$M_{\Psi} f(x) = \sup_{x\in Q}\|f\|_{\Psi,Q},$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$ containing x with sides parallel to the coordinate axes.

The generalized Hölder's inequality

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leq \|f\|_{\Phi,Q}\|g\|_{\Psi,Q}, \tag{2.3}$$

where Ψ is the complementary Young function associated to Φ, holds.

The main example that we are going to be using is $\Phi(t) = t(1+\log^+ t)$ with maximal function defined by $M_{L(\log L)}$. The complementary Young function is given by $\Psi(t) \approx e^t$ with the corresponding maximal function denoted by $M_{\exp L}$.

Recall the definition of quasinorm of Zygmund space:

$$\|f\|_{L(1+\log^+ L)} := \int_{\mathbb{R}^n} |f(x)|(1+\log^+|f(x)|)dx$$

The size of M^2 is given by the following.

Lemma 2.8 ([13], Lemma 1.6)**.** *There exists a positive constant C such that for any function f and for all $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : M^2 f(x) > \lambda\}| \leq C\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)dx. \tag{2.4}$$

Now we make some conventions. Throughout the paper, we always denote by c and C a positive constant which is independent of main parameters, but it may vary from line to line. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Constant, with subscript such as C_1, does not change in different occurrences.

For a measurable set E, χ_E denotes the characteristic function of E. Given $\lambda > 0$ and a cube Q, λQ denotes the cube with the same center as Q and whose side is λ times that of Q. For a fixed p with $p \in [1, \infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For any measurable set E and any integrable function f on E, we denote by f_Q the mean value of f over E, that is, $f_Q = (1/|Q|) \int_E f(x)dx$.

3. Pointwise estimates for maximal commutator

For $\delta > 0$ and $f \in L_\delta^{\rm loc}(\mathbb{R}^n)$ denote by

$$M_\delta f(x) := \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(y)|^\delta dy \right)^{\frac{1}{\delta}}.$$

The following Theorem is true.

Theorem 3.1. *Let* $b \in {\rm BMO}$ *and let* $0 < \delta < 1$. *Then, there exists a positive constant* $C = C_\delta$ *such that*

$$M_\delta(C_b(f))(x) \le C\|b\|_* M^2 f(x) \tag{3.1}$$

for all functions from $L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and fix a cube $x \in Q$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{3Q}$. Since for any $y \in \mathbb{R}^n$

$$\begin{aligned} C_b(f)(y) &= M((b - b(y))f)(y) = M((b - b_{3Q} + b_{3Q} - b(y))f)(y) \\ &\le M((b - b_{3Q})f_1)(y) + M((b - b_{3Q})f_2)(y) + |b(y) - b_{3Q}| Mf(y), \end{aligned}$$

we have

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q (C_b(f)(y))^\delta dy \right)^{\frac{1}{\delta}} &\lesssim \left(\frac{1}{|Q|} \int_Q |M((b - b_{3Q})f_1)(y)|^\delta dy \right)^{\frac{1}{\delta}} \\ &+ \left(\frac{1}{|Q|} \int_Q |M((b - b_{3Q})f_2)(y)|^\delta dy \right)^{\frac{1}{\delta}} \\ &+ \left(\frac{1}{|Q|} \int_Q ||b(y) - b_{3Q}|^\delta (Mf(y))^\delta dy \right)^{\frac{1}{\delta}} = {\rm I} + {\rm II} + {\rm III}. \end{aligned} \tag{3.2}$$

Since

$$\begin{aligned} \int_Q |M((b - b_{3Q})f_1)(y)|^\delta dy &= \int_0^{|Q|} \left[(M((b - b_{3Q})f_1))^* (t) \right]^\delta dt \\ &\le \left[\sup_{0<t<|Q|} t \left(M((b - b_{3Q})f_1) \right)^* (t) \right]^\delta \int_0^{|Q|} t^{-\delta} dt, \end{aligned}$$

using the boundedness of M from $L_1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ we have

$$\int_Q |M((b - b_{3Q})f_1)(y)|^\delta dy \lesssim \|(b - b_{3Q})f_1\|_{L_1(\mathbb{R}^n)}^\delta |Q|^{-\delta+1}$$

$$= \|(b-b_{3Q})f\|_{L_1(3Q)}^{\delta}|Q|^{-\delta+1}.$$

Thus

$$\mathrm{I} \lesssim \frac{1}{|Q|}\int_{3Q}|b(y)-b_{3Q}||f(y)|dy.$$

By generalized Hölder's inequality (2.3), we get

$$\mathrm{I} \lesssim \|b-b_{3Q}\|_{\exp L,3Q}\|f\|_{L\log L,3Q}.$$

Since by Lemma 2.5, there is a constant $C>0$ such that for any cube Q,

$$\|b-b_Q\|_{\exp L,Q} \le C\|b\|_*,$$

we arrive at

$$\mathrm{I} \lesssim \|b\|_* M_{L\log L}f(x). \tag{3.3}$$

Let us estimate II. Since II is comparable to $\inf_{y\in Q} M((b-b_{3Q})f)(y)$ (see [6], p. 160, for instance), then

$$\mathrm{II} \lesssim M((b-b_{3Q})f)(x).$$

Again by generalized Hölder's inequality and Lemma 2.5, we get

$$\mathrm{II} \lesssim \sup_{x\in Q}\|b-b_{3Q}\|_{\exp L,3Q}\|f\|_{L\log L,3Q} \lesssim \|b\|_* M_{L\log L}f(x). \tag{3.4}$$

Let $\delta<\varepsilon<1$. To estimate III we use Hölder's inequality with exponents r and r', where $r=\varepsilon/\delta>1$:

$$\mathrm{III} \le \left(\frac{1}{|Q|}\int_Q |b(y)-b_{3Q}|^{\delta r'}dy\right)^{\frac{1}{\delta r'}}\left(\frac{1}{|Q|}\int_Q (Mf(y))^{\delta r}dy\right)^{\frac{1}{\delta r}}.$$

By Lemma 2.4 we get

$$\mathrm{III} \lesssim \|b\|_*\left(\frac{1}{|Q|}\int_{3Q}(Mf(y))^{\varepsilon}dy\right)^{\frac{1}{\varepsilon}} \le \|b\|_* M_{\varepsilon}(Mf)(x). \tag{3.5}$$

Finally, since $M^2 \approx M_{L\log L}$ (see [13], p. 174 and [8], p. 159, for instance), by (3.2), (3.3), (3.4) and (3.5), we get

$$M_\delta(C_b(f))(x) \le C\|b\|_*\left(M_\varepsilon(Mf)(x)+M^2f(x)\right) \tag{3.6}$$

Since

$$M_\varepsilon(Mf)(x) \le M^2 f(x), \quad \text{when} \quad 0<\varepsilon<1,$$

we arrive at (3.1).

□

Remark 3.2. Note that the inequality (3.1) implies the inequality

$$M_\delta^{\#}(C_b(f))(x) \le C\|b\|_* M^2 f(x) \tag{3.7}$$

(see [9], Lemma 1, for instance). Indeed, assume that (3.1) holds. Then

$$M_\delta^{\#}(C_b(f))(x) \lesssim M_\delta(C_b(f))(x) \le C\|b\|_* M^2 f(x).$$

Theorem 3.3. *Let $b \in \mathrm{BMO}$. Then, there exists a positive constant C such that*

$$C_b(f)(x) \le C\|b\|_* M^2 f(x) \tag{3.8}$$

for all functions from $L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Since, by the Lebesgue differentiation theorem

$$C_b(f)(x) \le M_\delta(C_b(f))(x),$$

the statement follows from Theorem 3.1. □

4. Boundedness of the maximal commutator

Let us recall the following result on boundedness of maximal commutator in L_p-spaces.

Theorem 4.1. ([7], Theorem 2.4) *Let $1 < p < \infty$. Then the operator C_b is bounded in $L_p(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Using Theorem 3.3 and boundedness of the Hardy-Littlewood maximal operator M in $L_p(\mathbb{R}^n)$ we obtain new proof of sufficient part of Theorem 4.1.

In the following Theorem we give new proof of the result by Alphonse from [1] using Theorem 3.3 and Lemma 2.8. We also show necessity which is new. The fact that the operator C_b fails to be of weak type (1,1) follows from Lemma 5.1 and Example 5.7.

Theorem 4.2. *The following assertions are equivalent:*

(i) *There exists a positive constant C such that for each $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : C_b(f)(x) > \lambda\}| \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx \tag{4.1}$$

holds for all $f \in L(1 + \log^+ L)$.

(ii) *$b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Proof. (i) ⇒ (ii). Let Q_0 be any fixed cube and let $f = \chi_{Q_0}$. For any $\lambda > 0$ we have

$$\begin{aligned}
|\{x \in \mathbb{R}^n :\; C_b(f)(x) > \lambda\}| &= |\{x \in \mathbb{R}^n : \sup_{x \in Q} \frac{1}{|Q|} \int_{Q \cap Q_0} |b(x) - b(y)| dy > \lambda\}| \\
&\ge |\{x \in Q_0 : \sup_{x \in Q} \frac{1}{|Q|} \int_{Q \cap Q_0} |b(x) - b(y)| dy > \lambda\}| \\
&\ge |\{x \in Q_0 : \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| dy > \lambda\}| \\
&\ge |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}|,
\end{aligned}$$

since

$$|b(x) - b_{Q_0}| \le \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| dy.$$

By assumption the inequality (4.1) holds for f, thus we have

$$|\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| \le C|Q_0| \frac{1}{\lambda} \left(1 + \log^+ \frac{1}{\lambda}\right).$$

For $0 < \delta < 1$ we have

$$\begin{aligned}\int_{Q_0} |b - b_{Q_0}|^\delta &= \delta \int_0^\infty \lambda^{\delta-1} |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| d\lambda \\ &= \delta \left\{ \int_0^1 + \int_1^\infty \right\} \lambda^{\delta-1} |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| d\lambda \\ &\le \delta |Q_0| \int_0^1 \lambda^{\delta-1} d\lambda + C\delta |Q_0| \int_1^\infty \lambda^{\delta-1} \frac{1}{\lambda} \left(1 + \log^+ \frac{1}{\lambda}\right) d\lambda \\ &= |Q_0| + C\delta |Q_0| \int_1^\infty \lambda^{\delta-2} d\lambda = \left(1 + C\frac{\delta}{1-\delta}\right) |Q_0|.\end{aligned}$$

Thus $b \in \mathrm{BMO}_\delta(\mathbb{R}^n)$. Then by Lemma 2.4 we get that $b \in \mathrm{BMO}$.

(ii) $\Rightarrow$ (i). By Theorem 3.3 and Lemma 2.8, we have

$$\begin{aligned}&|\{x \in \mathbb{R}^n : C_b(f)(x) > \lambda\}| \\ &\qquad \le |\{x \in \mathbb{R}^n : M^2 f(x) > \frac{\lambda}{C\|b\|_*}\}| \\ &\qquad \le C \int_{\mathbb{R}^n} \frac{C\|b\|_* |f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{C\|b\|_* |f(x)|}{\lambda}\right)\right) dx.\end{aligned}$$

Since the inequality

$$1 + \log^+(ab) \le (1 + \log^+ a)(1 + \log^+ b) \tag{4.2}$$

holds for any a, $b > 0$, we get

$$\begin{aligned}&|\{x \in \mathbb{R}^n : C_b(f)(x) > \lambda\}| \\ &\qquad \le C\|b\|_* (1 + \log^+ \|b\|_*) \int_{\mathbb{R}^n} \frac{f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx.\end{aligned}$$

□

5. Boundedness of the commutator of maximal operator

In this section we obtain pointwise estimate of commutators of maximal operator by iterated maximal function. Using this estimate we obtain new proof of the result by M. Milman and T. Schonbek from [12]. Weak-type inequalities were also proved by means of this estimation.

We shall reduce the study of this commutator to that of C_b.

Lemma 5.1. *Let b be any non-negative locally integrable function on $\mathbb{R}^n$. Then*

$$|[M, b]f(x)| \le C_b(f)(x) \tag{5.1}$$

for all functions from $L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. It is easy to see that for any f, $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ the following pointwise estimate holds:

$$|Mf(x) - Mg(x)| \le M(f - g)(x). \tag{5.2}$$

Since b is non-negative, by (5.2) we can write

$$|[M,b]f(x)| = |M(bf)(x) - b(x)Mf(x)| = |M(bf)(x) - M(b(x)f)(x)|$$
$$\leq M(bf - b(x)f)(x) = M((b - b(x))f)(x) = C_b(f)(x).$$

□

Lemma 5.2. *Let b be any locally integrable function on $\mathbb{R}^n$. Then*

$$|[M,b]f(x)| \leq C_b(f)(x) + 2b^- Mf(x) \tag{5.3}$$

for all functions from $L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Since

$$|[M,b]f(x) - [M,|b|]f(x)| \leq 2b^- Mf(x)$$

(see [2], p. 3330, for instance), then

$$|[M,b]f(x)| \leq |[M,|b|]f(x)| + 2b^- Mf(x), \tag{5.4}$$

and by Lemma 5.1 we have

$$|[M,b]f(x)| \leq C_{|b|}f(x) + 2b^- Mf(x).$$

Since $||a| - |b|| \leq |a - b|$ holds for any $a, b \in \mathbb{R}$, we get $C_{|b|}f(x) \leq C_b f(x)$ for all $x \in \mathbb{R}^n$. □

Lemma 5.3. *Let b is in* $\mathrm{BMO}(\mathbb{R}^n)$*. Then, there exists a positive constant C such that*

$$|[M,b]f(x)| \leq C \left(\|b\|_* M^2 f(x) + b^-(x)Mf(x)\right) \tag{5.5}$$

for all functions from $L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Statement follows by Lemma 5.2 and Theorem 3.3.

□

Corollary 5.4. *Let b is in* $\mathrm{BMO}(\mathbb{R}^n)$ *such that $b^- \in L_\infty(\mathbb{R}^n)$. Then, there exists a positive constant C such that*

$$|[M,b]f(x)| \leq C \left(\|b\|_* M^2 f(x) + \|b^-\|_{L_\infty} Mf(x)\right) \tag{5.6}$$

for all functions from $L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 5.5. *Let b is in* $\mathrm{BMO}(\mathbb{R}^n)$ *such that $b^- \in L_\infty(\mathbb{R}^n)$. Then, there exists a positive constant C_1 such that*

$$|[M,b]f(x)| \leq C_1 \left(\|b^+\|_* + \|b^-\|_{L_\infty}\right) M^2 f(x) \tag{5.7}$$

for all functions from $L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Statement follows by Corollary 5.4, since $f \leq Mf$ and $\|b\|_* \leq \|b^+\|_* + \|b^-\|_* \lesssim \|b^+\|_* + \|b^-\|_{L_\infty}$. Note that by triangle inequality $\|b^+\|_* + \|b^-\|_\infty \lesssim \|b\|_* + \|b^-\|_\infty$ holds also. □

Corollary 5.6. *Let $1 < p \le \infty$ and let b is in* $\mathrm{BMO}(\mathbb{R}^n)$ *such that $b^- \in L_\infty(\mathbb{R}^n)$. Then the commutator $[M,b]$ is bounded in $L_p(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that*

$$\|[M,b]f\|_{L_p(\mathbb{R}^n)} \le C\left(\|b^+\|_* + \|b^-\|_{L_\infty}\right)\|f\|_{L_p(\mathbb{R}^n)} \tag{5.8}$$

for all functions from $L_p(\mathbb{R}^n)$.

Example 5.7. We show that $[M,b]$ fails to be of weak type $(1,1)$. Consider the BMO function $b(x) = \log|1+x|$ and let $f(x) = \chi_{(0,1)}(x)$. It is easy to see that for any $x < 0$

$$Mf(x) = \sup_{0<t<1} \frac{t}{t-x} = \frac{1}{1-x}.$$

On the other hand, for any $x < 0$

$$M(bf)(x) = \sup_{0<t<1} \frac{\int_0^t \log|1+y|dy}{t-x} = \sup_{0<t<1} \frac{(1+t)\log(1+t)-t}{t-x} = \frac{2\log 2 - 1}{1-x}.$$

Thus

$$[M,b]f(x) = \frac{2\log 2-1}{1-x} - \frac{\log|1+x|}{1-x}.$$

There is $\varepsilon_0 > 0$ such that for any $x < -\varepsilon_0$

$$\log|1+x| - (2\log 2 - 1) > \frac{1}{2}\log|x|.$$

Therefore, for any $\lambda > 0$,

$$\begin{aligned}
\lambda|\{x\in\mathbb{R} : |[M,b]f(x)| > \lambda\}| &\ge \lambda\left|\left\{x<0 : \left|\frac{2\log 2-1}{1-x} - \frac{\log|1+x|}{1-x}\right| > \lambda\right\}\right| \\
&\ge \lambda\left|\left\{x < -\varepsilon_0 : \frac{1}{2}\frac{\log|x|}{1-x} > \lambda\right\}\right| \\
&\ge \lambda\left|\left\{x < -\max\{e,\varepsilon_0\} : \frac{1}{2}\frac{\log|x|}{|x|} > \lambda\right\}\right| = \\
&= \lambda(\varphi^{-1}(-\max\{e,\varepsilon_0\}) - \varphi^{-1}(2\lambda)),
\end{aligned}$$

where φ is the increasing function $\varphi : (-\infty,-e) \to (0,e^{-1})$, given by $\varphi(x) = \log|x|/|x|$. To conclude observe that the right hand side of the estimate is unbounded as $\lambda \to 0$:

$$\lim_{\lambda\to 0} \lambda\varphi^{-1}(\lambda) = \lim_{\lambda\to\infty}\lambda\varphi(\lambda) = \infty.$$

Theorem 5.8. *Let b is in* $\mathrm{BMO}(\mathbb{R}^n)$ *such that $b^- \in L_\infty(\mathbb{R}^n)$. Then, there exists a positive constant C such that for each $\lambda > 0$,*

$$\begin{aligned}
&|\{x\in\mathbb{R}^n : |[M,b]f(x)| > \lambda\}| \\
&\qquad \le CC_0\left(1+\log^+ C_0\right)\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)dx
\end{aligned} \tag{5.9}$$

for all $f \in L\left(1+\log^+ L\right)$, where $C_0 = \|b^+\|_* + \|b^-\|_{L_\infty}$.

Proof. By Lemma 5.2, we have

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : |[M,b]f(x)| > \lambda\}| \\
&\qquad \le \left|\left\{x \in \mathbb{R}^n : C_b(f)(x) > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : |2b^-|Mf(x) > \frac{\lambda}{2}\right\}\right| \\
&\qquad \le \left|\left\{x \in \mathbb{R}^n : C_b(f)(x) > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : 2\|b^-\|_\infty Mf(x) > \frac{\lambda}{2}\right\}\right|.
\end{aligned}$$

By Theorem 4.2 using the inequality (4.2) we have

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : C_b(f)(x) > \frac{\lambda}{2}\right\}\right| \\
&\qquad \le CC_0\left(1+\log^+ C_0\right)\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)dx.
\end{aligned} \tag{5.10}$$

On the other hand, since the maximal operator M is a weak type (1,1), we get

$$\left|\left\{x \in \mathbb{R}^n : 2\|b^-\|_\infty Mf(x) > \frac{\lambda}{2}\right\}\right| \le C\|b^-\|_\infty \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}dx. \tag{5.11}$$

Combining (5.10) and (5.11) we get (5.9).

□

Remark 5.9. Unfortunately, in Theorem 5.8 we have only sufficient part, and we are not able to prove that the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is also necessary for inequality (5.9) to hold.

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