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Abstract. Recent results on purely electric (PE) or magnetic (PM) spacetimes in n dimensions are summarized. These include: Weyl types; diagonalizability; conditions under which direct (or warped) products are PE/PM.

1. Definition and general properties

The standard decomposition of the Maxwell tensor F_{ab} into its electric and magnetic parts $\vec{E}$ and $\vec{B}$ with respect to (wrt) an observer (i.e., a unit time-like vector $\boldsymbol{u}$) can be extended to any tensor in an n-dimensional spacetime [5, 10, 11]. Here we summarize the results of [5] about the Weyl tensor, and the connection with the null alignment classification [3, 7].

Consider the $\boldsymbol{u}$-orthogonal projector $h_{ab} = g_{ab} + u_a u_b$. The "electric" and "magnetic" parts of C_{abcd} can be defined, respectively, as [5]

$$(C_+)^{ab}{}_{cd} = h^{ae}h^{bf}h_c{}^g h_d{}^h C_{efgh} + 4u^{[a}u_{[c}C^{b]e}{}_{d]f}u_e u^f, \qquad (1)$$

$$(C_-)^{ab}{}_{cd} = 2h^{ae}h^{bf}C_{efk[c}u_{d]}u^k + 2u_k u^{[a}C^{b]kef}h_{ce}h_{df}. \qquad (2)$$

These extend the well-known 4D definitions [6, 12]. In any orthonormal frame adapted to $\boldsymbol{u}$ the electric [magnetic] part accounts for the Weyl components with an even [odd] number of indices u. *At a spacetime point (or region) the Weyl tensor is called "purely electric [magnetic]" (from now* *on, PE [PM]) wrt* $\boldsymbol{u}$ *if* $C_-=0$ $[C_+=0]$. The corresponding spacetime is also called PE [PM]. Several conditions on PE/PM Weyl tensors follow.

Proposition 1 (Bel-Debever-like criteria [5])**.** *A Weyl tensor* C_{abcd} *is: (i) PE wrt* $\boldsymbol{u}$ *iff* $u_a g^{ab} C_{bc[de} u_{f]} = 0$*; (ii) PM wrt* $\boldsymbol{u}$ *iff* $u_{[a} C_{bc][de} u_{f]} = 0$.

Proposition 2 (Eigenvalues [5])**.** *A PE [PM] Weyl operator*[1] *is diagonalizable, and possesses only real [purely imaginary] eigenvalues. Moreover, a PM Weyl operator has at least* $\frac{(n-1)(n-4)}{2}$ *zero eigenvalues.*

Proposition 3 (Algebraic type [5])**.** *A Weyl tensor which is PE/PM wrt a certain* $\boldsymbol{u}$ *can only be of type G,* I_i*, D or O. In the type* I_i *and D cases, the second null direction of the timelike plane spanned by* $\boldsymbol{u}$ *and any WAND is also a WAND (with the same multiplicity). Furthermore, a type D Weyl tensor is PE iff it is type D(d), and PM iff it is type D(abc).*

Proposition 4 (Uniqueness of $\boldsymbol{u}$ [5])**.** *A PE [PM] Weyl tensor is PE [PM] wrt: (i) a unique* $\boldsymbol{u}$ *(up to sign) in the type* I_i *and G cases; (ii) any* $\boldsymbol{u}$ *belonging to the space spanned by all double WANDs (and only wrt such* $\boldsymbol{u}$*s) in the type D case (noting also that if there are more than two double WANDs the Weyl tensor is necessarily PE (type D(d)) [13]).*

2. PE spacetimes

Proposition 5 ([5])**.** *All spacetimes admitting a shearfree, twistfree, unit timelike vector field* $\boldsymbol{u}$ *are PE wrt* $\boldsymbol{u}$*. In coordinates such that* $\boldsymbol{u} = V^{-1}\partial_t$*, the line-element reads*

$$\mathrm{d}s^2 = -V(t,x)^2\mathrm{d}t^2 + P(t,x)^2\xi_{\alpha\beta}(x)\mathrm{d}x^\alpha\mathrm{d}x^\beta. \tag{3}$$

The above metrics include, in particular, direct, warped and doubly warped products with a one-dimensional timelike factor, and thus all *static* spacetimes (see also [8]). For a warped spacetime $(M,\boldsymbol{g})$ with $M = M^{(n_1)} \times M^{(n_2)}$, one has $\boldsymbol{g} = e^{2(f_1+f_2)}\left(\boldsymbol{g}^{(n_1)} \oplus \boldsymbol{g}^{(n_2)}\right)$, where $\boldsymbol{g}^{(n_i)}$ is a metric on the factor space $M^{(n_i)}$ $(i=1,2)$ and f_i are functions on $M^{(n_i)}$ ($M^{(n_i)}$ has dimension n_i, $n=n_1+n_2$, and $M^{(n_1)}$ is Lorentzian).

Proposition 6 (Warps with $n_1 = 2$ [5,8])**.** *A (doubly) warped spacetime with* $n_1 = 2$ *is either type O, or type D(d) and PE wrt any* $\boldsymbol{u}$ *living in* $M^{(n_1)}$*; the uplifts of the null directions of the tangent space to* $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ *are double WANDs of* $(M,\boldsymbol{g})$*. If* $(M^{(n_2)}, \boldsymbol{g}^{(n_2)})$ *is Einstein the type specializes to D(bd), and if it is of constant curvature to D(bcd).*

In particular, all spherically, hyperbolically or plane symmetric spacetimes belong to the latter special case.

[1] In the sense of the Weyl operator approach of [1] (see also [2]).

Proposition 7 (Warps with $n_1 = 3$ [5,8])**.** *A (doubly) warped spacetime with $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ Einstein and $n_1 = 3$ is of type D(d) or O. The uplift of* any *null direction of the tangent space to $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ is a double WAND of $(M, \boldsymbol{g})$, which is PE wrt any $\boldsymbol{u}$ living in $M^{(n_1)}$.*

Proposition 8 (Warps with $n_1 > 3$ [5,8])**.** *In a (doubly) warped spacetime*

- *(i) if $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ is an Einstein spacetime of type D, $(M, \boldsymbol{g})$ can be only of type D (or O) and the uplift of a double WAND of $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ is a double WAND of $(M, \boldsymbol{g})$*
- *(ii) if $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ is of constant curvature, $(M, \boldsymbol{g})$ is of type D(d) (or O) and the uplifts of* any *null direction of the tangent space to $(M^{(n_1)}, \boldsymbol{g}^{(n_1)})$ is a double WAND of $(M, \boldsymbol{g})$; $(M, \boldsymbol{g})$ is PE wrt any $\boldsymbol{u}$ living in $M^{(n_1)}$.*

Proposition 9 (PE direct products [5])**.** *A direct product spacetime $M^{(n)} = M^{(n_1)} \times M^{(n_2)}$ is PE wrt a $\boldsymbol{u}$ that lives in $M^{(n_1)}$ iff $\boldsymbol{u}$ is an eigenvector of $R^{(n_1)}_{ab}$, and $M^{(n_1)}$ is PE wrt $\boldsymbol{u}$. ($\boldsymbol{u}$ is then also an eigenvector of the Ricci tensor R_{ab} of $M^{(n)}$, i.e., $R_{ui} = 0$.)*

A conformal transformation (e.g., to a (doubly) warped space) will not, of course, affect the above conclusions about the Weyl tensor. There exist also direct products which are PE wrt a vector $\boldsymbol{u}$ *not* living in $M^{(n_1)}$ [5].

Also the presence of certain (Weyl) isotropies (e.g., $SO(n-2)$ for $n > 4$) implies that the spacetime is PE, see [1, 5] for details and examples.

3. PM spacetimes

Proposition 10 (PM direct products [5])**.** *A direct product spacetime $M^{(n)} = M^{(n_1)} \times M^{(n_2)}$ is PM wrt a $\boldsymbol{u}$ that lives in $M^{(n_1)}$ iff all the following conditions hold (where $R_{(n_i)}$ is the Ricci scalar of $M^{(n_i)}$):*

- *i) $M^{(n_1)}$ is PM wrt $\boldsymbol{u}$ and has a Ricci tensor of the form $R^{(n_1)}_{ab} = \frac{R_{(n_1)}}{n_1} g^{(n_1)}_{ab} + u_{(a} q_{b)}$ (with $u^a q_a = 0$)*
- *ii) $M^{(n_2)}$ is of constant curvature and $n_2(n_2-1)R_{(n_1)} + n_1(n_1-1)R_{(n_2)} = 0$.*

Further, $M^{(n)}$ is PM Einstein iff $M^{(n_1)}$ is PM Ricci-flat and $M^{(n_2)}$ is flat.

See [5] for explicit (non-Einstein) examples. However, in general PM spacetimes are most elusive. For example,

Proposition 11 ([5])**.** *PM Einstein spacetimes of type D do not exist.*

In [5] also several results for PE/PM Ricci and Riemann tensors have been worked out, along with corresponding examples. In general, we observe that PE/PM tensors provide examples of *minimal tensors* [9]. Thanks to the *alignment theorem* [4], the latter are of special interest since they are precisely the *tensors characterized by their invariants* [4] (cf. also [5]). This in turn sheds new light on the classification of the Weyl tensor [3], providing a further invariant characterization that distinguishes the (minimal) types G/I/D from the (non-minimal) types II/III/N.

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