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Distributional Chaos for Linear Operators

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Abstract

We characterize distributional chaos for linear operators on Fréchet spaces in terms of a computable condition (DCC), and also as the existence of distributionally irregular vectors. A sufficient condition for the existence of dense uniformly distributionally irregular manifolds is presented, which is very general and can be applied to many classes of operators. Distributional chaos is also analyzed in connection with frequent hypercyclicity, and the particular cases of weighted shifts and composition operators are given as an illustration of the previous results.

1 Introduction and preliminaries

The study of the dynamics of linear operators on infinite-dimensional spaces has attracted attention of many researchers in recent years (we refer the reader to the books [4] and [11] for more information about this subject). Hypercyclicity, that is, the existence of vectors $x \in X$ whose orbit $\{x, Tx, T^2x, \dots\}$ is dense in X for an operator $T : X \to X$ on a topological vector space X, is certainly the most studied phenomenon in this context.

It seems that the first time that the term 'chaos' appeared in the mathematical literature was in Li and Yorke's paper [15] on the study of dynamics of interval maps. The notion of chaos derived from [15] concentrates on local aspects of dynamics of pairs.

Schweizer and Smítal introduced the concept of distributional chaos in [23] as a natural extension of the notion of chaos given by Li and Yorke.

Let $f : X \to X$ be a continuous map on a metric space X. For each pair $x, y \in X$ and each $n \in \mathbb{N}$, the *distributional function* $F^n_{xy} : \mathbb{R}^+ \to [0, 1]$ is defined by

$$F^n_{xy}(\tau) := \frac{1}{n} \operatorname{card}\{0 \le i \le n-1 : d(f^i(x), f^i(y)) < \tau\},$$

where card A denotes the cardinality of the set A. Moreover, define

$$F_{xy}(\tau) := \liminf_{n\to\infty} F^n_{xy}(\tau) \quad \text{and} \quad F^*_{xy}(\tau) := \limsup_{n\to\infty} F^n_{xy}(\tau).$$

The following notions were introduced in [23] and [21], respectively. They were considered for linear operators on Banach or Fréchet spaces in [6, 12, 13, 14, 16, 17, 20, 24].

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Definition 1. A continuous map $f : X \to X$ on a metric space X is *distributionally chaotic* if there exist an uncountable set $\Gamma \subset X$ and $\varepsilon > 0$ such that for every $\tau > 0$ and each pair of distinct points $x, y \in \Gamma$, we have that

$$F_{xy}(\varepsilon) = 0 \quad \text{and} \quad F^*_{xy}(\tau) = 1.$$

In this case, the set Γ is a *distributionally ε-scrambled set* and the pair (x, y) a *distributionally chaotic pair.*

We say that f is *densely distributionally chaotic* if the set Γ may be chosen to be dense in X.

Given $A \subset \mathbb{N}$, its *upper* and *lower densities* are defined by

$$\overline{\text{dens}}(A) := \limsup_{n\to\infty} \frac{\text{card}(A \cap [1, n])}{n} \quad \text{and} \quad \underline{\text{dens}}(A) := \liminf_{n\to\infty} \frac{\text{card}(A \cap [1, n])}{n},$$

respectively. With these concepts in mind, one can equivalently say that f is distributionally chaotic on Γ if there exists $\varepsilon > 0$ such that for any $x, y \in \Gamma$, $x \neq y$, we have

$$\underline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \varepsilon\} = 0 \quad \text{and} \quad \overline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \tau\} = 1,$$

for every $\tau > 0$.

The purpose of this paper is to study the above defined notions for continuous linear operators on Fréchet spaces.

Unless otherwise specified, X will denote an arbitrary Fréchet space, that is, a vector space X endowed with an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms (called a *fundamental sequence of seminorms*) that defines a metric

$$d(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x - y\|_k\} \quad (x, y \in X),$$

under which X is complete. Moreover, $B(X)$ will denote the set of all continuous linear operators $T : X \to X$.

The following concept is a generalization to Fréchet spaces of the one introduced by Beauzamy [5] for Banach spaces (see [7]).

Definition 2. Given $T \in B(X)$ and $x \in X$, we say that x is an *irregular vector* for T if there are $m \in \mathbb{N}$ and increasing sequences (n_k) and (j_k) of positive integers such that

$$\lim_{k\to\infty} T^{n_k} x = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{j_k} x\|_m = \infty.$$

Inspired by the notion of a distributionally irregular vector introduced in [6] in the context of Banach spaces, we consider the following generalization to Fréchet spaces.

Definition 3. Given $T \in B(X)$ and $x \in X$, we say that x is a *distributionally irregular vector* for T if there are $m \in \mathbb{N}$ and $A, B \subset \mathbb{N}$ with $\overline{\text{dens}}(A) = \overline{\text{dens}}(B) = 1$ such that

$$\lim_{n\in A} T^n x = 0 \quad \text{and} \quad \lim_{n\in B} \|T^n x\|_m = \infty.$$

We also recall that, given an operator $T \in B(X)$, a vector $x \in X$ is called *frequently hypercyclic* for T if for every non-empty open subset U of X, the set

$$\{n \in \mathbb{N} : T^n x \in U\}$$

has positive lower density. The operator T is called *frequently hypercyclic* if it possesses a frequently hypercyclic vector. T is called *topologically mixing* if for any pair U, V of non-empty open subsets of X, there exists some $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. T is *Devaney chaotic* if it is hypercyclic and it has a dense set of periodic points, that is, vectors $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$.

2 Distributional chaos and distributionally irregular vectors

Definition 4. Let $T \in B(X)$ and $x \in X$. The orbit of x is said to be *distributionally near to* 0 if there exists $A \subset \mathbb{N}$ with $\overline{\text{dens}}(A) = 1$ such that $\lim_{n\in A} T^n x = 0$. We say that x has a *distributionally unbounded orbit* if there exist $m \in \mathbb{N}$ and $B \subset \mathbb{N}$ with $\overline{\text{dens}}(B) = 1$ such that $\lim_{n\in B} \|T^n x\|_m = \infty$. Whenever we need to emphysize the number m in question, we say that x has a *distributionally m-unbounded orbit.*

With these definitions we have that a vector $x \in X$ is distributionally irregular if and only if its orbit is both distributionally unbounded and distributionally near to 0.

Assume X is a Banach space. If $T \in B(X)$ and $\|T^n\| \to \infty$, it follows from the Banach-Steinhaus Theorem that there exists a residual set of vectors in X with unbounded orbits. The next example shows that under these conditions we cannot guarantee the existence of a vector with distributionally unbounded orbit. Nevertheless, it was proved in [18] that if $\sum \frac{1}{\|T^n\|} < \infty$ then there exists a vector $x \in X$ such that $\|T^n x\| \to \infty$ (in particular, x has distributionally unbounded orbit).

Example 5. Assume $X = \ell^1(\mathbb{N})$. Given $\varepsilon \in (0, \frac{1}{5})$, there exists $T \in B(X)$ with

$$\|T^n\| = (n+1)^{(1-\varepsilon)} \quad (n \in \mathbb{N})$$

such that no $x \in X$ has distributionally unbounded orbit.

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be the standard basis for X. Define $T \in B(X)$ by $Te_1 = 0$ and $Te_k = (\frac{k}{k-1})^{1-\varepsilon} e_{k-1}$ for $k > 1$. It is easy to see that $\|T^n\| = \|T^n e_{n+1}\| = (n+1)^{(1-\varepsilon)}$ for all $n \in \mathbb{N}$. Suppose that there are $x \in X$, $\|x\| = 1$, and $N \in \mathbb{N}$ with

$$\text{card}\big\{1 \leq n \leq N : \|T^n x\| > \frac{3}{\varepsilon}\big\} \geq N(1-\varepsilon).$$

Then

$$\frac{1}{N}\sum_{n=1}^{N} \|T^n x\| \geq \frac{3}{\varepsilon}(1-\varepsilon) = \frac{3}{\varepsilon} - 3.$$

On the other hand, by writing $x = \sum_{j=1}^{\infty} \alpha_j e_j$ we obtain

$$\begin{aligned}
\sum_{n=1}^{N} \|T^n x\| &= \sum_{n=1}^{N} \sum_{j=n+1}^{\infty} |\alpha_j| \cdot \|T^n e_j\| \\
&= \sum_{n=1}^{N} \sum_{j=n+1}^{\infty} |\alpha_j| \Big(\frac{j}{j-n} \Big)^{1-\varepsilon} \\
&= \sum_{j=1}^{\infty} |\alpha_j| \, j^{1-\varepsilon} \sum_{n=1}^{\min\{N, j-1\}} (j-n)^{\varepsilon-1} \\
&\leq \sum_{j=1}^{2N} |\alpha_j| \, j^{1-\varepsilon} \sum_{n=1}^{j-1} (j-n)^{\varepsilon-1} + \sum_{j=2N+1}^{\infty} |\alpha_j| \sum_{n=1}^{N} \Big(\frac{j}{j-n} \Big)^{1-\varepsilon} \\
&\leq \sum_{j=1}^{2N} |\alpha_j| \, j^{1-\varepsilon} \frac{j^{\varepsilon}}{\varepsilon} + 2N,
\end{aligned}$$

where we have used the following estimations:

$$\sum_{n=1}^{j-1} (j-n)^{\varepsilon-1} = \sum_{k=1}^{j-1} k^{\varepsilon-1} \leq 1 + \int_1^{j-1} x^{\varepsilon-1} dx \leq \frac{j^{\varepsilon}}{\varepsilon}$$

and, for $j > 2N$, $(\frac{j}{j-n})^{1-\varepsilon} \leq 2^{1-\varepsilon} \leq 2$. So,

$$\frac{1}{N} \sum_{n=1}^{N} \|T^n x\| \leq \frac{1}{N} \sum_{j=1}^{2N} |\alpha_j| \frac{j}{\varepsilon} + 2 \leq 2 + \frac{2}{\varepsilon}.$$

Since $2 + \frac{2}{\varepsilon} < \frac{3}{\varepsilon} - 3$, we have a contradiction. □

Remark 6. In [17, Thm 2.1] it is provided an example of the same kind in a weighted ℓ^p-space such that $\|T^n\| \to \infty$ and

$$\lim_{n \to \infty} \frac{\operatorname{card}\{1 \leq j \leq n : \|T^j x\| < \varepsilon\}}{n} = 1$$

for all $x \in X$ and $\varepsilon > 1$, which in particular shows that T has no distributionally chaotic pair. Soon we will show that the above example admits no distributionally chaotic pair either.

Proposition 7. If $T \in B(X)$ and $m \in \mathbb{N}$, then the following assertions are equivalent:

(i) there exist $\varepsilon > 0$, a sequence (y_k) in X and an increasing sequence (N_k) in $\mathbb{N}$ such that $\lim_{k \to \infty} y_k = 0$ and

$$\operatorname{card}\big\{1 \leq j \leq N_k : \|T^j y_k\|_m > \varepsilon\big\} \geq N_k(1 - k^{-1})$$

for all $k \in \mathbb{N}$;

(ii) there exists $y \in X$ with distributionally m-unbounded orbit;

(iii) the set of all $y \in X$ with distributionally m-unbounded orbit is residual in X.

Proof. (iii) $\Rightarrow$ (ii): Trivial.
(ii) $\Rightarrow$ (i): Let $y \in X$ be a vector with distributionally m-unbounded orbit. By definition, there exists $A \subset \mathbb{N}$ with $\overline{\text{dens}}(A) = 1$ such that $\lim_{n\in A} \|T^n y\|_m = \infty$. Set $y_k := k^{-1}y$. Then $y_k \to 0$. Choose $\varepsilon > 0$ arbitrary. For each $k \in \mathbb{N}$, we have

$$\overline{\text{dens}}\{j \in \mathbb{N} : \|T^j y_k\|_m > \varepsilon\} = \overline{\text{dens}}\{j \in \mathbb{N} : \|T^j y\|_m > \varepsilon k\} \geq \overline{\text{dens}}(A) = 1.$$

So we can find $N_k \in \mathbb{N}$ satisfying (i).
(i) $\Rightarrow$ (iii): For each $k \in \mathbb{N}$, let

$$M_k := \Big\{x \in X : \exists n \in \mathbb{N} \text{ with } \operatorname{card}\big\{1 \leq j \leq n : \|T^j x\|_m > k\big\} \geq n(1 - k^{-1})\Big\}.$$

Clearly M_k is open. We show that M_k is dense. Let $x \in X$, $\delta > 0$ and $m_1 \in \mathbb{N}$. By (i), there exist $u \in \{y_1, y_2, \dots\}$ and $n \in \mathbb{N}$ such that $\|u\|_{m_1} < C := \frac{\delta\varepsilon}{4k^2}$ and

$$\operatorname{card}\{1 \leq j \leq n : \|T^j u\|_m > \varepsilon\} \geq n\Big(1 - \frac{1}{2k}\Big).$$

Consider the vectors

$$u_s := x + \frac{\delta s u}{2kC} \qquad (s = 0, 1, \dots, 2k - 1).$$

Clearly $\|u_s - x\|_{m_1} < \delta$ for all s. We show that there exists $s \in \{0, 1, \dots, 2k - 1\}$ with $u_s \in M_k$. Let $A := \{1 \leq j \leq n : \|T^j u\|_m > \varepsilon\}$. Then $\operatorname{card} A \geq n(1 - \frac{1}{2k})$. For each $s = 0, 1, \dots, 2k - 1$, let $B_s := \{1 \leq j \leq n : \|T^j u_s\|_m \leq k\}$. If $s, t \in \{0, 1, \dots, 2k - 1\}$ and $s \neq t$, then $B_s \cap B_t \cap A = \emptyset$. Indeed, suppose $s \neq t$ and $j \in B_s \cap B_t \cap A$. Then

$$\|T^j u_s - T^j u_t\|_m = \frac{|s - t| \cdot \delta \|T^j u\|_m}{2kC} > \frac{\delta\varepsilon}{2kC} = 2k.$$

However,

$$\|T^j u_s - T^j u_t\|_m \leq \|T^j u_s\|_m + \|T^j u_t\|_m \leq 2k,$$

a contradiction. So there exists $s_0 \in \{0, 1, \dots, 2k - 1\}$ with $\operatorname{card}(B_{s_0} \cap A) \leq \frac{\operatorname{card} A}{2k}$. Then $\operatorname{card}(A \setminus B_{s_0}) \geq n(1 - \frac{1}{2k})^2 \geq n(1 - k^{-1})$. For $j \in A \setminus B_{s_0}$ we have $\|T^j u_{s_0}\|_m > k$. Hence $u_{s_0} \in M_k$ and M_k is dense.

Thus $\bigcap_k M_k$ is a residual subset of X. Let $x \in \bigcap_k M_k$. For each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ with $\operatorname{card} A_k \geq n_k(1 - k^{-1})$, where $A_k := \{1 \leq j \leq n_k : \|T^j x\|_m > k\}$. Let $A := \bigcup_k A_k$. Then $\overline{\text{dens}}(A) = 1$ and $\lim_{j\in A} \|T^j x\|_m = \infty$. □

Proposition 8. If $T \in B(X)$ then the following assertions are equivalent:

(i) there exist $\varepsilon > 0$, a sequence (y_k) in X and an increasing sequence (N_k) in $\mathbb{N}$ such that $\lim_{k\to\infty} y_k = 0$ and

$$\lim_{k\to\infty} \frac{1}{N_k} \operatorname{card}\big\{1 \leq j \leq N_k : d(T^j y_k, 0) > \varepsilon\big\} = 1;$$

(ii) there exists $y \in X$ with distributionally unbounded orbit;

(iii) the set of all $y \in X$ with distributionally unbounded orbit is residual in X.

In the case X is a Banach space, the above assertions are also equivalent to the following:

(i') there exist $\varepsilon > 0$, a sequence (y_k) in X and an increasing sequence (N_k) in $\mathbb{N}$ such that $\lim_{k\to\infty} y_k = 0$ and

$$\operatorname{card}\big\{1 \le j \le N_k : \|T^j y_k\| > \varepsilon\big\} \ge \varepsilon N_k$$

for all $k \in \mathbb{N}$;

(ii') there exist $y \in X$ and $A \subset \mathbb{N}$ with $\overline{\operatorname{dens}}(A) > 0$ such that $\lim_{j\in A} \|T^j y\| = \infty$.

Proof. (iii) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii): Follow from the previous proposition.

Now, assume X is a Banach space.

(ii) $\Rightarrow$ (ii'): Obvious.

(ii') $\Rightarrow$ (i'): Choose $\varepsilon \in (0, \overline{\operatorname{dens}}\, A)$ and argue as in the proof of (ii) $\Rightarrow$ (i) in the previous proposition.

(i') $\Rightarrow$ (i): Let $\delta > 0$. We consider the following property $P(\delta)$: For every $L > 0$, there exist $y \in X$, $\|y\| = 1$, and $n \in \mathbb{N}$ such that

$$\operatorname{card}\{1 \le j \le n : \|T^j y\| > L\} \ge \delta n$$

(equivalently, there are $y' \in X$, $\|y'\| \le \frac{\varepsilon}{L}$, and $n \in \mathbb{N}$ with $\operatorname{card}\{j \le n : \|T^j y'\| > \varepsilon\} \ge \delta n$). Let δ_0 be the supremum of all δ for which $P(\delta)$ is true. By (i'), $\delta_0 \ge \varepsilon > 0$. Let $k \in \mathbb{N}$. Let $\delta_1 > 0$ satisfy

$$\frac{\delta_0 - \delta_1}{\delta_0 + \delta_1} \ge 1 - k^{-1}.$$

Since $P(\delta_0 + \delta_1)$ is not true, there exists $L > 0$ such that

$$\operatorname{card}\{1 \le j \le n : \|T^j y\| > L\} < (\delta_0 + \delta_1)n$$

for all $y \in X$, $\|y\| = 1$, and all $n \in \mathbb{N}$. Clearly we may assume that $L > \max\{k, \|T\|\}$. Since $P(\delta_0 - \delta_1)$ is true, there exist $u \in X$, $\|u\| = 1$, and $n_1 \in \mathbb{N}$ such that

$$\operatorname{card}\{1 \le j \le n_1 : \|T^j u\| > L^2\} \ge (\delta_0 - \delta_1)n_1.$$

Set

$$\begin{aligned}
A_1 &:= \{1 \le j \le n_1 : \|T^j u\| \le L\},\\
A_2 &:= \{1 \le j \le n_1 : L < \|T^j u\| \le L^2\},\\
A_3 &:= \{1 \le j \le n_1 : \|T^j u\| > L^2\}.
\end{aligned}$$

Since $\|T\| < L$, we have $1 \in A_1$. Let $A_1 = \{r_1, \dots, r_d\}$ with $r_1 < r_2 < \cdots < r_d$. Set formally $r_{d+1} := n_1 + 1$. For each $s = 1, \dots, d$, let $B_s := A_2 \cap \{r_s + 1, \dots, r_{s+1} - 1\}$ and $C_s := A_3 \cap \{r_s + 1, \dots, r_{s+1} - 1\}$. We have

$$\sum_{s=1}^{d} \operatorname{card} C_s = \operatorname{card} A_3 \ge (\delta_0 - \delta_1)n_1$$

and

$$\sum_{s=1}^{d} \operatorname{card}(C_s \cup B_s) = \operatorname{card}(A_3 \cup A_2) \le (\delta_0 + \delta_1)n_1.$$

So there exists s_0, $1 \leq s_0 \leq d$, such that $C_{s_0} \neq \emptyset$ and

$$\frac{\operatorname{card} C_{s_0}}{\operatorname{card}(C_{s_0} \cup B_{s_0})} \geq \frac{\delta_0 - \delta_1}{\delta_0 + \delta_1} \geq 1 - k^{-1}.$$

Set $v := \dfrac{\varepsilon T^{r_{s_0}} u}{k \| T^{r_{s_0}} u \|}$ and $n := r_{s_0+1} - r_{s_0} - 1$. For each $j \in C_{s_0}$, we have

$$\frac{\| T^{j - r_{s_0}} v \|}{\|v\|} = \frac{\|T^j u\|}{\|T^{r_{s_0}} u\|} \geq L.$$

So

$$\begin{aligned}
\operatorname{card}\big\{1 \leq j \leq n : \|T^j v\| > \varepsilon\big\} &= \operatorname{card}\Big\{1 \leq j \leq n : \frac{\|T^j v\|}{\|v\|} > k\Big\} \\
&\geq \operatorname{card}\big\{r_{s_0} \leq j < r_{s_0+1} : \|T^j u\| > k \|T^{r_{s_0}} u\|\big\} \\
&\geq \operatorname{card} C_{s_0} \\
&\geq n(1 - k^{-1}),
\end{aligned}$$

which completes the proof. □

Proposition 9. Let $T \in B(X)$ and suppose that there exists a dense subset X_0 of X such that the orbit of each $x \in X_0$ is distributionally near to 0. Then the set of all vectors with orbits distributionally near to 0 is residual.

Proof. For each $k, m \in \mathbb{N}$, let

$$M_{k,m} := \Big\{ x \in X : \exists n \in \mathbb{N} \text{ with } \operatorname{card}\{1 \leq j \leq n : \|T^j x\|_m < k^{-1}\} \geq n(1 - k^{-1}) \Big\}.$$

Clearly each $M_{k,m}$ is open and dense (since $M_{k,m} \supset X_0$). So the set $\bigcap_{k,m} M_{k,m}$ is residual and consists of vectors with orbits distributionally near to 0. □

Definition 10. Let $T \in B(X)$. We say that T satisfies the *Distributional Chaos Criterion* (DCC) if there exist sequences $(x_k), (y_k)$ in X such that:

(a) There exists $A \subset \mathbb{N}$ with $\overline{\operatorname{dens}}(A) = 1$ such that $\lim_{n \in A} T^n x_k = 0$ for all k.

(b) $y_k \in \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$, $\lim_{k \to \infty} y_k = 0$ and there exist $\varepsilon > 0$ and an increasing sequence (N_k) in $\mathbb{N}$ such that

$$\operatorname{card}\{1 \leq j \leq N_k : d(T^j y_k, 0) > \varepsilon\} \geq N_k(1 - k^{-1})$$

for all $k \in \mathbb{N}$.

Remarks 11. 1. We can assume that (x_k) and (y_k) are the same sequence because there exists a sequence $(\tilde{x}_k)$ in $\operatorname{span}\{x_n : n \in \mathbb{N}\} \setminus \{0\}$ that satisfies condition (a) by linearity and condition (b) by density.

2. In the case X is a Banach space, it follows from Proposition 8 that condition (b) in the above definition of (DCC) can be replaced by

(b') $y_k \in \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$, $\|y_k\| \to 0$ and there exist $\varepsilon > 0$ and an increasing sequence (N_k) in $\mathbb{N}$ such that

$$\operatorname{card}\{1 \leq j \leq N_k : \|T^j y_k\| > \varepsilon\} \geq \varepsilon N_k$$

for all $k \in \mathbb{N}$.

Theorem 12. If $T \in B(X)$ then the following assertions are equivalent:

(i) T satisfies (DCC);

(ii) T has a distributionally irregular vector;

(iii) T is distributionally chaotic;

(iv) T admits a distributionally chaotic pair.

Proof. (i) $\Rightarrow$ (ii): Let $X_0 := \{x \in X : \lim_{n\in A} T^n x = 0\}$. Then X_0 is a subspace of X, $T(X_0) \subset X_0$ and $T(\overline{X_0}) \subset \overline{X_0}$. Moreover, $x_k \in X_0$ and $y_k \in \overline{X_0}$ for all $k \in \mathbb{N}$.

By Proposition 9, the set of all vectors $x \in \overline{X_0}$ with orbits distributionally near to 0 is residual in $\overline{X_0}$. By Proposition 8, the set of all vectors $x \in \overline{X_0}$ with distributionally unbounded orbit is residual in $\overline{X_0}$. So the set of all distributionally irregular vectors is residual in $\overline{X_0}$. In particular, there exists a distributionally irregular vector.

(ii) $\Rightarrow$ (iii): Let $u \in X$ be a distributionally irregular vector. Then $\{\lambda u : \lambda \in \mathbb{K}\}$ is an uncountable distributionally ε-scrambled set for a certain $\varepsilon > 0$.

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (i): Let $(x, y) \in X \times X$ be a distributionally chaotic pair for T and set $u := x - y$. There exists $\varepsilon > 0$ such that

$$\overline{\text{dens}}\{j \in \mathbb{N} : d(T^j u, 0) > \varepsilon\} = 1 \tag{1}$$

and

$$\overline{\text{dens}}\{j \in \mathbb{N} : d(T^j u, 0) < \delta\} = 1$$

for each $\delta > 0$. So there is an increasing sequence (n_k) in $\mathbb{N}$ such that

$$\text{card}\{1 \le j \le n_k : d(T^j u, 0) < k^{-1}\} \ge n_k(1 - k^{-1}).$$

Let $A_k := \{1 \le j \le n_k : d(T^j u, 0) < k^{-1}\}$ $(k \in \mathbb{N})$ and $A := \bigcup_{k=1}^{\infty} A_k$. Then $\overline{\text{dens}}(A) = 1$ and $\lim_{n\in A} T^n u = 0$. For each $k \in \mathbb{N}$, let $x_k := T^k u$. Clearly

$$\lim_{n\in A} T^n x_k = 0 \quad (k \in \mathbb{N}).$$

Now, choose s_k such that $\|T^{s_k} u\|_k < k^{-1}$ and let $y_k := T^{s_k} u$. Then $y_k \to 0$. By (1), we have

$$\overline{\text{dens}}\{j \in \mathbb{N} : d(T^j y_k, 0) > \varepsilon\} = 1$$

for all $k \in \mathbb{N}$. Hence there is an increasing sequence (N_k) in $\mathbb{N}$ such that

$$\text{card}\{1 \le j \le N_k : d(T^j y_k, 0) > \varepsilon\} \ge N_k(1 - k^{-1})$$

for all $k \in \mathbb{N}$. This proves that T satisfies (DCC). $\square$

3 Dense distributional chaos

We devote this section to the existence of dense sets (manifolds) of distributionally irregular vectors. Concerning necessary conditions, we observed in [7] that the existence of a dense set of just irregular vectors for T implies that the adjoint operator T^* admits no eigenvalues λ with $|\lambda| \ge 1$. This result is sharp, even for dense distributional chaos.

Remark 13. There are operators $T : \ell^p(v) \to \ell^p(v)$ for certain weight sequences v such that every non-zero vector $x \in \ell^p(v)$ is distributionally irregular, and the set of eigenvalues of T^* is $\mathbb{D} := \{\lambda : |\lambda| < 1\}$. Indeed, let S be the bilateral forward shift on a weighted space $\ell^p(w, \mathbb{Z})$ such that $w_{i+1}/w_i \to 1$ and every non-zero vector $x \in \ell^p(w, \mathbb{Z})$ is distributionally irregular, as constructed in [17]. Let $v := (w_i)_{i\in\mathbb{N}}$ and consider the natural inclusion $\ell^p(v) \subset \ell^p(w, \mathbb{Z})$, $(x_1, x_2, \dots) \mapsto (\dots, 0, 0, x_1, x_2, \dots)$. The operator $T := S|_{\ell^p(v)}$ clearly satisfies the desired properties.

In what follows in this section we will consider several sufficient conditions for dense distributional chaos.

Definition 14. Given $T \in B(X)$, a vector subspace Y of X is called a *uniformly distributionally irregular manifold* for T if there exists $m \in \mathbb{N}$ such that the orbit of every nonzero vector y in Y is simultaneously distributionally m-unbounded and distributionally near to 0.

It is easy to see that such a Y is a distributionally 2^{-m+1}-scrambled set for T. Hence the existence of a dense uniformly distributionally irregular manifold implies dense distributional chaos.

Theorem 15. Assume X separable. Suppose that $T \in B(X)$ satisfies

$$T^n x \to 0 \quad \text{for all } x \in X_0,$$

where X_0 is a dense subset of X. Then the following assertions are equivalent:

(i) T is distributionally chaotic;

(ii) T is densely distributionally chaotic;

(iii) T admits a dense uniformly distributionally irregular manifold;

(iv) T admits a distributionally unbounded orbit.

We recall that Proposition 8 contains a very useful computable characterization of the existence of a distributionally unbounded orbit. It will be used several times in applications of the above theorem.

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i): Obvious.
(i) $\Rightarrow$ (iv): Follows from Theorem 12.
(iv) $\Rightarrow$ (iii): Without loss of generality we may assume that X_0 is a dense subspace of X and

$$\|Tx\|_k \le \|x\|_{k+1} \quad \text{for all } x \in X \text{ and } k \in \mathbb{N}.$$

By hypothesis, there is a vector $y \in X$ with distributionally unbounded orbit. So there exist $m \in \mathbb{N}$ and $B \subset \mathbb{N}$ with $\overline{\text{dens}}(B) = 1$ such that

$$\lim_{n\in B} \|T^n y\|_m = \infty.$$

Hence, for every $L > 0$ and $k \in \mathbb{N}$, there exist $x \in X$ as close to zero as we want and $n \in \mathbb{N}$ as large as we want so that

$$\text{card}\{1 \le i \le n : \|T^i x\|_m > L\} > n(1 - k^{-1}).$$

Clearly we can take $x \in X_0$ and assume, without loss of generality, that $m = 1$.

Thus we can construct inductively a sequence (x_k) of vectors in X_0 with $\|x_k\|_k \leq 1$, $k \in \mathbb{N}$, and an increasing sequence (n_k) of positive integers such that

$$\operatorname{card}\{1 \leq i \leq n_k : \|T^i x_k\|_1 > k2^k\} > n_k \left(1 - \frac{1}{k^2}\right), \tag{2}$$

$$\operatorname{card}\{1 \leq i \leq n_k : \|T^i x_s\|_k < \frac{1}{k}\} > n_k \left(1 - \frac{1}{k^2}\right), \; s = 1, \dots, k-1. \tag{3}$$

Given $\alpha, \beta \in \{0,1\}^{\mathbb{N}}$, we say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i \in \mathbb{N}$. Consider an increasing sequence (r_j) of positive integers such that

$$r_{j+1} \geq 1 + r_j + n_{r_j+1} \quad \text{for all } j \in \mathbb{N}. \tag{4}$$

Fix $\alpha \in \{0,1\}^{\mathbb{N}}$ defined by $\alpha_n = 1$ if and only if $n = r_j$ for some $j \in \mathbb{N}$. Given $\beta \in \{0,1\}^{\mathbb{N}}$ such that $\beta \leq \alpha$ and β contains an infinite number of 1's, we define the vector

$$x_\beta := \sum_i \frac{\beta_i}{2^i} x_i = \sum_j \frac{\beta_{r_j}}{2^{r_j}} x_{r_j}.$$

Observe that the above series is convergent since $\|x_i\|_i \leq 1$ for each $i \in \mathbb{N}$. We will show that the orbit of x_β is distributionally 1-unbounded and distributionally near to 0.

Let $k \in \mathbb{N}$ with $\beta_{r_k} = 1$. If $1 \leq i \leq n_{r_k}$, $\|T^i x_{r_k}\|_1 > r_k 2^{r_k}$ and $\|T^i x_s\|_{r_k} < \dfrac{1}{r_k}$ for each $s < r_k$, then

$$\begin{aligned}
\|T^i x_\beta\|_1 &\geq \frac{1}{2^{r_k}} \|T^i x_{r_k}\|_1 - \sum_{j \neq k} \frac{\beta_{r_j}}{2^{r_j}} \|T^i x_{r_j}\|_1 \\
&> r_k - \frac{1}{r_k} \sum_{j<k} \frac{1}{2^{r_j}} - \sum_{j>k} \frac{\|x_{r_j}\|_{1+i}}{2^{r_j}} \\
&\geq r_k - 1,
\end{aligned}$$

where we used the inequalities $\|x_{r_j}\|_{1+i} \leq 1$ for $j > k$, which hold because $1+i \leq 1+n_{r_k} \leq 1 + n_{r_{j-1}} \leq r_j$ (by (4)). Conditions (2) and (3) above imply that

$$\operatorname{card}\{1 \leq i \leq n_{r_k} : \|T^i x_\beta\|_1 > r_k - 1\} \geq n_{r_k}\left(1 - \frac{1}{r_k}\right),$$

and so the orbit of x_β is distributionally 1-unbounded.

On the other hand, if $1 \leq i \leq n_{r_k+1}$ and $\|T^i x_s\|_{r_k+1} < \dfrac{1}{r_k+1}$ for each $s < r_k + 1$, then

$$\begin{aligned}
\|T^i x_\beta\|_{r_k+1} &\leq \sum_{j \leq k} \frac{\beta_{r_j} \|T^i x_{r_j}\|_{r_k+1}}{2^{r_j}} + \sum_{j>k} \frac{\beta_{r_j} \|T^i x_{r_j}\|_{r_k+1}}{2^{r_j}} \\
&\leq \frac{1}{r_k+1} \sum_{j \leq k} \frac{1}{2^{r_j}} + \sum_{j>k} \frac{\|x_{r_j}\|_{1+r_k+i}}{2^{r_j}} \\
&< \frac{1}{r_k+1},
\end{aligned}$$

where we used the inequalities $\|x_{r_j}\|_{1+r_k+i} \le 1$ for $j > k$, which hold because $1 + r_k + i \le 1 + r_k + n_{r_k+1} \le 1 + r_{j-1} + n_{r_{j-1}+1} \le r_j$ (by (4)). Condition (3) above implies that

$$\operatorname{card}\Big\{1 \le i \le n_{r_k+1} : \|T^i x_\beta\|_{r_k+1} < \frac{1}{r_k+1}\Big\} \ge n_{r_k+1}\Big(1 - \frac{1}{r_k+1}\Big),$$

and so the orbit of x_β is distributionally near to 0.

Since X is separable, we can select a dense sequence (y_n) in X_0 and a countable collection $\gamma_n \in \{0,1\}^{\mathbb{N}}$ $(n \in \mathbb{N})$ such that each sequence γ_n contains an infinite number of 1's, $\gamma_n \le \alpha$ for every $n \in \mathbb{N}$, and the sequences γ_n have mutually disjoint supports.

We set the sequence of vectors $u_n := \sum_i \frac{\gamma_{n,i}}{2^i} x_i$, $n \in \mathbb{N}$. We already know that the orbit of each u_n is distributionally 1-unbounded and distributionally near to 0. Define now

$$z_n := y_n + \frac{1}{n} u_n, \quad n \in \mathbb{N}.$$

Since (y_n) is dense in X and the u_n's are uniformly bounded in X, we get that the sequence (z_n) is dense in X. We set $Y := \operatorname{span}\{z_n : n \in \mathbb{N}\}$, which is a dense subspace of X. If $u \in Y \setminus \{0\}$, then we can write

$$u = y_0 + \sum_k \frac{\rho_k}{2^k} x_k,$$

where $y_0 \in X_0$ and the sequence of scalars (ρ_k) takes only a finite number of values (each of them infinitely many times). As in the above proof we can show that the orbit of

$$v := \sum_k \frac{\rho_k}{2^k} x_k$$

is distributionally 1-unbounded and distributionally near to 0. Since $y = y_0 + v$ and $\lim_k T^k y_0 = 0$, the same is true for the orbit of y. Thus Y is a dense uniformly distributionally irregular manifold for T. □

Theorem 16. Assume X separable. Suppose that $T \in B(X)$ satisfies the following conditions:

(I) There exists a dense subset X_0 of X with $\lim_{n\to\infty} T^n x = 0$ for all $x \in X_0$.

(II) One of the following conditions is true:

- (a) X is a Fréchet space and there exists an eigenvalue λ with $|\lambda| > 1$.
- (b) X is a Banach space and $\sum \frac{1}{\|T^n\|} < \infty$ (in particular if $r(T) > 1$).
- (c) X is a Hilbert space and $\sum \frac{1}{\|T^n\|^2} < \infty$ (in particular if $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure).

Then T is densely distributionally chaotic.

Proof. In view of Theorem 15, it is enough to prove the existence of a vector $y \in X$ with distributionally unbounded orbit. In case (a), it is enough to get an eigenvector $y \ne 0$ associated to an eigenvalue λ with $|\lambda| > 1$. In cases (b) and (c), it was proved in [18] that there exists a vector $y \in X$ such that $\|T^n y\| \to \infty$. In particular, y has distributionally unbounded orbit.

By Theorem 1.1(i) in [9] (see also p. 239, Theorem 11 in [19]), $\sum \frac{1}{\|T^n\|^2} < \infty$ whenever $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure. □

In [10] it is proved that, if T satisfies the Godefroy-Shapiro Criterion, then T is hypercyclic. As consequence of (a) in Theorem 16 we also obtain that

Corollary 17. If T satisfies the Godefroy-Shapiro Criterion then T is densely distributionally chaotic.

Since every continuous linear operator on $H(\mathbb{C}^N)$ that commutes with any translation operator and is not a scalar multiple of the identity satisfies the Godefroy-Shapiro criterion [10], we get a natural class of densely distributionally chaotic operators.

Corollary 18. Every continuous linear operator on $H(\mathbb{C}^N)$ that commutes with any translation operator and is not a scalar multiple of the identity is densely distributionally chaotic.

A series $\sum_{k=1}^{\infty} x_k$ in X is said to be *unconditionally convergent* if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$d\Big(\sum_{k\in F} x_k, 0\Big) < \varepsilon,$$

whenever $F \subset \mathbb{N}$ is finite and $F \cap \{1, 2, \ldots, N\} = \emptyset$.

Theorem 19. Assume X separable and let $T \in B(X)$. Suppose that:

(a) There exists a dense subset X_0 of X with $\lim_{n\to\infty} T^n x = 0$ for all $x \in X_0$.

(b) There exist a subset Y of X, a mapping $S : Y \to Y$ with $TSy = y$ on Y, and a vector $z \in Y \setminus \{0\}$ such that $\sum_{n=1}^{\infty} T^n z$ and $\sum_{n=1}^{\infty} S^n z$ converge unconditionally.

Then T is densely distributionally chaotic.

Proof. If we define $w_{k_0} := \sum_{n=1}^{\infty} T^{k_0 n} z + z + \sum_{n=1}^{\infty} S^{k_0 n} z$, then $w_{k_0} \neq 0$ if k_0 is sufficient large and $T^{k_0} w_{k_0} = w_{k_0}$. Let $y_k := \sum_{n=k}^{\infty} S^{k_0 n} z$. Then $y_k \to 0$ and

$$T^{k_0 j} y_k = \sum_{n=1}^{j-k} T^{k_0 n} z + z + \sum_{n=1}^{\infty} S^{k_0 n} z \to w_{k_0}$$

as $j \to \infty$. For $0 \leq l < k_0$ we have

$$T^l w_{k_0} = \lim_{j\to\infty} T^{l+k_0 j} y_k.$$

Hence $\{T^l w_{k_0} : 0 \leq l < k_0\}$ are accumulation points of the orbit of y_k. Let $\varepsilon := \frac{1}{2}\min\{d(T^l w_{k_0}, 0) : 0 \leq l < k_0\}$. Then there exists an increasing sequence (N_k) of positive integers such that

$$\lim_{k\to\infty} \frac{1}{N_k} \operatorname{card}\{1 \leq j \leq N_k : d(T^j y_k, 0) > \varepsilon\} = 1.$$

By Proposition 8, T admits a distributionally unbounded orbit. Thus, by Theorem 15, T is densely distributionally chaotic. □

In [8] it was observed that the Frequent Hypercyclicity Criterion implies that T is frequently hypercyclic, Devaney chaotic and mixing. As a consequence of the above theorem, we obtain that it also implies dense distributional chaos.

Corollary 20. Assume X separable and let $T \in B(X)$. Suppose that there are a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that, for any $x \in X_0$,

(a) $\sum_{n=1}^{\infty} T^n x$ converges unconditionally,

(b) $\sum_{n=1}^{\infty} S^n x$ converges unconditionally,

(c) $TSx = x$.

Then T is frequently hypercyclic, Devaney chaotic, mixing and densely distributionally chaotic.

Definition 21. We say that a continuous linear operator T on a Banach space X has a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for every σ-measurable subset A of $\mathbb{T}$ of σ-measure equal to 1, the span of $\bigcup_{\lambda \in A} \ker(T - \lambda)$ is dense in X.

Theorem 22. Let T be a continuous linear operator on a Hilbert space X that has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues such that the measure σ can be chosen to be absolutely continuous with respect to the Lebesgue length measure on the unit circle. Then T is densely distributionally chaotic.

Proof. Under the conditions of the theorem, T satisfies the Hypercyclicity Criterion with respect to the sequence (n) (see Theorems 2.4 and 2.5 in [1]) and thus there exists a dense set X_0 such that $\lim_{n\to\infty} T^n x = 0$ for all $x \in X_0$. Moreover, $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure. Thus, by Theorem 16, T is densely distributionally chaotic. □

Problem 23. Is the above theorem true for Banach spaces?

As consequence of Corollary 20, we have a positive partial answer of the above problem under certain conditions. Indeed, if T has a σ-spanning set of $\mathbb{T}$-eigenvectors, where σ is the length measure on $\mathbb{T}$, then T satisfies the Frequent Hypercyclicity Criterion, and so T is densely distributionally chaotic. The same conclusion holds if the eigenvector fields are C^2-functions or X does not contain c_0 and the eigenvectors fields are α-Holderian for some $\alpha > \frac{1}{2}$ ([3, section 5.8]).

Since a composition operator with a parabolic or hyperbolic automorphism in $H^2(\mathbb{D})$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues with respect to the Lebesgue length measure on the unit circle [2], we obtain

Corollary 24. Let φ be an automorphism of $\mathbb{D}$. Then the composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is densely distributionally chaotic if and only if φ is not an elliptic automorphism.

4 Densely distributionally chaotic weighted shifts

The next result characterizes dense distributional chaos for unilateral weighted backward shifts on Fréchet sequence spaces in terms of the existence of a distributionally unbounded orbit.

Theorem 25. Let X be a Fréchet sequence space in which $(e_n)_{n\in\mathbb{N}}$ is a basis. Suppose that the unilateral weighted backward shift

$$B_\omega\big((x_n)_{n\in\mathbb{N}}\big) := (w_n x_{n+1})_{n\in\mathbb{N}}$$

is an operator on X. Then the following assertions are equivalent:

(i) B_ω is distributionally chaotic;

(ii) B_ω is densely distributionally chaotic;

(iii) B_ω admits a dense uniformly distributionally irregular manifold;

(iv) B_ω admits a distributionally unbounded orbit.

Proof. Let X_0 be the set of all finite linear combinations of the basis vectors e_n. Clearly X_0 is dense in X and $(B_\omega)^n x \to 0$ for all $x \in X_0$. Thus the result is just a special case of Theorem 15. □

As an application of the previous theorem, we have the following computable sufficient condition for dense distributional chaos for unilateral backward shifts on Fréchet sequence spaces.

Theorem 26. Let X be a Fréchet sequence space in which $(e_n)_{n\in\mathbb{N}}$ is a basis. Suppose that the unilateral backward shift B is an operator on X. If there exists a set $S \subset \mathbb{N}$ with $\overline{\text{dens}}(S) = 1$ such that

$$\sum_{n\in S} e_n \ \text{ converges in } X,$$

then B is densely distributionally chaotic.

Proof. For each $k \in \mathbb{N}$, let

$$y_k := \sum_{n\in S, n\geq k} e_n.$$

Then $\lim_{k\to\infty} y_k = 0$ and $B^n y_k = e_1 + \cdots$ for all $n \in S$ with $n \geq k$. Since the functional $x \to x_1$ is continuous, there exists $\varepsilon > 0$ such that $d(x, 0) \leq \varepsilon$ implies $|x_1| < 1$. Hence $d(B^n y_k, 0) > \varepsilon$ for all $n \in S$ with $n \geq k$. So there is a sequence (N_k) of positive integers increasing to ∞ such that

$$\lim_{k\to\infty} \frac{1}{N_k} \text{card}\big\{1 \leq j \leq N_k : d(B^j y_k, 0) > \varepsilon\big\} = 1.$$

By Proposition 8, T admits a distributionally unbounded orbit. Thus, by the previous theorem, B is densely distributionally chaotic. □

Corollary 27. Let X be a Fréchet sequence space in which $(e_n)_{n\in\mathbb{N}}$ is a basis. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of positive weights. Suppose that the unilateral weighted backward shift B_ω is an operator on X. If there exists a set $S \subset \mathbb{N}$ with $\overline{\text{dens}}(S) = 1$ such that

$$\sum_{n\in S} \Big(\prod_{\nu=1}^{n} \omega_\nu\Big)^{-1} e_n \ \text{ converges in } X, \tag{5}$$

then B_ω is densely distributionally chaotic.

Proof. Follows from the previous theorem by conjugacy. □

Corollary 28. Let X be a Fréchet sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Suppose that the unilateral weighted backward shift B_ω is an operator on X. If B_ω has a non-trivial periodic point, then B_ω is densely distributionally chaotic.

Proof. Under these hypothesis (5) holds ([11, Theorem 4.8(c)]). Hence it is enough to apply the previous corollary. □

Now we turn our attention to bilateral weighted forward shifts.

Theorem 29. Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Let $(\omega_n)_{n\in\mathbb{Z}}$ be positive weights and assume that

$$\lim_{n\to\infty}\Big(\prod_{\nu=1}^{n}\omega_\nu\Big)e_n = 0. \tag{6}$$

Suppose that the bilateral weighted forward shift

$$F_\omega\big((x_n)_{n\in\mathbb{Z}}\big) := (w_n x_{n-1})_{n\in\mathbb{Z}}$$

is an operator on X. Then the following assertions are equivalent:

(i) F_ω is distributionally chaotic;

(ii) F_ω is densely distributionally chaotic;

(iii) F_ω admits a dense uniformly distributionally irregular manifold;

(iv) F_ω admits a distributionally unbounded orbit.

Proof. Let X_0 be the set of all finite linear combinations of the basis vectors e_n. Clearly X_0 is dense in X. Moreover, $\lim_{n\to\infty}(F_\omega)^n x = 0$ for all $x \in X_0$, because $\lim_{n\to\infty}(F_\omega)^n e_j = 0$ for any $j \in \mathbb{Z}$ (by (6)). Hence it is enough to apply Theorem 15. □

Theorem 30. Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the bilateral forward shift F is an operator on X. If $\lim_{n\to\infty} e_n = 0$ and there exists a set $S \subset \mathbb{N}$ with $\overline{\text{dens}}(S) = 1$ such that

$$\sum_{n\in S} e_{-n} \quad \text{converges in } X,$$

then F is densely distributionally chaotic.

Proof. For each $k \in \mathbb{N}$, let

$$y_k := \sum_{n\in S, n\geq k} e_{-n}.$$

Then $\lim_{k\to\infty} y_k = 0$ and $F^n y_k = \cdots + e_1 + \cdots$ for all $n \in S$ with $n \geq k$. Since the functional $x \to x_1$ is continuous, there exists $\varepsilon > 0$ such that $d(x, 0) \leq \varepsilon$ implies $|x_1| < 1$. Hence $d(F^n y_k, 0) > \varepsilon$ for all $n \in S$ with $n \geq k$. So there is a sequence (N_k) of positive integers increasing to ∞ such that

$$\lim_{k\to\infty} \frac{1}{N_k}\,\text{card}\{1 \leq j \leq N_k : d(F^j y_k, 0) > \varepsilon\} = 1.$$

By Proposition 8, T admits a distributionally unbounded orbit. Thus, by the previous theorem, F is densely distributionally chaotic. □

Corollary 31. Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Let $(\omega_n)_{n\in\mathbb{Z}}$ be positive weights. Suppose that the bilateral weighted forward shift F_ω is an operator on X. If $\lim_{n\to\infty}(\prod_{\nu=1}^{n}\omega_\nu)e_n = 0$ and there exists a set $S \subset \mathbb{N}$ with $\overline{\text{dens}}(S) = 1$ such that

$$\sum_{n\in S}\Big(\prod_{\nu=-n+1}^{0}\omega_\nu\Big)^{-1} e_{-n} \ \text{ converges in } X,$$

then F_ω is densely distributionally chaotic.

Corollary 32. Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Suppose that the bilateral weighted forward shift F_ω is an operator on X. If F_ω has a non-trivial periodic point, then F_ω is densely distributionally chaotic.

5 Densely distributionally chaotic composition operators

We begin this section by recalling the following result.

Theorem 33 (The Denjoy-Wolff Iteration Theorem)**.** Suppose that φ is an analytic self-map of $\mathbb{D}$ that is not an elliptic automorphism.

1. If φ has a fixed point $p \in \mathbb{D}$, then (φ^n) converges uniformly on compact sets to p.

2. If φ has no fixed point in $\mathbb{D}$, then there is a fixed point $p \in \partial\mathbb{D}$ such that (φ^n) converges uniformly on compact sets to p.

Now we characterize dense distributional chaos for composition operators on the Fréchet space $H(\mathbb{D})$.

Theorem 34. Suppose that φ is an analytic self-map of $\mathbb{D}$. Then the composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is densely distributionally chaotic if and only if it has no fixed point in $\mathbb{D}$.

Proof. If φ is an elliptic automorphism, then φ is conjugated to a rotation, and so C_φ is not distributionally chaotic.

If φ is a non-elliptic automorphism with a fixed point $p \in \mathbb{D}$, then (φ^n) converges uniformly on compact sets to p. Thus $(f \circ \varphi^n)$ converges uniformly on compact sets to $f(p)$ and C_φ is not distributionally chaotic.

If φ has no fixed point in $\mathbb{D}$, let p be the Denjoy-Wolff point that belongs to $\partial\mathbb{D}$ such that (φ^n) converges uniformly on compact sets to p. Let X_0 denote the set of all functions that are continuous on $\overline{\mathbb{D}}$, analytic on $\mathbb{D}$ and vanish at p. Then X_0 is dense in $H(\mathbb{D})$ and $\lim_{n\to\infty} C_\varphi^n f = 0$ for all $f \in X_0$. For each $k \in \mathbb{N}$, let

$$g_k(z) := \frac{1}{k(p-z)}.$$

Then (g_k) converges to zero and there is a sequence (N_k) of positive integers increasing to ∞ such that

$$\lim_{k\to\infty}\frac{1}{N_k}\,\text{card}\big\{1 \le j \le N_k : d(C_\varphi^j g_k, 0) > \frac{1}{2}\big\} = 1.$$

By Proposition 8, T admits a distributionally unbounded orbit. Thus, by Theorem 15, C_φ is densely distributionally chaotic. □

Corollary 35. Let Ω be a simply connected domain in the complex plane and φ be an automorphism of Ω. For the composition operator $C_\varphi : H(\Omega) \to H(\Omega)$, the following assertions are equivalent:

(i) C_φ is chaotic;

(ii) C_φ is mixing;

(iii) C_φ is hypercyclic;

(iv) (φ^n) is a run-away sequence;

(v) φ has no fixed point in Ω;

(vi) C_φ is densely distributionally chaotic.

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are known.

If $\Omega \neq \mathbb{C}$ then the equivalence (v) $\Leftrightarrow$ (vi) follows from the above theorem by conjugation.

If $\Omega = \mathbb{C}$ and φ has no fixed point, then C_φ is a translation operator, and so it is densely distributionally chaotic. If φ has a fixed point, then C_φ is conjugated to C_ψ where $\psi(z) = az$ and thus it is not densely distributionally chaotic. □

6 Miscellanea

It is well-known that if T is invertible and hypercyclic (mixing), then T^{-1} is hypercyclic (mixing). In contrast, there exist operators T which are invertible and densely distributionally chaotic such that T^{-1} is not distributionally chaotic (see, e.g., [17]).

Also, there are densely distributionally chaotic operators on Banach spaces of the form $T = I + K$, where K is a compact operator (see [6]). Such a T is neither frequently hypercyclic nor Devaney chaotic.

Problem 36. Are there frequently hypercyclic operators which are not distributionally chaotic?

For example, in [3] the authors constructed a frequently hypercyclic operator on c_0 that is not Devaney chaotic (and not mixing). Is this operator distributionally chaotic?

Problem 37. Are there Devaney chaotic operators which are not distributionally chaotic?

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References

[1] F. Bayart and S. Grivaux, *Hypercyclicity and unimodular point spectrum*, J. Funct. Anal. **226** (2005), no. 2, 281-300.

[2] F. Bayart and S. Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 5083–5117.

[3] F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. London Math. Soc. (3) **94** (2007), no. 1, 181-210.

[4] F. Bayart and É. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, Cambridge, 2009.

[5] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988.

[6] T. Bermúdez, A. Bonilla, F. Martínez-Giménez and A. Peris, *Li-Yorke and distributionally chaotic operators*, J. Math. Anal. Appl. **373** (2011), no. 1, 83–93.

[7] N. C. Bernardes Jr., A. Bonilla, V. Müller and A. Peris, *Li-Yorke chaos in linear dynamics*. Preprint (2012).

[8] A. Bonilla and K.-G. Grosse-Erdmann, *Frequently hypercyclic operators and vectors*, Ergodic Theory Dynam. Systems **27** (2007), no. 2, 383-404. Erratum: Ergodic Theory Dynam. Systems **29** (2009), no. 6, 1993-1994.

[9] O. El-Fallah and T. Ransford, *Peripheral point spectrum and growth of powers of operators*, J. Operator Theory **52** (2004), no. 1, 89-101.

[10] G. Godefroy and J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal., **98** (1991), 229-269.

[11] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer, London, 2011.

[12] B. Hou, P. Cui and Y. Cao, *Chaos for Cowen-Douglas operators*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 929–936.

[13] B. Hou, G. Tian and L. Shi, *Some dynamical properties for linear operators*, Illinois J. Math. **53** (2009), no. 3, 857–864.

[14] B. Hou, G. Tian and S. Zhu, *Approximation of chaotic operators*, J. Operator Theory **67** (2012), no. 2, 469–493.

[15] T. Y. Li and J. A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), no. 10, 985–992.

[16] F. Martínez-Giménez, P. Oprocha and A. Peris, *Distributional chaos for backward shifts*, J. Math. Anal. Appl. **351** (2009), no. 2, 607–615.

[17] F. Martínez-Giménez, P. Oprocha and A. Peris, *Distributional chaos for operators with full scrambled sets*, Math. Z., to appear.

[18] V. Müller and J. Vrsovsky, *Orbits of linear operators tending to infinity*, Rocky Mountain J. Math. **39** (2009), no. 1, 219–230.

[19] N. K. Nikolskii, *Spectral synthesis for a shift operator and zeros in certain classes of analytic functions smooth up to the boundary*, Soviet Math. Dokl. **11** (1970), 206-209.

[20] P. Oprocha, *A quantum harmonic oscillator and strong chaos*, J. Phys. A **39** (2006), no. 47, 14559–14565.

[21] P. Oprocha, *Distributional chaos revisited*, Trans. Amer. Math. Soc. **361** (2009), no. 9, 4901–4925.

[22] G. T. Prăjitură, *Irregular vectors of Hilbert space operators*, J. Math. Anal. Appl. **354** (2009), no. 2, 689–697.

[23] B. Schweizer and J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 737–754.

[24] G. Tian and B. Hou, *Some remarks on distributional chaos for linear operators*, Commun. Math. Res. **27** (2011), no. 4, 307–314.

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