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Abstract: We prove an L^q theory result for generalized Stokes system on a $\mathcal{C}^{2,1}$ domain complemented with the perfect slip boundary conditions and under $\Phi-$growth conditions. Since the interior regularity was obtained in [3], a regularity up to the boundary is an aim of this paper. In order to get the main result, we use Calderón–Zygmund theory and the method developed in [1]. We obtain higher integrability of the first gradient of a solution.

Keywords: Generalized Stokes System; Perfect Slip Boundary Conditions; L^q theory

Mathematics Subject Classification (2000): 35J66, 76D03.

1 Introduction

This paper is concerned with steady flows of an incompressible fluid in a domain $\Omega \subset \mathbb{R}^n$, described by the system of equations

$$-\operatorname{div} \mathcal{S}(Du) + \nabla\pi = \operatorname{div} F \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $u = (u_1, \ldots, u_n)$ is the velocity, π represents the pressure, $\operatorname{div} F$ stands for the density of volume forces and $\mathcal{S}$ denotes the extra stress tensor. The symmetric part of the velocity gradient is denoted by Du, i.e. $Du = \frac{1}{2}[\nabla u + (\nabla u)^\top]$. By ν we denote an outward normal vector and τ stands for any tangent vector to $\partial\Omega$. The boundary conditions (1.3) are sometimes called perfect slip boundary conditions. We would like to remark that homogeneous Dirichlet boundary conditions and perfect slip boundary conditions are limit cases of Navier's slip boundary conditions:

$$u \cdot \nu = 0, \quad \alpha[\mathcal{S}(Du)\nu] \cdot \tau + (1-\alpha)u_\tau = 0, \quad \alpha \in [0,1], \quad \text{on } \partial\Omega.$$

We assume that we can construct the scalar potential $\Phi : [0,\infty) \mapsto [0,\infty)$ to the stress tensor $\mathcal{S}$, i.e.

$$\mathcal{S}_{ij}(A) = \partial_{ij}\Phi(|A|) = \Phi'(|A|)\frac{A_{ij}}{|A|} \quad \forall A \in \mathbb{R}^{n\times n}_{sym}, \quad A \neq 0. \tag{1.4}$$

By $f \sim g$ we mean that there are positive constants c and C such that $cf \le g \le Cf$. We require the following assumption to be fulfilled:

Assumption 1.1. *Suppose that $\Phi \in \mathcal{C}^{1,1}(0,\infty) \cap \mathcal{C}^1[0,\infty)$ is an N-function, $\Phi \in \Delta_2$, $\Phi^* \in \Delta_2$ and $\Phi'(s) \sim s\Phi''(s)$ holds for all $s > 0$.*

Some results are obtained under additional assumption that $\Phi''(s)$ is almost monotone, i.e. there exists $C > 0$ such that for all $s \in (0,t]$ either $\Phi''(s) \le C\Phi''(t)$ (almost increasing) or $\Phi''(s) \ge C\Phi''(t)$ (almost decreasing). If we consider power-law models, this situation corresponds to the case that we need to distinguish $p \ge 2$ and $p \in (1,2]$.

Assumption 1.1 gives non-standard $\Phi-$growth conditions, see [2, Lemma 21]. Specifically, there are constants $C, c > 0$ such that for all $A, B \in \mathbb{R}^{n\times n}_{sym}$ holds

$$\begin{aligned} \big(\mathcal{S}(A) - \mathcal{S}(B)\big) \cdot \big(A - B\big) &\ge C\Phi''(|A| + |B|)|A - B|^2, \\ |\mathcal{S}(A) - \mathcal{S}(B)| &\le c\Phi''(|A| + |B|)|A - B|. \end{aligned} \tag{1.5}$$

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In this paper we use standard notation for Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$, Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, $1 \le p \le \infty$, $k \in \mathbb{N}$, Orlicz spaces $(L^\Phi(\Omega), \|\cdot\|_\Phi)$ and Orlicz-Sobolev spaces $(W^{1,\Phi}(\Omega), \|\cdot\|_{1,\Phi})$, $\Omega \subset \mathbb{R}^n$ is a domain and $\Omega \in \mathcal{C}^{2,1}$. We define[1]

$$\begin{aligned} W^{1,\Phi}_\nu(\Omega)^n &= \{\varphi_i \in W^{1,\Phi}(\Omega),\ i = 1,\dots,n,\ \varphi\cdot\nu = 0 \text{ on } \partial\Omega\}, \\ W^{1,\Phi}_\sigma(\Omega)^n &= \{\varphi \in W^{1,\Phi}_\nu(\Omega)^n,\ \operatorname{div}\varphi = 0 \text{ in } \Omega\}. \end{aligned}$$

We begin with the definition of the weak solution of the problem (1.1), (1.2) and (1.3).

Definition 1.2. *We say that the pair $(u,\pi) \in W^{1,\Phi}_\sigma(\Omega)^n \times L^{\Phi^*}(\Omega)$ is a weak solution to* (1.1), (1.2) *and* (1.3) *if*

$$\int_\Omega \mathcal{S}(Du) : D\varphi \,\mathrm{d}x - \int_\Omega \pi \operatorname{div}\varphi \,\mathrm{d}x = \int_\Omega F : \nabla\varphi \,\mathrm{d}x$$

holds for all $\varphi \in W^{1,\Phi}_\nu(\Omega)^n$.

It is well known that the weak solution exists and is unique. It could be easily proven using the monotone operator theory.

Before stating our main result, we define function V and N-function Ψ (for the notion of an N-function and some properties see Section 2) which are very well suited for expressing differentiability properties of weak solutions. Definition of the function V in the framework of Orlicz spaces was first given in [2].

For given Φ we define the N-function Ψ by

$$\frac{\Psi'(s)}{s} = \sqrt{\frac{\Phi'(s)}{s}}. \tag{1.6}$$

and we define $V(A)$ such that $\Psi(|A|)$ is a scalar potential to $V(A)$, i.e.

$$V_{ij}(A) := \partial_{ij}\Psi(|A|) = \Psi'(|A|)\frac{A_{ij}}{|A|} \quad \forall A \in \mathbb{R}^{n\times n}_{sym}, \quad A \ne 0. \tag{1.7}$$

It is shown in [2, Lemma 25] that

$$\Psi''(s) \sim \sqrt{\Phi''(s)}. \tag{1.8}$$

Example 1.3. *Let us mention that growth conditions* (1.5) *allow us to consider models with a great deal of disparity, for example power-law models (including the singular case)*

$$\begin{aligned} \mathcal{S}(Du) &= \mu_0(1+|Du|^2)^{\frac{p-2}{2}}Du, \quad \Phi(|Du|) = \mu_0\int_0^{|Du|}(1+s^2)^{\frac{p-2}{2}}s\,\mathrm{d}s, \\ \mathcal{S}(Du) &= \mu_0(1+|Du|)^{p-2}Du, \quad \Phi(|Du|) = \mu_0\int_0^{|Du|}(1+s)^{p-2}s\,\mathrm{d}s, \\ \mathcal{S}(Du) &= \mu_0|Du|^{p-2}Du, \quad \Phi(|Du|) = \mu_0\int_0^{|Du|}s^{p-1}\,\mathrm{d}s, \end{aligned}$$

where $\mu_0 \in \mathbb{R}^+$ and $p \in (1,\infty)$. In this case the function V and N-function Ψ have following structure:

$$\begin{aligned} V(Du) &= \mu_0(1+|Du|^2)^{\frac{p-2}{4}}Du, \quad \Psi(|Du|) = \mu_0\int_0^{|Du|}(1+s^2)^{\frac{p-2}{4}}s\,\mathrm{d}s, \\ V(Du) &= \mu_0(1+|Du|)^{\frac{p-2}{2}}Du, \quad \Psi(|Du|) = \mu_0\int_0^{|Du|}(1+s)^{\frac{p-2}{2}}s\,\mathrm{d}s, \\ V(Du) &= \mu_0|Du|^{\frac{p-2}{2}}Du, \quad \Psi(|Du|) = \mu_0\int_0^{|Du|}s^{\frac{p}{2}}\,\mathrm{d}s. \end{aligned}$$

Before formulating the main result we would like to mention some previous results which motivated us to our work. In [7] T. Iwaniec showed L^q theory result for linear problem based on local comparison with the solution to the problem with the zero right hand side. He proved that the regularity properties of solutions transfers from the homogeneous problem to the original one. One year later in [8] he extended this result also for $p-$Laplace equations. Among lots of papers based on the comparison problem we mention especially [1]. The approach of L. Caffarelli and A. Peral presented in [1] will be used to prove our main result. In connection with Orlicz spaces we refer to [14], for results concerning the problem with growth described with variable exponent c.f. [6]. To

[1] As we use the notation $W^{1,\Phi}_\sigma(\Omega)^n$ for vector-valued functions with components in the function space $W^{1,\Phi}_\sigma(\Omega)$, we use analogically the notation $W^{1,\Phi}_\sigma(\Omega)^{n\times n}$ for tensor-valued functions.

our knowledge the first result about L^q regularity for Stokes type system with growth described by N-function was given in [3]. L. Diening and P. Kaplický showed interior L^q regularity of generalized Stokes system in $\mathbb{R}^3$ under Assumption 1.1. The key part of the proof was Theorem 3.2, where for the problem with zero right hand side gradient of function $V(Du)$ is controlled by oscillations of $V(Du)$. In [11] the authors showed that, for the Stokes type problem with non-zero right hand side under homogeneous perfect slip boundary conditions, gradient of function $V(Du)$ is controlled by the constant from the first apriori estimate. They presented the global result based on a different approach than it is used in this paper. Instead of flattening the boundary and reflection the solution beyond the boundary in a suitable way they worked on a general boundary from the beginning.

The main result of this paper concerns with higher integrability of the first gradient of solutions to (1.1) – (1.3). In order to write down a local estimate, we use a function $H_{x,R}$ which describes the boundary on some neighborhood of a point $x \in \partial\Omega$. The precise definition of a function H can be found on the beginning of the Section 4.

Theorem 1.4 (Main Result)**.** *Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^{2,1}$ domain, Assumption* 1.1 *be fulfilled and u be a weak solution to* (1.1)*–*(1.3)*. Let $x_0 \in \partial\Omega$ be arbitrary. Then there exist neighborhoods $\mathcal{V}_{x_0}$ and $\mathcal{U}_{x_0}$ of point x_0 such that $\mathcal{U}_{x_0} \subset \mathcal{V}_{x_0}$ and following implication holds*

$$(\Phi^*(|F|) \in L^q(\mathcal{V}_{x_0})) \Rightarrow (\Phi(|Du|) \in L^q(\mathcal{U}_{x_0}))\,,$$

provided $q \in (1,\infty)$ for $n = 2$ and $q \in \left(1, \frac{2n}{n-2}\right)$, resp. $q \in \left(1, \frac{2n}{n-2} + \delta\right)$ for $n \geq 3$ and some $\delta > 0$ in case Φ'' is almost monotone.

Moreover, for a cube $Q \subset \mathbb{R}^n_+$ and $R > 0$ sufficiently small, it holds

$$\fint_{H_{x_0,R}(Q)} \Phi(|\nabla u|)^q \,\mathrm{d}x \leq c\left(\fint_{H_{x_0,R}(4Q)\cap\Omega} \Phi(|u|)^q \,\mathrm{d}x + \fint_{H_{x_0,R}(4Q)\cap\Omega} \Phi^*(|F|)^q \,\mathrm{d}x + \left(\fint_{H_{x_0,R}(Q)} \Phi(|\nabla u|) \,\mathrm{d}x\right)^q\right), \tag{1.9}$$

This theorem provides a local regularity of solution near boundary. However, the interior regularity of solution was proven in [3] and thus one may easily derive global regularity of solution as well as global estimates in case Ω is a bounded domain.

The method of the proof is basically the same as in [3] and it is based on the approach published in [1]. The validity of two hypothesis (H1') and (H2) from [1] has to be shown. The paper is organized as follows. In Section 2 we remind some basic properties of Orlicz spaces which are needed later. In this section, we also introduce a generalization of Lemma 1.3 from [1].

Section 3 is devoted to the homogeneous system near the flat boundary and also the hypothesis (H1') is verified there. Instead of working on the general smooth boundary like in [11], we use the special structure of perfect slip boundary conditions in order to extend the solution in a suitable way beyond the flat boundary.

Finally, in Section 4 we flatten the general $\mathcal{C}^{2,1}$ boundary and we complete the proof of the main theorem by showing the validity of hypothesis (H2). Unlike the usual comparison problem we compare the properties of homogeneous system on a flat boundary with non-homogeneous system on a general boundary.

2 Preliminaries

A real function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called N-function if the derivative Φ' exists and is right continuous for $s \geq 0$, positive for $s > 0$, non-decreasing and satisfies $\Phi'(0) = 0$ and $\lim_{s\to\infty} \Phi'(s) = \infty$. N-function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if there exists a positive constant C, such that $\Phi(2s) \leq C\Phi(s)$ for $s > 0$. By $\Delta_2(\Phi)$ we denote the smallest such constant C.

By $(\Phi')^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ we denote the function

$$(\Phi')^{-1}(s) := \sup\{t \in \mathbb{R}^+ : \Phi'(t) \leq s\}.$$

The complementary function of Φ is defined as

$$\Phi^*(s) := \int_0^s (\Phi')^{-1}(t)\,\mathrm{d}t.$$

It is again an N-function and for all $\delta > 0$ there exists $c(\delta) > 0$ such that for all $s, t \geq 0$ holds so called Young's inequality

$$st \leq \delta\Phi(s) + c(\delta)\Phi^*(t). \tag{2.1}$$

For a measurable function f we can define gauge norm as

$$\|f\|_\Phi := \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \le 1\right\}.$$

The Orlicz space $L^\Phi(\Omega)$ is defined as the set $\{f : \|f\|_{\Phi,\Omega} < \infty\}$. It holds

$$\Phi^*(\Phi'(s)) \sim \Phi(s). \tag{2.2}$$

For more details concerning Orlicz spaces see for example [13]. So called shifted N-function Φ_a is for $a \ge 0$ defined as follows

$$\Phi_a'(s) := \Phi'(a+s)\frac{s}{a+s}. \tag{2.3}$$

This basically states that $\Phi_a''(s) \sim \Phi''(a+s)$. Moreover, $\{\Phi_a, \Phi_a^*\} \in \Delta_2$ uniformly in a, see [2, Appendix].

For the average integral we use the notation

$$\langle f\rangle_Q = \fint_Q f\,\mathrm{d}x = \frac{1}{|Q|}\int_Q f\,\mathrm{d}x.$$

Now we collect several useful lemmas which will be used later.

Lemma 2.1 (Shift change)**.** *[4, Lemma 5.15] Let Φ fulfill Assumption* 1.1*. Then for any $\delta > 0$ there exists $c(\delta) > 1$ such that for all $A, B \in \mathbb{R}^{n\times n}$ and $s \ge 0$*

$$\Phi_{|A|}(s) \le c(\delta)\Phi_{|B|}(s) + \delta|V(A) - V(B)|^2.$$

Lemma 2.2. *[2, Lemma 31] Let Φ be an N-function with $\Delta_2(\{\Phi^*, \Phi\}) < \infty$. Then there exist $\delta > 0$, $c > 0$ which depend only on $\Delta_2(\{\Phi^*, \Phi\})$ such that for all $t > 0$ and all $s \in [0, 1]$*

$$\Phi_a(st) \le cs^{1+\delta}\Phi_a(t).$$

Lemma 2.3. *[3, Lemma 2.7] For all $A \in L^\Phi(Q)^{n\times n}$ it holds*

$$\fint_Q |V(A) - V(\langle A\rangle_Q)|^2\,\mathrm{d}x \sim \fint_Q |V(A) - \langle V(A)\rangle_Q|^2\,\mathrm{d}x.$$

Lemma 2.4. *[3, Lemma 2.4] Let Φ satisfy Assumption* 1.1 *and V be defined as in* (1.7)*. Then for all $P, Q \in \mathbb{R}^{n\times n}$ we have*

$$(A(P) - A(Q)) : (P - Q) \sim |V(P) - V(Q)|^2 \sim \Phi_{|P|}(|P - Q|) \sim \Phi''(|P| + |Q|)|P - Q|^2$$

and

$$|A(P) - A(Q)| \le C\Phi_{|P|}(|P - Q|).$$

Lemma 2.5 (Korn's inequalities)**.** *Let Φ be an N-function with $\Delta_2(\{\Phi, \Phi^*\}) < \infty$. There exists a positive constant C such that for any cube $Q \subset \mathbb{R}^n$ and function $u \in W^{1,\Phi}(Q)^n$ it holds that*

$$\int_Q \Phi(|\nabla u|)\,\mathrm{d}x \le C\left(\int_Q \Phi(|Du|)\,\mathrm{d}x + \int_Q \Phi\left(\frac{|u|}{\operatorname{diam} Q}\right)\mathrm{d}x\right), \tag{2.4}$$

$$\fint_Q \Phi_a(|\nabla u - \langle\nabla u\rangle_Q|)\,\mathrm{d}x \le C\fint_Q \Phi_a(|Du - \langle Du\rangle_Q|)\,\mathrm{d}x, \tag{2.5}$$

where a is a positive constant or $|Du|$. Moreover, if $u|_{\partial Q} = 0$, it holds that

$$\int_Q \Phi(|\nabla u|)\,\mathrm{d}x \le c\int_Q \Phi(|Du|)\,\mathrm{d}x. \tag{2.6}$$

Proof. The inequality (2.4) folows from [5]. Namely, one should focus on Lemma 5.17, Proposition 6.1 and Theorem 6.13 given there. The inequality (2.5) for $a = |Du|$ is proven in [3, Lemma 2.9] and in [5]. For the proof of (2.6) see Theorem 6.10 in [5]. □

Lemma 2.6 (Bogovskiĭ's Lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be a rectangle. Let Φ be N-function with $\Delta_2(\{\Phi^*, \Phi\}) < \infty$, $g \in L^\Phi(\Omega)$, $h \in W^{1,\Phi}(\Omega)^n$ and Γ be one side of Ω. Then there exists $z \in W^{1,\Phi}(\Omega)^n$ solving*

$$\operatorname{div} z = g \quad \text{in } \Omega, \tag{2.7}$$

$$z \cdot \nu = h \cdot \nu \quad \text{on } \Gamma, \tag{2.8}$$

Moreover, there exists $C > 0$ depending only on $\Delta_2(\Phi)$ and $\Delta_2(\Phi^*)$ such that

$$\int_\Omega \Phi(|\nabla z|)\,\mathrm{d}x \le C\left(\int_\Omega \Phi(|g|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla h|)\,\mathrm{d}x\right). \tag{2.9}$$

Proof. Without loss of generality, we may assume that Γ is a part of a hyperplane $\{x; x_n = 0\}$. It is enough to consider equation

$$\begin{aligned} \operatorname{div} \tilde{z} &= g - \operatorname{div} h - \fint_\Omega (g - \operatorname{div} h)\,\mathrm{d}x \text{ in } \Omega, \\ \tilde{z} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Furthermore, we define an affine function $b : \Omega \mapsto \mathbb{R}^3$ as follows

$$b_i(x) = \begin{cases} 0 & \text{for } i \in \{1, \dots, n-1\} \text{ and } x \in \Omega, \\ 0 & \text{for } i = n \text{ and } x \in \Gamma, \\ x_n \fint_\Omega (g - \operatorname{div} h)\,\mathrm{d}x & \text{for } i = n \text{ and } x \in \Omega. \end{cases}$$

Then $z = \tilde{z} + h + b$ solves (2.7) and (2.8). According to [5, Theorem 6.6] there exists a positive constant c independent of $\operatorname{diam}\Omega$ such that

$$\int_\Omega \Phi(|\nabla \tilde{z}|)\,\mathrm{d}x \le c \int_\Omega \Phi\left(\left| g - \operatorname{div} h - \fint_\Omega (g - \operatorname{div} h)\,\mathrm{d}x \right| \right) \mathrm{d}x.$$

The estimate (2.9) follows easily. □

For a cube Q and $\alpha > 0$ we define a cube αQ as a cube with the same center as Q whose edges are parallel and have length α times length of edges of Q. Furthermore, for a dyadic cube Q_k we denote its predecessor by $\tilde{Q}_k$. This notation is effective throughout the paper. The proof of the main theorem is based on the following lemma.

Lemma 2.7. *Let $\mathcal{O} \subset \mathbb{R}^n$, $1 \le p < q < s < \infty$, $f \in L^{q/p}(\mathcal{O})$, $g \in L^{q/p}(\mathcal{O})$ and $w \in L^p(\mathcal{O})^n$. Further, let $Q \subset \mathcal{O}$ be a cube and Q_k be dyadic cubes obtained from Q. Then there exists $\varepsilon_0 > 0$ such that the following implication holds:*

If there exists $\varepsilon \in (0, \varepsilon_0)$ such that for every dyadic cube $Q_k \subset Q$ there exists $w_a \in L^p(4\tilde{Q}_k \cap \mathcal{O})^n$ with following properties:

$$\left(\fint_{2\tilde{Q}_k \cap \mathcal{O}} |w_a|^s\,\mathrm{d}x \right)^{\frac{1}{s}} \le \frac{C}{2} \left(\fint_{4\tilde{Q}_k \cap \mathcal{O}} |w_a|^p\,\mathrm{d}x \right)^{\frac{1}{p}}, \tag{2.10}$$

$$\fint_{4\tilde{Q}_k \cap \mathcal{O}} |w_a|^p\,\mathrm{d}x \le C \fint_{4\tilde{Q}_k \cap \mathcal{O}} |w|^p\,\mathrm{d}x + C \fint_{4\tilde{Q}_k \cap \mathcal{O}} |g|\,\mathrm{d}x, \tag{2.11}$$

$$\fint_{4\tilde{Q}_k \cap \mathcal{O}} |w - w_a|^p\,\mathrm{d}x \le \varepsilon \fint_{4\tilde{Q}_k \cap \mathcal{O}} |w|^p\,\mathrm{d}x + C \fint_{4\tilde{Q}_k \cap \mathcal{O}} |f|\,\mathrm{d}x, \tag{2.12}$$

then $w \in L^q(Q)^n$. Positive constants C and ε are independent on Q_k, w_a and w.
Furthermore, there exists a positive constant c independent of f, g and w such that

$$\fint_Q |w|^q\,\mathrm{d}x \le c \left(\fint_{4Q \cap \mathcal{O}} |f|^{\frac{q}{p}}\,\mathrm{d}x + \fint_{4Q \cap \mathcal{O}} |g|^{\frac{q}{p}}\,\mathrm{d}x + \left(\fint_Q |w|^p \right)^{\frac{q}{p}} \right). \tag{2.13}$$

The proof itself is based on Calderon–Zygmund theory and the considerations presented in [1]. L. A . Caffarelli and I. Peral proved Lemma 2.7 in [1, Theorem A] in case $f, g = 0$. The Lemma was later used by L. Diening and P. Kaplický in [3] for $f \neq 0$, however, authors did not provide any proof.

Throughout the proof we suppose that the functions w_a and w are defined by zero outside the domain $\mathcal{O}$. Since volume of $4\tilde{Q}_k \cap \mathcal{O}$ is proportional to $4\tilde{Q}_k$, the estimates (2.10), (2.11) and (2.12) still hold true for slightly changed constants when we replace $4\tilde{Q}_k \cap \mathcal{O}$ with $4\tilde{Q}_k$ and $2\tilde{Q}_k \cap \mathcal{O}$ with $2\tilde{Q}_k$.

We introduce Hardy-Littlewood maximal operator

$$M(f)(x) = \sup \left\{ \fint_P |f(y)|\,\mathrm{d}y, P \subset 4Q \text{ is a cube containing } x \right\},$$

which satisfies the weak type $(1, 1)$ inequality. In order to prove Lemma 2.7 we present the following observation

Lemma 2.8. *There exists $K_0 > 2^{n(p+1)}$ such that for all $K > K_0$ and for every $\delta \in (0, 1)$ there exists $L \in \left(0, \frac{K_0}{2}\right)$ and $\varepsilon > 0$ such that for every $\lambda > 0$, for $A = \{x \in Q, M(|w|^p) > K\lambda, M(|f|) + M(|g|) \le L\lambda\}$ and $B = \{x \in Q, M(|w|^p) > \lambda\}$ it holds, that if (2.10), (2.11) and (2.12) hold with ε, then following implication is true*

$$|Q_k \cap A| > (\delta + C_B K^{-s/p})|Q_k| \Rightarrow \tilde{Q}_k \subset B,$$

where C_B is a constant coming from (2.10), (2.11) and from strong type (r, r) estimate for Hardy-Littlewood maximal operator.

Proof. We proceed in a similar way like in [1]. We suppose, for contradiction, that $|Q_k \cap A| > (\delta + C_B K^{-s/p})|Q_k|$ and it is not true that $\tilde{Q}_k \subset B$. Thus there are points $x_0 \in \tilde{Q}_k$ and $x_1 \in (Q_k \cap A) \subset \tilde{Q}_k$ such that

$$M(|w|^p)(x_0) \le \lambda \text{ and } M(|f|)(x_1) + M(|g|)(x_1) \le L\lambda. \tag{2.14}$$

Then $⨍_{4\tilde{Q}_k} |w_a|^p \,\mathrm{d}x \le C\lambda$ due to (2.11) and (2.14). From (2.10) and (2.12) we get

$$⨍_{2\tilde{Q}_k} |w_a|^s \,\mathrm{d}x \le C'\lambda^{s/p}, \qquad ⨍_{4\tilde{Q}_k} |w - w_a|^p \,\mathrm{d}x \le \varepsilon\lambda + C ⨍_{4\tilde{Q}_k} |f| \,\mathrm{d}x \le (\varepsilon + L)C\lambda. \tag{2.15}$$

We define an operator M^* as follows:

$$M^*(f)(x) = \sup \left\{ ⨍_P f(y) \,\mathrm{d}y, P \text{ is a cube containig } x, P \subset 2\tilde{Q}_k \right\}. \tag{2.16}$$

Due to (2.14), it holds for every $x \in Q_k$ that $M\{|w|^p\} \le \max\{M^*(|w|^p), 2^n\lambda\}$. For K sufficiently large it follows that

$$M(|w|^p) > K\lambda \Rightarrow M^*(|w|^p) > K.\lambda \tag{2.17}$$

We use (2.15), Tchebyshev inequality and a strong type $\left(\frac{s}{p}, \frac{s}{p}\right)$ estimate for M^* in order to obtain following estimate

$$\begin{aligned}\left|\left\{x \in Q_k, M^*(|w_a|^p) > \frac{K}{2^{p+1}}\lambda\right\}\right| &= \left|\left\{x \in Q_k, M^*(|w_a|^p)^{s/p} > \left(\frac{K}{2^{p+1}}\lambda\right)^{s/p}\right\}\right| \\ &\le 2^{s+s/p}(\lambda K)^{-s/p} \||w_a|^p\|_{s/p}^{s/p} = 2^{s+s/p}(\lambda K)^{-s/p} |2\tilde{Q}_k| ⨍_{2\tilde{Q}_k} |w_a|^s \,\mathrm{d}x \le C_B K^{-s/p}|Q_k|.\end{aligned}$$

Due to (2.17)

$$|\{x \in Q_k, M(|w|^p) > K\lambda, M(|f|) + M(|g|) \le L\lambda\}| \le |\{x \in Q_k, M^*(|w|^p) \ge K\lambda, M(|f|) + M(|g|) \le L\lambda\}|$$

and, furhter,

$$\begin{aligned}&|\{x \in Q_k, M^*(|w|^p) > K\lambda, M(|f|) + M(|g|) \le L\lambda\}| \\ &\le \left|\left\{x \in Q_k, M^*(|w - w_a|^p) + M^*(|w_a|^p) > \frac{K}{2^p}\lambda, M(|f|) + M(|g|) \le L\lambda\right\}\right| \\ &\le \left|\left\{x \in Q_k, M^*(|w - w_a|^p) > \frac{K}{2^{p+1}}\lambda, M(|f|) + M(|g|) \le L\lambda)\right\}\right| + \left|\left\{x \in Q_k, M^*(|w_a|^p) \ge \frac{K}{2^{p+1}}\lambda\right\}\right| \\ &\le \left|\left\{x \in Q_k, M^*(|w - w_a|^p) > \frac{K}{2^{p+1}}\lambda, M(|f| + |g|) \le L\lambda\right\}\right| + C_B K^{-s/p}|Q_k|.\end{aligned}$$

Thus, using weak type $(1,1)$ estimates and (2.15), we get

$$\begin{aligned}|\{x \in Q_k, M^*(|w|^p) > K\lambda, M(|f| + |g|) \le L\lambda\}| &\le C\frac{2^{p+1}}{K\lambda} \int_{4\tilde{Q}_k} |w - w_a|^p \,\mathrm{d}x + C_B K^{-s/p}|Q_k| \\ &\le C\frac{2^{p+1}}{K}(\varepsilon + L)|Q_k| + C_B K^{-s/p}|Q_k|.\end{aligned}$$

By a suitable choice of constants L and ε we get the contradiction with the very first assumption of this proof. ☐

The previous lemma and Calderon-Zygmund theory, c.f. [1, Lemma 1.2], imply the following claim.

Corollary 2.9. *Let A and B be defined as in Lemma 2.8. Let K be so large and δ so small, that $K^{\frac{q}{p}} C_K < 1$ where $C_K := (\delta + C_B K^{-\frac{s}{p}})$, moreover, let $\lambda > K$ be such large that*

$$|\{x \in Q, M(|w|^p) > K\lambda\}| \le \frac{C}{K\lambda} \int_Q |w|^p = \delta|Q|. \tag{2.18}$$

Then $|A| < C_K|B|$.

Proof of Lemma 2.7. We set $h = M(|f| + |g|)$ and $l = M(|w|^p)$. By μ_h we denote the distribution function of a function h:

$$\mu_h(s) = |\{x \in Q : |h(x)| > s\}|, \quad s \geq 0.$$

From Corollary 2.9 one may derive

$$\mu_l(K\lambda) - \mu_h(L\lambda) \leq C_K \mu_l(\lambda). \tag{2.19}$$

By arguments of a measure theory it is true that $l \in L^{q/p}(Q)$ if and only if

$$\sum_{k=1}^{\infty} K^{k\frac{q}{p}} \mu_l(K^k\lambda) < \infty.$$

Due to (2.19) we get $\mu_l(K^k\lambda) \leq C_K \mu_l(K^{k-1}\lambda) + \mu_h(LK^{k-1}\lambda)$ for every $k \in \mathbb{N}$. Thus

$$\mu_l(K^k\lambda) \leq C_K^k \mu_l(\lambda) + \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda).$$

Consequently,

$$\sum_{k=1}^{\infty} K^{k\frac{q}{p}} \mu_l(K^k\lambda) \leq \mu_l(\lambda) \sum_{k=1}^{\infty} K^{k\frac{q}{p}} C_K^k + \sum_{k=1}^{\infty} K^{k\frac{q}{p}} \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda) \leq$$
$$\mu_l(\lambda) \sum_{k=1}^{\infty} \left(K^{q/p} C_K\right)^k + \sum_{k=1}^{\infty} K^{k\frac{q}{p}} \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda). \tag{2.20}$$

It suffices to choose δ and K such that $K^{q/p} C_K < 1$. This choice is possible due to definition of C_K. Thus

$$\mu_l(\lambda) \sum_{k=1}^{\infty} \left(K^{q/p} C_K\right)^k < \infty.$$

It holds that

$$\sum_{k=1}^{\infty} K^{k\frac{q}{p}} \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda) = \sum_{j=0}^{\infty} C_K^j \sum_{k=j+1}^{\infty} K^{k\frac{q}{p}} \mu_h(LK^{k-1-j}\lambda) = \sum_{j=0}^{\infty} \mu_h(LK^j\lambda) \sum_{i=0}^{\infty} (K^{\frac{q}{p}})^{j+i+1} C_K^i$$
$$= \sum_{j=0}^{\infty} (K^{q/p})^j \mu_h(LK^j\lambda) K^{\frac{q}{p}} \sum_{i=0}^{\infty} \left(K^{q/p} C_K\right)^i = \frac{K^{q/p}}{1 - K^{q/p} C_K} \sum_{j=0}^{\infty} (K^{q/p})^j \mu_h(K^j L\lambda) < \infty,$$

provided $h \in L^{\frac{q}{p}}(Q)$. Thus from (2.20) we have $l \in L^{\frac{q}{p}}(Q)$ and, consequently, $w \in L^q(Q)^n$.

Further,

$$\int_Q |w|^q \, \mathrm{d}x = \int_Q (|w|^p)^{q/p} \, \mathrm{d}x \leq \int_Q (M(|w|^p))^{q/p} \, \mathrm{d}x = \int_{\mathbb{R}^+} \underbrace{\frac{q}{p} t^{\frac{q}{p}-1} \mu_l(t)}_{=:\nu(t)} \, \mathrm{d}t \leq \int_0^{\lambda} \nu(t) \, \mathrm{d}t + \sum_{k=0}^{\infty} \int_{K^k\lambda}^{K^{k+1}\lambda} \nu(t) \, \mathrm{d}t$$
$$\leq \int_0^{\lambda} \nu(t) \, \mathrm{d}t + \sum_{k=0}^{\infty} K^k \lambda (K-1) \frac{q}{p} (K^{k+1}\lambda)^{\frac{q}{p}-1} \mu_l(K^k\lambda). \tag{2.21}$$

Using Tchebyshev inequality, we get

$$\int_0^{\lambda} \nu(t) \, \mathrm{d}t = \int_0^{\lambda} \frac{q}{p} t^{\frac{q}{p}-1} \mu_l(t) \, \mathrm{d}t \leq \int_0^{\lambda} \frac{q}{p} t^{\frac{q}{p}-1} \frac{1}{t} \int_Q |w|^p \, \mathrm{d}x \, \mathrm{d}t \leq c \lambda^{\frac{q}{p}-1} \int_Q |w|^p \, \mathrm{d}x.$$

Moreover, due to (2.20), we have

$$\sum_{k=0}^{\infty} K^k \lambda (K-1) \frac{q}{p} \left(K^{k+1}\lambda\right)^{\frac{q}{p}-1} \mu_l(K^k\lambda) \leq \frac{K-1}{K} \frac{q}{p} K^{\frac{q}{p}} \sum_{k=0}^{\infty} \left(K^k\lambda\right)^{\frac{q}{p}} \mu_l(K^k\lambda)$$
$$\leq C K^{\frac{q}{p}} \left(\mu_l(\lambda) + \int_Q \left(M(f)^{\frac{q}{p}} + M(g)^{\frac{q}{p}}\right) \, \mathrm{d}x\right).$$

We put these estimates into (2.21) and, since $\mu_l(\lambda) \le \frac{C}{\lambda} \int_Q |w|^p \,\mathrm{d}x$, we get

$$\int_Q |w|^q \,\mathrm{d}x \le C \int_{4Q} \left(|f|^{\frac{q}{p}} + |g|^{\frac{q}{p}}\right) \mathrm{d}x + C \left(\frac{K^{\frac{q}{p}}}{\lambda} + \lambda^{\frac{q}{p}-1}\right) \int_Q |w|^p \,\mathrm{d}x, \tag{2.22}$$

where a strong-type $\left(\frac{q}{p}, \frac{q}{p}\right)$ estimate for the maximal operator was applied. From (2.18) we know that $⨍_Q |w|^p \,\mathrm{d}x \sim \lambda$. Dividing (2.22) by $|Q|$ leads to

$$⨍_Q |w|^q \,\mathrm{d}x \le C \left(⨍_{4Q} |f|^{\frac{q}{p}} \,\mathrm{d}x + ⨍_{4Q} |g|^{\frac{q}{p}} \,\mathrm{d}x + \left(⨍_Q |w|^p \,\mathrm{d}x\right)^{\frac{q}{p}}\right),$$

where we used that $K < \lambda$. □

3 Flat boundary

In this section we put some ideas in the case when the boundary $\partial\Omega$ is flat. At first, consider the homogeneous system on the half–space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$

$$\begin{aligned} -\operatorname{div} \mathcal{S}(Dv) + \nabla p = 0 \qquad & \text{in } \mathbb{R}^n_+, \\ \operatorname{div} v = 0 \qquad & \text{in } \mathbb{R}^n_+, \\ v \cdot \nu = 0, \quad [\mathcal{S}(Dv)\nu] \cdot \tau = 0 \qquad & \text{on } \{x : x_n = 0\}. \end{aligned} \tag{3.1}$$

The aim of this section is to prove the following theorem.

Theorem 3.1. *Let $\Omega = \mathbb{R}^n_+$, $x_0 \in \Omega$ and Q be a cube with a center x_0. Assume that v is a solution to (3.1). Then there exists a constant C depending only on $\Delta_2(\{\Phi, \Phi^*\})$ and constants in (1.5) such that*

$$⨍_{\Omega\cap Q} |\nabla V(Dv)|^2 \,\mathrm{d}x \le \frac{C}{R^2} \Big(⨍_{\Omega\cap 2Q} |V(Dv)|^2 \,\mathrm{d}x\Big). \tag{3.2}$$

Moreover, if Φ'' is almost monotone, the estimate (3.2) can be improved to

$$⨍_{\Omega\cap Q} |\nabla V(Dv)|^2 \,\mathrm{d}x \le \frac{C}{R^2} \Big(⨍_{\Omega\cap 2Q} |V(Dv) - \langle V(Dv)\rangle_{2Q}|^2 \,\mathrm{d}x\Big). \tag{3.3}$$

By e_i we denote the unit vector in the direction x_i, $i = 1, \ldots, n$. Since the boundary is flat, $\tau^\alpha = e_\alpha$ for $\alpha = 1, \ldots, n-1$ and $\nu = -e_n$, $\partial_\nu = \partial_n$ and $\partial_{\tau^\alpha} = \partial_\alpha$ for $\alpha = 1, \ldots, n-1$. By x' we denote the first $n-1$ components of x, i.e. $x = (x', x_n)$. At first suppose that Φ'' is bounded from below and from above. The space $W^{1,\Phi}_\sigma(\mathbb{R}^n_+)$ reduces to $W^{1,2}_\sigma(\mathbb{R}^n_+)$.

Lemma 3.2. *Let Assumption 1.1 be fulfilled and $\Phi'' \in [c_3, c_4] \subset (0, \infty)$. Then for every weak solution to the problem (3.1) it holds*

$$v \in W^{2,2}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})^n.$$

Proof. We omit the proof. We only point out the main idea. Up to some modifications because of the boundary conditions we would follow the method used in [12, Section 3] where the authors are dealing with the evolution case in $\mathbb{R}^3$ under homogeneous Dirichlet boundary conditions. The authors are interested in the power-law model for the case $p \ge 2$.

The standard approach is to show the interior regularity first and then the regularity up to the boundary. The interior regularity and boundary regularity in tangent direction would be obtained easily using difference quotient technique. Our situation is easier since the boundary is flat. This computation for stationary problem in $\Omega \subset \mathbb{R}^2$ is also done in articles [9, Section 4] and [10, Theorem 3.19, Step 3]. □

If Φ'' is not bounded we consider the following truncation

$$(\Phi^\varepsilon)''(s) = \min\left(\max\left(\Phi''(s), \varepsilon\right), \frac{1}{\varepsilon}\right), \quad \varepsilon \in (0, 1).$$

We can construct $\mathcal{S}^\varepsilon$ by (1.4), define Ψ^ε by (1.6) and consider V^ε such that Ψ^ε is a scalar potential to V^ε. As one can easily check, Assumption 1.1 holds if we replace Φ and $\mathcal{S}$ by Φ^ε and $\mathcal{S}^\varepsilon$. Moreover, it holds

Proposition 3.3 (Proposition 4.1 in [11]). *If $\Phi \in \Delta_2$, $\Phi^* \in \Delta_2$ then also $\Phi^\varepsilon \in \Delta_2$, $(\Phi^\varepsilon)^* \in \Delta_2$ and $\Delta_2(\{\Phi^\varepsilon, (\Phi^\varepsilon)^*\})$ does not depend on ε.*

Instead of (3.1) we consider regularized boundary value problem

$$\begin{aligned} -\operatorname{div} \mathcal{S}^\varepsilon(Dv^\varepsilon) + \nabla p^\varepsilon = 0 \qquad & \text{in } \mathbb{R}^n_+, \\ \operatorname{div} v^\varepsilon = 0 \qquad & \text{in } \mathbb{R}^n_+, \\ v^\varepsilon \cdot \nu = 0, \quad [\mathcal{S}^\varepsilon(Dv^\varepsilon)\nu]\cdot\tau = 0 \qquad & \text{on } \{x : x_n = 0\}, \end{aligned} \tag{3.4}$$

Now we extend the solution from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. For $\alpha = 1,\ldots,n-1$ define $\tilde{v}^\varepsilon$ as follows

$$\tilde{v}^\varepsilon_\alpha(x', x_n) = \begin{cases} v^\varepsilon_\alpha(x', x_n) & \text{for } x_n > 0, \\ v^\varepsilon_\alpha(x', -x_n) & \text{for } x_n < 0, \end{cases} \tag{3.5}$$

$$\tilde{v}^\varepsilon_n(x', x_n) = \begin{cases} v^\varepsilon_n(x', x_n) & \text{for } x_n > 0, \\ -v^\varepsilon_n(x', -x_n) & \text{for } x_n < 0. \end{cases} \tag{3.6}$$

Using (3.5) and (3.6) we compute components of the symmetric gradient of $\tilde{v}^\varepsilon$ on $\mathbb{R}^n_-$. For $x_n > 0$ observe

$$\begin{aligned} D_{\alpha\alpha}\tilde{v}^\varepsilon(x', -x_n) &= D_{\alpha\alpha}\tilde{v}^\varepsilon(x', x_n), \\ D_{\alpha n}(x', -x_n) &= -D_{\alpha n}\tilde{v}^\varepsilon(x', x_n), \\ D_{nn}\tilde{v}^\varepsilon(x', -x_n) &= D_{nn}\tilde{v}^\varepsilon(x', x_n). \end{aligned}$$

Note that for $v^\varepsilon \in W^{1,\Phi}_\sigma(\mathbb{R}^n_+)^n$ the extended solution $\tilde{v}^\varepsilon$ belongs to $W^{1,\Phi}_\sigma(\mathbb{R}^n)^n$ since $\tilde{v}^\varepsilon$ is absolutely continuous on lines a.e. and the derivative of $\tilde{v}^\varepsilon$ is in $L^\Phi(\mathbb{R}^n)^{n\times n}$ pointwisely. For a test function $\varphi \in W^{1,\Phi}_\sigma(\mathbb{R}^n)^n$ we define φ^+ by components

$$\begin{aligned} \varphi^+_\alpha &= \frac{1}{2}\big(\varphi_\alpha(x', x_n) + \varphi_\alpha(x', -x_n)\big), \quad \alpha = 1,\ldots,n-1, \\ \varphi^+_n &= \frac{1}{2}\big(\varphi_n(x', x_n) - \varphi_n(x', -x_n)\big), \end{aligned}$$

and similarly

$$\begin{aligned} \varphi^-_\alpha &= \frac{1}{2}\big(\varphi_\alpha(x', x_n) - \varphi_\alpha(x', -x_n)\big), \quad \alpha = 1,\ldots,n-1, \\ \varphi^-_n &= \frac{1}{2}\big(\varphi_n(x', x_n) + \varphi_n(x', -x_n)\big), \end{aligned}$$

One can easily check that $\operatorname{div}\varphi^+ = \operatorname{div}\varphi^- = 0$ holds in $\mathbb{R}^n$. Thus

$$\int_{\mathbb{R}^n} \mathcal{S}^\varepsilon(D\tilde{v}^\varepsilon) : D\varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{S}^\varepsilon(D\tilde{v}^\varepsilon) : D\varphi^+ \,\mathrm{d}x + \int_{\mathbb{R}^n} \mathcal{S}^\varepsilon(D\tilde{v}^\varepsilon) : D\varphi^- \,\mathrm{d}x = \mathcal{I}_1 + \mathcal{I}_2 = 0 \quad \forall \varphi \in W^{1,\Phi}_\sigma(\mathbb{R}^n)^n, \tag{3.7}$$

where $\mathcal{I}_1$ is equal to zero due to the equation (3.4) and $\mathcal{I}_2$ is equal to zero because of the symmetry. We can see that now we can consider (3.7) instead of the weak formulation of (3.4) on the half-space.

Remark 3.4. *In the result of this section there appeared average integrals over cube Q and its multiple αQ. There could appear a problem if we were close to the boundary. Nevertheless, due to the fact the solution is extended beyond the boundary in the way we presented, results of this section holds also in the case we consider $\alpha Q \cap \mathbb{R}^n_+$ instead of αQ.*

Lemma 3.5. *For fixed $\alpha, \alpha' \in (1,2), \alpha < \alpha'$, there exists constant C depending on α, α' and $\Delta_2(\{\Phi, \Phi^*\})$ such that uniformly in $\varepsilon \in (0,1)$ holds*

$$⨍_{\alpha Q} |\nabla V^\varepsilon(D\tilde{v}^\varepsilon)|^2 \,\mathrm{d}x \le \frac{C}{R^2} ⨍_{\alpha' Q} |V^\varepsilon(D\tilde{v}^\varepsilon)|^2 \,\mathrm{d}x. \tag{3.8}$$

and under the additional assumption that Φ'' is almost monotone:

$$⨍_{\alpha Q} |\nabla V^\varepsilon(D\tilde{v}^\varepsilon)|^2 \,\mathrm{d}x \le \frac{C}{R^2} ⨍_{\alpha' Q} |V^\varepsilon(D\tilde{v}^\varepsilon) - \langle V^\varepsilon(D\tilde{v}^\varepsilon)\rangle_{\alpha' Q}|^2 \,\mathrm{d}x. \tag{3.9}$$

Proof. This lemma is proven in [3, Lemma 3.5] in case $n = 3$. The test function was constructed to take advantage of the operator curl, which is in $\mathbb{R}^3$ defined by $\operatorname{curl} g = (\partial_2 g_3 - \partial_3 g_2, \partial_3 g_1 - \partial_1 g_3, \partial_1 g_2 - \partial_2 g_1)$. Since the authors are not aware of any straightforward generalization of the curl operator to n dimensions, we use the language of exterior differential calculus to construct the right test function.

At first we state some notation. Although we denoted by $\{e_i, i = 1, \dots, n\}$ the orthonormal basis in R^n before, we need to distinguish vectors and forms now and therefore we use $\{\partial_i, i = 1, \dots, n\}$ to represent an orthonormal basis for vectors in $\mathbb{R}^n$, whereas $\{dx^i, i = 1, \dots, n\}$ denotes corresponding dual 1-form basis. In order to follow the standard notation, we use in this section the upper indices for components of vectors whereas the lower indices indicates components of form of any order. In the next section we won't work with such forms, therefore all indices will be the lower ones.

We will use so called musical isomorphisms $\sharp$ and $\flat$, where $\sharp$ raise the indices of a 1-form β to give the vector $\beta^\sharp$ whereas $\flat$ lowers the indices of a vector z to produces a 1-form $z^\flat$, i.e.

$$\beta = \sum_{i=1}^n \beta_i dx^i, \quad \beta^\sharp = \sum_{i=1}^n \beta^i \partial_i, \quad z = \sum_{i=1}^n z^i \partial_i, \quad z^\flat = \sum_{i=1}^n z_i dx^i.$$

By d we mean the exterior derivative and the symbol $\wedge$ denotes the wedge product. Let us denote

$$\begin{aligned} \widehat{dx^i} &:= dx^1 \cdots \wedge dx^{i-1} \wedge dx^{i+1} \cdots \wedge dx^n, \\ \widehat{dx^i \wedge dx^j} &:= dx^1 \cdots \wedge dx^{i-1} \wedge dx^{i+1} \cdots \wedge dx^{j-1} \wedge dx^{j+1} \cdots \wedge dx^n. \end{aligned}$$

The Hodge map $\star$ is linear isomorphism between the vector spaces of differential $k-$ and $(n-k)-$ forms. In Riemannian metric it holds

$$\star(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n},$$

where $(i_1, \dots, i_n)$ is any even permutation of $(1, 2, \dots, n)$. Let $\xi \in \mathcal{C}_0^\infty(\alpha' Q)$ be a cut-off function with $\chi_{\alpha Q} \le \xi \le \chi_{\alpha' Q}$ and $\|\nabla^j \xi\|_\infty \le C/R^j$ for $j = 1, 2$. Let $q : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function with $\nabla q = \langle \nabla \tilde{v}^\varepsilon \rangle_{\alpha' Q}$. We test (3.7) by

$$\varphi = \big(\star d[\xi^2 \star d(\tilde{v}^\varepsilon - q)^\flat]\big)^\sharp. \tag{3.10}$$

Note that the test function is well defined. $\flat$ converts the vector field $(\tilde{v}^\varepsilon - q)$ into a 1-form $(\tilde{v}^\varepsilon - q)^\flat$. d computes something like a curl but, but expressed as a 2-form $d(\tilde{v}^\varepsilon - q)^\flat$. $\star$ turns this 2-form into a $(n-2)$-form. After multiplication by ξ^2 and application of the derivative d we obtain $(n-1)-$form and Hodge star $\star$ create $1-$form, which is by $\sharp$ converted to the vector.

Moreover, one can easily see that $\operatorname{div} \varphi = 0$, since $\operatorname{div} \varphi = \star d \star \varphi^\flat$ and $dd\gamma = 0$ for an arbitrary degree differential form.

Let's see how (3.10) looks in components. For better lucidity we define $z = \tilde{v}^\varepsilon - q$. At first we compute derivative of $z^\flat = \sum_{i=1}^n z_i dx^i$ and apply the Hodge map:

$$\begin{aligned} dz^\flat &= \sum_{i,j=1}^n \partial_j z_i dx^j \wedge dx^i = \sum_{i<j} (\partial_i z_j - \partial_j z_i) dx^i \wedge dx^j, \\ \xi^2 \star dz^\flat &= \sum_{i<j} \xi^2 (\partial_i z_j - \partial_j z_i)(-1)^{i+j-3} \widehat{dx^i \wedge dx^j}, \end{aligned}$$

where we used that $dx^i \wedge dx^j = -dx^j \wedge dx^i$ for $i \neq j$ and $dx^i \wedge dx^i = 0$. Further,

$$\begin{aligned} d(\xi^2 \star dz^\flat) =& \sum_{i<j} [\xi^2(\partial_i^2 z_j - \partial_i \partial_j z_i) + 2\xi \partial_i \xi (\partial_i z_j - \partial_j z_i)](-1)^{i+j-3} dx^i \wedge \widehat{dx^i \wedge dx^j} + \\ & \sum_{i<j} [\xi^2(\partial_j \partial_i z_j - \partial_j^2 z_i) + 2\xi \partial_j \xi (\partial_i z_j - \partial_j z_i)](-1)^{i+j-3} dx^j \wedge \widehat{dx^i \wedge dx^j}. \end{aligned} \tag{3.11}$$

We can change the summation indices in the second sum in (3.11), move dx^i to the i-th position in the product and finally put these two sums together.

$$\begin{aligned} d(\xi^2 \star dz^\flat) =& \sum_{i<j} [\xi^2(\partial_i^2 z_j - \partial_i \partial_j z_i) + 2\xi \partial_i \xi (\partial_i z_j - \partial_j z_i)](-1)^{i+j-3+i-1} \widehat{dx^j} \\ &+ \sum_{i>j} [\xi^2(-\partial_i^2 z_j + \partial_i \partial_j z_i) + 2\xi \partial_i \xi (-\partial_i z_j + \partial_j z_i)](-1)^{i+j-3+i-2} \widehat{dx^j} \\ =& \sum_{i \neq j} [\xi^2(\partial_i^2 z_j - \partial_i \partial_j z_i) + 2\xi \partial_i \xi (\partial_i z_j - \partial_j z_i)](-1)^j \widehat{dx^j}. \end{aligned} \tag{3.12}$$

Thus, applying the Hodge star and going back from forms to vectors

$$[\star d(\xi^2 \star dz^\flat)]^\sharp = \sum_{i\neq j}[\xi^2(\partial_i^2 z^j - \partial_i\partial_j z^i) + 2\xi\partial_i\xi(\partial_i z^j - \partial_j z^i)](-1)^{j+j-1}\partial_j.$$

As one can easily check, z is divergence-free, therefore $\sum_{i\neq j}\partial_i\partial_j z_i = -\partial_j\partial_j z_j$ and we finally obtain

$$\varphi = \sum_{i,j=1}^{n}\Big(-\xi^2\partial_i^2(\tilde v^\varepsilon)^j + 2\xi\partial_i\xi[-\partial_i(\tilde v^\varepsilon - q)^j + \partial_j(\tilde v^\varepsilon - q)^i]\Big)\partial_j, \tag{3.13}$$

where we moreover used that q is a linear function, thus $\partial_i^2 q = 0$. Inserting (3.13) into (3.7) we get

$$\begin{aligned} -\int_{\alpha' Q}[\mathcal{S}^\varepsilon(D\tilde v^\varepsilon) - \mathcal{S}^\varepsilon(Dq)] : \nabla(\xi^2\Delta\tilde v^\varepsilon)\,\mathrm{d}x \\ + \int_{\alpha' Q}[\mathcal{S}^\varepsilon(D\tilde v^\varepsilon) - \mathcal{S}^\varepsilon(Dq)] : \nabla\big(2\xi\nabla\xi(\nabla\tilde v^\varepsilon - \nabla q) - (\nabla\tilde v^\varepsilon - \nabla q)^T\big)\,\mathrm{d}x = \mathcal{J}_1 + \mathcal{J}_2 = 0. \end{aligned} \tag{3.14}$$

From this point we proceed almost in the same way as in [3, Proof of Lemma 3.5], where the authors due to the different regularization estimated more terms. For the sake of completeness we reproduce the computation also here.

Lets start with proving (3.9). We proceed in a different way when Φ'' is almost decreasing or almost increasing. At first let us assume that Φ'' is almost decreasing. After some manipulation involving integrating by parts in the first term we have

$$\begin{aligned} \mathcal{J}_1 = \int_{\alpha' Q}\nabla\mathcal{S}^\varepsilon(D\tilde v^\varepsilon)\xi^2\nabla^2\tilde v^\varepsilon\,\mathrm{d}x - \int_{\alpha' Q}[\mathcal{S}^\varepsilon(D\tilde v^\varepsilon) - \mathcal{S}^\varepsilon(Dq)]\operatorname{div}(\nabla\xi^2\otimes(\nabla\tilde v^\varepsilon - \nabla q))\,\mathrm{d}x \\ + \int_{\alpha' Q}[\mathcal{S}^\varepsilon(D\tilde v^\varepsilon) - \mathcal{S}^\varepsilon(Dq)]\nabla[(\nabla\tilde v^\varepsilon - \nabla q)\nabla\xi^2]\,\mathrm{d}x = \mathcal{J}_{1.1} + \mathcal{J}_{1.2} + \mathcal{J}_{1.3}. \end{aligned} \tag{3.15}$$

Assumption 1.1, symmetry of $\mathcal{S}^\varepsilon$, the relation between Ψ'' and Φ'' (1.8) and the definition of the function V (1.7) are used to gain the following information from $\mathcal{J}_{1.1}$:

$$\mathcal{J}_{1.1} \geq C\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon|)\xi|\nabla D\tilde v^\varepsilon|^2\,\mathrm{d}x \geq C\int_{\alpha Q}|\nabla V^\varepsilon(D\tilde v^\varepsilon)|^2\,\mathrm{d}x. \tag{3.16}$$

Notice that $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and the term $\mathcal{J}_2$ have similar structure and can be estimated together as follows

$$|\mathcal{J}_{1.2}| + |\mathcal{J}_{1.3}| + |\mathcal{J}_2| \leq C\int_{\alpha' Q}|\mathcal{S}^\varepsilon(D\tilde v^\varepsilon) - \mathcal{S}^\varepsilon(Dq)|\Big(\frac{1}{R^2}|\nabla\tilde v^\varepsilon - \nabla q)| + \frac{1}{R}\xi|\nabla^2\tilde v^\varepsilon|\Big)\,\mathrm{d}x = \mathcal{J}_3 + \mathcal{J}_4.$$

In order to estimate $\mathcal{J}_3$ we use Young's inequality (2.1) together with (2.2), Korn's inequality (2.5) and Lemma 2.4.

$$\begin{aligned} \mathcal{J}_3 \leq \frac{C}{R^2}\int_{\alpha' Q}(\Phi^\varepsilon)'_{|Dq|}(|D\tilde v^\varepsilon - Dq|)|\nabla\tilde v^\varepsilon - \nabla q|\,\mathrm{d}x \leq \frac{C}{R^2}\int_{\alpha' Q}\Phi^\varepsilon_{|Dq|}(|D\tilde v^\varepsilon - Dq|)\,\mathrm{d}x \\ + \frac{C}{R^2}\int_{\alpha' Q}\Phi^\varepsilon_{|Dq|}(|\nabla\tilde v^\varepsilon - \nabla q|)\,\mathrm{d}x \leq \frac{C}{R^2}\int_{\alpha' Q}|V^\varepsilon(D\tilde v^\varepsilon) - V^\varepsilon(Dq)|^2\,\mathrm{d}x. \end{aligned}$$

In estimates of the last integral $\mathcal{J}_4$ the assumption on almost monotonicity of $\Phi^{\varepsilon\prime\prime}$ will be needed. Using Assumption 1.1, the classical Young's inequality and Lemma 2.4 we get

$$\begin{aligned} \mathcal{J}_4 \leq \frac{C}{R}\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon| + |Dq|)|D\tilde v^\varepsilon - Dq|\xi|\nabla^2\tilde v^\varepsilon|\,\mathrm{d}x \leq \frac{C(\delta)}{R^2}\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon| + |Dq|)|D\tilde v^\varepsilon - Dq|^2\,\mathrm{d}x \\ +\delta\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon| + |Dq|)|\nabla^2\tilde v^\varepsilon|^2\xi^2\,\mathrm{d}x \leq \frac{C}{R^2}\int_{\alpha' Q}|V^\varepsilon(D\tilde v^\varepsilon) - V^\varepsilon(Dq)|^2\,\mathrm{d}x + \delta\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon|)|\nabla^2\tilde v^\varepsilon|^2\xi^2\,\mathrm{d}x, \end{aligned}$$

where in the last term we used the fact that Φ^ε is almost decreasing. Since $\delta > 0$ can be chosen arbitrarily small, this term can be subsumed into (3.16).

In case Φ'' is almost increasing, it suffices to estimate $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and $\mathcal{J}_2$ in a different way, other estimates remains the same. We integrate by parts in $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and $\mathcal{J}_2$ and obtain

$$\begin{aligned} |\mathcal{J}_{1.2}| + |\mathcal{J}_{1.3}| + |\mathcal{J}_2| \leq \frac{C}{R}\int_{\alpha' Q}|\nabla\mathcal{S}^\varepsilon(D\tilde v^\varepsilon)||\nabla\tilde v^\varepsilon - \nabla q|\xi\,\mathrm{d}x \\ \leq \delta\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon|)|\nabla^2\tilde v^\varepsilon|^2\xi^2\,\mathrm{d}x + \frac{C(\delta)}{R^2}\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde v^\varepsilon|)|\nabla\tilde v^\varepsilon - \nabla q|^2\,\mathrm{d}x = \mathcal{J}_5 + \mathcal{J}_6, \end{aligned}$$

where we moreover used the classical Young's inequality in the last step. The term $\mathcal{J}_5$ can be subsumed into (3.16). Since $(\Phi^\varepsilon)''$ is almost increasing, we can add $|\nabla\tilde{v}^\varepsilon - \nabla q|$ to the argument of $\mathcal{J}_6$, use the definition of shifted N-functions (2.3) and apply Lemma 2.4.

$$\begin{aligned}\mathcal{J}_6 &\le \frac{C(\delta)}{R^2}\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde{v}^\varepsilon| + |\nabla\tilde{v}^\varepsilon - \nabla q|)|\nabla\tilde{v}^\varepsilon - \nabla q)|^2\,\mathrm{d}x\\ &\le \frac{C(\delta)}{R^2}\int_{\alpha' Q}\Phi^\varepsilon{}_{|D\tilde{v}^\varepsilon|}(|\nabla\tilde{v}^\varepsilon - \nabla q|)\,\mathrm{d}x \le \frac{C}{R^2}\int_{\alpha' Q}|V^\varepsilon(D\tilde{v}^\varepsilon) - V^\varepsilon(Dq)|^2\,\mathrm{d}x.\end{aligned}$$

We gather all estimates above, use that $Dq = \langle D\tilde{v}^\varepsilon\rangle_{2Q}$ and apply Lemma 2.3 to obtain

$$\int_{\alpha' Q}|V^\varepsilon(D\tilde{v}^\varepsilon) - V^\varepsilon(\langle D\tilde{v}^\varepsilon\rangle_{2Q})|^2\,\mathrm{d}x \le \int_{\alpha' Q}|V^\varepsilon(D\tilde{v}^\varepsilon) - \langle V^\varepsilon(D\tilde{v}^\varepsilon)\rangle_{2Q}|^2\,\mathrm{d}x,$$

which concludes the proof of (3.9). To prove (3.8) it is enough to focus on estimates of $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and $\mathcal{J}_2$ where the assumption of almost monotonicity was used. Considering the same test function and omitting the term $\mathcal{S}^\varepsilon(Dq)$ in (3.14) we have

$$|\mathcal{J}_{1.2}| + |\mathcal{J}_{1.3}| + |\mathcal{J}_2| \le C\int_{\alpha' Q}|\mathcal{S}^\varepsilon(D\tilde{v}^\varepsilon)|\Big(\frac{1}{R^2}|\nabla\tilde{v}^\varepsilon - \nabla q)| + \frac{1}{R}\xi|\nabla^2\tilde{v}^\varepsilon|\Big)\,\mathrm{d}x = \mathcal{J}_7 + \mathcal{J}_8.$$

In $\mathcal{J}_7$ we proceed like in $\mathcal{J}_3$. The situation is easier since we don't need to deal with shifted N-functions.

$$\begin{aligned}\mathcal{J}_7 &\le \frac{C}{R^2}\int_{\alpha' Q}(\Phi^\varepsilon)'(|D\tilde{v}^\varepsilon|)|\nabla\tilde{v}^\varepsilon - \nabla q|\,\mathrm{d}x \le \frac{C}{R^2}\int_{\alpha' Q}\Phi^\varepsilon(|D\tilde{v}^\varepsilon|)\,\mathrm{d}x\\ &\quad + \frac{C}{R^2}\int_{\alpha' Q}\Phi^\varepsilon(|\nabla\tilde{v}^\varepsilon - \nabla q|)\,\mathrm{d}x \le \frac{C}{R^2}\int_{\alpha' Q}|V^\varepsilon(D\tilde{v}^\varepsilon)|^2\,\mathrm{d}x.\end{aligned}$$

The term $\mathcal{J}_8$ can be handled in a similar way like $\mathcal{J}_4$:

$$\begin{aligned}\mathcal{J}_8 &\le \frac{C}{R}\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde{v}^\varepsilon|)|D\tilde{v}^\varepsilon|\xi|\nabla^2\tilde{v}^\varepsilon|\,\mathrm{d}x \le \frac{C(\delta)}{R^2}\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde{v}^\varepsilon|)|D\tilde{v}^\varepsilon|^2\,\mathrm{d}x\\ &+\delta\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde{v}^\varepsilon|)|\nabla^2\tilde{v}^\varepsilon|^2\xi^2\,\mathrm{d}x \le \frac{C}{R^2}\int_{\alpha' Q}|V^\varepsilon(D\tilde{v}^\varepsilon)|^2\,\mathrm{d}x + \delta\int_{\alpha' Q}(\Phi^\varepsilon)''(|D\tilde{v}^\varepsilon|)|\nabla^2\tilde{v}^\varepsilon|^2\xi^2\,\mathrm{d}x,\end{aligned}$$

where the last term can be subsumed to (3.16). Thus, the estimate (3.8) holds. □

Proof of Theorem 3.1. It remains to pass with $\varepsilon \to 0^+$. Two limit passages need to be shown. First, we need to go from the regularized equations (3.4) to the original one (3.1). Second, we need to obtain estimates (3.2) and (3.3) from (3.8) and (3.9). Since we used the same regularization of Φ'' as in [11], the structure of the limit passages is the same and therefore we mention only important steps. For details we refer to [11, Section 5].

To pass from (3.4) to (3.1) we need at first almost everywhere convergence of symmetric gradients. We use

Lemma 3.6. *[11, Lemma 5.1] Let $-1 < \beta < 0 < \alpha$ and $c > 0$. We define $m(s) = cs^\alpha$ for $s \in (0,1)$ and $m(s) = cs^\beta$ for $s \ge 1$. Let there exist $C > 0$ such that the sequence $\{A_k\}_{k=1}^\infty$, $A_k : Q \to \mathbb{R}^3$ fulfills*

$$\int_Q m(|A_k|)(|A_k|^2 + |\nabla A_k|^2)\,\mathrm{d}x \le C. \tag{3.17}$$

Then there exist a subsequence $\{A_{k_l}\}_{l=1}^\infty$ and A such that $A_{kl} \to A$ a.e. in Ω as $l \to \infty$.

From Lemma 2.4 we know that $|V^\varepsilon(Du)|^2 \sim (\Phi^\varepsilon)''(|Du|)|Du|^2$, therefore (3.8) and (3.9) can be rewritten to the form

$$\int_Q(\Phi^\varepsilon)''(|D\tilde{v}^\varepsilon|)(|D\tilde{v}^\varepsilon|^2 + |\nabla D\tilde{v}^\varepsilon|^2)\,\mathrm{d}x \le C,$$

Due to [13, Chapter 2, Corollary 5] and $\Phi^\varepsilon(s) \sim s^2(\Phi^\varepsilon)''(s)$ existence of a function m, such that (3.17) holds, is guaranteed. Lemma 3.6 gives $D\tilde{v}^\varepsilon \to A$. To identify A with $D\tilde{v}$ we consider an N-function $\tilde{\Psi}$ with $m(|B|) = \tilde{\Psi}''(|B|)$ for all $B \in \mathbb{R}^{n\times n}$. From (3.17) and Korn's inequality we have the uniform estimate $\int_Q \tilde{\Psi}(|\tilde{v}^\varepsilon|)\,\mathrm{d}x + \int_Q \tilde{\Psi}(|\nabla\tilde{v}^\varepsilon|)\,\mathrm{d}x \le C$. Thus, there is $\tilde{v}^\varepsilon \in W^{1,\tilde{\Psi}}(Q)^n$ such that $\nabla\tilde{v}^\varepsilon \rightharpoonup \nabla\tilde{v}$ in $L^{\tilde{\Psi}}(Q)^{n\times n}$ up to a subseguence. Clearly, $A = D\tilde{v}$ and

$$D\tilde{v}^\varepsilon \to D\tilde{v} \quad a.e. \tag{3.18}$$

Since Φ^ε is a scalar potential to $\mathcal{S}^\varepsilon$, the second ingredient to pass in the equation is the uniform integrability of $(\Phi^\varepsilon)'$ which is provided by the following lemma.

Lemma 3.7. *[11, Lemma 5.2] Let $\int_\Omega \Phi^\varepsilon(|D\tilde{v}^\varepsilon|)\,\mathrm{d}x \le C$. Then there exists $\delta > 0$ such that for all $\varepsilon, \sigma \in (0,1)$ and all $E \subset \Omega$ such that $|E| < \delta$*

$$\int_E (\Phi^\varepsilon)'(|D\tilde{v}^\varepsilon|)\,\mathrm{d}x \le \sigma.$$

Now it is straightforward to pass to the limit in (3.4) as $\varepsilon \to 0^+$ and get by Vitali's theorem (3.1). Lets focus on passing from (3.8) to (3.2) and (3.9) to (3.3). The family $\{V^\varepsilon(D\tilde{v}^\varepsilon)\}_\varepsilon$ is bounded in $W^{1,2}(Q)^n$, so up to a subsequence due to the embedding $V^\varepsilon(D\tilde{v}^\varepsilon) \to \chi$ in $L^2(Q)^{n\times n}$. To identify χ with $V(D\tilde{v})$ and pass in the left hand sides of (3.8) and (3.9) it is sufficient to show locally uniform convergence of V^ε since we already have (3.18).

Lemma 3.8. *[11, Lemma 5.3] Let K be a compact subset of $\mathbb{R}^{n\times n}$. Then $V^\varepsilon \rightrightarrows V$ on K as $\varepsilon \to 0^+$.*

To pass in the right hand side of (3.8) and (3.9) and finish the limit process we need the uniform integrability of $|V^\varepsilon|^2$:

Lemma 3.9. *Let $\int_\Omega (|\nabla V^\varepsilon(D\tilde{v}^\varepsilon)|^2 + |V^\varepsilon(D\tilde{v}^\varepsilon)|^2)\,\mathrm{d}x \le C$. Then there exists $\delta > 0$ such that for all $\varepsilon, \sigma \in (0,1)$ and all $E \subset \Omega$ such that $|E| < \delta$ holds*

$$\int_E |V^\varepsilon(D\tilde{v}^\varepsilon)|^2\,\mathrm{d}x \le \sigma.$$

Proof. It follows easily from

$$\int_E |V^\varepsilon(D\tilde{v}^\varepsilon)|^2\,\mathrm{d}x \le c\|\chi_E\|_{\frac{n}{2}}\|V^\varepsilon(D\tilde{v}^\varepsilon)\|_{\frac{2n}{n-2}} \to 0 \text{ as } |E| \to 0.$$

□

This concludes the proof of Theorem 3.1.

□

4 Proof of the main theorem

In order to handle a general $\mathcal{C}^{2,1}$ non-flat boundary, we present its following description. Throughout this section, we assume that $x_0 \in \partial\Omega$ is fixed. The boundary can be understood on a neighborhood of the point x_0 as a graph of a function $a : \mathbb{R}^{n-1} \mapsto \mathbb{R}^n$, $a(0) = x_0$ such that $\partial_\alpha a = e_\alpha$ for $\alpha = 1, \ldots, n-1$. We set[2] $n(x') = \partial_1 a \times \ldots \times \partial_{n-1} a(x')$ and we introduce a function $H_{x_0} : \mathbb{R}^n \mapsto \mathbb{R}^n$ which is defined as

$$H_{x_0}(x) := a(x') + n(x')x_n.$$

For $R > 0$ we also consider restrictions $H_{x_0,R}$ of the function H_{x_0} on a half–ball $B_R^+ := B_R \cap \mathbb{R}^n_+$, i.e.:

$$H_{x_0,R}(x) = H_{x_0}(x)|_{B_R^+}.$$

Since x_0 is fixed, we use H_R instead of $H_{x_0,R}$ and H instead of H_{x_0} throughout this chapter. It holds that $H_R(0) = x_0$. It follows, that $\nabla H = I$ and smoothness of the boundary implies that $H_R \in \mathcal{C}^{1,1}$ and, consequently, $\nabla H_R(x) - \nabla H_R(0) = R\omega$ where ω is a function bounded independently of R. Similarly, also $\nabla H_R^{-1}(x) - \nabla H_R^{-1}(0) = R\omega$. Hereinafter, ω stands for a matrix valued function and ω'' for a real-valued function which express a perturbation arising from a curvature of the boundary. These functions may vary from line to line, however they are bounded independently of R.

The function H_R maps B_R^+ into Ω for all $R < R_0$ sufficiently small. Furthermore, we set $y = H_R(x)$, $\Omega_R := H_R(B_R^+)$ and $\Gamma_R = \overline{\Omega_R} \cap \partial\Omega$.

For a general function $f : \Omega_R \mapsto \mathbb{R}$ we state a function $\overline{f} : B_R^+ \mapsto \mathbb{R}$ defined as $\overline{f}(x) = f(H_R(x)) = f(y)$. It holds that

$$\nabla_y f = \nabla_x \overline{f} \nabla H_R^{-1} = \nabla_x \overline{f} + R\nabla_x \overline{f}\omega.$$

In case $f : \Omega_R \mapsto \mathbb{R}^n$ it also holds

$$\begin{aligned} 2D_y f &= \left(\nabla_x \overline{f} \nabla H_R^{-1}\right) + \left(\nabla_x \overline{f} \nabla H_R^{-1}\right)^T = 2D_x \overline{f} + Z_{\overline{f}} \\ \operatorname{div}_y f &= \operatorname{Tr}\left(\nabla_x \overline{f} \nabla H_R^{-1}\right) = \operatorname{div}_x \overline{f} + R\operatorname{Tr}\left(\nabla_x \overline{f}\omega\right), \end{aligned}$$

where

$$Z_{\overline{f}} = \left(\nabla_x \overline{f}(\nabla H_R^{-1} - I) + (\nabla H_R^{-1} - I)^T(\nabla_x \overline{f})^T\right).$$

[2] Recall that by x' we denote the first $n-1$ coordinates of x, i.e. $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$

We consider a function $\pi_{\Omega_R} := \pi - \pi_c$ where a constant π_c will be determined later. From the Definition 1.2 we have

$$\int_{\Omega_R} \mathcal{S}(Du) : D\varphi \,\mathrm{d}y - \int_{\Omega_R} \pi_{\Omega_R} \operatorname{div}\varphi \,\mathrm{d}y = \int_{\Omega_R} F : D\varphi \,\mathrm{d}y, \tag{4.1}$$

whenever $\varphi \in W^{1,\Phi}_{\nu}(\Omega)^n$, $\varphi = 0$ on $\partial\Omega_R \setminus \partial\Omega$. The equation (4.1) can be transformed using the function H into the following identity

$$\begin{aligned}\int_{B_R^+} \mathcal{S}(D\overline{u} + Z_{\overline{u}}) : (D\overline{\varphi} + Z_{\overline{\varphi}})(\det \nabla H_R) \,\mathrm{d}x - \int_{B_R^+} \overline{\pi}_{\Omega_R}(\operatorname{div}\overline{\varphi} + \operatorname{Tr}(\nabla\overline{\varphi}(\nabla H_R^{-1} - I)))(\det \nabla H_R) \,\mathrm{d}x \\ = \int_{B_R^+} \overline{F} : (\nabla\overline{\varphi} + \overline{\varphi}(\nabla H_R^{-1} - I))(\det \nabla H_R) \,\mathrm{d}x. \end{aligned}\tag{4.2}$$

which holds for all $\overline{\varphi} \in W^{1,\Phi}(B_R^+)^n$ satisfying

$$\overline{\varphi} = 0 \text{ on } \partial B_R^+ \setminus \Gamma_{B_R^+}, \tag{4.3}$$

$$\overline{\varphi}\cdot\nu = 0 \text{ on } \Gamma_{B_R^+}, \tag{4.4}$$

where $\Gamma_{B_R^+} = \partial B_R^+ \cap \{x; x_n = 0\}$. We point out that $\det \nabla H_R = 1 + R\omega''$. Let us also emphasize that $\overline{u}$ satisfies (4.4).

We choose a constant π_c such that, using considerations presented in [3, Section 4.2], we can derive that

$$\int_{\Omega_R} \Phi^*(|\pi_{\Omega_R}|) \,\mathrm{d}y \le C\left(\int_{\Omega_R} \Phi^*(|\mathcal{S}(Du)|) \,\mathrm{d}y + \int_{\Omega_R} \Phi^*(|F|) \,\mathrm{d}y\right),$$

and for sufficiently small $R > 0$, using Lemma 2.5, we also get

$$\int_{B_R^+} \Phi^*(|\overline{\pi}_{\Omega_R}|) \,\mathrm{d}x \le C\left(\int_{B_R^+} \Phi(|D\overline{u}|) \,\mathrm{d}x + \int_{B_R^+} \Phi^*(|\overline{F}|) \,\mathrm{d}x + \int_{B_R^+} \Phi(|\overline{u}|) \,\mathrm{d}x\right), \tag{4.5}$$

where we employed $|\mathcal{S}(Du)| = \Phi'(|Du|)$, Lemma 2.2 and (2.2).

In what follows, we verify that for a solution $\overline{u}$ there exists an approximative function v, which is solution to (3.1), such that $D\overline{u}$ and Dv satisfy hypothesis of Lemma 2.7.

From Theorem 3.1 and from Poincaré inequality we get for every cube $Q' \subset \mathbb{R}^n$

$$\left(\fint_{Q'} |V(Dv)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{2Q'} |V(Dv)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \tag{4.6}$$

for $q \le \frac{2n}{n-2}$ provided $n \ge 3$ and $q > 2$ arbitrary for $n = 2$. In case Φ'' is almost monotone, $n \ge 3$, we can even allow $q = \frac{rn}{n-r}$ for some $r > 2$. This improvement follows from Sobolev-Poincaré and Reverse Hölder inequalities, c.f. [3, Theorem 3.6]. Thus, the condition for approximative function (2.10) is verified.

The verification of (2.11) and (2.12) is presented in the following lemma. In this lemma we work with cubes Q, however this lemma holds true even for rectangles which appear in (2.11) and (2.12).

Lemma 4.1. *Let R be sufficiently small. Let $\overline{u} \in W^{1,\Phi}(B_R^+)^n$ be a solution to (4.2) and let Q be a cube*[3] *contained in B_R^+. Then there exists a weak solution $v \in W^{1,\Phi}(Q)^n$ to (3.1) and a positive constant C independent of $\overline{u}$, v, Q and R such that*

$$\int_Q |V(Dv)|^2 \,\mathrm{d}x \le C\int_Q |V(D\overline{u})|^2 \,\mathrm{d}x + C\int_Q \Phi(|\overline{u}|) \,\mathrm{d}x. \tag{4.7}$$

Furthermore, for all δ there exists a positive constant C_δ independent of v, $\overline{u}$, Q and R such that

$$\int_Q |V(D\overline{u}) - V(Dv)|^2 \,\mathrm{d}x \le C_\delta \int_Q \Phi^*(|F|) \,\mathrm{d}x + \big(\delta + C(R + R^\alpha)\big)\int_Q |V(D\overline{u})|^2 \,\mathrm{d}x + C\int_Q \Phi(|\overline{u}|) \,\mathrm{d}x, \tag{4.8}$$

for some $\alpha > 0$.

Proof. In what follows we assume that $\Gamma_Q := \partial Q \cap \{x; x_n = 0\} \ne \emptyset$. In case $\Gamma_Q = \emptyset$ it is enough to consider zero Dirichlet boundary condition instead of (4.10). Since this case is similar to the one solved in [3], we prove the lemma only under the assumption $\Gamma_Q \ne \emptyset$.

[3] We may also assume that Q is a rectangle which appears in Lemma 2.7, i.e. there exists a cube $Q' \subset B_R^+$ such that $Q = 4Q' \cap \mathbb{R}^+_n$.

We choose $R' \in (0,R)$ such that $\operatorname{diam} Q$ is proportional to R' and $Q \subset B^+_{R'}$ and we consider equations (4.2) and (4.4) on $B^+_{R'}$. This means that we use $H_{R'}$ instead of H_R in (4.2) We define a function $\overline{u}_2$ as a function which satisfies following equation

$$\begin{aligned} \operatorname{div} \overline{u}_2 &= R' \operatorname{Tr}(\nabla \overline{u} \omega) \text{ in } Q, && (4.9)\\ \overline{u}_2 \cdot \nu &= 0 \text{ on } \Gamma_Q. && (4.10) \end{aligned}$$

It follows from Lemmas 2.6, 2.2 and 2.5 that

$$\int_Q \Phi(|\nabla \overline{u}_2|)\,\mathrm{d}x \le C R'^{\alpha} \int_Q \Phi(|D\overline{u}|)\,\mathrm{d}x + C \int_Q \Phi(|\overline{u}|)\,\mathrm{d}x. \tag{4.11}$$

We set $\overline{u}_1 = \overline{u} - \overline{u}_2$ and from (4.11) we get

$$\int_Q \Phi(|\nabla \overline{u}_1|)\,\mathrm{d}x \le C \int_Q \Phi(|D\overline{u}|)\,\mathrm{d}x + C \int_Q \Phi(|\overline{u}|)\,\mathrm{d}x. \tag{4.12}$$

We consider a solution v to (3.1) such that $v = \overline{u}_1$ on ∂Q. It is worth emphasizing that $\overline{u}_1 \cdot \nu = 0$ on Γ_Q and $\operatorname{div} \overline{u}_1 = 0$ on Q. Following [3], we test the weak formulation of (3.1) by a function $\overline{u}_1 - v$ and obtain

$$\int_Q \mathcal{S}(Dv) : Dv\,\mathrm{d}x = \int_Q \mathcal{S}(Dv) : D\overline{u}_1\,\mathrm{d}x. \tag{4.13}$$

We point out that due to (4.9) and (4.10) we have $\operatorname{div}(\overline{u}_1 - v) = 0$ and $(\overline{u}_1 - v)\cdot \nu = 0$ on ∂Q. Whereas the left hand side of (4.13) can be estimated from below by $\int_Q |V(Dv)|^2\,\mathrm{d}x$ due to Lemma 2.4, we estimate the right hand side of (4.13) as follows

$$\begin{aligned} \int_Q \mathcal{S}(Dv) : D\overline{u}_1\,\mathrm{d}x &\le \delta \int_Q \Phi(|Dv|)\,\mathrm{d}x + C_\delta \int_Q \Phi(|\nabla \overline{u}_1|)\,\mathrm{d}x \\ &\le C\delta \int_Q |V(Dv)|^2\,\mathrm{d}x + C_\delta \int_Q |V(D\overline{u})|^2\,\mathrm{d}x + C_\delta \int_Q \Phi(|\overline{u}|)\,\mathrm{d}x, \end{aligned}$$

where we used Young's inequality (2.1), (4.12) and Lemma 2.4. Thus, for sufficiently small $\delta > 0$ we have

$$\int_Q |V(Dv)|^2\,\mathrm{d}x \le C \int_Q |V(D\overline{u})|^2\,\mathrm{d}x + C \int_Q \Phi(|\overline{u}|)\,\mathrm{d}x, \tag{4.14}$$

which proves (4.7).

In order to conclude the proof of Lemma 4.1, it remains to prove (4.8). The function $\overline{u}_1 - v$ can be taken as a test function in (4.2). With the knowledge $\int_Q S(Dv) : (D\overline{u}_1 - Dv)\,\mathrm{d}x = 0$ we derive

$$\begin{aligned} &\int_Q S(D\overline{u} + Z_{\overline{u}}) : (D\overline{u}_1 - Dv + Z_{\overline{u}_1 - v})(1 + R'\omega'')\,\mathrm{d}x - \int_Q S(Dv) : (D\overline{u}_1 - Dv)\,\mathrm{d}x \\ &\quad + \int_Q \overline{\pi}_{\Omega_{R'}} \operatorname{Tr}(\nabla(\overline{u}_1 - v)R'\omega)(1 + R\omega'')\,\mathrm{d}x = \int_Q \overline{F} : \big(\nabla \overline{u}_1 - \nabla v + R'\omega \nabla(\overline{u}_1 - v)\big)(1 + R'\omega'')\,\mathrm{d}x =: \mathcal{I}_1. \end{aligned}$$

We can rewrite this identity as follows

$$\begin{aligned} &\int_Q \big(S(D\overline{u}_1) - S(Dv)\big) : (D\overline{u}_1 - Dv)\,\mathrm{d}x = \mathcal{I}_1 + R' \int_Q S(D\overline{u} + Z_{\overline{u}}) : (D\overline{u}_1 - Dv + Z_{\overline{u}_1 - v})\omega''\,\mathrm{d}x \\ &\quad + R' \int_Q S(D\overline{u} + Z_{\overline{u}}) : (Z_{\overline{u}_1 - v})\,\mathrm{d}x + \int_Q \big(S(D\overline{u}) - S(D\overline{u} + Z_{\overline{u}})\big) : (D\overline{u}_1 - Dv)\,\mathrm{d}x \\ &\quad + \int_Q \overline{\pi}_{\Omega_{R'}} \operatorname{Tr}\big((\nabla \overline{u}_1 - \nabla v)R'\omega\big)(1 + R'\omega'')\,\mathrm{d}x + \int_Q \big(S(D\overline{u}_1) - S(D\overline{u})\big) : (D\overline{u}_1 - Dv)\,\mathrm{d}x = \sum_{i=1}^{6} \mathcal{I}_i. \end{aligned} \tag{4.15}$$

The left hand side can be estimated from below due to Lemma 2.4 as

$$\begin{aligned} &\int_Q \big(\mathcal{S}(Dv) - \mathcal{S}(D\overline{u}_1)\big) : (Dv - D\overline{u}_1)\,\mathrm{d}x \ge C \int_Q |V(Dv) - V(D\overline{u}_1)|^2\,\mathrm{d}x \\ &\ge C \int_Q |V(Dv) - V(D\overline{u})|^2\,\mathrm{d}x - C \int_Q |V(D\overline{u}) - V(D\overline{u}_1)|^2\,\mathrm{d}x = C \int_Q |V(Dv) - V(D\overline{u})|^2\,\mathrm{d}x - \mathcal{I}_7. \end{aligned} \tag{4.16}$$

Lemmas 2.1 and 2.4 yield

$$
\begin{aligned}
\mathcal{I}_7 &= C\int_Q |V(D\overline{u}) - V(D\overline{u}_1)|^2 \,\mathrm{d}x \le c\int_Q \Phi_{|D\overline{u}|}(|D\overline{u} - D\overline{u}_1|)\,\mathrm{d}x \\
&\le C_\delta \int_Q \Phi(|D\overline{u}_2|)\,\mathrm{d}x + \delta\int_Q |V(D\overline{u})|^2\,\mathrm{d}x \le (C_\delta R'^\alpha + \delta)\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + C\int_Q \Phi(|\overline{u}|)\,\mathrm{d}x.
\end{aligned}
$$

Further, using Lemma 2.4, 2.1 and (4.14), we have

$$
\begin{aligned}
|\mathcal{I}_6| &\le \int_Q |S(D\overline{u}) - S(D\overline{u}_1)||Dv - D\overline{u}_1|\,\mathrm{d}x \le C_\delta \int_Q \Phi'_{|D\overline{u}|}(|D\overline{u}_2|)|Dv - D\overline{u}_1|\,\mathrm{d}x \\
&\le C_\delta \int_Q \Phi^*_{|D\overline{u}|}(\Phi'_{D\overline{u}}(|D\overline{u}_2|))\,\mathrm{d}x + \delta\int_Q \Phi_{|D\overline{u}|}(|Dv - D\overline{u}_1|)\,\mathrm{d}x \\
&\le C_\delta \int_Q \Phi_{|D\overline{u}|}(|D\overline{u}_2|)\,\mathrm{d}x + \delta\int_Q (|V(D\overline{u})|^2 + |V(D\overline{u}_1 - Dv)|^2)\,\mathrm{d}x \\
&\le C_\delta \int_Q \Phi(|D\overline{u}_2|)\,\mathrm{d}x + \delta\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + \delta\int_Q |V(D\overline{u}_1)|^2\,\mathrm{d}x + \delta\int_Q |V(Dv)|^2\,\mathrm{d}x \\
&\le (C_\delta R'^\alpha + \delta)\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + \int_Q \Phi(|\overline{u}|)\,\mathrm{d}x.
\end{aligned}
$$

The term $\mathcal{I}_4$ can be estimated in the same way as term $\mathcal{I}_6$. Briefly

$$
\begin{aligned}
|\mathcal{I}_4| &\le C_\delta \int_Q \Phi(R'|(\omega\nabla\overline{u} + (\nabla\overline{u})^T\omega^T)|)\,\mathrm{d}x + \delta\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + \delta\int_Q |V(Dv)|^2\,\mathrm{d}x + \delta\int_Q |V(D\overline{u}_1)|^2\,\mathrm{d}x \\
&\le CR'^\alpha \int_Q |V(D\overline{u})|^2\,\mathrm{d}x + \delta\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + C\int_Q \Phi(|\overline{u}|)\,\mathrm{d}x.
\end{aligned}
$$

The term $\mathcal{I}_5$ can be estimated using Young's inequality (2.1), Lemma 2.2, Korn's inequality (2.6), (4.12), (4.14), (4.5) and Lemma 2.4 as follows

$$
\begin{aligned}
|\mathcal{I}_5| &\le C\int_Q |\overline{\pi}_{\Omega_{R'}}||R'(\nabla\overline{u}_1 - \nabla v)|\,\mathrm{d}x \le C_\delta \int_Q \Phi(R'|\nabla\overline{u}_1 - \nabla v|)\,\mathrm{d}x + \delta\int_Q \Phi^*(|\overline{\pi}_{\Omega_{R'}}|)\,\mathrm{d}x \\
&\le (\delta + CR'^\alpha)\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + \delta\int_Q \Phi^*(|F|)\,\mathrm{d}x + (\delta + C)\int_Q \Phi(|\overline{u}|)\,\mathrm{d}x.
\end{aligned}
$$

Terms $\mathcal{I}_2$ and $\mathcal{I}_3$ can be estimated easily as follows

$$
|\mathcal{I}_2 + \mathcal{I}_3| \le R'C\int_Q |V(D\overline{u})|^2\,\mathrm{d}x + C\int_Q \Phi(|\overline{u}|)\,\mathrm{d}x.
$$

Finally, in the same spirit as before

$$
\begin{aligned}
|\mathcal{I}_1| \le C\int_Q |\overline{F}||\nabla\overline{u}_1 - \nabla v|\,\mathrm{d}x &\le C_\delta \int_Q \Phi^*(|\overline{F}|)\,\mathrm{d}x + \delta\int_Q \Phi(|Du_1 - Dv|)\,\mathrm{d}x \\
&\le C_\delta \int_Q \Phi^*(|\overline{F}|)\,\mathrm{d}x + \delta\int_Q |V(D\overline{u})|^2\,\mathrm{d}x.
\end{aligned}
$$

Putting these estimates into (4.16) and (4.15), we get (4.8). □

Proof of Theorem 1.4. Let $x_0 \in \partial\Omega$ with neighborhoods $\mathcal{U}_{x_0}$ and $\mathcal{V}_{x_0}$ such that $\mathcal{U}_{x_0} \subset \mathcal{V}_{x_0}$. Let $\Phi^*(|F|) \in L^q(\mathcal{V}_{x_0})$. The estimate (2.10) is true due to (4.6) and from Lemma 4.1 we get that (2.11) and (2.12) hold true for sufficiently small R. It also holds that $\Phi^*(|\overline{F}|) \in L^q(B_R^+)$. Since $u \in W^{1,\Phi}(B_R^+)$, it holds

$$
\int_{B_R^+} |\nabla\Phi(|u|)|\,\mathrm{d}x = \int_{B_R^+} \Phi'(|u|)\,|\nabla u|\,\mathrm{d}x \le C\int_{B_R^+} \Phi^*\Phi'(|u|)\,\mathrm{d}x + C\int_{B_R^+} \Phi(|\nabla u|)\,\mathrm{d}x \le C. \tag{4.17}
$$

Thus, from Orlicz–Sobolev embedding we know that $\Phi(|u|) \in L^{\frac{n}{n-1}}(\mathcal{V}_{x_0})$ and, consequently, also $\Phi(|\overline{u}|) \in L^{\frac{n}{n-1}}(B_R)$. All assumptions of Lemma 2.7 are met (with $g = \Phi(\overline{u})$ and $f = \Phi^*(|F|)+\Phi(|\overline{u}|)$) and therefore we get $V(D\overline{u}) \in L^{\tilde{q}}(B_R)^{n\times n}$ for $\tilde{q} = \min\{2q, \frac{2n}{n-1}\}$. Consequently, due to the change of variables $V(Du) \in L^{\tilde{q}}(\Omega_R)^{n\times n}$ and also $\Phi(|Du|) \in L^{\tilde{q}/2}(\mathcal{U}_{x_0})$. If $\frac{\tilde{q}}{2} = q$, we are done, otherwise, we use (4.17) on an N-function $\Psi := \Phi^{\frac{n}{n-1}}$ in order to get $\Phi(|u|) \in L^{\left(\frac{n}{n-1}\right)^2}(\mathcal{U}_{x_0})$. We may again use Lemma 2.7 with the same setting in order to get $V(D\overline{u}) \in L^{\tilde{q}}(B_R)^{n\times n}$ for $\tilde{q} = \min\{2q, \frac{2n^2}{(n-1)^2}\}$. Again, if $\tilde{q} = 2q$ we are done, otherwise we iterate this process till $\tilde{q} = 2q$.

The estimate (1.9) follows easily from (2.13) □

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