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Jiří Neustupa

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A Contribution to the Theory of Regularity of a Weak Solution to the Navier–Stokes Equations via One Component of Velocity and Other Related Quantities

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Abstract

We deal with a suitable weak solution $(\mathbf{v}, p)$ to the Navier–Stokes equations in $\Omega \times (0,T)$, where Ω is a domain in $\mathbb{R}^3$, $T > 0$ and $\mathbf{v} = (v_1, v_2, v_3)$. We show that the regularity of $(\mathbf{v}, p)$ at a point $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$ is essentially determined by the Serrin–type integrability of the positive part of a certain linear combination of v_1^2, v_2^2, v_3^2 and p in a backward neighborhood of $(\mathbf{x}_0, t_0)$. An appropriate choice of coefficients in the linear combination leads to the Serrin–type condition on one component of $\mathbf{v}$ or, alternatively, on the positive part of the Bernoulli pressure $\frac{1}{2}|\mathbf{v}|^2 + p$ or the negative part of p, etc.

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1 Introduction

1.1. The Navier–Stokes system. Let Ω be either the whole space $\mathbb{R}^3$ or a half-space or a bounded or exterior domain with the boundary of the class $C^{2+\varsigma}$ ($\varsigma > 0$) and let $T > 0$. We deal with the Navier–Stokes problem

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$\mathbf{v} = \mathbf{v}_0 \qquad \text{in } \Omega \times \{0\} \tag{1.3}$$

for the unknown velocity $\mathbf{v} = (v_1, v_2, v_3)$ and pressure p. Symbol ν denotes the coefficient of viscosity, which is supposed to be a positive constant. If $\partial\Omega \neq \emptyset$ then we consider the problem (1.1), (1.2), (1.3) with the homogeneous Dirichlet boundary condition

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T). \tag{1.4}$$

1.2. Weak and suitable weak solution, regular and singular points. The definition of a weak solution to the system (1.1), (1.2) and its basic properties are explained e.g. in the books by Ladyzhenskaya [8], Temam [21], Sohr [19] and in the survey paper [6] by Galdi. Here, we only recall that the weak solution satisfies (1.1), (1.2) in the sense of distributions in $\Omega \times (0,T)$ and belongs to $L^\infty(0,T;\, \mathbf{L}^2(\Omega)) \cap L^2(0,T;\, \mathbf{W}_0^{1,2}(\Omega))$.

The existence of a weak solution to (1.1), (1.2), (1.5) is known on an arbitrarily long time interval $(0,T)$ (provided that the initial velocity $\mathbf{v}_0$ is an appropriate space, see [8], [19], [21] or [6]), but its regularity and uniqueness are generally open problems. Since, roughly speaking, regular solutions are unique, the question of uniqueness also leads to the question of regularity.

The definition of the so called suitable weak solution to the system (1.1), (1.2), with many related results, can be found e.g. in papers [1], [9], [10] and [22]. Recall that a weak solution $\mathbf{v}$ of system (1.1), (1.2) is called a *suitable weak solution* if an associated pressure p belongs to $L^{3/2}(\Omega \times (0,T))$ and the pair $(\mathbf{v}, p)$ satisfies the so called *generalized energy inequality*

$$2\nu \int_0^T \int_\Omega |\nabla \mathbf{v}|^2 \, \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \leq \ \int_0^T \int_\Omega \big[\, |\mathbf{v}|^2 \left(\partial_t \phi + \nu \Delta \phi \right) + \left(|\mathbf{v}|^2 + 2p \right) \mathbf{v} \cdot \nabla \phi \big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \tag{1.5}$$

for every non–negative function ϕ from $C_0^\infty(\Omega \times (0,T))$. (Some authors use different conditions on the pressure in their definitions. Our class $L^{3/2}(\Omega \times (0,T))$ is the same as in [9], [10] and [22].) By the definition from [1], the point $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$ is said to be a *regular point* of weak solution $\mathbf{v}$ if there exists a neighborhood U of $(\mathbf{x}_0, t_0)$ such that $\mathbf{v} \in \mathbf{L}^\infty(U)$. Points in $\Omega \times (0,T)$ that are not regular are called *singular*. It is shown in [1] that the set of singular points of a suitable weak solution has the 1–dimensional parabolic measure (which dominates the 1–dimensional Hausdorff measure) equal to zero.

1.3. On some local regularity criteria. There exist many so called local regularity criteria, saying that if a suitable weak solution a posteriori satisfies certain conditions in a backward neighborhood of point $(\mathbf{x}_0, t_0)$ then $(\mathbf{x}_0, t_0)$ is a regular point. (See e.g. papers [1], [4], [9], [10], [15], [22], etc. In this paper, we use a criterion from [22] (by Wolf). The criterion is formulated more generally, but it particularly says that there exists $\varepsilon > 0$ such that if

$$\frac{1}{\delta^2} \int_{t_0 - \delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \leq \ \varepsilon \tag{1.6}$$

holds for at least one $\delta > 0$ then $(\mathbf{x}_0, t_0)$ is a regular point of the solution $\mathbf{v}$. (Here, $B_\delta(\mathbf{x}_0)$ naturally denotes the ball of radius δ and center $\mathbf{x}_0$.)

Let us also note that Takahashi [20] proved that if the norm of a weak solution $\mathbf{v}$ in $L^r_w(t_0 - \rho^2, t_0; \mathbf{L}^s(B_\rho(\mathbf{x}_0))$ (where L^r_w denotes the weak L^r–space and $2/r + 3/s \leq 1$, $3 < s \leq \infty$) is less than or equal to ε then $(\mathbf{x}_0, t_0)$ is a regular point of $\mathbf{v}$. Takahashi's criterion has been refined in [16] and [17]. In [17], $\mathbf{v}$ is supposed to be integrable with powers $r \in [3, \infty)$ (in time) and $s \in (3, \infty)$ (in space) not necessarily in some backward neighbourhood of $(\mathbf{x}_0, t_0)$, but only in the intersection of such a neighbourhood with the exterior of the space–time paraboloid

$$P_a: \quad a(t_0 - t) = |\mathbf{x} - \mathbf{x}_0|^2. \tag{1.7}$$

Exponents r and s are required to satisfy the condition $2/r + 3/s \leq 1$ and number a is supposed to satisfy the inequalities $0 < a < 4\nu\lambda_S(B_1)$, where $\lambda_S(B_1)$ is the least eigenvalue of the Dirichlet–Stokes operator in the unit ball B_1 in $\mathbb{R}^3$.

1.4. More on one–component regularity criteria. The studies of regularity of a suitable weak solution $\mathbf{v}$ in dependence on one component of $\mathbf{v}$ were started by paper [12] (Neustupa, Penel), where the authors proved the regularity of $\mathbf{v}$ in $D \times (t_1, t_2)$ (where D was a sub–domain of Ω and $0 \leq t_1 < t_2 \leq T$) under the assumption that the component v_1 was essentially bounded in D. The condition on v_1 has been successively improved in a series of further papers: 1) [14] (by Neustupa, Penel and Novotný; here, v_1 is only assumed to be in $L^r(t_1, t_2; \, L^s(D))$ where $2/r + 3/s \leq \frac{1}{2}$), 2) [7] (by Kukavica and Ziane; the case $D = \mathbb{R}^3$, v_1 is assumed to be in

$L^r(0,T;\ L^s(\mathbb{R}^3))$ where $2/r+3/s=\frac{5}{8}$ for $r\in[\frac{16}{5},\infty)$ and $s\in(\frac{24}{5},\infty])$, 3) [2] (by Cao and Titi); here the authors consider the spatially periodic problem in $\mathbb{R}^3$ and use the condition $2/r+3/s<\frac{2}{3}+2/(3s)$, $s>\frac{7}{2}$), 4) [23] (by Zhou and Pokorný; the exponents r, s are supposed to satisfy the conditions $2/r+3/s\leq\frac{3}{4}+1/(2s)$, $s>\frac{10}{3}$). One can observe that none of these papers reaches the natural Serrin level $2/r+3/s\leq 1$. This level was in a certain sense reached by Chemin, Zhang and Zhang [3], where the regularity of solution $\mathbf{v}$ has been proven under the assumption that $v_1\in L^r(0,T;\ \dot{H}^{1/2+2/r}(\mathbb{R}^3))$, where $r\in(4,\infty)$. The homogeneous Sobolev space $\dot{H}^{1/2+2/r}(\mathbb{R}^3)$ is continuously imbedded to $L^{3r/(r-2)}(\mathbb{R}^3)$. Hence the condition $v_1\in L^r(0,T;\ \dot{H}^{1/2+2/r}(\mathbb{R}^3))$ implies that $v_1\in L^r(0,T;\ L^{3r/(r-2)}(\mathbb{R}^3))$, and the exponents r and $s:=3/(r-2)$ now satisfy Serrin's condition $2/r+3/s\leq 1$. Nevertheless, the requirement $v_1\in L^r(0,T;\ \dot{H}^{1/2+2/r}(\mathbb{R}^3))$ includes the condition on the fractional derivative of v_1 and it is stronger than just the condition $v_1\in L^r(0,T;\ L^{3r/(r-2)}(\mathbb{R}^3))$. Thus, we may conclude that, to our best knowledge, the question whether the condition $v_1\in L^r(t_1,t_2;\ L^s(D))$ for r and s, basically satisfying the condition $2/r+3/s\leq 1$, is sufficient for regularity of solution $\mathbf{v}$ in $D\times(t_1,t_2)$, is still open.

1.5. On the results of this paper. We provide a partial answer to the question formulated at the end of the previous subsection. Our answer concerns the regularity of a suitable weak solution $\mathbf{v}$ at a chosen point $(\mathbf{x}_0,t_0)\in\Omega\times(0,T)$. For $\rho\in(0,\sqrt{t_0})$ and $a\geq 1$, we denote

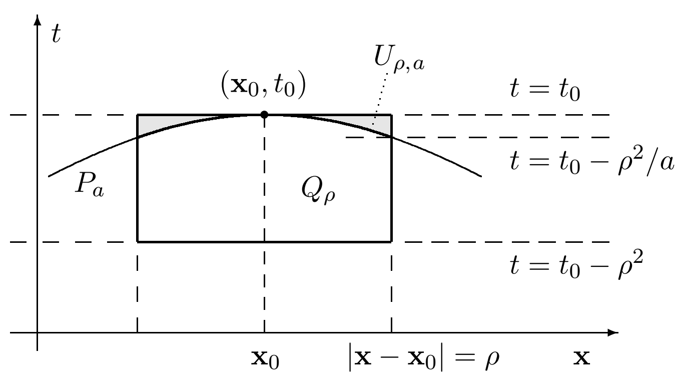
$$\begin{aligned} Q_\rho &:= \{(\mathbf{x},t);\ |\mathbf{x}-\mathbf{x}_0|<\rho,\ t_0-\rho^2<t<t_0\},\\ U_{\rho,a} &:= \big\{(\mathbf{x},t);\ \theta(t)<|\mathbf{x}-\mathbf{x}_0|<\rho,\ t_0-\rho^2/a<t<t_0\big\}, \end{aligned}$$

where

$$\theta(t) := \sqrt{a(t_0-t)}. \tag{1.8}$$

Q_ρ is a ρ–backward parabolic neighborhood of point $(\mathbf{x}_0,t_0)$. Set $U_{\rho,a}$ is separated from the interior of $Q_\rho\smallsetminus U_{\rho,a}$ by the space–time paraboloid P_a, see (1.7). It should be noted that parameter a can be chosen arbitrarily large. Consequently, paraboloid P_a may be arbitrarily wide and set $U_{\rho,a}$ can be proportionally an arbitrarily small part of Q_ρ.

Fig. 1: The sets Q_ρ, $U_{\rho,a}$ and paraboloid P_a.



We suppose that $\mathbf{v}$ satisfies Serrin's integrability condition in $U_{\rho,a}$ and the component v_1 of $\mathbf{v}$ satisfies Serrin's condition in $Q_\rho\smallsetminus U_{\rho,a}$, which is the major part of Q_ρ. We show that these assumptions imply that $(\mathbf{x}_0,t_0)$ is a regular point of solution $\mathbf{v}$ (see Theorem 1). Theorem 2 generalizes Theorem 1 so that the assumption on v_1 is replaced by an assumption on the positive part of a certain linear combination of v_1^2, v_2^2, v_3^2 and p. Our method is especially based on the transformation of the system (1.1), (1.2) to new coordinates $\mathbf{x}'$, t' (subsection 2.3), application of the generalized energy inequality in the $(\mathbf{x}', t')$–space (subsection 2.7), estimates of appropriate quantities and on the precise evaluation of critical integrals, where the directions of the velocity at various points also play an important role (subsections 3.6 and 3.7). Although we still need the assumption on the Serrin–type integrability of all components of $\mathbf{v}$ in set $U_{\rho,a}$, we believe that the presented results shed (in addition to the papers [2], [3], [7], [12], [14], [23]) another light on the mechanism how the behavior of just one component of $\mathbf{v}$ (or more generally, a linear combination of v_1^2, v_2^2, v_3^2 and p) influences the regularity of solution $\mathbf{v}$.

For $r > 1$, $s > 1$, we abbreviate $L^{r,s}(Q_\rho) := L^r(t_0 - \rho^2, t_0;\, L^s(B_\rho(\mathbf{x}_0))$ and we denote by $|\!|\!| \cdot |\!|\!|_{r,s;\, Q_\rho}$ the corresponding norm. More generally, if M is a measurable set in $\Omega \times (0,T)$, $I(M)$ is the orthogonal projection of M into the t–axis and $M_t := \{\mathbf{x} \in \Omega;\ (\mathbf{x}, t) \in D\}$ then we denote by $L^{r,s}(D)$ the space of functions f with the finite norm

$$|\!|\!| f |\!|\!|_{r,s;\, M} \ := \ \left[\int_{I(M)} \left(\int_{M_t} |f(\mathbf{x},t)|^s \, \mathrm{d}\mathbf{x} \right)^{\frac{r}{s}} \mathrm{d}t \right]^{\frac{1}{r}}.$$

We also denote by $\mathbf{L}^{r,s}(D)$ the corresponding space of vector functions.

The next theorem shows that the local regularity of a suitable weak solution $\mathbf{v}$ at a space–time point $(\mathbf{x}_0, t_0)$ is essentially determined just by one component of $\mathbf{v}$:

Theorem 1. *Let $\mathbf{v} = (v_1, v_2, v_3)$ be a suitable weak solution of the system (1.1), (1.2) in $\Omega \times (0,T)$, $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$, $a \geq 1$ and $\rho \in (0, \sqrt{t_0})$. Suppose that*

- *(a) there exist $r \in [3, \infty)$ and $s \in (3, \infty)$ satisfying $2/r + 3/s = 1$, such that $\mathbf{v} \in \mathbf{L}^{r,s}(U_{\rho,a})$ and*
- *(b) there exist $r^* \in [2, \infty)$ and $s^* \in (3, \infty]$ satisfying $2/r^* + 3/s^* = 1$, such that $v_1 \in L^{r^*,s^*}(Q_\rho \smallsetminus U_{\rho,a})$.*

Then $(\mathbf{x}_0, t_0)$ is a regular point of solution $\mathbf{v}$.

Denote

$$\mathcal{F}[\mathbf{v}, p, \gamma_1, \gamma_2, \gamma_3] \ := \ \big[(1+\gamma_1) v_1^2 + (1+\gamma_2) v_2^2 + (1+\gamma_3) v_3^2 + (\gamma_1 + \gamma_2 + \gamma_3) p \big]_+$$

for γ_1, γ_2, $\gamma_3 \in \mathbb{R}$. (The subscript "$+$" denotes the positive part.) The next theorem is a generalization of Theorem 1:

Theorem 2. *Let $\mathbf{v} = (v_1, v_2, v_3)$ be a suitable weak solution of the system (1.1), (1.2) in $\Omega \times (0,T)$, p be an associated pressure, $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$, $a \geq 1$ and $\rho \in (0, \sqrt{t_0})$. Assume that $\mathbf{v}$ satisfies condition (a) of Theorem 1 and also the condition*

- *(c) there exist $r^{**} \in [1, \infty)$, $s^{**} \in (\frac{3}{2}, \infty]$ satisfying $2/r^{**} + 3/s^{**} = 2$ and γ_1, γ_2, $\gamma_3 \in \mathbb{R}$, such that*

$$1 - \frac{5\pi}{128} (\gamma_1 + \gamma_2 + \gamma_3) + \frac{15\pi}{128} \gamma_k \ > \ 0 \qquad \textit{(for } k = 1, 2, 3\textit{)} \tag{1.9}$$

  and $\mathcal{F}[\mathbf{v}, p, \gamma_1, \gamma_2, \gamma_3] \in L^{r^{**},s^{**}}(Q_\rho \smallsetminus U_{\rho,a})$.

Then $(\mathbf{x}_0, t_0)$ is a regular point of solution $\mathbf{v}$.

Observe that if $\gamma_1 = 2$ and $\gamma_2 = \gamma_3 = -1$ then condition (c) reduces to condition (b). On the other hand, if $\gamma_1 = \gamma_2 = \gamma_3 = -1$ then condition (c) requires $[-3p]_+ \in L^{r^{**},s^{**}}(Q_\rho \smallsetminus U_{\rho,a})$. It is equivalent to the condition $p_- \in L^{r^{**},s^{**}}(Q_\rho \smallsetminus U_{\rho,a})$, which has already been used in paper [11]. (Here, p_- denotes the negative part of p.) Thus, our Theorem 2 generalizes Theorem 1 from [11]. Finally, if $\gamma_1 = \gamma_2 = \gamma_3 = 2$ then condition (c) just requires that the positive part of the so called Bernoulli pressure $\frac{1}{2}|\mathbf{v}|^2 + p$ is in $L^{r^{**},s^{**}}(Q_\rho \smallsetminus U_{\rho,a})$.

As Theorem 1 is a special case of Theorem 2, we will further prove Theorem 2.

2 Proof of Theorem 2 – part I

2.1. The used regularity criterion. We will show that

$$\lim_{\delta\to 0+} \frac{1}{\delta^2} \iint_{U_{\delta,a}} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \;=\; 0 \tag{2.1}$$

and there exists a sequence $\{\delta_n\}$, such that $\delta_n \searrow 0$ for $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{\delta_n^2} \iint_{V_{\delta_n,a}} |\mathbf{v}|^3 \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \;=\; 0, \tag{2.2}$$

where

$$V_{\delta,a} \;:=\; \bigl\{(\mathbf{x},t);\; |\mathbf{x}-\mathbf{x}_0| < \theta(t),\; t_0 - \delta^2/a < t < t_0\bigr\}.$$

We will show in subsection 2.8 that (2.1) and (2.2) imply (1.6).

2.2. The proof of (2.1). Applying Hölder's inequality, we get

$$\begin{aligned} \frac{1}{\delta^2}\, |||\mathbf{v}|||^3_{3,3;\,U_{\delta,a}} \;&\le\; \frac{1}{\delta^2} \int_{t_0-\delta^2/a}^{t_0} \biggl(\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}|^s \, \mathrm{d}\mathbf{x}\biggr)^{\frac{3}{s}} \Bigl(\frac{4\pi\delta^3}{3}\Bigr)^{1-\frac{3}{s}} \mathrm{d}t \\ &\le\; \Bigl(\frac{4\pi}{3}\Bigr)^{1-\frac{3}{s}} a^{\frac{3}{r}-1} \biggl[\int_{t_0-\delta^2/a}^{t_0} \biggl(\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}|^s \, \mathrm{d}\mathbf{x}\biggr)^{\frac{r}{s}} \mathrm{d}t\biggr]^{\frac{3}{r}}. \end{aligned}$$

Since $\mathbf{v}$ belongs to $L^{r,s}(U_{\rho,a})$, the right hand side tends to zero as $\delta \to 0+$. Hence (2.1) holds.

2.3. Transformation to the new coordinates $\mathbf{x}', t'$. In order to prove (2.2), we transform the system (1.1), (1.2) to the new coordinates $\mathbf{x}'$ and t', which are related to $\mathbf{x}$ and t through the formulas

$$\mathbf{x}' = \frac{\mathbf{x}-\mathbf{x}_0}{\theta(t)}, \qquad t' = \int_{t_0-\rho^2/a}^{t} \frac{\mathrm{d}\tau}{\theta^2(\tau)} = \frac{1}{a}\, \ln \frac{\rho^2}{a(t_0-t)}. \tag{2.3}$$

Then

$$t \;=\; t_0 - \frac{\rho^2}{a}\, \mathrm{e}^{-at'} \qquad \text{and} \qquad \theta(t) \;=\; \rho\, \mathrm{e}^{-\frac{1}{2}at'}. \tag{2.4}$$

The time interval $(t_0 - \rho^2/a, t_0)$ on the t–axis now corresponds to the interval $\bigl(0, \infty\bigr)$ on the t'–axis. Equations (2.3) represent a one–to–one transformation of the parabolic region $V_{\rho,a}$ in the $\mathbf{x}, t$–space onto the infinite stripe

$$V'_a \;:=\; \bigl\{(\mathbf{x}',t') \in \mathbb{R}^4;\; t' > 0 \text{ and } |\mathbf{x}'| < 1\bigr\}$$

in the $\mathbf{x}', t'$–space. Similarly, (2.3) is a one–to–one mapping of the set $U_{\rho,a}$ in the $\mathbf{x}, t$–space onto

$$U'_a \ :=\ \big\{(\mathbf{x}', t') \in \mathbb{R}^4;\ t' > 0 \text{ and } 1 < |\mathbf{x}'| < \mathrm{e}^{\frac{1}{2}at'}\big\}$$

in the $\mathbf{x}', t'$–space. We denote

$$t'_\delta \ :=\ \frac{2}{a}\, \ln \frac{\rho}{\delta}. \tag{2.5}$$

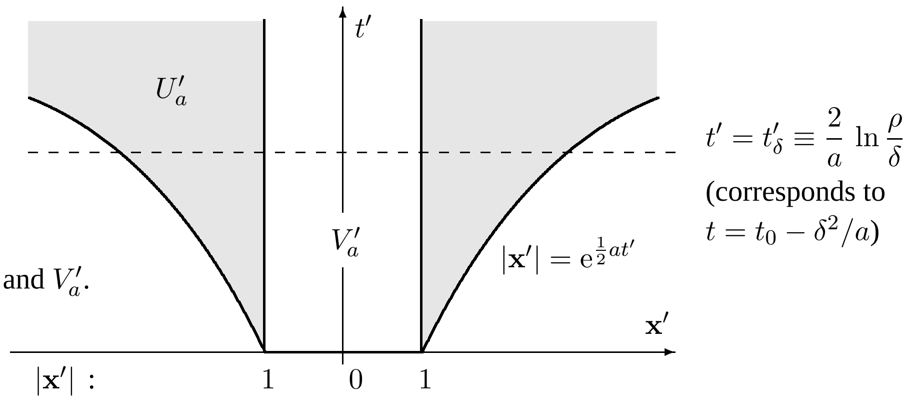


Fig. 2:
The sets U'_a and V'_a.

Then $t' = t'_\delta$ corresponds to $t = t_0 - \delta^2/a$. Numbers δ and t'_δ are also related through the formula $\delta = \rho\, \mathrm{e}^{-\frac{1}{2}at'_\delta}$ and $\delta \to 0+$ corresponds to $t'_\delta \to \infty$. (The transformation (2.3) has also been used in [17]. However, while a was supposed to satisfy certain condition of smallness in [17], here it can be arbitrarily large.) If we put

$$\begin{aligned}
\mathbf{v}(\mathbf{x}, t) \ &=\ \frac{1}{\theta(t)}\, \mathbf{v}'\Big(\frac{\mathbf{x} - \mathbf{x}_0}{\theta(t)},\ \frac{1}{a}\ \ln \frac{\rho^2}{a(t_0 - t)}\Big), \\
p(\mathbf{x}, t) \ &=\ \frac{1}{\theta^2(t)}\, p'\Big(\frac{\mathbf{x} - \mathbf{x}_0}{\theta(t)},\ \frac{1}{a}\ \ln \frac{\rho^2}{a(t_0 - t)}\Big)
\end{aligned}$$

then the functions $\mathbf{v}'$, p' represent a suitable weak solution of the system of equations

$$\partial_{t'}\mathbf{v}' + \mathbf{v}' \cdot \nabla'\mathbf{v}' \ =\ -\nabla' p' + \nu\Delta'\mathbf{v}' - \tfrac{1}{2}a\mathbf{v}' - \tfrac{1}{2}a\mathbf{x}' \cdot \nabla'\mathbf{v}', \tag{2.6}$$

$$\mathrm{div}'\, \mathbf{v}' \ =\ 0 \tag{2.7}$$

in any bounded sub–domain of $Q'_a := \big\{(\mathbf{x}', t') \in \mathbb{R}^4;\ t' > 0 \text{ and } |\mathbf{x}'| < \mathrm{e}^{\frac{1}{2}at'}\big\}$. (The symbols ∇' and Δ' denote the nabla operator and the Laplace operator with respect to the spatial variable $\mathbf{x}'$.)

One can simply calculate that condition (a) implies that $\mathbf{v}' \in \mathbf{L}^{r,s}(U'_a)$ and condition (c) implies that $\mathcal{F}[\mathbf{v}', p', \gamma_1, \gamma_2, \gamma_3] \in L^{r^{**}, s^{**}}(V'_a)$.

2.4. Notation. Let $0 < d_1 < d_2$. We denote by A_{d_1,d_2} and A'_{d_1,d_2} the annuli $\{\mathbf{x} \in \mathbb{R}^3;\ d_1 < |\mathbf{x} - \mathbf{x}_0| < d_2\}$ and $\{\mathbf{x}' \in \mathbb{R}^3;\ d_1 < |\mathbf{x}'| < d_2\}$, respectively. We also denote by B'_{d_1} the ball $\{\mathbf{x}' \in \mathbb{R}^3;\ |\mathbf{x}'| < d_1\}$. The mapping $\mathbf{x} \mapsto \mathbf{x}' = (\mathbf{x} - \mathbf{x}_0)/\theta(t)$ is a one–to–one transformation of $A_{\theta(t)d_1, \theta(t)d_2}$ onto A'_{d_1,d_2} and $B_{\theta(t)d_1}(\mathbf{x}_0)$ onto B'_{d_1} at each time instant $t \in (t_0 - \rho^2/a, t_0)$.

2.5. The cut–off functions ψ and φ. Let $d > 1$. (Number d will be finally specified to be "sufficiently large" in subsection 3.8.) Let ψ be an infinitely differentiable function in the interval $(-\infty, \infty)$, such that

$$\psi(\xi)\ \begin{cases} = 1 & \text{for } \xi \le d, \\ = 0 & \text{for } \ 2d < \xi, \end{cases}$$

ψ is non-increasing in $[d, 2d]$ and there exists $c_1 > 0$ (independent of d) such that

$$|\dot{\psi}(\xi)| \le \frac{c_1}{d} \qquad \text{and} \qquad |\ddot{\psi}(\xi)| \le \frac{c_1}{d^2} \tag{2.8}$$

for $d \le \xi \le 2d$. Put

$$\varphi \ := \ \sqrt{\psi}.$$

We will further use $\psi(\xi)$ and $\varphi(\xi)$ with $\xi = |\mathbf{x}|$, and we shall mostly write only ψ or φ instead of $\psi(|\mathbf{x}|)$ or $\varphi(|\mathbf{x}|)$, respectively.

2.6. The first estimate of $\delta^{-2}\, |\!|\!|\mathbf{v}|\!|\!|^3_{3,3;\, V_{\delta,a}}$. Recall that $t' = t'_\delta = 2a^{-1}\, \ln(\rho/\delta)$ corresponds to $t = t_0 - \delta^2/a$ (see formulas (2.3)–(2.5)). Transforming $\delta^{-2}\, |\!|\!|\mathbf{v}|\!|\!|^3_{3,3;\, V_{\delta,a}}$ to the variables $\mathbf{x}'$, t', we get

$$\begin{aligned} &\frac{1}{\delta^2} \iint_{V_{\delta,\, a}} |\mathbf{v}|^3\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \ = \ \frac{\rho^2}{\delta^2} \int_{t'_\delta}^{\infty} \!\! \int_{B'_1} |\mathbf{v}'|^3\, \mathrm{d}\mathbf{x}'\, \mathrm{e}^{-at'}\, \mathrm{d}t' \ \le \ \frac{\rho^2}{\delta^2} \int_{t'_\delta}^{\infty} \|\mathbf{v}'\|^{\frac{3}{2}}_{6;\, B'_1}\, \|\mathbf{v}'\|^{\frac{3}{2}}_{2;\, B'_1}\, \mathrm{e}^{-at'}\, \mathrm{d}t' \\ &\le\ \frac{\rho^2}{\delta^2} \int_{t'_\delta}^{\infty} \|\varphi\mathbf{v}'\|^{\frac{3}{2}}_{6;\, B'_{2d}}\, \|\varphi\mathbf{v}'\|^{\frac{3}{2}}_{2;\, B'_{2d}}\, \mathrm{e}^{-at'}\, \mathrm{d}t' \ \le\ c_2\, \frac{\rho^2}{\delta^2} \int_{t'_\delta}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^{\frac{3}{2}}_{2;\, B'_{2d}}\, \|\varphi\mathbf{v}'\|^{\frac{3}{2}}_{2;\, B'_{2d}}\, \mathrm{e}^{-at'}\, \mathrm{d}t' \\ &\le\ c_2\, \frac{\rho^2}{\delta^2} \left(\int_{t'_\delta}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;\, B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}at'}\, \mathrm{d}t' \right)^{\frac{3}{4}} \left(\int_{t'_\delta}^{\infty} \|\varphi\mathbf{v}'\|^6_{2;\, B'_{2d}}\, \mathrm{e}^{-2at'}\, \mathrm{d}t' \right)^{\frac{1}{4}} \\ &=\ c_2 \left(\int_{t'_\delta}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;\, B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(t'-t'_\delta)}\, \mathrm{d}t' \right)^{\frac{3}{4}} \left(\int_{t'_\delta}^{\infty} \|\varphi\mathbf{v}'\|^6_{2;\, B'_{2d}}\, \mathrm{e}^{-2a(t'-t'_\delta)}\, \mathrm{d}t' \right)^{\frac{1}{4}}. \end{aligned} \tag{2.9}$$

Here, c_2 is an absolute constant, coming from Sobolev's inequality. (See e.g. [5, p. 54].) In order to estimate the integrals on the right hand side of (2.9), we use the generalized energy inequality in the $\mathbf{x}', t'$–space.

2.7. The generalized energy inequality in the $\mathbf{x}'$, t'–space. Since $\mathbf{v}'$, p' is a suitable weak solution to the system (2.6), (2.7), it satisfies (by analogy with (1.5)) the generalized energy inequality

$$\begin{aligned} 2\nu \int_{Q'_a} |\nabla'\mathbf{v}'|^2\, \phi\, \mathrm{d}\mathbf{x}'\, \mathrm{d}t' \ \le\ \int_{Q'_a} \big[\, &|\mathbf{v}'|^2 \left(\partial_{t'}\phi + \nu\Delta'\phi \right) + \left(|\mathbf{v}'|^2 + 2p' \right) \mathbf{v}' \cdot \nabla'\phi \\ &+ \tfrac{1}{2}\, a\, |\mathbf{v}'|^2\, \phi + \tfrac{1}{2}\, a\, (\mathbf{x}' \cdot \nabla'\phi)\, |\mathbf{v}'|^2 \big]\, \mathrm{d}\mathbf{x}'\, \mathrm{d}t' \end{aligned} \tag{2.10}$$

for every non–negative function ϕ from $C_0^\infty(Q'_a)$.

Consider function ϕ in the form $\phi(\mathbf{x}', t') = [\mathcal{R}_{1/m}\psi](|\mathbf{x}'|)\, \mathrm{e}^{\kappa(t'-t'_\delta)}\, [\mathcal{R}_{1/m}\chi](t')$, where $\kappa \in \mathbb{R}$, χ is the characteristic function of the interval (t'_δ, t') and $\mathcal{R}_{1/m}$ is a one–dimensional mollifier with the kernel supported in $(-1/m, 1/m)$. Then the term $\frac{1}{2}\, a\, (\mathbf{x}' \cdot \nabla'\phi)\, |\mathbf{v}'|^2$ on the right hand side of (2.10) can be omitted, because $\mathbf{x}' \cdot \nabla'\phi \le 0$. The limit for $m \to \infty$ yields

$$\|\varphi\mathbf{v}'(\,.\,, t')\|^2_{2;\, B'_{2d}}\, \mathrm{e}^{\kappa(t'-t'_\delta)} + 2\nu \int_{t'_\delta}^{t'} \|\nabla'(\varphi\mathbf{v}'(\,.\,, \tau))\|^2_{2;\, B'_{2d}}\, \mathrm{e}^{\kappa(\tau-t'_\delta)}\, \mathrm{d}\tau$$

$$\leq \|\varphi \mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B'_{2d}} + \Big(\frac{a}{2}+\kappa\Big) \int_{t'_\delta}^{t'} \|\varphi \mathbf{v}'\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{\kappa(\tau-t'_\delta)}\, \mathrm{d}\tau$$
$$+ \int_{t'_\delta}^{t'} \int_{A'_{d,2d}} \big[2\nu\, |\nabla'\varphi|^2\, |\mathbf{v}'|^2 + \big(|\mathbf{v}'|^2+2p'\big)\, (\mathbf{v}'\cdot\nabla'\psi)\big]\, \mathrm{e}^{\kappa(\tau-t'_\delta)}\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\tau. \tag{2.11}$$

(A similar limit procedure has been used in [17].) Inequality (2.11) holds for a.a. $t' \geq t'_\delta$, where t'_δ is for technical reasons supposed to be greater than $t'_* := 2a^{-1}\, \ln 2d$. Choosing $\kappa = -\frac{2}{3}a$, we get

$$\|\varphi \mathbf{v}'(\,.\,,t')\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(t'-t'_\delta)} + \frac{a}{6} \int_{t'_\delta}^{t'} \|\varphi \mathbf{v}'(\,.\,,\tau)\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(\tau-t'_\delta)}\, \mathrm{d}\tau$$
$$+ 2\nu \int_{t'_\delta}^{t'} \|\nabla'(\varphi \mathbf{v}'(\,.\,,\tau))\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(\tau-t'_\delta)}\, \mathrm{d}\tau$$
$$\leq \|\varphi \mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B'_{2d}} + K^I(\delta) + K^{II}(\delta), \tag{2.12}$$

where

$$K^I(\delta) \;:=\; \int_{t'_\delta}^{\infty} \int_{A'_{d,2d}} \big(2\nu\, |\nabla'\varphi|^2\, |\mathbf{v}'|^2 + |\mathbf{v}'|^2\, |\mathbf{v}'\cdot\nabla'\psi|\big)\, \mathrm{d}\mathbf{x}'\, \mathrm{e}^{-\frac{2}{3}a(\tau-t'_\delta)}\, \mathrm{d}\tau,$$
$$K^{II}(\delta) \;:=\; \int_{t'_\delta}^{\infty} \int_{A'_{d,2d}} \big|2p'\, (\mathbf{v}'\cdot\nabla'\psi)\big|\, \mathrm{d}\mathbf{x}'\, \mathrm{e}^{-\frac{2}{3}a(\tau-t'_\delta)}\, \mathrm{d}\tau.$$

The next lemma is proven in [17]:

Lemma 1. *Assume that* $\mathbf{v}' \in \mathbf{L}^{r^{**},s^{**}}(U'_a)$, $0<\alpha\leq r,\ \ 0<\beta\leq s,\ \ R>1,\ \ t'_\delta > 2a^{-1}\,\ln R$, *and at least one of the two conditions 1)* $\alpha = r,\ \omega\geq 0$, *2)* $\alpha<r,\ \omega>0$ *holds. Then*

$$\int_{t'_\delta}^{\infty} \bigg(\int_{A'_{1,R}} |\mathbf{v}'|^\beta\, \mathrm{d}\mathbf{x}'\bigg)^{\frac{\alpha}{\beta}}\, \mathrm{e}^{-\omega a(t'-t'_\delta)}\, \mathrm{d}t' \;\longrightarrow\; 0 \qquad \textit{as}\ t'_\delta\to\infty. \tag{2.13}$$

Applying Lemma 1, we can show that $K^I(\delta)\to 0$ for $\delta\to 0+$. (Note that in the case of the integral containing $|\mathbf{v}'|^2\, |\mathbf{v}'\cdot\nabla'\varphi^2|$, we apply Lemma 1 with $\alpha=\beta=3$ and $\omega=\frac{2}{3}$. Here, we use the assumption $r\geq 3$.) As to the term $K^{II}(\delta)$, we refer to [17], where $K^{II}(\delta)$ is estimated as follows:

$$K^{II}(\delta) \;\leq\; c_3(\delta)\, \|\varphi \mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B'_{2d}} + c_4(\delta), \tag{2.14}$$

where $c_3(\delta)\to 0$ and $c_4(\delta)\to 0$ for $\delta\to 0+$. (In [17], the author considers an infinitely differentiable function φ with values in $[0,1]$ such that $\varphi=1$ in B'_3 and $\varphi=0$ outside B'_4 instead of our φ, but this difference plays no role.) The proof of (2.14) is relatively laborious especially because it requires to estimate the transformed pressure p'. Note that both $c_3(\delta)$ and $c_4(\delta)$ also depend on parameter a. Thus, inequalities (2.12) and (2.14) yield

$$\|\varphi \mathbf{v}'(\,.\,,t')\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(t'-t'_\delta)} + \frac{a}{6} \int_{t'_\delta}^{t'} \|\varphi \mathbf{v}'(\,.\,,\tau)\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(\tau-t'_\delta)}\, \mathrm{d}\tau$$
$$+ 2\nu \int_{t'_\delta}^{t'} \|\nabla'(\varphi \mathbf{v}'(\,.\,,\tau))\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(\tau-t'_\delta)}\, \mathrm{d}\tau$$

$$\leq\ [1+c_3(\delta)]\, \|\varphi\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B'_{2d}} + c_5(\delta), \tag{2.15}$$

where $c_5(\delta)\to 0$ for $\delta\to 0+$.

2.8. A conditional completion of the proof of Theorem 1. Suppose that

(i) *there exists a sequence* $\{\delta_n\}$ *such that* $\delta_n \searrow 0$ *and* $\lim_{n\to\infty} \|\varphi\mathbf{v}'(\,.\,,t'_{\delta_n})\|_{2;\,B'_{2d}} = 0.$

Then the proof of Theorem 1 can be completed as follows: the identity (2.1) is proven in subsection 2.2. The inequalities (2.9), (2.15) and condition (i) imply that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\delta_n^2} \iint_{V_{\delta_n,a}} |\mathbf{v}|^3\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\
&\quad \leq\ \lim_{n\to\infty} c_2 \left(\int_{t'_{\delta_n}}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(t'-t'_{\delta_n})}\, \mathrm{d}t' \right)^{\frac{3}{4}} \left(\int_{t'_{\delta_n}}^{\infty} \|\varphi\mathbf{v}'\|^6_{2;\,B'_{2d}}\, \mathrm{e}^{-2a(t'-t'_{\delta_n})}\, \mathrm{d}t' \right)^{\frac{1}{4}} \\
&\quad \leq\ c_2 \lim_{n\to\infty} \left(\int_{t'_{\delta_n}}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(t'-t'_{\delta_n})}\, \mathrm{d}t' \right)^{\frac{3}{4}} \\
&\qquad\quad \cdot \left(\operatorname*{ess\,sup}_{t'_{\delta_n}<t'<\infty} \|\varphi\mathbf{v}'(\,.\,,t')\|_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{1}{3}a(t'-t'_{\delta_n})} \right) \left(\int_{t'_{\delta_n}}^{\infty} \|\varphi\mathbf{v}')\|^2_{2;\,B'_{2d}}\, \mathrm{e}^{-\frac{2}{3}a(t'-t'_{\delta_n})}\, \mathrm{d}t' \right)^{\frac{1}{4}} \\
&\quad \leq\ c_2 \lim_{n\to\infty} \left(\frac{1}{2\nu}\right)^{\frac{3}{4}} \left(\frac{6}{a}\right)^{\frac{1}{4}} \left([1+c_3(\delta_n)]\, \|\varphi\mathbf{v}'(\,.\,,t'_{\delta_n})\|^2_{2;\,B'_{2d}} + c_5(\delta_n) \right)^{\frac{3}{2}} \ =\ 0.
\end{aligned}$$

This proves (2.2).

For all $m\in\mathbb{N}$ and $n\in\mathbb{N}$ such that $\delta_n \geq \sqrt{a}\,\delta_m$, we have $Q_{\delta_m} \subset \big(U_{\delta_n,a}\cup V_{\delta_n,a}\big)$. Denote by n_m the maximum of all $n\in\mathbb{N}$ such that $\delta_{n_m} \geq \sqrt{a}\,\delta_m$. Then $\delta_m\to\infty$ implies $\delta_{n_m}\to\infty$ (for $m\to\infty$). Hence, using also (2.1) and (2.2), we have

$$\lim_{m\to\infty} \frac{1}{\delta_m^2} \iint_{Q_{\delta_m}} |\mathbf{v}|^3\, \mathrm{d}\mathbf{x}\, \mathrm{d}t\ \leq\ \lim_{m\to\infty} \frac{1}{\delta_{n_m}^2} \iint_{U_{\delta_{n_m},a}\cup V_{\delta_{n_m},a}} |\mathbf{v}|^3\, \mathrm{d}\mathbf{x}\, \mathrm{d}t\ =\ 0.$$

This implies (1.6), which means that $(\mathbf{x}_0,t_0)$ is a regular point of the solution $\mathbf{v}$, p..

3 Proof of Theorem 2 – part II

The purpose of this section is to show that condition (i) holds, provided that assumption (c) of Theorem 2 is satisfied. Recall that $t'_{\delta_n} = 2a^{-1}\ln(\rho/\delta_n)$. We observe that $\delta_n\searrow 0$ is equivalent to $t'_{\delta_n}\nearrow\infty$. In order to simplify the notation, we further write only t'_n instead of t'_{δ_n}. The existence of a sequence $\{t'_n\}$ such that $t'_n\nearrow\infty$ and $\|\varphi\mathbf{v}'(\,.\,,t'_n)\|_{2;\,B'_{2d}}\to 0$ (for $n\to\infty$) will be established in this section.

3.1. The integrals of $({v'_k}^2+p')\,\psi$ ($k=1,2,3$). Here, we show that the integrals of $({v'_k}^2+p')\,\psi$ in B'_{2d} are equal to certain integrals over $A'_{d,2d}$. Assume, for example, that $k=1$. Let us multiply equation (2.7) by $\nabla(\frac{1}{2}{x'_1}^2\psi)\equiv(x'_1,0,0)\,\psi+\frac{1}{2}{x'_1}^2\,\nabla\psi$ and integrate in B'_{2d}. Since $\mathbf{v}'$ is a suitable weak solution to the system (2.7), (2.10) and the set of its singular points has 1D–Hausdorff measure equal to zero, the integral has a sense for a.a. $t'>t'_*$. We obtain

$$0\ =\ \int_{B'_{2d}} [\mathbf{v}'\cdot\nabla'\mathbf{v}'+\nabla'p']\cdot\big[(x'_1,0,0)\,\psi+\tfrac{1}{2}{x'_1}^2\,\nabla\psi\big]\ \mathrm{d}\mathbf{x}',$$

$$
0 \;=\; \int_{B'_{2d}} \big\{ [v'_j(\partial'_j v'_1) x'_1 + (\partial'_1 p') x'_1]\, \psi + [v'_j(\partial'_j v'_i)\, \tfrac{1}{2} {x'_1}^2\, \partial'_i \psi + \partial'_i p'\, \tfrac{1}{2} {x'_1}^2\, \partial'_i \psi \big\}\ \mathrm{d}\mathbf{x}',
$$

$$
\begin{aligned}
0 \;=\; \int_{B'_{2d}} \big[& {v'_1}^2 \psi + v'_j v'_1 x'_1\, \partial'_j \psi + p'\, \psi + p' x'_1\, \partial'_1 \psi + v'_1 v'_i x'_1\, \partial'_i \psi + v'_j v'_i\, \tfrac{1}{2} {x'_1}^2\, \partial'_i \partial'_j \psi \\
& + p' x'_1\, \partial'_1 \psi + p'\, \tfrac{1}{2} {x'_1}^2\, \Delta' \psi \big]\ \mathrm{d}\mathbf{x}'.
\end{aligned}
$$

Similar identities also hold for $k = 2$ and $k = 3$. Thus, we have

$$
\begin{aligned}
0 \;=\; & \int_{B'_{2d}} [{v'_k}^2 + p']\, \psi\ \mathrm{d}\mathbf{x}' + \int_{A'_{d,2d}} \Big[2 v'_k x'_k\, (\mathbf{v}' \cdot \nabla' \psi) + 2 p' x'_k\, \partial'_k \psi + \tfrac{1}{2} x_k^2\, \mathbf{v}' \cdot {\nabla'}^2 \psi \cdot \mathbf{v}' \\
& \qquad + p'\, \tfrac{1}{2} x_k^2\, \Delta' \psi \Big]\ \mathrm{d}\mathbf{x}',
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \int_{B'_1} [{v'_k}^2 + p']\, \psi\ \mathrm{d}\mathbf{x}' + \int_{A'_{1,2d}} [{v'_k}^2 + p']\, \psi\ \mathrm{d}\mathbf{x}' + \int_{A'_{d,2d}} \Big[2 v'_k x'_k\, \frac{(\mathbf{v}' \cdot \mathbf{x}')}{|\mathbf{x}'|}\, \dot{\psi} + 2 p'\, \frac{{x'_k}^2}{|\mathbf{x}'|}\, \dot{\psi} \\
& + \frac{{x'_k}^2}{2} \Big(\frac{|\mathbf{v}'|^2}{|\mathbf{x}'|}\, \dot{\psi} - \frac{(\mathbf{v}' \cdot \mathbf{x}')^2}{|\mathbf{x}'|^3}\, \dot{\psi} + \frac{(\mathbf{v}' \cdot \mathbf{x}')^2}{|\mathbf{x}'|^2}\, \ddot{\psi} \Big) + p'\, \frac{{x'_k}^2}{2} \Big(\frac{2}{|\mathbf{x}'|}\, \dot{\psi} + \ddot{\psi} \Big) \Big]\ \mathrm{d}\mathbf{x}' \qquad (3.1)
\end{aligned}
$$

for $k = 1, 2, 3$. (One does not sum over k in (3.1). Moreover, the second integral is considered only in $A'_{d,2d}$, because the derivatives of ψ are supported in the closure of $A'_{d,2d}$.)

3.2. The integral of $|\mathbf{v}'|^2 + 3p'$ in B'_1. In this subsection, we express the integral of $(|\mathbf{v}'|^2 + 3p')$ in B'_1 by means of some other integrals over $A'_{1,2d}$. Define function ϕ in the interval $(-\infty, \infty)$ by the formulas

$$
\phi(\xi) \quad \begin{cases} = 1 & \text{for } \xi \leq 1, \\ = \Big(-\dfrac{1}{3} + \dfrac{4}{3\xi^3} \Big)\, \psi(\xi) & \text{for } \ \xi > 1. \end{cases}
$$

ϕ is continuous and piecewise continuously differentiable. Moreover, $\phi(\xi) = 0$ for $\xi \geq 2d$ and

$$
\xi\, \dot{\phi}(\xi) + 3\phi(\xi) \;=\; -1 \qquad \text{for } 1 < \xi < d. \tag{3.2}
$$

By analogy with functions ψ and φ, ϕ will further mostly mean $\phi(|\mathbf{x}'|)$. We multiply equation (2.7) by $\mathbf{x}\, \phi(|\mathbf{x}'|)$ and integrate in $\mathbb{R}^3$. Since $\mathbf{x}\, \phi(|\mathbf{x}|) = \nabla' \Phi(|\mathbf{x}'|)$, where $\Phi(\xi)$ is an antiderivative to $\xi\, \phi(\xi)$, we get

$$
\begin{aligned}
0 \;&=\; \int_{B'_{2d}} \big[\mathbf{v}' \cdot \nabla' \mathbf{v}' \cdot \mathbf{x}'\, \phi + \nabla' p' \cdot \mathbf{x}'\, \phi \big]\ \mathrm{d}\mathbf{x}', \\
0 \;&=\; \int_{B'_{2d}} \Big[|\mathbf{v}'|^2\, \phi + \frac{(\mathbf{v}' \cdot \mathbf{x}')^2}{|\mathbf{x}'|}\, \dot{\phi} + p' \big(3\phi + |\mathbf{x}'|\, \dot{\phi} \big) \Big]\ \mathrm{d}\mathbf{x}'.
\end{aligned}
$$

Using the concrete form of function ϕ and applying (3.2), we further obtain

$$
\begin{aligned}
0 \;=\; & \int_{B'_1} \big(|\mathbf{v}'|^2 + 3p' \big)\ \mathrm{d}\mathbf{x}' + \int_{A'_{1,d}} \Big[|\mathbf{v}'|^2 \Big(-\frac{1}{3} + \frac{4}{3|\mathbf{x}'|^3} \Big) - \frac{4 (\mathbf{v}' \cdot \mathbf{x}')^2}{|\mathbf{x}'|^5} \Big]\ \mathrm{d}\mathbf{x}' - \int_{A'_{1,d}} p'\ \mathrm{d}\mathbf{x}' \\
& + \int_{A'_{d,2d}} \Big[|\mathbf{v}'|^2 \Big(-\frac{\psi}{3} + \frac{4\psi}{3|\mathbf{x}'|^3} \Big) + \frac{(\mathbf{v}' \cdot \mathbf{x}')^2}{|\mathbf{x}|} \Big(-\frac{\dot{\psi}}{3} + \frac{4\dot{\psi}}{3|\mathbf{x}'|^3} \Big) - \frac{4\, (\mathbf{v}' \cdot \mathbf{x}')^2}{|\mathbf{x}'|^5}\, \psi \Big]\ \mathrm{d}\mathbf{x}' \\
& - \int_{A'_{d,2d}} p' \Big(\psi + \frac{|\mathbf{x}'|}{3}\, \dot{\psi} - \frac{4\dot{\psi}}{3|\mathbf{x}'|^2} \Big)\ \mathrm{d}\mathbf{x}'. \qquad (3.3)
\end{aligned}
$$

3.3. Condition (i) – the beginning. In order to fulfill condition (i), we need an information on the behavior of $\|\varphi\mathbf{v}'(\,.\,,t'_\delta)\|^2_{2;\,B'_{2d}}$ for $t'_\delta\to\infty$. Therefore we multiply formula (3.1) by α_k, sum over $k=1,2,3$ and add the sum to the equation $\|\varphi\mathbf{v}'\|^2_{2;\,B'_{2d}}=\|\varphi\mathbf{v}'\|^2_{2;\,B'_1}+\|\varphi\mathbf{v}'\|^2_{2;\,A'_{1,2d}}$. Furthermore, we multiply formula (3.3) by $\beta:=\alpha_1+\alpha_2+\alpha_3$ and also add the product to $\|\varphi\mathbf{v}'\|^2_{2;\,B'_{2d}}$. (The real numbers $\alpha_1,\,\alpha_2,\,\alpha_3$ will be specified later, see (3.5).) Due to the choice of β, the factor $\alpha_1+\alpha_2+\alpha_3-\beta$ in front of $\int_{A'_{1,2d}}p'\,\psi\,\mathrm{d}\mathbf{x}'$ is equal to zero. Thus, we obtain

$$\begin{aligned}
\|\varphi\mathbf{v}'\|^2_{2;\,B'_{2d}} \;=\;& \|\varphi\mathbf{v}'\|^2_{2;\,A'_{1,2d}}+\sum_{k=1}^3\int_{B'_1}\big[(1+\alpha_k){v'_k}^2+\alpha_k p'\big]\,\mathrm{d}\mathbf{x}'+\beta\int_{B'_1}\big(|\mathbf{v}'|^2+3p'\big)\,\mathrm{d}\mathbf{x}'\\
&+\sum_{k=1}^3\int_{A'_{1,d}}\alpha_k{v'_k}^2\,\mathrm{d}\mathbf{x}'+\beta\int_{A'_{1,d}}\Big[|\mathbf{v}'|^2\Big(-\frac{1}{3}+\frac{4}{3|\mathbf{x}'|^3}\Big)-\frac{4(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^5}\Big]\,\mathrm{d}\mathbf{x}'\\
&+\sum_{k=1}^3\alpha_k\int_{A'_{d,2d}}\Big[{v'_k}^2\,\psi+2v'_kx'_k\,\frac{\mathbf{v}'\cdot\mathbf{x}'}{|\mathbf{x}'|}\,\dot\psi+2p'\,\frac{{x'_k}^2}{|\mathbf{x}'|}\,\dot\psi\\
&\qquad+\frac{{x'_k}^2}{2}\Big(\frac{|\mathbf{v}'|^2}{|\mathbf{x}'|}\,\dot\psi-\frac{(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^3}\,\dot\psi+\frac{(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^2}\,\ddot\psi\Big)+p'\,\frac{{x'_k}^2}{2}\Big(\frac{2}{|\mathbf{x}'|}\,\dot\psi+\ddot\psi\Big)\Big]\,\mathrm{d}\mathbf{x}'\\
&+\beta\int_{A'_{d,2d}}\Big[|\mathbf{v}'|^2\Big(-\frac{\psi}{3}+\frac{4\psi}{3|\mathbf{x}'|^3}\Big)+\frac{(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}|}\Big(-\frac{\dot\psi}{3}+\frac{4\dot\psi}{3|\mathbf{x}'|^3}\Big)-\frac{4\,(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^5}\,\psi\Big]\,\mathrm{d}\mathbf{x}'\\
&-\beta\int_{A'_{d,2d}}p'\,\Big(\frac{|\mathbf{x}'|}{3}\,\dot\psi-\frac{4\dot\psi}{3|\mathbf{x}'|^2}\Big)\,\mathrm{d}\mathbf{x}'.
\end{aligned}$$

Subtracting $\|\varphi\mathbf{v}'\|^2_{2;\,A'_{1,2d}}$ from both sides and taking into account that $\varphi=1$ in B'_1, we get

$$\|\mathbf{v}'\|^2_{2;\,B'_1}\;=\;\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5, \tag{3.4}$$

where

$$\begin{aligned}
\Psi_1 :=\;& \int_{B'_1}\big[(1+2\alpha_1+\alpha_2+\alpha_3)\,{v'_1}^2+(1+\alpha_1+2\alpha_2+\alpha_3)\,{v'_2}^2\\
&\quad+(1+\alpha_1+\alpha_2+2\alpha_3)\,{v'_3}^2+4(\alpha_1+\alpha_2+\alpha_3)\,p'\big]\,\psi\,\mathrm{d}\mathbf{x}',\\
\Psi_2 :=\;& \sum_{k=1}^3\int_{A'_{1,d}}\alpha_k{v'_k}^2\,\mathrm{d}\mathbf{x}'+\beta\int_{A'_{1,d}}\Big[|\mathbf{v}'|^2\Big(-\frac{1}{3}+\frac{4}{3|\mathbf{x}'|^3}\Big)-\frac{4(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^5}\Big]\,\mathrm{d}\mathbf{x}',\\
\Psi_3 :=\;& \sum_{k=1}^3\alpha_k\int_{A'_{d,2d}}\Big[{v'_k}^2\psi+2v'_kx'_k\,\frac{\mathbf{v}'\cdot\mathbf{x}'}{|\mathbf{x}'|}\,\dot\psi+\frac{{x'_k}^2}{2}\Big(\frac{|\mathbf{v}'|^2}{|\mathbf{x}'|}\,\dot\psi-\frac{(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^3}\,\dot\psi+\frac{(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^2}\,\ddot\psi\Big)\Big]\,\mathrm{d}\mathbf{x}'\\
&+\beta\int_{A'_{d,2d}}\Big[|\mathbf{v}'|^2\Big(-\frac{\psi}{3}+\frac{4\psi}{3|\mathbf{x}'|^3}\Big)+\frac{(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}|}\Big(-\frac{\dot\psi}{3}+\frac{4\dot\psi}{3|\mathbf{x}'|^3}\Big)-\frac{4\,(\mathbf{v}'\cdot\mathbf{x}')^2}{|\mathbf{x}'|^5}\,\psi\Big]\,\mathrm{d}\mathbf{x}',\\
\Psi_4 :=\;& \sum_{k=1}^3\frac{\alpha_k}{2}\int_{A'_{d,2d}}{x'_k}^2\,p'\,\Big(\ddot\psi+\frac{6}{|\mathbf{x}'|}\,\dot\psi\Big)\,\mathrm{d}\mathbf{x}',\\
\Psi_5 :=\;& -\beta\int_{A'_{d,2d}}p'\,\Big(\frac{|\mathbf{x}'|}{3}\,\dot\psi-\frac{4\dot\psi}{3|\mathbf{x}'|^2}\Big)\,\mathrm{d}\mathbf{x}'.
\end{aligned}$$

Let us now choose $\alpha_1,\ \alpha_2,\ \alpha_3$ so that $2\alpha_1+\alpha_2+\alpha_3=\gamma_1,\ \alpha_1+2\alpha_2+\alpha_3=\gamma_2$ and $\alpha_1+\alpha_2+2\alpha_3=\gamma_3$. Then

$$\alpha_k \ = \ \gamma_k-\frac{1}{4}(\gamma_1+\gamma_2+\gamma_3) \qquad \text{(for } k=1,2,3). \tag{3.5}$$

Thus, function Ψ_1 satisfies

$$\begin{aligned}\Psi_1 \ &= \ \int_{B'_1}\bigl[(1+\gamma_1)\,{v'_1}^2+(1+\gamma_2)\,{v'_2}^2+(1+\gamma_3)\,{v'_3}^2+(\gamma_1+\gamma_2+\gamma_3)\,p'\bigr]\,\psi\;\mathrm{d}\mathbf{x}' \\ &\leq \ \widetilde{\Psi}_1 \ := \ \int_{B'_1}\mathcal{F}[\mathbf{v}',p',\gamma_1,\gamma_2,\gamma_3]\,\psi\;\mathrm{d}\mathbf{x}',\end{aligned}$$

where

$$\begin{aligned}\int_{t'_*}^{\infty}|\widetilde{\Psi}_1|^{r^{**}}\,\mathrm{d}t' \ &\leq \ \Bigl(\frac{4\pi}{3}\Bigr)^{\frac{s^{**}-1}{s^{**}}\,r^{**}}\int_{t'_*}^{\infty}\biggl(\int_{B'_1}\mathcal{F}[\mathbf{v}',p',\gamma_1,\gamma_2,\gamma_3]^{s^{**}}\,\mathrm{d}\mathbf{x}'\biggr)^{\frac{r^{**}}{s^{**}}}\mathrm{d}t' \\ &= \ \Bigl(\frac{4\pi}{3}\Bigr)^{\frac{s^{**}-1}{s^{**}}\,r^{**}}\int_{t_*}^{t_0}\biggl(\int_{|\mathbf{x}-\mathbf{x}_0|<\theta(t)}\mathcal{F}[\mathbf{v},p,\gamma_1,\gamma_2,\gamma_3]^{s^{**}}\,\mathrm{d}\mathbf{x}\biggr)^{\frac{r^{**}}{s^{**}}}\mathrm{d}t \ < \ \infty\end{aligned} \tag{3.6}$$

due to assumption (c), provided that $\gamma_1,\ \gamma_2,\ \gamma_3$ satisfy the restrictions formulated in this condition. (Here, we denote $t_*:=t_0-(\rho^2/a)\,\mathrm{e}^{-at'_*}$ – compare with (2.4).) Hence $\widetilde{\Psi}_1\in L^{r^{**}}(t'_*,\infty)$. Since

$$\begin{aligned}\int_{t'_*}^{\infty}|\Psi_2|^{\frac{r}{2}}\,\mathrm{d}t' \ &\leq \ C\int_{t'_*}^{\infty}\biggl(\int_{A'_{1,d}}|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\biggr)^{\frac{r}{2}}\mathrm{d}t' \ \leq \ C(d)\int_{t'_*}^{\infty}\biggl(\int_{A'_{1,d}}|\mathbf{v}'|^s\,\mathrm{d}\mathbf{x}'\biggr)^{\frac{r}{s}}\mathrm{d}t' \\ &= \ C(d)\int_{t_*}^{t_0}\biggl(\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<d\theta(t)}|\mathbf{v}|^s\,\mathrm{d}\mathbf{x}\biggr)^{\frac{r}{s}}\mathrm{d}t \ < \ \infty\end{aligned} \tag{3.7}$$

(due to condition (a)), we observe that $\Psi_2\in L^{r/2}(t'_*,\infty)$. Function Ψ_3 can be treated in the same way, with a small difference, i.e. that we integrate in $A'_{d,2d}$ instead of $A'_{1,d}$ in the $\mathbf{x}'$–space and in the region $d\theta(t)<|\mathbf{x}-\mathbf{x}_0|<2d\theta(t)$ instead of $\theta(t)<|\mathbf{x}-\mathbf{x}_0|<d\theta(t)$ in the $\mathbf{x}$–space. However, we also deduce that $\Psi_3\in L^{r/2}(t'_*,\infty)$.

3.4. The estimates of Ψ_4. The functions Ψ_4 and Ψ_5 need a special treatment, because they contain the pressure p' and we not have an explicit additional information on the integrability of p' in $A'_{d,2d}$ (in contrast to $\mathbf{v}'$, which is due to assumption (a) of Theorem 1 in $\mathbf{L}^{r,s}(U'_a)$). Nevertheless, if η is an appropriate cut–off function in $\mathbb{R}^3$ then p' satisfies the obvious identity

$$\eta(\mathbf{x}')\,p'(\mathbf{x}',t') \ = \ -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}'-\mathbf{y}'|}\,\bigl[\Delta'(\eta p')\bigr](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}' \tag{3.8}$$

for $\mathbf{x}'\in\mathbb{R}^3$. Concretely, we assume that $0<\kappa<1$ and choose η so that it is infinitely differentiable and satisfies

$$\eta(\mathbf{x}')\ \begin{cases} =1 & \text{for } |\mathbf{x}'|\leq\kappa\,\mathrm{e}^{\frac{1}{2}at'}, \\ \in[0,1] & \text{for } \kappa\,\mathrm{e}^{\frac{1}{2}at'}<|\mathbf{x}'|\leq\mathrm{e}^{\frac{1}{2}at'}, \\ =0 & \text{for } \mathrm{e}^{\frac{1}{2}at'}<|\mathbf{x}'|, \end{cases}$$

$$|\nabla'\eta| \;\leq\; \frac{2}{1-\kappa}\,\mathrm{e}^{-\frac{1}{2}at'} \qquad \text{and} \qquad |\nabla'^2\eta| \;\leq\; \frac{8}{(1-\kappa)^2}\,\mathrm{e}^{-at'}.$$

Integrating by parts in (3.8) and using the formula $\Delta' p' = -\partial'_i\partial'_j(v'_i v'_j)$, we derive that

$$\eta(\mathbf{x}')\,p'(\mathbf{x}',t') \;=\; p'_1(\mathbf{x}',t') + p'_2(\mathbf{x}',t') + p'_3(\mathbf{x}',t'), \tag{3.9}$$

where

$$\begin{aligned}
p'_1(\mathbf{x}',t') \;&=\; \frac{1}{4\pi}\int_{B'_1} \frac{\partial^2}{\partial y'_i\,\partial y'_j}\Big(\frac{1}{|\mathbf{x}'-\mathbf{y}'|}\Big)\,[v'_i v'_j](\mathbf{y}',t')\;\mathrm{d}\mathbf{y}', \\
p'_2(\mathbf{x}',t') \;&=\; \frac{1}{4\pi}\int_{A'_{1,\,\mathrm{e}^{at'/2}}} \frac{\partial^2}{\partial y'_i\,\partial y'_j}\Big(\frac{1}{|\mathbf{x}'-\mathbf{y}'|}\Big)\,[\eta v'_i v'_j](\mathbf{y}',t')\;\mathrm{d}\mathbf{y}', \\
p'_3(\mathbf{x}',t') \;&=\; -\frac{1}{2\pi}\int_{A'_{\kappa\mathrm{e}^{at'/2},\,\mathrm{e}^{at'/2}}} \frac{x'_i-y'_i}{|\mathbf{x}'-\mathbf{y}'|^3}\,\Big(\frac{\partial\eta}{\partial y'_j}\,v'_i v'_j\Big)(\mathbf{y}',t')\;\mathrm{d}\mathbf{y}' \\
&\quad+\frac{1}{4\pi}\int_{A'_{\kappa\mathrm{e}^{at'/2},\,\mathrm{e}^{at'/2}}} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}\,\Big(\frac{\partial^2\eta}{\partial y'_i\,\partial y'_j}\,v'_i v'_j\Big)(\mathbf{y}',t')\;\mathrm{d}\mathbf{y}' \\
&\quad+\frac{1}{2\pi}\int_{A'_{\kappa\mathrm{e}^{at'/2},\,\mathrm{e}^{at'/2}}} \frac{x'_i-y'_i}{|\mathbf{x}'-\mathbf{y}'|^3}\,\Big(\frac{\partial\eta}{\partial y'_i}\,p'\Big)(\mathbf{y}',t')\;\mathrm{d}\mathbf{y}' \\
&\quad+\frac{1}{4\pi}\int_{A'_{\kappa\mathrm{e}^{at'/2},\,\mathrm{e}^{at'/2}}} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}\,[\Delta'\eta\,p'](\mathbf{y}',t')\;\mathrm{d}\mathbf{y}'.
\end{aligned}$$

We can now split Ψ_4 to the sum $\Psi_{41}+\Psi_{42}+\Psi_{43}$, where

$$\Psi_{4l} \;:=\; \sum_{k=1}^{3}\frac{\alpha_k}{2}\int_{A'_{d,2d}} {x'_k}^2\,p'_l\,\Big(\ddot{\psi}(|\mathbf{x}'|)+\frac{6}{|\mathbf{x}'|}\,\dot{\psi}(|\mathbf{x}'|)\Big)\;\mathrm{d}\mathbf{x}' \qquad \text{for } \; l=1,2,3.$$

3.5. Estimates of Ψ_{42} and Ψ_{43}. Let us at first deal with the "easy" terms Ψ_{42} and Ψ_{43}. . Applying the Calderon–Zygmund theorem, we obtain

$$\int_{A'_{1,\,\mathrm{e}^{at'/2}}} |p'_2(\mathbf{y}',t')|^{\frac{s}{2}}\;\mathrm{d}\mathbf{y}' \;\leq\; C\int_{A'_{1,\,\mathrm{e}^{at'/2}}} |\mathbf{v}'(\mathbf{y}',t')|^s\;\mathrm{d}\mathbf{y}'. \tag{3.10}$$

Using also (2.8), we can show that $\Psi_{42}\in L^{r/2}(t'_*,\infty)$:

$$\begin{aligned}
\int_{t'_*}^{\infty}|\Psi_{42}|^{\frac{r}{2}}\;\mathrm{d}t' \;&\leq\; C\int_{t'_*}^{\infty}\Big(\int_{A'_{d,2d}}|p'_2|\;\mathrm{d}\mathbf{x}'\Big)^{\frac{r}{2}}\mathrm{d}t' \;\leq\; C\int_{t'_*}^{\infty}\Big(\int_{A'_{d,2d}}|p'_2|^{\frac{s}{2}}\;\mathrm{d}\mathbf{x}'\Big)^{\frac{r}{s}} d^{3\,\frac{s-2}{s}\,\frac{r}{2}}\;\mathrm{d}t' \\
&\leq\; C\int_{t'_*}^{\infty}\Big(\int_{A'_{1,\mathrm{e}^{at'/2}}}|p'_2|^{\frac{s}{2}}\;\mathrm{d}\mathbf{x}'\Big)^{\frac{r}{s}}\;\mathrm{d}t' \;\leq\; C\int_{t'_*}^{\infty}\Big(\int_{A'_{1,\mathrm{e}^{at'/2}}}|\mathbf{v}'|^s\;\mathrm{d}\mathbf{x}'\Big)^{\frac{r}{s}}\;\mathrm{d}t' \\
&=\; C\int_{t_*}^{t_0}\Big(\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\rho}|\mathbf{v}|^s\;\mathrm{d}\mathbf{x}\Big)^{\frac{r}{s}}\mathrm{d}t \;<\; \infty.
\end{aligned} \tag{3.11}$$

In order to derive an analogous information on Ψ_{43}, we estimate p'_3 as follows: if $\mathbf{x}'\in A'_{d,2d}$ then

$$|p'_3(\mathbf{x}',t')| \;\leq\; C\,\mathrm{e}^{-\frac{3}{2}at'}\int_{A'_{\kappa\mathrm{e}^{at'/2},\,\mathrm{e}^{at'/2}}}\big[|\mathbf{v}'|^2+|p'|\big]\;\mathrm{d}\mathbf{x}'. \tag{3.12}$$

(Note that the generic constant C in (3.11) and (3.12) depends on α_1, α_2, α_3 and d.) Since the set of possible singular points of solution $\mathbf{v}$ has the 1–dimensional Hausdorff measure equal to zero, we can assume without loss of generality that ρ (respectively $\kappa\in(0,1)$) are chosen so small (respectively close to 1) that $\mathbf{v}$ has no singular points in the region $\kappa\rho-\sigma<|\mathbf{x}-\mathbf{x}_0|<\rho+\sigma$, $t_0-\rho^2-\sigma^2<t<t_0+\sigma^2$ for some $\sigma>0$. Then $\mathbf{v}$, with all its spatial derivatives, is essentially bounded in $\{(\mathbf{x},t)\in\mathbb{R}^4;\ \kappa\rho<|\mathbf{x}-\mathbf{x}_0|<\rho,\ t_0-\rho^2<t<t_0\}$. The known results on the interior regularity of pressure (Lemma 2 from [17] or Theorem 4 from [18]) imply that if $\Omega=\mathbb{R}^3$ then p (together with all its spatial derivatives) is also essentially bounded in the same region. If Ω satisfies the assumptions from Section 1, but it differs from $\mathbb{R}^3$, then p (together with all its spatial derivatives) is only in $L^{\lambda}(t_0-\rho^2,t_0;\ L^{\infty}(A_{\kappa\rho,\rho})$ for each $\lambda\in(1,2)$. (See Lemma 2 in [13] or Theorem 2 in [18]). Denote, for a while, by $p_\infty(t)$ (respectively $v_\infty(t)$) the L^∞–norm of $p(\,.\,,t)$ (respectively $\mathbf{v}(\,.\,,t)$) in $A_{\kappa\rho,\rho}$. Put $t'_{**}:=(2/a)\,\ln(2d/\kappa)$ and $t_{**}:=t_0-(\rho^2/a)\,\mathrm{e}^{-at'_{**}}$. (Then $2d<\kappa\,\mathrm{e}^{at'/2}$ for $t'>t'_{**}$.) Since $r\geq 3$, we have $r/(r-1)\leq\frac{3}{2}$. Choose μ and λ so that $1<\mu<\lambda<2$. Then

$$\begin{aligned}
\int_{t'_{**}}^{\infty}|\Psi_{43}|^{\mu}\,\mathrm{d}t' \ &\leq\ C\int_{t'_{**}}^{\infty}\biggl(\int_{A'_{d,2d}}|p'_3|\,\mathrm{d}\mathbf{x}'\biggr)^{\mu}\mathrm{d}t'\\
&\leq\ C\int_{t'_{**}}^{\infty}\biggl(\mathrm{e}^{-\frac{3}{2}at'}\int_{A'_{\kappa\mathrm{e}^{at'/2},\,\mathrm{e}^{at'/2}}}\bigl(|\mathbf{v}'|^2+|p'|\bigr)\,\mathrm{d}\mathbf{x}'\biggr)^{\mu}\mathrm{d}t'\\
&=\ C\,\rho^{-3\mu}\int_{t_{**}}^{t_0}\biggl(\int_{\kappa\rho<|\mathbf{x}-\mathbf{x}_0|<\rho}\bigl(|\mathbf{v}|^2+|p|\bigr)\,\mathrm{d}\mathbf{x}\biggr)^{\alpha}\theta^{2\mu-2}(t)\,\mathrm{d}t\\
&\leq\ C\,\rho^{-3\mu}\int_{t_{**}}^{t_0}\bigl(v_\infty^{2\mu}(t)+p_\infty^{\mu}(t)\bigr)\,\theta^{2\alpha-2}(t)\,\mathrm{d}t\\
&\leq\ C\,\rho^{-3\mu}\biggl[\int_{t_{**}}^{t_0}\bigl(v_\infty^{2\lambda}(t)+p_\infty^{\lambda}(t)\bigr)\,\mathrm{d}t\biggr]^{\frac{\mu}{\lambda}}\biggl[\int_{t_{**}}^{t_0}\theta^{(2\mu-2)\,\frac{\lambda}{\lambda-\mu}}(t)\,\mathrm{d}t\biggr]^{\frac{\lambda-\mu}{\lambda}},
\end{aligned}$$

where $C=C(\alpha_1,\alpha_2,\alpha_3,d)$. The first integral on the last line is finite because $1<\lambda<2$ and the second integral is finite because the exponent $(2\mu-2)\,\frac{\lambda}{\lambda-\mu}$ is greater than -2. Hence $\Psi_{43}\in L^{\mu}(t'_{**},\infty)$ for each $\mu\in(1,2)$.

3.6. The function Ψ_{41}. Here, we deal with the "most difficult" part of Ψ_4, which is the term Ψ_{41}. Function p'_1 satisfies:

$$\begin{aligned}
p'_1(\mathbf{x}',t')\ &=\ \frac{1}{4\pi}\int_{B'_1}\Bigl(3\,\frac{(x'_i-y'_i)(x'_j-y'_j)}{|\mathbf{x}'-\mathbf{y}'|^5}-\frac{\delta_{ij}}{|\mathbf{x}'-\mathbf{y}'|^3}\Bigr)\,[v'_iv'_j](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'\\
&=\ \frac{1}{4\pi}\int_{B'_1}\Bigl(3\,\frac{[\mathbf{v}(\mathbf{y}',t')\cdot(\mathbf{x}'-\mathbf{y}')]^2}{|\mathbf{x}'-\mathbf{y}'|^5}-\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{|\mathbf{x}'-\mathbf{y}'|^3}\Bigr)\,[v'_iv'_j](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'.
\end{aligned}$$

One can calculate that

$$\begin{aligned}
3\,\frac{[\mathbf{v}(\mathbf{y}',t')\cdot(\mathbf{x}'-\mathbf{y}')]^2}{|\mathbf{x}'-\mathbf{y}'|^5}-\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{|\mathbf{x}'-\mathbf{y}'|^3}\ &=3\,\frac{[\mathbf{v}(\mathbf{y}',t')\cdot\mathbf{x}']^2}{|\mathbf{x}'|^5}-\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{|\mathbf{x}'|^3}\\
&\quad+O\bigl(d^{-4}|\mathbf{v}(\mathbf{y}',t')|^2\bigr)
\end{aligned}$$

for $\mathbf{x}\in B'_1$ and $\mathbf{y}\in A'_{d,2d}$. Hence

$$p'_1(\mathbf{x}',t')\ =\frac{1}{4\pi}\int_{B'_1}\Bigl(3\,\frac{[\mathbf{v}(\mathbf{y}',t')\cdot\mathbf{x}']^2}{|\mathbf{x}'|^5}-\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{|\mathbf{x}'|^3}\Bigr)\,\mathrm{d}\mathbf{y}'$$

$$+\frac{1}{4\pi}\int_{B'_1} O\big(d^{-4}|\mathbf{v}(\mathbf{y}',t')|^2\big)\,\mathrm{d}\mathbf{y}'. \tag{3.13}$$

The contribution of the second term on the right hand side to Ψ_{41} (let us denote it by Ψ_{412}) satisfies

$$\begin{aligned} |\Psi_{412}| \;&\leq\; \frac{C}{d^4}\sum_{k=1}^{3}\frac{\alpha_k}{2}\,\|\mathbf{v}'(\,.\,,t')\|^2_{2;\,B'_1}\int_{A'_{d,2d}} {x'_k}^2\,\Big|\ddot{\psi}(|\mathbf{x}'|)+\frac{6}{|\mathbf{x}'|}\,\dot{\psi}(|\mathbf{x}'|)\Big|\,\mathrm{d}\mathbf{x}' \\ &\leq\; \frac{C}{d}\,\|\mathbf{v}'(\,.\,,t')\|^2_{2;\,B'_1}. \end{aligned} \tag{3.14}$$

The contribution of the first integral on the right hand side of (3.13) to Ψ_{41} (we denote it by Ψ_{411}) can be split to the sum:

$$\Psi_{411} \;:=\; \sum_{k=1}^{3}\frac{\alpha_k}{2}\,\Psi_{411k}, \tag{3.15}$$

where

$$\Psi_{411k} \;=\int_{A'_{d,2d}} {x'_k}^2\Big(\ddot{\psi}(|\mathbf{x}'|)+\frac{6}{|\mathbf{x}'|}\,\dot{\psi}(|\mathbf{x}'|)\Big)\,\frac{1}{4\pi}\int_{B'_1}\Big(3\,\frac{[\mathbf{v}'(\mathbf{y}',t')\cdot\mathbf{x}']^2}{|\mathbf{x}'|^5}-\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{|\mathbf{x}'|^3}\Big)\,\mathrm{d}\mathbf{y}'\,\mathrm{d}\mathbf{x}'.$$

The term Ψ_{411} will be finally (after we substitute for $\Psi_1,\,\dots,\,\Psi_5$ to (3.4)) compared with the left hand side of (3.4), i.e. with $\|\mathbf{v}'\|^2_{2;\,B'_1}$. Hence we cannot just estimate Ψ_{411k} by a constant times $\|\mathbf{v}'\|^2_{2;\,B'_1}$, but we must evaluate it precisely. Assume at first that $k=1$. The integral over $A'_{d,2d}$ (with respect to $\mathbf{x}'$) can be split (by Fubini's theorem) to the iterated integral $\int_d^{2d}\int_{S_\xi}\dots\;\mathrm{d}S_\xi\,\mathrm{d}\xi$, where S_ξ is the sphere in $\mathbb{R}^3$ with the center at the point $\mathbf{0}$ and radius ξ. Furthermore, the surface integral over S_ξ is equal to the iterated integral $\int_{-\xi}^{\xi}\int_{C_\xi(x'_1)}\dots\;\mathrm{d}l\,\mathrm{d}x'_1$, where $C_\xi(x'_1)$ is a circle on S_ξ, corresponding to fixed x'_1. (Hence the radius h of $C_\xi(x'_1)$ is $h=(\xi^2-{x'_1}^2)^{1/2}$.) Finally, the line integral over $C_\xi(x'_1)$ can be expressed as the integral from 0 to 2π with respect to σ, using the parametric equations $x'_2=h\,\cos\sigma$, $x'_3=h\,\sin\sigma$. (Then the Cartesian coordinates of point $\mathbf{x}'$ on $C_\xi(x'_1)$ are $\big(x'_1,h\,\cos\sigma,h\,\sin\sigma\big)$ and $\mathrm{d}l$ transforms to $h\,\mathrm{d}\sigma$.) Thus,

$$\begin{aligned} \Psi_{4111} \;&=\int_d^{2d}\int_{-\xi}^{\xi}\int_0^{2\pi} {x'_1}^2\Big[\ddot{\psi}(\xi)+\frac{6}{\xi}\,\dot{\psi}(\xi)\Big]\,\frac{1}{4\pi}\int_{B'_1}\Big(3\,\frac{[\mathbf{v}'(\mathbf{y}',t')\cdot(x'_1,h\cos\sigma,h\sin\sigma)]^2}{\xi^5} \\ &\qquad\qquad -\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{\xi^3}\Big)\,\mathrm{d}\mathbf{y}'\,h\,\mathrm{d}\sigma\,\mathrm{d}x'_1\,\mathrm{d}\xi \\ &=\int_{B'_1}\int_d^{2d}\int_{-\xi}^{\xi}{x'_1}^2\Big[\ddot{\psi}(\xi)+\frac{6}{\xi}\,\dot{\psi}(\xi)\Big]\,\frac{h}{4\pi}\int_0^{2\pi}\Big(3\,\frac{[\mathbf{v}'(\mathbf{y}',t')\cdot(x'_1,h\cos\sigma,h\sin\sigma)]^2}{\xi^5} \\ &\qquad\qquad -\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{\xi^3}\Big)\,\mathrm{d}\sigma\,\mathrm{d}x'_1\,\mathrm{d}\xi\,\mathrm{d}\mathbf{y}'. \end{aligned}$$

Since $\mathbf{v}'\equiv\mathbf{v}'(\mathbf{y}',t')$ is independent of σ, the inside integral can be explicitly calculated:

$$\begin{aligned} &\int_0^{2\pi}\Big(3\,\frac{[\mathbf{v}'\cdot(x'_1,h\cos\sigma,h\sin\sigma)]^2}{\xi^5}-\frac{|\mathbf{v}'|^2}{\xi^3}\Big)\,\mathrm{d}\sigma \\ &\qquad=\;\frac{\pi}{\xi^3}\Big[\Big(\frac{6{x'_1}^2}{\xi^2}-2\Big)\,{v'_1}^2+\Big(\frac{3h^2}{\xi^2}-2\Big)\,{v'_2}^2+\Big(\frac{3h^2}{\xi^2}-2\Big)\,{v'_3}^2\Big] \end{aligned}$$

$$= \frac{\pi}{\xi^3}\left(\frac{3{x_1'}^2}{\xi^2}-1\right)\left(2{v_1'}^2-{v_2'}^2-{v_3'}^2\right).$$

(We have used the formula $h^2=\xi^2-{x_1'}^2$.) Hence, calculating also the integral from $-\xi$ to ξ with respect to x_1':

$$\int_{-\xi}^{\xi}{x_1'}^2\,\frac{h}{4\pi}\,\frac{\pi}{\xi^3}\left(\frac{3{x_1'}^2}{\xi^2}-1\right)\mathrm{d}x_1' \;=\; \frac{1}{4\xi^3}\int_{-\xi}^{\xi}{x_1'}^2\sqrt{\xi^2-{x_1'}^2}\left(\frac{3{x_1'}^2}{\xi^2}-1\right)\mathrm{d}x_1' \;=\; \frac{\pi\xi}{64},$$

we get

$$\begin{aligned}\Psi_{4111} \;&=\; \int_{B_1'}\left(\int_d^{2d}\Big[\ddot{\psi}(\xi)+\frac{6}{\xi}\,\dot{\psi}(\xi)\Big]\,\frac{\pi\xi}{64}\,\mathrm{d}\xi\right)\left(2{v_1'}^2-{v_2'}^2-{v_3'}^2\right)\mathrm{d}\mathbf{y}'\\ &=\; I(\psi)\int_{B_1'}\left(2{v_1'}^2-{v_2'}^2-{v_3'}^2\right)\mathrm{d}\mathbf{y}' \;=\; I(\psi)\int_{B_1'}\left(3{v_1'}^2-|\mathbf{v}'|^2\right)\mathrm{d}\mathbf{y}',\end{aligned}$$

where

$$I(\psi) \;:=\; \frac{\pi}{64}\int_d^{2d}\Big[\ddot{\psi}(\xi)+\frac{6}{\xi}\,\dot{\psi}(\xi)\Big]\,\xi\,\mathrm{d}\xi \;=\; \frac{\pi}{64}\int_d^{2d}\Big[\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\xi\,\dot{\psi}(\xi)\big)+5\dot{\psi}(\xi)\Big]\,\mathrm{d}\xi \;=\; -\frac{5\pi}{64}.$$

We obtain similarly the formulas

$$\Psi_{411k} \;=\; \frac{5\pi}{64}\int_{B_1'}\left(|\mathbf{v}'|^2-3{v_k'}^2\right)\mathrm{d}\mathbf{y}' \qquad \text{for } k=2,3.$$

Substituting now for Ψ_{411k} ($k=1,2,3$) to (3.15), we get

$$\Psi_{411} \;=\; \frac{5\pi}{128}\int_{B_1'}\left[(\alpha_1+\alpha_2+\alpha_3)\,|\mathbf{v}'|^2-3\,(\alpha_1{v_1'}^2+\alpha_2{v_2'}^2+\alpha_3{v_3'}^2)\right]\mathrm{d}\mathbf{y}'. \tag{3.16}$$

3.7. The function Ψ_5. By analogy with Ψ_4, we write $\Psi_5=\Psi_{51}+\Psi_{52}+\Psi_{53}$, where

$$\Psi_{5l} \;:=\; -\beta\int_{A'_{d,2d}} p_l'\left(\frac{|\mathbf{x}'|}{3}\,\dot{\psi}-\frac{4\dot{\psi}}{3|\mathbf{x}'|^2}\right)\mathrm{d}\mathbf{x}' \qquad (\text{for } l=1,2,3)$$

and similarly as in the cases of Ψ_{42} and Ψ_{43}, we can also prove that $\Psi_{52}\in L^{r/2}(t_*',\infty)$ and $\Psi_{53}\in L^{\mu}(t_{**}',\infty)$ for all $\mu\in(1,2)$. The most difficult part of Ψ_5 is again Ψ_{51}, which contains p_1'. If we express p_1' by formula (3.13), we get $\Psi_{51}=\Psi_{511}+\Psi_{512}$, where

$$\begin{aligned}\Psi_{511} \;&:=\; -\frac{\beta}{4\pi}\int_{A'_{d,2d}}\left(\frac{|\mathbf{x}'|}{3}\,\dot{\psi}-\frac{4\dot{\psi}}{3|\mathbf{x}'|^2}\right)\int_{B_1'}\left(3\,\frac{[\mathbf{v}(\mathbf{y}',t')\cdot\mathbf{x}']^2}{|\mathbf{x}'|^5}-\frac{|\mathbf{v}'(\mathbf{y}',t')|^2}{|\mathbf{x}'|^3}\right)\mathrm{d}\mathbf{y}'\,\mathrm{d}\mathbf{x}',\\ \Psi_{512} \;&:=\; -\frac{\beta}{4\pi}\int_{A'_{d,2d}}\left(\frac{|\mathbf{x}'|}{3}\,\dot{\psi}-\frac{4\dot{\psi}}{3|\mathbf{x}'|^2}\right)\int_{B_1'} O\big(d^{-4}|\mathbf{v}(\mathbf{y}',t')|^2\big)\mathrm{d}\mathbf{y}'\,\mathrm{d}\mathbf{x}'.\end{aligned}$$

Hence

$$\Psi_{511} \;=\; -\frac{\beta}{4\pi}\int_{B_1'}\int_{A'_{d,2d}}\left(\frac{|\mathbf{x}'|}{3}\,\dot{\psi}-\frac{4\dot{\psi}}{3|\mathbf{x}'|^2}\right)\left(3\,\frac{[\mathbf{v}\cdot\mathbf{x}']^2}{|\mathbf{x}'|^5}-\frac{|\mathbf{v}'|^2}{|\mathbf{x}'|^3}\right)\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{y}', \tag{3.17}$$

$$|\Psi_{512}| \; \leq \; \frac{C}{d}\, \|\mathbf{v}'\|^2_{2;\,B'_1}, \tag{3.18}$$

where $\mathbf{v}' \equiv \mathbf{v}(\mathbf{y}',t')$. Transforming the inside integral in (3.17) to the spherical coordinates R, ζ, ϑ, we calculate:

$$\int_{A'_{d,2d}} \Big(\frac{|\mathbf{x}'|}{3}\, \dot{\psi} - \frac{4\dot{\psi}}{3|\mathbf{x}'|^2} \Big) \Big(3\, \frac{[\mathbf{v}\cdot\mathbf{x}']^2}{|\mathbf{x}'|^5} - \frac{|\mathbf{v}'|^2}{|\mathbf{x}'|^3} \Big)\, \mathrm{d}\mathbf{x}' \; = \; I_1 \cdot I_2,$$

where

$$\begin{aligned}
I_1 \; &= \; \int_d^{2d} \Big(\frac{R}{3}\, \dot{\psi}(R) - \frac{4\dot{\psi}(R)}{3R^2} \Big)\, \frac{1}{R}\, \mathrm{d}R, \\
I_2 \; &= \; \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \big[3\, [\mathbf{v}' \cdot (\cos\zeta\, \cos\vartheta,\, \sin\zeta\, \cos\vartheta,\, \sin\vartheta)]^2 - |\mathbf{v}'|^2 \big]\; \cos\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\zeta \\
&= \; \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \big[3{v'_1}^2\, \cos^2\zeta\, \cos^3\vartheta + 3{v'_2}^2\, \sin^2\zeta\, \cos^3\vartheta + 3{v'_3}^2\, \sin^2\vartheta\, \cos\vartheta \\
&\qquad + 6v'_1v'_2\, \cos\zeta\, \sin\zeta\, \cos^3\vartheta + 6v'_1v'_3\, \cos\zeta\, \cos^2\vartheta\, \sin\vartheta \\
&\qquad + 6v'_2v'_3\, \sin\zeta \cos^2\vartheta\, \sin\vartheta - \big({v'_1}^2 + {v'_2}^2 + {v'_3}^2 \big)\, \cos\vartheta \big]\; \mathrm{d}\vartheta\, \mathrm{d}\zeta \\
&= \; \pi \int_{-\pi/2}^{\pi/2} \big[3{v'_1}^2\, \cos^3\vartheta + 3{v'_2}^2\, \cos^3\vartheta + 6{v'_3}^2\, \sin^2\vartheta\, \cos\vartheta \\
&\qquad - 2\big({v'_1}^2 + {v'_2}^2 + {v'_3}^2 \big)\, \cos\vartheta \big]\; \mathrm{d}\vartheta.
\end{aligned}$$

The last integral equals zero. Hence $\Psi_{511} = 0$. (This is in fact not surprising, because for each $\mathbf{x}$ on the sphere $S_R(\mathbf{0})$, the difference $3\,[\mathbf{v}\cdot\mathbf{x}']^2/|\mathbf{x}'|^2 - |\mathbf{v}'|^2$ is equal to the second power of the component of $\mathbf{v}$ in the direction of $\mathbf{x}$ (multiplied by 2) minus the second power of the component of $\mathbf{v}$ in the direction perpendicular to $\mathbf{x}$, and when one integrates with respect to $\mathbf{x}$ over the sphere $S_R(\mathbf{0})$, it yields zero.)

3.8. Condition (i) – the completion. If we now use formulas (3.4), (3.16) and the identity $\Psi_{511} = 0$, we obtain

$$\begin{aligned}
\|\mathbf{v}'\|^2_{2;\,B'_1} \; &\leq \; \widetilde{\Psi}_1 + \Psi_2 + \Psi_3 + \Psi_{411} + \Psi_{412} + \Psi_{42} + \Psi_{43} + \Psi_{512} + \Psi_{52} + \Psi_{53} \\
&= \; \Psi_{412} + \Psi_{512} + \widetilde{\Psi} + \frac{5\pi}{128} \int_{B'_1} \Big[(\alpha_1+\alpha_2+\alpha_3)\, |\mathbf{v}'|^2 - 3 \sum_{k=1}^3 \alpha_k\, {v'_k}^2 \Big]\; \mathrm{d}\mathbf{y}',
\end{aligned}$$

where $\widetilde{\Psi} \; = \; \widetilde{\Psi}_1 + \Psi_2 + \Psi_3 + \Psi_{42} + \Psi_{43} + \Psi_{52} + \Psi_{53}$. Hence

$$\int_{B'_1} \Big[\Big(1 - \frac{5\pi}{128} (\alpha_1+\alpha_2+\alpha_3) \Big)\, |\mathbf{v}'|^2 + \frac{15\pi}{128} \sum_{k=1}^3 \alpha_k {v'_k}^2 \Big]\; \mathrm{d}\mathbf{x}' \; \leq \; \Psi_{412} + \Psi_{512} + \widetilde{\Psi}.$$

Substituting for α_1, α_2 and α_3 from (3.5), we obtain exactly the same inequality, only with γ_1, γ_2, γ_3 instead of α_1, α_2, α_3:

$$\int_{B'_1} \Big[\Big(1 - \frac{5\pi}{128} (\gamma_1+\gamma_2+\gamma_3) \Big)\, |\mathbf{v}'|^2 + \frac{15\pi}{128} \sum_{k=1}^3 \gamma_k {v'_k}^2 \Big]\; \mathrm{d}\mathbf{x}' \; \leq \; \Psi_{412} + \Psi_{512} + \widetilde{\Psi}.$$

Due to inequalities (1.9), there exists $\epsilon > 0$ such that the left hand side is greater than or equal to $\epsilon\, \|\mathbf{v}'\|^2_{2;\,B'_1}$. Due to (3.14) and (3.18), one can choose d so large that $|\Psi_{412} + \Psi_{512}| \leq \frac{1}{2}\epsilon\, \|\mathbf{v}'\|^2_{2;\,B'_1}$. Then we have $\frac{1}{2}\epsilon\, \|\mathbf{v}'\|^2_{2;\,B'_1} \leq |\widetilde{\Psi}|$, which implies that

$$\frac{\epsilon}{2}\, \|\varphi\mathbf{v}'\|^2_{2;\,B'_{2d}} \;\leq\; \widetilde{\Psi} + \frac{\epsilon}{2}\, \|\varphi\mathbf{v}'\|^2_{2;\,A'_{1,2d}}.$$

$\widetilde{\Psi}$ is a sum of terms which are either in $L^{r^{**}}(t'_*, \infty)$ (like $\widetilde{\Psi}_1$), or in $L^{r/2}(t'_*, \infty)$ (this concerns Ψ_2, Ψ_3 and Ψ_{42}), or in $L^{\mu}(t'_{**}, \infty)$ (like Ψ_{43} and Ψ_{53}). Moreover, by analogy with Ψ_2 or Ψ_3, one can show that $\|\varphi\mathbf{v}'\|^2_{2;\,A'_{1,2d}} \in L^{r/2}(t'_*, \infty)$, too. Consequently, there exists a sequence $\{t'_n\}$ such that $t'_n \nearrow \infty$ and $\widetilde{\Psi}(t'_n) + \frac{1}{2}\epsilon\, \|\varphi\mathbf{v}'(\,.\,, t'_n)\|^2_{2;\,A'_{1,2d}} \longrightarrow 0$ (for $n \to \infty$). This verifies condition (i) from Section 2.

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Author's address: Czech Academy of Sciences, Institute of Mathematics
Žitná 25, 115 67 Praha 1
Czech Republic
e-mail: neustupa@math.cas.cz