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Universal AF-algebras

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UNIVERSAL AF-ALGEBRAS

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Abstract. We study the approximately finite-dimensional (AF) C^*-algebras that appear as inductive limits of sequences of finite-dimensional C^*-algebras and left-invertible embeddings. We show that there is such a separable AF-algebra $\mathcal{A}_{\mathfrak{F}}$ with the property that any separable AF-algebra is isomorphic to a quotient of $\mathcal{A}_{\mathfrak{F}}$. Equivalently, by Elliott's classification of separable AF-algebras, there are surjectively universal countable scaled (or with order-unit) dimension groups. This universality is a consequence of our result stating that $\mathcal{A}_{\mathfrak{F}}$ is the Fraïssé limit of the category of all finite-dimensional C^*-algebras and left-invertible embeddings.

With the help of Fraïssé theory we describe the Bratteli diagram of $\mathcal{A}_{\mathfrak{F}}$ and provide conditions characterizing it up to isomorphisms. $\mathcal{A}_{\mathfrak{F}}$ belongs to a class of separable AF-algebras which are all Fraïssé limits of suitable categories of finite-dimensional C^*-algebras, and resemble $C(2^{\mathbb{N}})$ in many senses. For instance, they have no minimal projections, tensorially absorb $C(2^{\mathbb{N}})$ (i.e. they are $C(2^{\mathbb{N}})$-stable) and satisfy similar homogeneity and universality properties as the Cantor set.

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1. Introduction

Operator algebraists often refer to (for good reasons, of course) the UHF-algebras such as CAR-algebra as the noncommutative analogues of the Cantor set $2^{\mathbb{N}}$, or more precisely the commutative C^*-algebra $C(2^{\mathbb{N}})$. We will introduce a different class of separable AF-algebras, we call them "AF-algebras with Cantor property" (Definition 4.1), which in some contexts are more suitable noncommutative analogues of $C(2^{\mathbb{N}})$. One of the main features of AF-algebras with Cantor property is that they are direct limits of sequences of finite-dimensional C^*-algebras where the connecting maps are left-invertible homomorphisms. This property, for example, guarantees that if the algebra is infinite-dimensional, it has plenty of nontrivial ideals and quotients, while UHF-algebras are simple. The Cantor set is a "special and unique" space in the category of all compact (zero-dimensional) metrizable spaces in the sense that it bears some universality and homogeneity properties; it maps onto any compact (zero-dimensional) metrizable space and it has the homogeneity property that any homeomorphism between finite quotients lifts to a homeomorphism of the Cantor set (see [7]). Moreover, Cantor set is the unique compact zero-dimensional metrizable space with the property that (stated algebraically): for

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every $m, n \in \mathbb{N}$ and unital embeddings $\phi : \mathbb{C}^n \to \mathbb{C}^m$ and $\alpha : \mathbb{C}^n \to C(2^{\mathbb{N}})$ there is an embedding $\beta : \mathbb{C}^m \to C(2^{\mathbb{N}})$ such that the diagram

$$\begin{array}{ccc} & & C(2^{\mathbb{N}}) \\ & \overset{\alpha}{\nearrow} & \uparrow \beta \\ \mathbb{C}^n & \xhookrightarrow{\phi} & \mathbb{C}^m \end{array}$$

commutes. The AF-algebras with Cantor property satisfy similar universality and homogeneity properties in their corresponding categories of finite-dimensional C^*-algebras and left-invertible homomorphisms. Recall that a homomorphism $\phi : \mathcal{B} \to \mathcal{A}$ is left-invertible if there is a homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that $\pi \circ \phi = \mathrm{id}_{\mathcal{B}}$. To further justify the resemblance between the Cantor set and these algebras, note that the map ϕ in the above must be left-invertible and if α is left-invertible then β can be chosen to be left-invertible. In general, AF-algebras with Cantor property and the maps in the corresponding categories are not assumed to be unital. Although, when restricted to the categories with unital maps, one can obtain the unital AF-algebras with same properties subject to the condition that maps are unital. For instance, the "truly" noncommutative AF-algebra with Cantor property $\mathcal{A}_{\mathfrak{F}}$, that was mentioned in the abstract, is the unique (nonunital) AF-algebra which is the limit of a sequence of finite-dimensional C^*-algebras and left-invertible homomorphisms (necessarily embeddings), with the property that for every finite-dimensional C^*-algebras $\mathcal{D}, \mathcal{E}$ and (not necessarily unital) left-invertible embeddings $\phi : \mathcal{D} \to \mathcal{E}$ and $\alpha : \mathcal{D} \to \mathcal{A}_{\mathfrak{F}}$ there is a left-invertible embedding $\beta : \mathcal{E} \to \mathcal{A}_{\mathfrak{F}}$ such that the diagram

$$\begin{array}{ccc} & & \mathcal{A}_{\mathfrak{F}} \\ & \overset{\alpha}{\nearrow} & \uparrow \beta \\ \mathcal{D} & \xhookrightarrow{\phi} & \mathcal{E} \end{array}$$

commutes (Theorem 8.5). One of our main results (Theorem 8.1) states that $\mathcal{A}_{\mathfrak{F}}$ maps surjectively onto any separable AF-algebra. However, this universality property is not unique to $\mathcal{A}_{\mathfrak{F}}$ (Remark 8.2).

These properties of the Cantor set can be viewed as consequences of the fact that it is the Fraïssé limit of the category of all nonempty finite spaces and surjective maps (as well as the category of all nonempty compact metric spaces and non-expansive quotient maps); see [8]. The theory of Fraïssé limits was introduced by R. Fraïssé [6] in 1954 as a model-theoretic approach to the back-and-forth argument. Roughly speaking, Fraïssé theory establishes a correspondence between classes of finite (or finitely generated) models of a first order language with certain properties (the joint-embedding property, the amalgamation property and having countably many isomorphism types), known as *Fraïssé classes*, and the unique (ultra-)homogeneous and universal countable structure, known as the *Fraïssé limit*, which can be represented as the union of a chain of models from the class. Fraïssé theory has been recently extended way beyond the countable first-order structures, in particular, covering some topological spaces, Banach spaces and, even more recently, some C^*-algebras. Usually in these extensions the classical Fraïssé theory is replaced by its "approximate" version. Approximate Fraïssé theory was developed by Ben Yaacov [1] in continuous model theory (an earlier approach was developed in [13]) and independently, in the framework of metric-enriched categories, by the second author [8]. The Urysohn metric space, the separable infinite-dimensional Hilbert space [1], and the Gurariĭ space [9] are some of the other well known examples of Fraïssé limits of metric spaces (see also [10] for more on Fraïssé limits in functional analysis).

The Fraïssé limits of C^*-algebras are studied in [4] and [11], where it has been shown that the Jiang-Su algebra, all UHF algebras, and the hyperfinite II_1-factor are Fraïssé limits of suitable classes of finitely generated C^*-algebras with distinguished traces. Here we investigate the separable AF-algebras that arise as limits of Fraïssé classes of finite-dimensional C^*-algebras. Apart from $C(2^{\mathbb{N}})$, which is the Fraïssé limit of the class of all commutative finite-dimensional C^*-algebras and unital (automatically left-invertible) embeddings, the UHF-algebras are also Fraïssé limits of such classes ([4]). In [4, Theorem 3.9] the authors describe another class of AF-algebras whose members are Fraïssé limits of classes of finite-dimensional C^*-algebras (again with distinguished traces). These AF-algebras are, in particular, simple and have unique traces. In general, however, obstacles arising from existence of (non-unique) traces prevent many classes of finite-dimensional C^*-algebras from having the amalgamation property ([4, Proposition 3.3]), therefore making it difficult to realize AF-algebras as Fraïssé limits. The AF-algebra $C(2^{\mathbb{N}})$ is neither a UHF-algebra nor it is among AF-algebras considered in [4, Theorem 3.9]. Therefore, it is natural to ask whether $C(2^{\mathbb{N}})$ belongs to any larger (non-trivial) class of AF-algebras whose elements are Fraïssé limits of some class of finite-dimensional C^*-algebras. This was our initial motivation behind introducing the class of separable AF-algebras with Cantor property (Definition 4.1). This class (properly) contains the AF-algebras of the form $\mathcal{D} \otimes C(2^{\mathbb{N}})$, for any finite-dimensional C^*-algebra $\mathcal{D}$.

If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, we say $\mathcal{B}$ is a *retract* of $\mathcal{A}$ if there is a left-invertible embedding from $\mathcal{B}$ into $\mathcal{A}$. In Section 3 we consider the (direct) sequences of finite-dimensional C^*-algebras

$$\mathcal{A}_1 \xrightarrow{\phi_1^2} \mathcal{A}_2 \xrightarrow{\phi_2^3} \mathcal{A}_3 \xrightarrow{\phi_3^4} \dots$$

where each ϕ_n^{n+1} is a left-invertible embedding. The AF-algebra $\mathcal{A}$ that arises as the (direct) limit of this sequence has the property that every matrix algebra M_n appearing as a direct-sum component (an ideal) of some $\mathcal{A}_n$ is a retract of $\mathcal{A}$ (in particular $\mathcal{A}$ maps onto this matrix algebra). Moreover, every retract of $\mathcal{A}$ which is a matrix algebra, appears as a direct-sum component of some $\mathcal{A}_n$ (Lemma 3.5). The AF-algebras with Cantor property are defined and studied in Section 4. They have the property that they are characterized by the set of their matrix algebra retracts. That is, two AF-algebras with Cantor property are isomorphic if and only if they have exactly the same matrix algebras as their retracts (Corollary 7.9).

We will use the Fraïssé-theoretic framework of (*metric-enriched*) categories described in [8], rather than the (metric) model-theoretic approach to the Fraïssé theory. A brief introduction to Fraïssé categories is provided in Section 5. We show that (Theorem 7.2) any category of finite-dimensional C^*-algebras and (not necessarily unital) left-invertible embeddings, which is closed under taking direct sums and ideals of its objects (we call these categories $\oplus$-stable) is a Fraïssé category. Moreover, the Fraïssé limits of these categories have the Cantor property (Lemma 7.4) and in fact any AF-algebra $\mathcal{A}$ with Cantor property can be realized as the Fraïssé limit of such a category, where the objects of the category are the finite-dimensional retracts of $\mathcal{A}$ (Definition 3.1 and Theorem 7.7).

In particular, the category $\mathfrak{F}$ of *all* finite-dimensional C^*-algebras and left-invertible embeddings is a Fraïssé category (Section 8). A priori, the Fraïssé limit of this category $\mathcal{A}_{\mathfrak{F}}$ is a separable AF-algebra with has the universality property that any separable AF-algebra $\mathcal{A}$ which is the limit of a sequence of finite-dimensional C^*-algebras with left-invertible embeddings as connecting maps, can be embedded into $\mathcal{A}_{\mathfrak{F}}$ via a left-invertible embedding. In particular, there is a surjective homomorphism $\theta : \mathcal{A}_{\mathfrak{F}} \twoheadrightarrow \mathcal{A}$. Also any separable AF-algebra is isomorphic to a quotient (by an essential ideal) of an AF-algebra which is the limit of a sequence of finite-dimensional C^*-algebras with left-invertible embeddings (Proposition 3.7). Combining the two quotient maps, we have the following result, which is later restated as Theorem 8.1.

Theorem 1.1. *The category of all finite-dimensional C^*-algebras and left-invertible embeddings is a Fraïssé category. Its Fraïssé limit $\mathcal{A}_{\mathfrak{F}}$ is a separable AF-algebra such that there is a surjective homomorphism from $\mathcal{A}_{\mathfrak{F}}$ onto any separable AF-algebra.*

The Bratteli diagram of $\mathcal{A}_{\mathfrak{F}}$ is described in Proposition 8.4, using the fact that it has the Cantor property. It is the unique AF-algebra with the Cantor property such that every finite-dimensional C^*-algebra is its retract. The unital versions of these results are given in Section 9 (with a bit of extra work, since unlike $\mathfrak{F}$, the category of all finite-dimensional C^*-algebras and *unital* left invertible maps is not a Fraïssé class).

Separable AF-algebras are famously characterized [5] by their K_0-invariants which are scaled countable dimension groups (with order-unit, in the unital case). By applying the K_0-functor to Theorem 1.1 we have the following result.

Corollary 1.2. *There is a scaled countable dimension group (with order-unit) which maps onto any scaled countable dimension group (with order-unit).*

The corresponding characterizations of these dimension groups are mentioned in Section 10.

Finally, this paper could have been written entirely in the language of partially ordered abelian groups, where the categories of "simplicial groups" and left-invertible positive embeddings replace our categories. However, we do not see any clear advantage in doing so.

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2. Preliminaries

Recall that an approximately finite-dimensional (AF) algebra is a C^*-algebra which is an inductive limit of a sequence of finite-dimensional C^*-algebras. We review a few basic facts about separable AF-algebras. The background needed regarding AF-algebras is quite elementary and [3] is more than sufficient. The AF-algebras that are considered here are always separable and therefore by "AF-algebra" we always mean "separable AF-algebra". AF-algebras can be characterized up to isomorphisms by their Bratteli diagrams [2]. However, there is no efficient way (at least visually) to decide whether two Bratteli diagrams are isomorphic, i.e., they correspond to isomorphic AF-algebras. A much better characterization of AF-algebras uses K-theory. To each C^*-algebra K_0-functor assigns a partially ordered abelian group (its K_0-group) which turns out to be a complete invariant for AF-algebras [5]. Moreover, there is a complete description of all possible K_0-groups of AF-algebras. Namely, a partially ordered abelian group is isomorphic to the K_0-group of an AF-algebra if and only if it is a countable dimension group.

We mostly use the notation from [3] with some minor adjustments. Let M_k denote the C^*-algebra of all $k \times k$ matrices over $\mathbb{C}$. Suppose $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_n^m)$ is an AF-algebra with Bratteli diagram $\mathfrak{D}$ such that each $\mathcal{A}_n \cong \mathcal{A}_{n,1} \oplus \cdots \oplus \mathcal{A}_{n,\ell}$, is a finite-dimensional C^*-algebra and each $\mathcal{A}_{n,s}$ is a full matrix algebra. The node of $\mathfrak{D}$ corresponding to $\mathcal{A}_{n,s}$ is "officially" denoted by the expression $\dim(n,s) = (n,s)$, where $\dim(n,s)$ is the dimension of the matrix algebra $\mathcal{A}_{n,s}$, for $1 \leq s \leq \ell$. However, we only write (n,s) to represent the node corresponding to $\mathcal{A}_{n,s}$, knowing that (n,s) intrinsically carries over the natural number $\dim(n,s)$. For $(n,s),(m,t) \in \mathfrak{D}$ we write $(n,s) \to (m,t)$ if (n,t) is connected (m,t) by at least one path in $\mathfrak{D}$, i.e. if ϕ_n^m sends $\mathcal{A}_{n,s}$ faithfully into $\mathcal{A}_{m,t}$.

The ideals of AF-algebras are also AF-algebras and they can be recognized from the Bratteli diagram of the algebra. Namely, the Bratteli diagrams of ideals correspond to *directed* and *hereditary* subsets of the Bratteli diagram of the algebra (see [3, Theorem III.4.2]). Recall that an essential ideal $\mathcal{J}$ of $\mathcal{A}$ has nonzero intersections with every nonzero ideal of $\mathcal{A}$. Suppose $\mathfrak{D}$ is the Bratteli diagram for an AF-algebra $\mathcal{A}$ and $\mathcal{J}$ is an ideal of $\mathcal{A}$ whose Bratteli diagram corresponds to $\mathfrak{J} \subseteq \mathfrak{D}$. Then $\mathcal{J}$ is essential if and only if for every $(n,s) \in \mathfrak{D}$ there is $(m,t) \in \mathfrak{J}$ such that $(n,s) \to (m,t)$.

If $\mathcal{D} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_l$ and $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ are finite-dimensional C^*-algebras where $\mathcal{D}_i$ and $\mathcal{E}_j$ are matrix algebras and $\phi : \mathcal{D} \to \mathcal{E}$ is a homomorphism, we denote the "multiplicity of $\mathcal{D}_i$ in $\mathcal{E}_j$ along ϕ" by $\mathrm{Mult}_\phi(\mathcal{D}_i, \mathcal{E}_j)$. Also let $\mathrm{Mult}_\phi(\mathcal{D}, \mathcal{E}_j)$ denote the tuple

$$(\mathrm{Mult}_\phi(\mathcal{D}_1, \mathcal{E}_j), \ldots, \mathrm{Mult}_\phi(\mathcal{D}_l, \mathcal{E}_j)) \in \mathbb{N}^l.$$

Suppose $\pi_j : \mathcal{E} \to \mathcal{E}_j$ is the canonical projection. If $\mathrm{Mult}_\phi(\mathcal{D}, \mathcal{E}_j) = (x_1, \ldots, x_l)$ then the group homomorphism $K_0(\pi_j \circ \phi) : \mathbb{Z}^l \to \mathbb{Z}$ sends $(y_1, \ldots, y_l)$ to $\sum_{i \leq l} x_i y_i$. Therefore if $\phi, \psi : \mathcal{D} \to \mathcal{E}$ are homomorphisms, we have $K_0(\phi) = K_0(\psi)$ if and only if $\mathrm{Mult}_\phi(\mathcal{D}, \mathcal{E}_j) = \mathrm{Mult}_\psi(\mathcal{D}, \mathcal{E}_j)$ for every $j \leq k$.

The following well known facts about AF-algebras will be used several times throughout the article. We denote the unitization of $\mathcal{A}$ by $\widetilde{\mathcal{A}}$ and if u is a unitary in $\widetilde{\mathcal{A}}$, then Ad_u denotes the automorphisms of $\mathcal{A}$ given by $a \to u^* a u$.

Lemma 2.1. [3, Lemma III.3.2] *Suppose $\epsilon > 0$ and $\{\mathcal{A}_n\}$ is an increasing sequence of finite-dimensional C^*-algebras such that $\mathcal{A} = \overline{\bigcup \mathcal{A}_n}$. If $\mathcal{F}$ is a finite-dimensional subalgebra of $\mathcal{A}$, then there are $m \in \mathbb{N}$ and a unitary u in $\widetilde{\mathcal{A}}$ such that $u^* \mathcal{F} u \subseteq \mathcal{A}_m$ and $\|1 - u\| < \epsilon$.*

Lemma 2.2. *Suppose $\mathcal{D}$ is a finite-dimensional C^*-algebra, $\mathcal{A}$ is a separable AF-algebra and $\phi, \psi : \mathcal{D} \to \mathcal{A}$ are homomorphisms such that $\|\phi - \psi\| < 1$. Then there is a unitary $u \in \widetilde{\mathcal{A}}$ such that $\mathrm{Ad}_u \circ \psi = \psi$.*

Proof. We have $K_0(\phi) = K_0(\psi)$, since otherwise for some nonzero projection p in $\mathcal{D}$ the dimensions of the projections $\phi(p)$ and $\psi(p)$ differ and hence $\|\psi - \phi\| \geq 1$. Therefore there is a unitary u in $\widetilde{\mathcal{A}}$ such that $\mathrm{Ad}_u \circ \psi = \phi$, by [12, Lemma 7.3.2]. □

Lemma 2.3. *Suppose $\mathcal{D} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_l$ is a finite-dimensional C^*-algebra, where each $\mathcal{D}_i$ is a matrix algebra. Assume $\gamma : \mathcal{D} \hookrightarrow M_k$ and $\phi : \mathcal{D} \hookrightarrow M_\ell$ are embeddings. The following are equivalent.*

1. *There is an embedding $\delta : M_k \hookrightarrow M_\ell$ such that $\delta \circ \gamma = \phi$.*
2. *There is an embedding $\delta : M_k \hookrightarrow M_\ell$ such that $\|\delta \circ \gamma - \phi\| < 1$.*
3. *There is a natural number $c \geq 1$ such that $\ell \geq ck$ and $\mathrm{Mult}_\phi(\mathcal{D}, M_\ell) = c\,\mathrm{Mult}_\gamma(\mathcal{D}, M_k)$.*

Proof. (1) trivially implies (2). To see (2)⇒(3), note that we have

$$\mathrm{Mult}_\phi(\mathcal{D}_i, M_\ell) = \mathrm{Mult}_\delta(M_k, M_\ell)\,\mathrm{Mult}_\gamma(\mathcal{D}_i, M_k),$$

for every $i \leq l$, since otherwise $\|\delta \circ \gamma - \phi\| \geq 1$. Let $c = \mathrm{Mult}_\delta(M_k, M_\ell)$. To see (3)⇒(1), let $\delta' : M_k \to M_l$ be the embedding which sends an element of M_k to c many identical copies of it along the diagonal of M_ℓ. Then we have $K_0(\phi) = K_0(\delta' \circ \gamma)$, by the assumption of (3). Therefore there is a unitary u in M_ℓ such that $\mathrm{Ad}_u \circ \delta' \circ \gamma = \phi$. Let $\delta = \mathrm{Ad}_u \circ \delta'$. □

3. AF-algebras with left-invertible connecting maps

Suppose $\mathcal{A}, \mathcal{B}$ are C^*-algebras. A homomorphism $\phi : \mathcal{B} \to \mathcal{A}$ is *left-invertible* if there is a (surjective) homomorphism $\pi : \mathcal{A} \twoheadrightarrow \mathcal{B}$ such that $\pi \circ \phi = \mathrm{id}_\mathcal{B}$. Clearly a left-invertible homomorphism is necessarily an embedding.

Definition 3.1. We say $\mathcal{B}$ is a *retract* of $\mathcal{A}$ if there is a left-invertible embedding from $\mathcal{B}$ into $\mathcal{A}$. We say a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is an *inner* retract if and only if there is a homomorphism $\theta : \mathcal{A} \twoheadrightarrow \mathcal{B}$ such that $\theta|_\mathcal{B} = \mathrm{id}_\mathcal{B}$.

The image of a left-invertible embedding $\phi : \mathcal{B} \hookrightarrow \mathcal{A}$ is an inner retract of $\mathcal{A}$. Next proposition contains some elementary facts about retracts of finite-dimensional C^*-algebras and left-invertible maps between them. They follow from elementary facts about finite-dimensional C^*-algebras, e.g., matrix algebras are simple.

Proposition 3.2. *A C^*-algebra $\mathcal{D}$ is a retract of a finite-dimensional C^*-algebra $\mathcal{E}$, if and only if $\mathcal{E} \cong \mathcal{D} \oplus \mathcal{F}$, for some finite-dimensional C^*-algebra $\mathcal{F}$. In other words, $\mathcal{D}$ is a retract of $\mathcal{E}$, if and only if $\mathcal{D}$ is isomorphic to an ideal of $\mathcal{E}$.*

Suppose $\phi : \mathcal{D} \hookrightarrow \mathcal{E}$ is a (unital) left-invertible embedding and $\pi : \mathcal{E} \twoheadrightarrow \mathcal{D}$ is a left inverse of ϕ. Then $\mathcal{E}$ can be written as $\mathcal{E}_0 \oplus \mathcal{E}_1$ and there are ϕ_0, ϕ_1 such that $\phi_0 : \mathcal{D} \to \mathcal{E}_0$ is an isomorphism, $\phi_1 : \mathcal{D} \to \mathcal{E}_1$ is a (unital) homomorphism and

- *$\phi(d) = (\phi_0(d), \phi_1(d))$, for every $d \in \mathcal{D}$,*
- *$\pi(e_0, e_1) = \phi_0^{-1}(e_0)$ for every $(e_0, e_1) \in \mathcal{E}_0 \oplus \mathcal{E}_1$.*

Suppose $(\mathcal{A}_n, \phi_n^m)$ is a sequence where each connecting map $\phi_n^m : \mathcal{A}_n \hookrightarrow \mathcal{A}_m$ is left-invertible. Let $\pi_n^{n+1} : \mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n$ be a left inverse of ϕ_n^{n+1}, for each n. For $m > n$ define $\pi_n^m : \mathcal{A}_m \twoheadrightarrow \mathcal{A}_n$ by $\pi_n^m = \pi_n^{n+1} \circ \cdots \circ \pi_{m-1}^m$. Then π_n^m is a left inverse of ϕ_n^m which satisfies $\pi_n^m \circ \pi_m^k = \pi_n^k$, for every $n \leq m \leq k$.

Definition 3.3. We say $(\mathcal{A}_n, \phi_n^m)$ is a *left-invertible sequence* if each ϕ_n^m is left-invertible and $\phi_n^n = \mathrm{id}_{\mathcal{A}_n}$. We call (π_n^m) a *compatible* left inverse of the left-invertible sequence $(\mathcal{A}_n, \phi_n^m)$ if $\pi_n^m : \mathcal{A}_m \twoheadrightarrow \mathcal{A}_n$ are surjective homomorphisms such that $\pi_n^m \circ \pi_m^k = \pi_n^k$ and $\pi_n^m \circ \phi_n^m = \mathrm{id}_{\mathcal{A}_n}$, for every $n \leq m \leq k$.

The following simple lemma is true for arbitrary categories, see [7, Lemma 6.2].

Lemma 3.4. *Suppose $(\mathcal{A}_n, \phi_n^m)$ is a left-invertible sequence of C^*-algebras with a compatible left inverse (π_n^m) and $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_n^m)$. Then for every n there are surjective homomorphisms $\pi_n^\infty : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ such that $\pi_n^\infty \circ \phi_n^\infty = \mathrm{id}_{\mathcal{A}_n}$ and $\pi_n^m \circ \pi_m^\infty = \pi_n^\infty$ for each $n \leq m$.*

Proof. First define π_n^∞ on $\bigcup_i \phi_i^\infty[\mathcal{A}_i]$, which is dense in $\mathcal{A}$. If $a = \phi_m^\infty(a_m)$ for some m and $a_m \in \mathcal{A}_m$ then let

$$\pi_n^\infty(a) = \pi_n^m(a_m) \text{ if } n \leq m,$$
$$\pi_n^\infty(a) = \phi_m^n(a_m) \text{ if } n > m.$$

These maps are well-defined (norm-decreasing) homomorphism, so they extend to $\mathcal{A}$ and satisfy the requirements of the lemma. □

In particular, each $\mathcal{A}_n$ or any retract of it, is a retract of $\mathcal{A}$. The converse of this is also true.

Lemma 3.5. *Suppose $(\mathcal{A}_n, \phi_n^m)$ is a left-invertible sequence of finite-dimensional C^*-algebras with $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_n^m)$.*

(1) If $\mathcal{D}$ is a finite-dimensional subalgebra of $\mathcal{A}$, then $\mathcal{D}$ is contained in an inner retract of $\mathcal{A}$.
(2) If $\mathcal{D}$ is a finite-dimensional retract of $\mathcal{A}$, then there is $m \in \mathbb{N}$ such that $\mathcal{D}$ is a retract of $\mathcal{A}_{m'}$ for every $m' \geq m$.

Proof. Let (π_n^m) be a compatible left inverse of $(\mathcal{A}_n, \phi_n^m)$.

(1) If $\mathcal{D}$ is a finite-dimensional subalgebra of $\mathcal{A}$, then for some $m \in \mathbb{N}$ and a unitary $u \in \widetilde{\mathcal{A}}$, it is contained in $u\phi_m^\infty[\mathcal{A}_m]u^*$ (Lemma 2.1). The latter is an inner retract of $\mathcal{A}$.

(2) If $\mathcal{D}$ is a retract of $\mathcal{A}$, there is an embedding $\phi : \mathcal{D} \hookrightarrow \mathcal{A}$ with a left inverse $\pi : \mathcal{A} \twoheadrightarrow \mathcal{D}$. Find m and a unitary u in $\widetilde{\mathcal{A}}$ such that $u^*\phi[\mathcal{D}]u \subseteq \phi_m^\infty[\mathcal{A}_m]$. This implies that

$$\pi_n^\infty \circ \pi_n^\infty(u^*\phi(d)u) = u^*\phi(d)u$$

for every $d \in \mathcal{D}$. Define $\psi : \mathcal{D} \hookrightarrow \mathcal{A}_m$ by $\psi(d) = \pi_m^\infty(u^*\phi(d)u)$. Then ψ has a left inverse $\theta : \mathcal{A}_m \twoheadrightarrow \mathcal{D}$ defined by $\theta(x) = \pi(u\phi_m^\infty(x)u^*)$, since for every $d \in \mathcal{D}$ we have

$$\theta(\psi(d)) = \theta(\pi_n^\infty(u^*\phi(d)u)) = \pi(u\phi_n^\infty \circ \pi_n^\infty(u^*\phi(d)u)u^*) = \pi(\phi(d)) = d.$$

Because $\mathcal{A}_m$ is a retract of $\mathcal{A}_{m'}$, for every $m' \geq m$, we conclude that $\mathcal{D}$ is also a retract of $\mathcal{A}_{m'}$. □

It is not a surprise that many AF-algebras are not limits of left-invertible sequences of finite-dimensional C^*-algebras. Because, for instance, such AF-algebra has infinitely many ideals (unless it is finite-dimensional), and admits finite traces, as it maps onto finite-dimensional C^*-algebras. Therefore, for example, the C^* algebra of all compact operators on ℓ_2 or infinite-dimensional (infinite-type) UHF-algebras are not limits of left-invertible sequences of finite-dimensional C^*-algebras. The following proposition gives another criteria to distinguish these AF-algebras. For example, it can be directly used to show that infinite-dimensional UHF-algebras are not limits of such sequences.

Proposition 3.6. *Suppose $\mathcal{A}$ is an AF-algebra isomorphic to the limit of a left-invertible sequence of finite-dimensional C^*-algebras and $\mathcal{A} = \overline{\bigcup_n \mathcal{B}_n}$ for an increasing sequence of finite-dimensional subalgebras $(\mathcal{B}_n)$. Then there is an increasing sequence (n_i) of natural numbers and an increasing sequence $(\mathcal{C}_i)$ of finite-dimensional* *subalgebras of* $\mathcal{A}$ *such that* $\mathcal{B}_{n_i} \subseteq \mathcal{C}_i \subseteq \mathcal{B}_{n_{i+1}}$ *and* $\mathcal{C}_i$ *is an inner retract of* $\mathcal{C}_{i+1}$ *for every* $i \in \mathbb{N}$*.*

Proof. Suppose $\mathcal{A}$ is the limit of a left-invertible direct sequence $(\mathcal{A}_n, \phi_n^m)$ of finite-dimensional C^*-algebras. Theorem III.3.5 of [3], applied to sequences $(\mathcal{B}_n)$ and $(\phi_n^\infty[\mathcal{A}_n])$, shows that there are sequences (n_i), (m_i) of natural numbers and a unitary $u \in \widetilde{A}$ such that

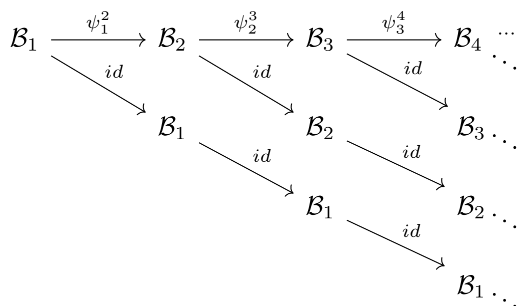
$$\mathcal{B}_{n_i} \subseteq u^* \phi_{m_i}^\infty[\mathcal{A}_{m_i}] u \subseteq \mathcal{B}_{n_{i+1}}$$

for every $i \in \mathbb{N}$. Finally, let $\mathcal{C}_i = u^* \phi_{m_i}^\infty[\mathcal{A}_{m_i}] u$. □

However, next proposition shows that any AF-algebra is a quotient of an AF-algebra which is the limit of a left-invertible sequence of finite-dimensional C^*-algebras.

Proposition 3.7. *For every (unital) AF-algebra* $\mathcal{B}$ *there is a (unital) AF-algebra* $\mathcal{A} \supseteq \mathcal{B}$ *which is the limit of a (unital) left-invertible sequence of finite-dimensional* C^**-algebras and* $\mathcal{A}/\mathcal{J} \cong \mathcal{B}$ *for an essential ideal* $\mathcal{J}$ *of* $\mathcal{A}$*.*

Proof. Suppose $\mathcal{B}$ is the limit of the sequence $(\mathcal{B}_n, \psi_n^m)$ of finite-dimensional C^*-algebras and homomorphisms. Let $\mathcal{A}$ denote the limit of the following diagram:



Then $\mathcal{A}$ is an AF-algebra which contains $\mathcal{B}$ and the connecting maps are left-invertible embeddings. The ideal $\mathcal{J}$ corresponding to the (directed and hereditary) subdiagram of the above diagram which contains all the nodes except the ones on the top line is essential and clearly $\mathcal{A}/\mathcal{J} \cong \mathcal{B}$. □

4. AF-algebras with Cantor property

We define the notion of the "Cantor property" for an AF-algebra. These algebras have properties which are, in a sense, generalizations of the ones satisfied (sometimes trivially) by $C(2^{\mathbb{N}})$. It is easier to state these properties using the notation for Bratteli diagrams that we fixed in Section 2. For example, every node of the Bratteli diagram of $C(2^{\mathbb{N}})$ splits in two, which here is generalized to each node splits into (at least two) nodes with the same dimension at some further stage, which of course guarantees that there are no minimal projections in the limit algebra.

Definition 4.1. We say an AF-algebra $\mathcal{A}$ has *the Cantor property* if there is a sequence $(\mathcal{A}_n, \phi_n^m)$ of finite-dimensional C^*-algebras and embeddings such that $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_n^m)$ and the Bratteli diagram $\mathfrak{D}$ of $(\mathcal{A}_n, \phi_n^m)$ has the following properties:

(D0) For every $(n, s) \in \mathfrak{D}$ there is $(n+1, t) \in \mathfrak{D}$ such that $\dim(n, s) = \dim(n+1, t)$ and $(n, s) \to (n + 1, t)$.

(D1) For every $(n, s) \in \mathfrak{D}$ there are distinct nodes $(m, t), (m, t') \in \mathfrak{D}$, for some $m > n$, such that $\dim(n, s) = \dim(m, t) = \dim(m, t')$ and $(n, s) \to (m, t)$ and $(n, s) \to (m, t')$.

(D2) For every $(n, s_1), \dots, (n, s_k), (n', s') \in \mathfrak{D}$ and $\{x_1, \dots, x_k\} \subseteq \mathbb{N}$ such that $\sum_{i=1}^{k} x_i \dim(n, s_i) \leq \dim(n', s')$, there is $m \geq n$ such that for some $(m, t) \in \mathfrak{D}$ we have $\dim(m, t) = \dim(n', s')$ and there are exactly x_i distinct paths from (n, s_i) to (m, t) in $\mathfrak{D}$.

The Bratteli diagram of $C(2^{\mathbb{N}})$ trivially satisfies these conditions and therefore $C(2^{\mathbb{N}})$ has the Cantor property.

Remark 4.2. Condition (D0) states exactly that $(\mathcal{A}_n, \phi_n^m)$ is a left-invertible sequence. Dropping (D0) from Definition 4.1 does not change the definition (i.e., $\mathcal{A}$ has the Cantor property if and only if it has a representing sequence satisfying (D1) and D(2)). This is because (D1) alone implies the existence of a left-invertible sequence with limit $\mathcal{A}$ that still satisfies (D1). However, we add (D0) for simplicity to make sure that $(\mathcal{A}_n, \phi_n^m)$ is already a left-invertible sequence, since, as we shall see later, being the limit of a left-invertible direct sequence of finite-dimensional C^*-algebras is a crucial property of AF-algebras with Cantor property. Condition (D2) can be rewritten as

(D2′) For every ideal $\mathcal{D}$ of $\mathcal{A}_n$, if M_ℓ is a retract of $\mathcal{A}$ and $\gamma : \mathcal{D} \hookrightarrow M_\ell$ is an embedding, then there is $m \geq n$ and $\mathcal{A}_{m,t} \subseteq \mathcal{A}_m$ such that $\mathcal{A}_{m,t} \cong M_\ell$ and $\mathrm{Mult}_{\phi_n^m}(\mathcal{D}, \mathcal{A}_{m,t}) = \mathrm{Mult}_\gamma(\mathcal{D}, M_\ell)$.

Definition 4.1 may be adjusted for unital AF-algebras where all the maps are considered to be unital.

Definition 4.3. A unital AF-algebra $\mathcal{A}$ has the Cantor property if and only if it satisfies the conditions of Definition 4.1, where ϕ_n^m are unital and in condition (D2) the inequality $\sum_{i=1}^{k} x_i \dim(n, s_i) \leq \dim(n', s')$ is replaced with equality.

Proposition 4.4. *Suppose $\mathcal{A}$ is an AF-algebra with Cantor property. If $\mathcal{D}, \mathcal{E}$ are finite-dimensional retracts of $\mathcal{A}$, then so is $\mathcal{D} \oplus \mathcal{E}$.*

Proof. Suppose $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_l$ and $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_k$, where $\mathcal{D}_i, \mathcal{E}_i$ are isomorphic to matrix algebras. By Lemma 3.5 both $\mathcal{D}$ and $\mathcal{E}$ are retracts of some $\mathcal{A}_m$, which means all $\mathcal{D}_i$ and $\mathcal{E}_i$ appear in $\mathcal{A}_m$ as retracts (or ideals). By (D1) and enlarging m if necessary, we can make sure these retracts in $\mathcal{A}_m$ are orthogonal, meaning that $\mathcal{A}_m \cong \mathcal{D} \oplus \mathcal{E} \oplus \mathcal{F}$, for some finite-dimensional C^*-algebra $\mathcal{F}$. Therefore $\mathcal{D} \oplus \mathcal{E}$ is a retract of $\mathcal{A}_m$ and as a result, it is a retract of $\mathcal{A}$. □

Lemma 4.5. *Suppose $\mathcal{A}$ is an AF-algebra with Cantor property, witnessed by $(\mathcal{A}_n, \phi_n^m)$ satisfying Definition 4.1 and $\mathcal{E}$ is a finite-dimensional retract of $\mathcal{A}$. If $\gamma : \mathcal{A}_n \hookrightarrow \mathcal{E}$ is a left-invertible embedding then there are $m \geq n$ and a left-invertible embedding $\delta : \mathcal{E} \hookrightarrow \mathcal{A}_m$ such that $\delta \circ \gamma = \phi_n^m$.*

Proof. Suppose $\mathcal{A}_n = \mathcal{A}_{n,1} \oplus \cdots \oplus \mathcal{A}_{n,l}$ and $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_k$ where $\mathcal{E}_i$ and $\mathcal{A}_{n,j}$ are matrix algebras. Let π_i denote the canonical projection from $\mathcal{E}$ onto $\mathcal{E}_i$. For every $i \leq k$ put

$$Y_i = \{j \leq l : \gamma[\mathcal{A}_{n,j}] \cap \mathcal{E}_i \neq 0\},$$

and let $\mathcal{A}_{n,Y_i} = \bigoplus_{j\in Y_i} \mathcal{A}_{n,j}$. Then $\mathcal{A}_{n,Y_i}$ is an ideal (a retract) of $\mathcal{A}_n$ and the map $\gamma_i : \mathcal{A}_{n,Y_i} \hookrightarrow \mathcal{E}_i$, the restriction of γ to $\mathcal{A}_{n,Y_i}$ composed with π_i, is an embedding. Since $\mathcal{E}$ is a finite-dimensional retract of $\mathcal{A}$, it is a retract of some $\mathcal{A}_{n'}$ (Lemma 3.5). So each $\mathcal{E}_i$ is a retract of $\mathcal{A}_{n'}$. By applying (D2) for each $i \le k$ there are $m_i \ge n$ and $(m_i, t_i) \in \mathfrak{D}$ such that $\dim(m_i, t_i) = \dim(\mathcal{E}_i)$ and $\mathrm{Mult}_{\phi_n^{m_i}}(\mathcal{A}_{n,Y_i}, \mathcal{A}_{m_i,t_i}) = \mathrm{Mult}_{\gamma_i}(\mathcal{A}_{n,Y_i}, \mathcal{E}_i)$. Let $m = \max\{m_i : i \le k\}$ and by (D0) find (m, s_i) such that $\dim(m_i, t_i) = \dim(m, s_i)$ and $(m_i, t_i) \to (m, s_i)$. Applying (D1) and possibly increasing m allows us to make sure that $(m, s_i) \neq (m, s_j)$ for distinct i, j and therefore $\mathcal{A}_{m,s_i}$ are pairwise orthogonal. Then $\{\mathcal{A}_{m,s_i} : i \le k\}$ is a sequence of pairwise orthogonal subalgebras (retracts) of $\mathcal{A}_m$ such that $\mathcal{A}_{m,s_i} \cong \mathcal{E}_i$ and

$$\mathrm{Mult}_{\phi_n^m}(\mathcal{A}_{n,Y_i}, \mathcal{A}_{m,s_i}) = \mathrm{Mult}_{\gamma_i}(\mathcal{A}_{n,Y_i}, \mathcal{E}_i).$$

By Lemma 2.3 there are isomorphisms $\delta_i : \mathcal{E}_i \hookrightarrow \mathcal{A}_{m,s_i}$ such that $\gamma_i \circ \delta_i$ is equal to the restriction of ϕ_n^m to $\mathcal{A}_{n,Y_i}$ projected onto $\mathcal{A}_{m,s_i}$.

Suppose 1_m is the unit of $\mathcal{A}_m$ and q_i is the unit of $\mathcal{A}_{m,s_i}$. Each q_i is a central projection of $\mathcal{A}_m$, because $\mathcal{A}_{m,s_i}$ are ideals of $\mathcal{A}_m$. Since γ is left-invertible, for each $j \le l$ there is $k(j) \le k$ such that $\mathcal{A}_{n,j} \cong \mathcal{E}_{k(j)}$ and $\hat{\gamma}_j = \pi_{k(j)} \circ \gamma|_{\mathcal{A}_{n,j}}$ is an isomorphism. Also for $j \le l$ let

$$X_j = \{i \le k : \gamma[\mathcal{A}_{n,j}] \cap \mathcal{E}_i \neq 0\}.$$

Note that

(1) $k(j) \in X_j$,
(2) $k(j') \notin X_j$ if $j \neq j'$,
(3) $i \in X_j \Leftrightarrow j \in Y_i$.

Let $\hat{\delta}_j : \mathcal{E}_{k(j)} \to (1_m - \sum_{i\in X_j} q_i)\mathcal{A}_m(1_m - \sum_{i\in X_j} q_i)$ be the homomorphism defined by

$$\hat{\delta}_j(e) = (1_m - \sum_{i\in X_j} q_i)\phi_n^m(\hat{\gamma}_j^{-1}(e))(1_m - \sum_{i\in X_j} q_i).$$

Define $\delta : \mathcal{E} \hookrightarrow \mathcal{A}_m$ by

$$\delta(e_1, \dots, e_k) = \hat{\delta}_1(e_{k(1)}) + \dots + \hat{\delta}_l(e_{k(l)}) + \delta_1(e_1) + \dots + \delta_k(e_k).$$

Since each δ_i is an isomorphism, it is clear that δ is left-invertible. To check that $\delta \circ \gamma = \phi_n^m$, by linearity of the maps it is enough to check it only for $\bar{a} = (0, \dots, 0, a_j, 0, \dots, 0) \in \mathcal{A}_n$. If $\gamma(\bar{a}) = (e_1, \dots, e_k)$ then

$$e_i = \begin{cases} 0 & i \notin X_j \\ \gamma_i(\bar{a}) & i \in X_j \end{cases}$$

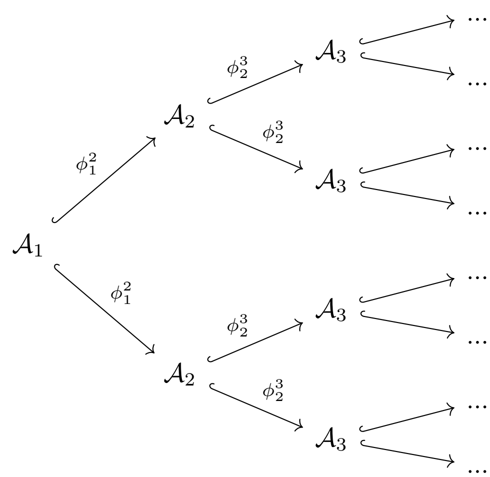
for $i \le k$. Also note that $e_{k(j)} = \hat{\gamma}_j(a_j)$. Assume $X_j = \{r_1, \dots, r_\ell\}$. Then by (1)-(3) we have

$$\begin{aligned} \delta \circ \gamma(\bar{a}) &= \hat{\delta}_j(\hat{\gamma}_j(a_j)) + \delta_{r_1}(\gamma_{r_1}(\bar{a})) + \dots + \delta_{r_\ell}(\gamma_{r_\ell}(\bar{a})) \\ &= (1_m - \sum_{i\in X_j} q_i)\phi_n^m(\bar{a})(1_m - \sum_{i\in X_j} q_i) + q_{r_1}\phi_n^m(\bar{a})q_{r_1} + \dots + q_{r_\ell}\phi_n^m(\bar{a})q_{r_\ell} \\ &= \phi_n^m(\bar{a}). \end{aligned}$$

This completes the proof. $\square$

4.1. **AF-algebras with Cantor property are $C(2^{\mathbb{N}})$-absorbing.** Suppose $\mathcal{A}$ is an AF-algebra with Cantor property. Define $\mathcal{A}^{\mathcal{C}}$ to be the limit of the sequence $(\mathcal{B}_n, \psi_n^m)$ such that $\mathcal{B}_n = \bigoplus_{i \leq 2^{n-1}} \mathcal{A}_n \cong \mathbb{C}^{2^{n-1}} \otimes \mathcal{A}_n$ and $\psi_n^{n+1} = \bigoplus_{i \leq 2^n} \phi_n^{n+1}$, as shown in the following diagram

(4.1)



It is straightforward to check that $\mathcal{A}^{\mathcal{C}} \cong \mathcal{A} \otimes C(2^{\mathbb{N}}) \cong C(2^{\mathbb{N}}, \mathcal{A})$.

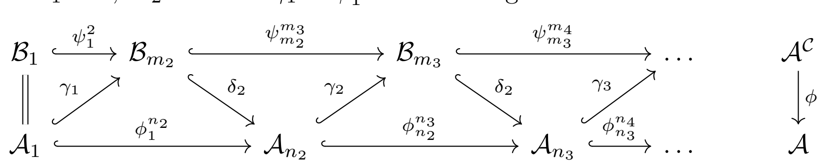
Lemma 4.6. *$\mathcal{A}^{\mathcal{C}}$ has the Cantor property.*

Proof. We check that $(\mathcal{B}_n, \psi_n^m)$ satisfies (D0)–(D2). Each ψ_n^{n+1} is left-invertible, by Proposition 3.2 and since ϕ_n^{n+1} is left-invertible, therefore (D0) holds. Conditions (D1) and (D2) are trivially satisfied by analyzing the Bratteli diagram (4.1), since $\mathcal{A}$ satisfies them. □

Lemma 4.7. *Suppose $\mathcal{A}$ is an AF-algebra with Cantor property. Then $\mathcal{A} \otimes C(2^{\mathbb{N}})$ is isomorphic to $\mathcal{A}$.*

Proof. Identify $\mathcal{A} \otimes C(2^{\mathbb{N}})$ with $\mathcal{A}^{\mathcal{C}}$. Find sequences (m_i) and (n_i) of natural numbers and left-invertible embeddings $\gamma_i : \mathcal{A}_{n_i} \hookrightarrow \mathcal{B}_{m_{i+1}}$ and $\delta_i : \mathcal{B}_{n_i} \hookrightarrow \mathcal{A}_{m_i}$ such that $n_1 = m_1 = 1$, $m_2 = 2$ and $\gamma_1 = \psi_1^2$ and the diagram below is commutative.

(4.2)



The existence of such γ_i and δ_i is guaranteed by Lemma 4.5, since each $\mathcal{B}_i$ is a retract of $\mathcal{A}$, by Proposition 4.4 and Lemma 3.5, and of course each $\mathcal{A}_i$ is a retract of $\mathcal{B}_i$. The universal property of inductive limits implies the existence of an isomorphism between $\mathcal{A}$ and $\mathcal{A}^{\mathcal{C}}$. □

4.2. **Ideals.** Let $\mathcal{A} = \varinjlim_n (\mathcal{A}_n, \phi_n^m)$ be an AF-algebra with Cantor property, such that the Bratteli diagram $\mathfrak{D}$ of $(\mathcal{A}_n, \phi_n^m)$ satisfies (D0)–(D2) of Definition 4.1. Let $\mathfrak{J} \subseteq \mathfrak{D}$ denote the Bratteli diagram of an ideal $\mathcal{J} \subseteq \mathcal{A}$. Put $\mathcal{J}_n = \bigoplus_{(n,s) \in \mathfrak{J}} \mathcal{A}_{n,s}$, which is an ideal (a retract) of $\mathcal{A}_n$. Then $\mathcal{J} = \varinjlim_n (\mathcal{J}_n, \phi_n^m|_{\mathcal{J}_n})$. It is automatic from the fact that $\mathfrak{J}$ is a directed subdiagram of $\mathfrak{D}$ that each $\phi_n^m|_{\mathcal{J}_n} : \mathcal{J}_n \hookrightarrow \mathcal{J}_m$ is left-invertible and that $(\mathcal{J}_n, \phi_n^m|_{\mathcal{J}_n})$ satisfies (D0)–(D2). In particular:

Proposition 4.8. *Any ideal of an AF-algebra with Cantor property also has the Cantor property.*

Here is another elementary fact about $C(2^{\mathbb{N}})$ that is (essentially by Lemma 4.7) passed on to AF-algebras with Cantor property.

Proposition 4.9. *Suppose $\mathcal{A}$ is an AF-algebra with Cantor property and $\mathcal{Q}$ is a quotient of $\mathcal{A}$. Then there is a surjection $\eta : \mathcal{A} \twoheadrightarrow \mathcal{Q}$ such that $ker(\eta)$ is an essential ideal of $\mathcal{A}$.*

Proof. It is enough to show that there is an essential ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{J}$ is isomorphic to $\mathcal{A}$. In fact, we will show that there is an essential ideal $\mathcal{J}$ of $\mathcal{A}^{\mathcal{C}}$ such that $\mathcal{A}^{\mathcal{C}}/\mathcal{J}$ is isomorphic to $\mathcal{A}$. This is enough since $\mathcal{A}^{\mathcal{C}}$ is isomorphic to $\mathcal{A}$ (Lemma 4.7). Let $\mathfrak{D}$ be the Bratteli diagram of $\mathcal{A}^{\mathcal{C}}$ as in Diagram (4.1). Let $\mathfrak{J}$ be the directed and hereditary subdiagram of $\mathfrak{D}$ containing all the nodes in Diagram (4.1) except the lowest line. Being directed and hereditary, $\mathfrak{J}$ corresponds to an ideal $\mathcal{J}$, which intersects any other directed and hereditary subdiagram of $\mathfrak{D}$. Therefore $\mathcal{J}$ is an essential ideal of $\mathcal{A}^{\mathcal{C}}$ and $\mathcal{A}^{\mathcal{C}}/\mathcal{J}$ is isomorphic to the limit of the sequence $\mathcal{A}_1 \xrightarrow{\phi_1^2} \mathcal{A}_2 \xrightarrow{\phi_2^3} \mathcal{A}_3 \xrightarrow{\phi_3^4} \dots$ in the lowest line of Diagram (4.1), which is $\mathcal{A}$. □

5. Fraïssé categories

Suppose $\mathfrak{L}$ is a category of metric structures with non-expansive (1-Lipschitz) morphisms. We refer to objects and morphisms (arrows) of $\mathfrak{L}$ by $\mathfrak{L}$-objects and $\mathfrak{L}$-arrows, respectively. We write $A \in \mathfrak{L}$ if A is an $\mathfrak{L}$-object and $\mathfrak{L}(A, B)$ to denote the set of all $\mathfrak{L}$-arrows from A to $B \in \mathfrak{L}$. The category $\mathfrak{L}$ is *metric-enriched* if for every $\mathfrak{L}$-objects A and B there is a metric d on $\mathfrak{L}(A, B)$ satisfying

$$d(\psi_0 \circ \phi, \psi_1 \circ \phi) \leq d(\psi_0, \psi_1) \qquad \text{and} \qquad d(\psi \circ \phi_0, \psi \circ \phi_0) \leq d(\phi_0, \phi_1)$$

whenever the compositions make sense. An $\mathfrak{L}$-*sequence* is a direct sequence in $\mathfrak{K}$.

A category $\mathfrak{K}$ is a full subcategory of $\mathfrak{L}$ if $\mathfrak{K} \subseteq \mathfrak{L}$ and $\mathfrak{L}(A, B) = \mathfrak{K}(A, B)$ for every $A, B \in \mathfrak{K}$. Equipped with the same metrics, $\mathfrak{K}$ is also a metric-enriched category. Assume that $\mathfrak{K}$ is a full subcategory of a metric-enriched category $\mathfrak{L}$. Consider the following conditions.

- (L0) $\mathfrak{L}$-arrows are monics.
- (L1) Every $\mathfrak{K}$-sequence has its limit in $\mathfrak{L}$.
- (L2) Every $\mathfrak{L}$-object is the limit of a $\mathfrak{K}$-sequence.
- (L3) For every $\epsilon > 0$, for every $\mathfrak{K}$-sequence (B_n, ψ_n^m) with $B = \varinjlim(B_n, \psi_n^m)$ in $\mathfrak{L}$, every $\mathfrak{K}$-object D and for every $\mathfrak{L}$-arrow $\phi : D \to B$, there is a natural number n and a $\mathfrak{K}$-arrow $\psi : D \to B_n$ such that $d(\psi_n^\infty \circ \psi, \phi) < \epsilon$.

In the conditions above and later on, by a *limit* we mean inductive limit (called also *colimit*). In the category of C*-algebras or, more generally, Banach spaces, the limit of a sequence of isometric embeddings always exists and is isometric to the completion of the union of the corresponding chain of spaces.

Definition 5.1. Suppose $\mathfrak{K}$ is a metric-enriched category. We say $\mathfrak{K}$ is a *Fraïssé category* if

- (JEP) $\mathfrak{K}$ has *the joint embedding property*: for $A, B \in \mathfrak{K}$ there is $C \in \mathfrak{K}$ such that $\mathfrak{K}(A, C)$ and $\mathfrak{K}(B, C)$ are nonempty.

(NAP) $\mathfrak{K}$ has *the near amalgamation property*: for every $\epsilon > 0$, objects $A, B, C \in \mathfrak{K}$, arrows $\phi \in \mathfrak{K}(A, B)$ and $\psi \in \mathfrak{K}(A, C)$, there are $D \in \mathfrak{K}$ and $\phi' \in \mathfrak{K}(B, D)$ and $\psi' \in \mathfrak{K}(C, D)$ such that $d(\phi' \circ \phi, \psi' \circ \psi) < \epsilon$.

(SEP) $\mathfrak{K}$ is *separable*: there is a countable *dominating* subcategory $\mathfrak{C}$, that is,
- for every $A \in \mathfrak{K}$ there is $C \in \mathfrak{C}$ and a $\mathfrak{K}$-arrow $\phi : A \to C$,
- for every $\epsilon > 0$ and a $\mathfrak{K}$-arrow $\phi : A \to B$ with $A \in \mathfrak{C}$, there exist a $\mathfrak{K}$-arrow $\psi : B \to C$ with $\mathcal{C} \in \mathfrak{C}$ and a $\mathfrak{C}$-arrow $\alpha : A \to C$ such that $d(\alpha, \psi \circ \phi) < \epsilon$.

Theorem 5.2. [8, Theorem 3.3] *Suppose $\mathfrak{K}$ is a Fraïssé category. Then there exists a sequence (U_n, ϕ_n^m) in $\mathfrak{K}$ satisfying*

(F) for every $n \in \mathbb{N}$, for every $\epsilon > 0$ and for every $\mathfrak{K}$-arrow $\gamma : U_n \to D$, there are $m \geq n$ and a $\mathfrak{K}$-arrow $\delta : D \to U_m$ such that $d(\phi_n^m, \delta \circ \gamma) < \epsilon$.

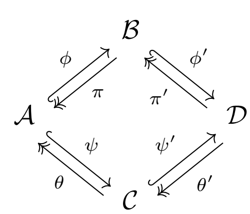
If $\mathfrak{K}$ is a Fraïssé category, the $\mathfrak{K}$-sequence (U_n, ϕ_n^m) from Theorem 5.2, is uniquely determined by "Fraïssé condition" (F). That is, any two $\mathfrak{K}$-sequences satisfying (F) can be approximately intertwined (there is an approximate back-and-forth between them), and hence the limits of the sequences must be isomorphic (see [8, Theorem 3.5]). Therefore the $\mathfrak{K}$-sequence satisfying (F) is usually referred to as "the" *Fraïssé sequence*. The limit of the Fraïssé sequence is called the *Fraïssé limit* of the category $\mathfrak{K}$.

Theorem 5.3. [8] *Assume $\mathfrak{K} \subseteq \mathfrak{L}$ satisfy (L0)-(L3), $\mathfrak{K}$ is a Fraïssé category and $U \in \mathfrak{L}$ is the Fraïssé limit of $\mathfrak{K}$. Then*
- *(uniqueness) U is unique, up to isomorphisms.*
- *(universality) For every $\mathfrak{L}$-object B there is an $\mathfrak{L}$-arrow $\phi : B \to U$.*
- *(almost $\mathfrak{K}$-homogeneity) For every $\epsilon > 0$, $\mathfrak{K}$-object A and $\mathfrak{L}$-arrows $\phi_i : A \to U$ $(i = 0, 1)$, there is an automorphism $\eta : U \to U$ such that $d(\eta \circ \phi_0, \phi_1) < \epsilon$.*

In the following suppose $\mathfrak{K}$ is a (naturally metric-enriched by the norm) category of C^*-algebras such that every $\mathfrak{K}$-arrow is a left-invertible embedding.

Definition 5.4. Let $\ddagger\mathfrak{K}$ denote the category with the same objects as $\mathfrak{K}$, but a $\ddagger\mathfrak{K}$-arrow from $\mathcal{D}$ to $\mathcal{E}$ is a pair (ϕ, π) where $\phi : \mathcal{D} \to \mathcal{E}$ is left-invertible and $\pi : \mathcal{E} \to \mathcal{D}$ is a left inverse of ϕ. We will denote such $\ddagger\mathfrak{K}$-arrow by $(\phi, \pi) : \mathcal{D} \to \mathcal{E}$. The composition is $(\phi, \pi) \circ (\phi', \pi') = (\phi \circ \phi', \pi' \circ \pi)$. The category $\ddagger\mathfrak{K}$ is usually called the category of *embedding-projection pairs* or briefly EP-pairs (see [7]) over $\mathfrak{K}$.

Definition 5.5. We say $\ddagger\mathfrak{K}$ has the *near proper amalgamation property* if for every $\epsilon > 0$, objects $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathfrak{K}$, arrows $(\phi, \pi) \in \ddagger\mathfrak{K}(\mathcal{A}, \mathcal{B})$ and $(\psi, \theta) \in \ddagger\mathfrak{K}(\mathcal{A}, \mathcal{C})$, there are $D \in \ddagger\mathfrak{K}$ and $(\phi', \pi') \in \ddagger\mathfrak{K}(\mathcal{B}, \mathcal{D})$ and $(\psi', \theta') \in \ddagger\mathfrak{K}(\mathcal{C}, \mathcal{D})$ such that the diagram



"fully commutes" up to ϵ, meaning that, $\|\phi' \circ \phi - \psi' \circ \psi\|$, $\|\pi' \circ \pi - \theta' \circ \theta\|$, $\|\phi \circ \theta - \pi' \circ \psi'\|$ and $\|\psi \circ \pi - \theta' \circ \phi'\|$ are all less than or equal to ϵ. We say $\ddagger\mathfrak{K}$ has the "proper amalgamation property" if ϵ could be 0.

6. Categories of finite-dimensional C^*-algebras and left-invertible mappings

In this section $\mathfrak{K}$ always denotes a (metric-enriched) category whose objects are (not necessarily all) finite-dimensional C^*-algebras, closed under isomorphisms, and $\mathfrak{K}$-arrows are left-invertible embeddings. For such $\mathfrak{K}$, let $\mathfrak{L}\mathfrak{K}$ denote the "category of limits" of $\mathfrak{K}$; a category whose objects are limits of $\mathfrak{K}$-sequences and if $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{L}\mathfrak{K}$-objects, then an $\mathfrak{L}\mathfrak{K}$-arrow from $\mathcal{B}$ into $\mathcal{C}$ is a left-invertible embedding $\phi : \mathcal{B} \hookrightarrow \mathcal{C}$. Clearly $\mathfrak{L}\mathfrak{K}$ contains $\mathfrak{K}$ as a full subcategory. The metric defined between $\mathfrak{L}\mathfrak{K}$-arrows ϕ and ψ with the same domain and codomain is $\|\phi - \psi\|$. The corresponding category of EP-pairs $\ddagger\mathfrak{K}$ is defined as in the previous section.

Remark 6.1. If $\mathfrak{K}$ is a category of finite-dimensional C^*-algebras and embeddings, then it has the near amalgamation property (NAP) if and only if it has the amalgamation property ([4, Lemma 3.2]), namely, with $\epsilon = 0$. Similarly, the near proper amalgamation property of $\ddagger\mathfrak{K}$ is equivalent to the proper amalgamation property of $\ddagger\mathfrak{K}$. Also in this case, the Fraïssé sequence $(\mathcal{U}_n, \phi_n^m)$, whenever it exists for $\mathfrak{K}$, satisfies the Fraïssé condition (F) of Theorem 5.2 with $\epsilon = 0$. Therefore in this section (F) refers the following condition.

(F) for every $n \in \mathbb{N}$ and for every $\mathfrak{K}$-arrow $\gamma : \mathcal{U}_n \to \mathcal{D}$, there are $m \geq n$ and $\mathfrak{K}$-arrow $\delta : \mathcal{D} \to \mathcal{U}_m$ such that $\phi_n^m = \delta \circ \gamma$.

Lemma 6.2. $\mathfrak{K} \subseteq \mathfrak{L}\mathfrak{K}$ *satisfy (L0)–(L3).*

Proof. Conditions (L0)–(L2) are trivially satisfied. In order to show (L3), suppose $\mathcal{B} \in \mathfrak{L}\mathfrak{K}$ is the limit of the $\mathfrak{K}$-sequence $(\mathcal{B}_n, \psi_n^{n+1})$ and (θ_n^m) is a compatible left inverse of (ψ_n^{n+1}). Assume $\mathcal{D}$ is a $\mathfrak{K}$-object and $\phi : \mathcal{D} \hookrightarrow \mathcal{B}$ is an $\mathfrak{L}\mathfrak{K}$-arrow with a left inverse $\pi : \mathcal{B} \twoheadrightarrow \mathcal{D}$. For given $\epsilon > 0$, find n and a unitary u in $\mathcal{B}$ such that $u^*\phi[\mathcal{D}]u \subseteq \psi_n^\infty[\mathcal{B}_n]$ and $\|u - 1\| < \epsilon/2$ (Lemma 2.1). Define $\psi : \mathcal{D} \hookrightarrow \mathcal{B}_n$ by $\psi(d) = \theta_n^\infty(u^*\phi(d)u)$. Then ψ has a left inverse $\theta : \mathcal{B}_n \twoheadrightarrow \mathcal{D}$ defined by $\theta(x) = \pi(u\psi_n^\infty(x)u^*)$ (see the proof of Lemma 3.5 (2)). The condition $\|u-1\| < \epsilon/2$ implies that $\|\psi_n^\infty(\psi(d)) - \phi(d)\| < \epsilon$, for every d in the unit ball of $\mathcal{D}$. □

Lemma 6.3. $\mathfrak{K}$ *is separable.*

Proof. There are, up to isomorphisms, countably many $\mathfrak{K}$-objects, namely finite sums of matrix algebras. The set of all embeddings between fixed two finite-dimensional C*-algebras is a separable metric space. Thus, $\mathfrak{K}$ trivially has a countable dominating subcategory. □

Lemma 6.4. *Suppose $\mathfrak{K}$ is a Fraïssé category with Fraïssé limit $\mathcal{U}$ and $\ddagger\mathfrak{K}$ has the proper amalgamation property. Every AF-algebra $\mathcal{B}$ in $\mathfrak{L}\mathfrak{K}$, is a retract of $\mathcal{U}$. In particular, $\mathcal{U}$ maps onto any AF-algebra in $\mathfrak{L}$.*

Proof. Suppose $(\mathcal{U}_n, \phi_n^m)$ is a Fraïssé sequence in $\mathfrak{K}$. That is, it satisfies condition (F) of Remark 6.1 and its limit is automatically $\mathcal{U}$.

Now suppose $(\mathcal{B}_n, \psi_n^m)$ is a $\mathfrak{K}$-sequence whose direct limit is $\mathcal{B}$. Pick left inverses θ_n^m compatible with ψ_n^m and form a $\ddagger\mathfrak{K}$-sequence $(\mathcal{B}_n, (\psi_n^m, \theta_n^m))$, whose limit is, of course, again $\mathcal{B}$. Using (JEP) of $\mathfrak{K}$ and fixing arbitrary left inverses, find $\mathcal{F}_1 \in \mathfrak{K}$ and $\ddagger\mathfrak{K}$-arrows $(\gamma_1, \eta_1) : \mathcal{U}_1 \to \mathcal{F}_1$ and $(\mu_1, \nu_1) : \mathcal{B}_1 \to \mathcal{F}_1$. By (F) and again fixing arbitrary left inverses, there are $n_1 \geq 1$ and a $\ddagger\mathfrak{K}$-arrow $(\delta_1, \lambda_1) : \mathcal{F}_1 \to \mathcal{U}_{n_1}$ such that $\phi_1^{n_1} = \delta_1 \circ \gamma_1$ (see Diagram (6.3) below).

Consider the composition map $(\delta_1\circ\mu_1, \nu_1\circ\lambda_1) : \mathcal{B}_1 \to \mathcal{U}_{n_1}$ and $(\psi_1^2, \theta_1^2) : \mathcal{B}_1 \to \mathcal{B}_2$ and use the proper amalgamation property to find $\mathcal{F}_2 \in \mathfrak{K}$ and $\ddagger\mathfrak{K}$-arrows $(\mu_2, \nu_2) : \mathcal{B}_2 \to \mathcal{F}_2$ and $(\gamma_2, \eta_2) : \mathcal{U}_{n_1} \to \mathcal{F}_2$ such that

$$\gamma_2 \circ \delta_1 \circ \mu_1 = \mu_2 \circ \psi_1^2 \qquad \text{and} \qquad \nu_2 \circ \gamma_2 = \psi_1^2 \circ \nu_1 \circ \lambda_1. \tag{6.1}$$

Again using (F) we can find $n_2 \geq n_1$ and $(\delta_2, \lambda_2) : \mathcal{F}_2 \to \mathcal{U}_{n_2}$ such that

$$\phi_{n_1}^{n_2} = \delta_2 \circ \gamma_2. \tag{6.2}$$

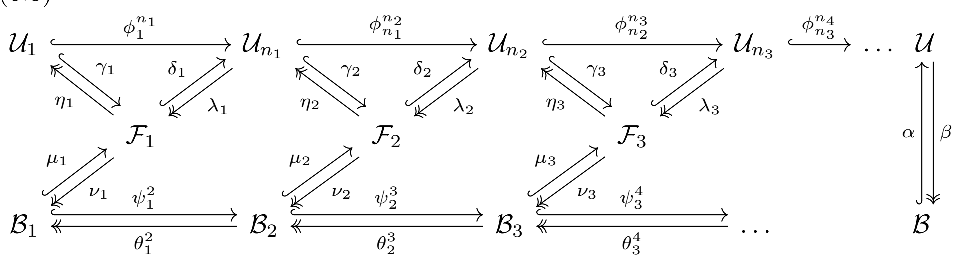
Combining the equations in (6.1) and (6.2) we have (also can be easily checked in Diagram (6.3))

$$\phi_{n_1}^{n_2} \circ \delta_1 \circ \mu_1 = \delta_2 \circ \mu_2 \circ \psi_1^2 \qquad \text{and} \qquad \psi_1^2 \circ \nu_1 \circ \lambda_1 = \nu_2 \circ \lambda_2 \circ \phi_{n_1}^{n_2}$$

Again use the proper amalgamation property to find $\mathcal{F}_3 \in \mathfrak{K}$ and $(\mu_3, \nu_3) : \mathcal{B}_3 \to \mathcal{F}_3$ and $(\gamma_3, \eta_3) : \mathcal{U}_{n_2} \to \mathcal{F}_3$. Follow the procedure, by finding $\ddagger\mathfrak{K}$-arrow $(\delta_3, \lambda_3) : \mathcal{F}_3 \to \mathcal{U}_{n_3}$, for some $n_3 \geq n_2$ such that

$$\phi_{n_2}^{n_3} \circ \delta_2 \circ \mu_2 = \delta_3 \circ \mu_3 \circ \psi_2^3 \qquad \text{and} \qquad \psi_2^3 \circ \nu_2 \circ \lambda_2 = \nu_3 \circ \lambda_3 \circ \phi_{n_2}^{n_3}$$

(6.3)



Let $\alpha_i = \delta_i \circ \mu_i$ and $\beta_i : \nu_i \circ \lambda_i$. By the construction, for every $i \in \mathbb{N}$ we have

$$\phi_{n_i}^{n_{i+1}} \circ \alpha_i = \alpha_{i+1} \circ \psi_i^{i+1} \qquad \text{and} \qquad \psi_1^2 \circ \beta_i = \beta_{i+1} \circ \phi_{n_i}^{n_{i+1}}$$

and β_i is a left inverse of α_i. Then $\alpha = \lim_i \alpha_i$ is a well-defined embedding from $\mathcal{B}$ to $\mathcal{U}$ and $\beta = \lim_i \beta_i$ is a well-defined surjection from $\mathcal{U}$ onto $\mathcal{B}$ such that $\beta \circ \alpha = \mathrm{id}_{\mathcal{B}}$. □

7. AF-algebras with Cantor property as Fraïssé limits

Suppose $\mathfrak{K}$ is a category of (not necessarily all) finite-dimensional C^*-algebras, closed under isomorphisms, and $\mathfrak{K}$-arrows are left-invertible embeddings.

Definition 7.1. We say $\mathfrak{K}$ is $\oplus$-*stable* if it satisfies the following conditions.

(1) If $\mathcal{D}$ is a $\mathfrak{K}$-object, then so is any retract (ideal) of $\mathcal{D}$,
(2) $\mathcal{D} \oplus \mathcal{E} \in \mathfrak{K}$ whenever $\mathcal{D}, \mathcal{E} \in \mathfrak{K}$.

In general 0 is a retract of any C^*-algebra and therefore it is the initial object of any $\oplus$-stable category, unless, when working with the unital categories (when all the $\mathfrak{K}$-arrows are unital), which in that case 0 is not a $\mathfrak{K}$-object anymore. Unital categories are briefly discussed in Section 9.

Theorem 7.2. *Suppose $\mathfrak{K}$ is a $\oplus$-stable category. Then $\ddagger\mathfrak{K}$ has proper amalgamation property. In particular, $\mathfrak{K}$ is a Fraïssé category.*

Proof. Suppose $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are $\mathfrak{K}$-objects and $\ddagger\mathfrak{K}$-arrows $(\phi, \pi) : \mathcal{D} \to \mathcal{E}$ and $(\psi, \theta) : \mathcal{D} \to \mathcal{F}$ are given. Since ϕ and ψ are left-invertible, by Proposition 3.2 we can identify $\mathcal{E}$ and $\mathcal{F}$ with $\mathcal{E}_0 \oplus \mathcal{E}_1$ and $\mathcal{F}_0 \oplus \mathcal{F}_1$, respectively, and find $\phi_0, \phi_1, \psi_0, \psi_1$ such that

- $\phi_0 : \mathcal{D} \to \mathcal{E}_0$ and $\psi_0 : \mathcal{D} \to \mathcal{F}_0$ are isomorphisms,
- $\phi_1 : \mathcal{D} \to \mathcal{E}_1$ and $\psi_1 : \mathcal{D} \to \mathcal{F}_1$ are homomorphisms,
- $\phi(d) = (\phi_0(d), \phi_1(d))$ and $\psi(d) = (\psi_0(d), \psi_1(d))$ for every $d \in \mathcal{D}$,
- $\pi(e_0, e_1) = \phi_0^{-1}(e_0)$ and $\theta(f_0, f_1) = \psi_0^{-1}(f_0)$.

Define homomorphisms $\mu : \mathcal{E} \to \mathcal{F}_1$ and $\nu : \mathcal{F} \to \mathcal{E}_1$ by $\mu = \psi_1 \circ \pi$ and $\nu = \phi_1 \circ \theta$ (see Diagram (7.1)). Since $\mathfrak{K}$ is $\oplus$-stable $\mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1$ is a $\mathfrak{K}$-object. Define $\mathfrak{K}$-arrows $\phi' : \mathcal{E} \hookrightarrow \mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1$ and $\psi' : \mathcal{F} \hookrightarrow \mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1$ by

$$\phi'(e_0, e_1) = (\phi_0^{-1}(e_0), e_1, \mu(e_0, e_1))$$

and

$$\psi'(f_0, f_1) = (\psi_0^{-1}(f_0), \nu(f_0, f_1), f_1).$$

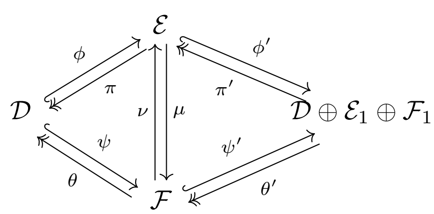
For every $d \in \mathcal{D}$ we have

$$\phi'(\phi(d)) = \phi'(\phi_0(d), \phi_1(d)) = (d, \phi_1(d), \mu(\phi(d))) = (d, \phi_1(d), \psi_1(d))$$

and

$$\psi'(\psi(d)) = \psi'(\psi_0(d), \psi_1(d)) = (d, \nu(\phi(d)), \psi_1(d)) = (d, \phi_1(d), \psi_1(d)).$$

(7.1)



Therefore $\phi' \circ \phi = \psi' \circ \psi$. The map $\pi' : \mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1 \to \mathcal{E}$ defined by $\pi'(d, e_1, f_1) = (\phi_0(d), e_1)$ is a left inverse of ϕ'. Similarly the map $\theta' : \mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1 \to \mathcal{F}$ defined by $\theta'(d, e_1, f_1) = (\psi_0(d), f_1)$ is a left inverse of ψ'. Therefore $(\phi', \pi') : \mathcal{E} \to \mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1$ and $(\psi', \theta') : \mathcal{E} \to \mathcal{D} \oplus \mathcal{E}_1 \oplus \mathcal{F}_1$ are $\mathfrak{K}$-arrows. We have

$$\pi \circ \pi'(d, e_1, f_1) = \pi(\phi_0(d), e_1) = d,$$

$$\theta \circ \theta'(d, e_1, f_1) = \theta(\psi_0(d), e_1) = d.$$

Hence $\pi \circ \pi' = \theta \circ \theta'$. Also

$$\begin{aligned} \theta' \circ \phi'(e_0, e_1) &= \theta'(\phi_0^{-1}(e_0), e_1, \mu(e_0, e_1)) = (\psi_0(\phi_0^{-1}(e_0)), \mu(e_0, e_1)) \\ &= (\psi_0(\pi(e_0, e_1)), \psi_1(\pi(e_0, e_1))) = \psi(\pi(e_0, e_1)). \end{aligned}$$

So $\theta' \circ \phi' = \psi \circ \pi$ and similarly we have $\phi \circ \theta = \pi' \circ \psi'$. This shows that $\ddagger\mathfrak{K}$ has proper amalgamation property. Since $\mathfrak{K}$ is separable and has an initial object, in particular, it is a Fraïssé category. □

Therefore any $\oplus$-stable category $\mathfrak{K}$ has a unique Fraïssé sequence; a $\mathfrak{K}$-sequence which satisfies (F).

Notation. Let $\mathcal{A}_{\mathfrak{K}}$ denote the Fraïssé limit of the $\oplus$-stable category $\mathfrak{K}$.

The AF-algebra $\mathcal{A}_{\mathfrak{K}}$ is $\mathfrak{K}$-universal and almost $\mathfrak{K}$-homogeneous (Theorem 5.3). In fact, $\mathcal{A}_{\mathfrak{K}}$ is $\mathfrak{K}$-homogeneous (where ϵ is zero). To see this, suppose $\mathcal{F}$ is a finite-dimensional C^*-algebra in $\mathfrak{K}$ and $\phi_i : \mathcal{F} \hookrightarrow \mathcal{A}_{\mathfrak{K}}$ $(i = 0, 1)$ are left-invertible embeddings. By almost $\mathfrak{K}$-homogeneity, there is an automorphism $\eta : \mathcal{A}_{\mathfrak{K}} \to \mathcal{A}_{\mathfrak{K}}$ such that $\|\eta \circ \phi_0 - \phi_1\| < 1$. There exists (Lemma 2.2) a unitary $u \in \widetilde{\mathcal{A}}$ such that $\mathrm{Ad}_u \circ \eta \circ \phi_0 = \phi_1$. The automorphism $\mathrm{Ad}_u \circ \eta$ witnesses the $\mathfrak{K}$-homogeneity.

Moreover, since $\ddagger\mathfrak{K}$ has the proper amalgamation property, every AF-algebra in $\mathfrak{L}\mathfrak{K}$, is a retract of $\mathcal{A}_{\mathfrak{K}}$ (Lemma 6.4).

Corollary 7.3. *Suppose $\mathfrak{K}$ is a $\oplus$-stable category, then*

- *(universality) Every AF-algebra which is the limit of a $\mathfrak{K}$-sequence, is a retract of $\mathcal{A}_{\mathfrak{K}}$.*
- *($\mathfrak{K}$-homogeneity) For every finite-dimensional C^*-algebra $\mathcal{F} \in \mathfrak{K}$ and left-invertible embeddings $\phi_i : \mathcal{F} \hookrightarrow \mathcal{A}_{\mathfrak{K}}$ $(i = 0, 1)$, there is an automorphism $\eta : \mathcal{A}_{\mathfrak{K}} \to \mathcal{A}_{\mathfrak{K}}$ such that $\eta \circ \phi_0 = \phi_1$.*

We will describe the structure of $\mathcal{A}_{\mathfrak{K}}$ by showing that it has the Cantor property.

Lemma 7.4. *Suppose $\mathfrak{K}$ is a $\oplus$-stable category, then $\mathcal{A}_{\mathfrak{K}}$ has the Cantor property.*

Proof. Suppose $\mathcal{A}_{\mathfrak{K}} = \varinjlim_n (\mathcal{A}_n, \phi_n^m)$, where $(\mathcal{A}_n, \phi_n^m)$ is a $\mathfrak{K}$-sequence, i.e., $(\mathcal{A}_n, \phi_n^m)$ is a left-invertible sequence of finite-dimensional C^*-algebras in $\mathfrak{K}$. Since $\mathcal{A}_{\mathfrak{K}}$ is the Fraïssé limit of $\mathfrak{K}$, $(\mathcal{A}_n, \phi_n^m)$ satisfies (F). We claim that $(\mathcal{A}_n, \phi_n^m)$ satisfies (D0)–(D2) of Definition 4.1. Suppose $\mathfrak{D}$ is the Bratteli diagram of $(\mathcal{A}_n, \phi_n^m)$ and $\mathcal{A}_n = \mathcal{A}_{n,1} \oplus \cdots \oplus \mathcal{A}_{n,k_n}$ for every n, such that each $\mathcal{A}_{n,s}$ is a matrix algebra.

The condition (D0) is trivial since ϕ_n^m are left-invertible. To see (D1), fix $\mathcal{A}_{n,s}$. Note that since $\mathcal{A}_n$ is a $\mathfrak{K}$-object and $\mathfrak{K}$ is $\oplus$-stable, we have $\mathcal{A}_n \oplus \mathcal{A}_n \in \mathfrak{K}$. Let $\gamma : \mathcal{A}_n \hookrightarrow \mathcal{A}_n \oplus \mathcal{A}_n$ be the left-invertible embedding defined by $\gamma(a) = (a, a)$. Use the Fraïssé condition (F) to find $\delta : \mathcal{A}_n \oplus \mathcal{A}_n \hookrightarrow \mathcal{A}_m$, for some $m \geq n$, such that $\delta \circ \gamma = \phi_n^m$. Since δ is left-invertible, there are distinct (m, t) and (m, t') in $\mathfrak{D}$ such that $\mathcal{A}_{m,t} \cong \mathcal{A}_{m,t'} \cong \mathcal{A}_{n,s}$. Then $\delta \circ \gamma = \phi_n^m$ implies that $(n, s) \to (m, t)$ and $(n, s) \to (m, t')$ in $\mathfrak{D}$.

To see (D2) assume $\mathcal{D} \subseteq \mathcal{A}_n$ is an ideal of $\mathcal{A}_n$ and M_ℓ is a retract of $\mathcal{A}_{\mathfrak{K}}$ and there is an embedding $\gamma : \mathcal{D} \hookrightarrow M_\ell$. Suppose $\mathcal{A}_n = \mathcal{D} \oplus \mathcal{E}$ for some $\mathcal{E}$. Since $\mathfrak{K}$ is $\oplus$-stable, $\mathcal{D} \oplus \mathcal{E} \oplus M_\ell$ is a $\mathfrak{K}$-object. Therefore $\gamma' : \mathcal{D} \oplus \mathcal{E} \hookrightarrow \mathcal{D} \oplus \mathcal{E} \oplus M_\ell$ defined by $\gamma'(d, e) = (d, e, \gamma(d))$ is a $\mathfrak{K}$-arrow. Then by (F) there is a left-invertible embedding $\delta' : \mathcal{D} \oplus \mathcal{E} \oplus M_\ell \hookrightarrow \mathcal{A}_m$ for some $m \geq n$, such that

$$\delta' \circ \gamma' = \phi_n^m. \tag{7.2}$$

Since δ' is left-invertible, there is (m, t) such that $\dim(\mathcal{A}_{m,t}) = \ell$ and

$$\delta_{m,t} = \pi_{\mathcal{A}_{m,t}} \circ \delta|_{M_\ell} : M_\ell \hookrightarrow \mathcal{A}_{m,t}$$

is an isomorphism, where $\pi_{\mathcal{A}_{m,t}} : \mathcal{A}_m \twoheadrightarrow \mathcal{A}_{m,t}$ is the canonical projection. Let

$$\phi_{m,t} = \pi_{\mathcal{A}_{m,t}} \circ \phi_n^m|_{\mathcal{D}} : \mathcal{D} \to \mathcal{A}_{m,t}.$$

By definition of γ' and (7.2) it is clear that $\phi_{m,t} = \delta_{m,t} \circ \gamma$ and that $\phi_{m,t}$ is also an embedding. By Lemma 2.3 we have $\mathrm{Mult}_{\phi_{m,t}}(\mathcal{D}, \mathcal{A}_{m,t}) = c\,\mathrm{Mult}_\gamma(\mathcal{D}, M_\ell)$ for some natural number $c \geq 1$. Since $\delta_{m,t}$ is an isomorphism, we have $c = 1$. This proves (D2). ∎

Next we show that every AF-algebra with Cantor property can be realized as the Fraïssé limit of a suitable $\oplus$-stable category of finite-dimensional C^*-algebras and left-invertible embeddings.

7.1. **The category $\mathfrak{K}_{\mathcal{A}}$.** Suppose $\mathcal{A}$ is an AF-algebra with Cantor property. Let $\mathfrak{K}_{\mathcal{A}}$ denote the category whose objects are finite-dimensional retracts of $\mathcal{A}$ and $\mathfrak{K}_{\mathcal{A}}$-arrows are left-invertible embeddings. Let $\mathfrak{L}_{\mathcal{A}}$ be the category whose objects are limits of $\mathfrak{K}_{\mathcal{A}}$-sequences. If $\mathcal{B}$ and $\mathcal{C}$ are $\mathfrak{L}_{\mathcal{A}}$-objects, an $\mathfrak{L}_{\mathcal{A}}$-arrow from $\mathcal{B}$ into $\mathcal{C}$ is a left-invertible embedding $\phi : \mathcal{B} \hookrightarrow \mathcal{C}$.

Lemma 7.5. *$\mathfrak{K}_{\mathcal{A}}$ is a Fraïssé category and $\ddagger\mathfrak{K}_{\mathcal{A}}$ has the proper amalgamation property.*

Proof. By Theorem 7.2, it is enough to show that $\mathfrak{K}_{\mathcal{A}}$ is a $\oplus$-stable category. Condition (1) of Definition 7.1 is trivial. Condition (2) follows from Proposition 4.4. □

Lemma 7.6. *$\mathfrak{K}_{\mathcal{A}} \subseteq \mathfrak{L}_{\mathcal{A}}$ satisfy (L0)–(L3).*

Proof. Similar to Lemma 6.2. □

Again, Theorem 5.3 guarantees the existence of a unique $\mathfrak{K}_{\mathcal{A}}$-universal and $\mathfrak{K}_{\mathcal{A}}$-homogeneous AF-algebra in $\mathfrak{L}_{\mathcal{A}}$, namely the Fraïssé limit of $\mathfrak{K}_{\mathcal{A}}$.

Theorem 7.7. *The Fraïssé limit of $\mathfrak{K}_{\mathcal{A}}$ is $\mathcal{A}$.*

Proof. There is a sequence $(\mathcal{A}_n, \phi_n^m)$ of finite-dimensional C^*-algebras and embeddings such that $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_n^m)$ satisfies (D0)–(D2) of Definition 4.1. First note that by (D0), $(\mathcal{A}_n, \phi_n^m)$ is a $\mathfrak{K}_{\mathcal{A}}$-sequence and therefore $\mathcal{A}$ is an $\mathfrak{L}_{\mathcal{A}}$-object. In order to show that $\mathcal{A}$ is the Fraïssé limit of $\mathfrak{K}_{\mathcal{A}}$, we need to show that $(\mathcal{A}_n, \phi_n^m)$ satisfies condition (F). This is Lemma 4.5. □

Theorem 7.8. *Suppose $\mathfrak{K}$ is a $\oplus$-stable category. $\mathcal{A}_{\mathfrak{K}}$ is the unique AF-algebra such that*

(1) it has the Cantor property,
(2) a finite-dimensional C^*-algebra is a retract of $\mathcal{A}_{\mathfrak{K}}$ if and only if it is a $\mathfrak{K}$-object.

Proof. We have already shown that $\mathcal{A}_{\mathfrak{K}}$ has the Cantor property (Lemma 7.4). By Lemma 3.5(2), every finite-dimensional retract of $\mathcal{A}_{\mathfrak{K}}$ is a $\mathfrak{K}$-object and every finite-dimensional C^*-algebra in $\mathfrak{K}$ is a retract of $\mathcal{A}_{\mathfrak{K}}$, by the $\mathfrak{K}$-universality of $\mathcal{A}_{\mathfrak{K}}$. If $\mathcal{A}$ is an AF-algebra satisfying (1) and (2), then by definition $\mathfrak{K}_{\mathcal{A}} = \mathfrak{K}$. The uniqueness of the Fraïssé limit and Theorem 7.7 imply that $\mathcal{A} \cong \mathcal{A}_{\mathfrak{K}}$. □

Corollary 7.9. *Two AF-algebras with Cantor property are isomorphic if and only if they have the same set of matrix algebras as retracts.*

8. Surjectively universal AF-algebras

Let $\mathfrak{F}$ denote the category of *all* finite-dimensional C^*-algebras and left-invertible embeddings. The category $\mathfrak{F}$ is $\oplus$-stable and therefore it is Fraïssé by Theorem 7.2. The Fraïssé limit $\mathcal{A}_{\mathfrak{F}}$ of this category has the universality property (Corollary 7.3) that any AF-algebra which is the limit of a left-invertible sequence of finite-dimensional C^*-algebras can be embedded via a left-invertible embedding into $\mathcal{A}_{\mathfrak{F}}$. In fact, $\mathcal{A}_{\mathfrak{F}}$ is surjectively universal in the category of all (separable) AF-algebras.

Theorem 8.1. *There is a surjective homomorphism from $\mathcal{A}_{\mathfrak{F}}$ onto any separable AF-algebra.*

Proof. Suppose $\mathcal{B}$ is a separable AF-algebra. Proposition 3.7 states that there is an AF-algebra $\mathcal{A}$, which is the limit of a left-invertible sequence of finite-dimensional C^*-algebras and $\mathcal{A}/\mathcal{J} \cong \mathcal{B}$, for some ideal $\mathcal{J}$. By the universality of $\mathcal{A}_{\mathfrak{F}}$ (Corollary 7.3) there is a left-invertible embedding $\phi : \mathcal{A} \hookrightarrow \mathcal{A}_{\mathfrak{F}}$. If $\theta : \mathcal{A}_{\mathfrak{F}} \twoheadrightarrow \mathcal{A}$ is a left inverse of ϕ then its composition with the quotient map $\pi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{J}$ gives a surjective homomorphism from $\mathcal{A}_{\mathfrak{F}}$ onto $\mathcal{B}$. □

Remark 8.2. Since $\mathcal{A}_{\mathfrak{F}}$ has the Cantor property (Lemma 7.4), it does not have any minimal projections. Therefore, for example, it cannot be isomorphic to $\mathcal{A}_{\mathfrak{F}} \oplus \mathbb{C}$. Hence the property of being surjectively universal AF-algebra is not unique to $\mathcal{A}_{\mathfrak{F}}$.

Corollary 8.3. *An AF-algebra $\mathcal{A}$ is surjectively universal if and only if $\mathcal{A}_{\mathfrak{F}}$ is a quotient of $\mathcal{A}$.*

Theorem 7.8 provides a characterization of $\mathcal{A}_{\mathfrak{F}}$, up to isomorphism, in terms of its structure.

Corollary 8.4. *$\mathcal{A}_{\mathfrak{F}}$ is the unique separable AF-algebra with Cantor property such that every matrix algebra M_k is a retract of $\mathcal{A}$.*

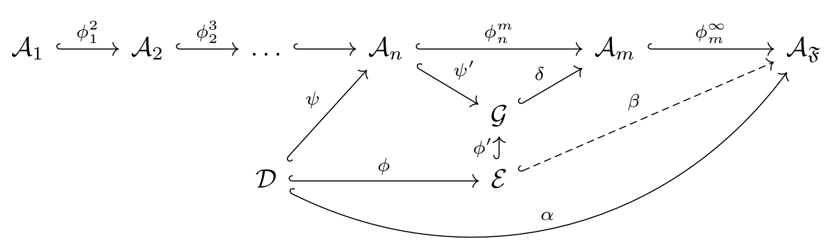
Equivalently, an AF-algebra $\mathcal{A}$ is isomorphic to $\mathcal{A}_{\mathfrak{F}}$ if and only if there is a sequence $(\mathcal{A}_n, \phi_n^m)$ of finite-dimensional C^*-algebras and embeddings such that $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_n^m)$ and the Bratteli diagram $\mathfrak{D}$ of $(\mathcal{A}_n, \phi_n^m)$ satisfies (D0)-(D2) and

(D3) for every k there is $(n, s) \in \mathfrak{D}$ such that $\dim(n, s) = k$.

Theorem 8.5. *$\mathcal{A}_{\mathfrak{F}}$ is the unique AF-algebra that is the limit of a left-invertible sequence of finite-dimensional C^*-algebras and for any finite-dimensional C^*-algebras $\mathcal{D}, \mathcal{E}$ and left-invertible embeddings $\phi : \mathcal{D} \to \mathcal{E}$ and $\alpha : \mathcal{D} \to \mathcal{A}_{\mathfrak{F}}$ there is a left-invertible embedding $\beta : \mathcal{E} \to \mathcal{A}_{\mathfrak{F}}$ such that $\beta \circ \phi = \alpha$.*

Proof. Suppose $\mathcal{A}_{\mathfrak{F}}$ is the limit of the Fraïssé $\mathfrak{F}$-sequence $(\mathcal{A}_n, \phi_n^m)$. By definition, α and ϕ are $\mathfrak{F}$-arrows. There is (by (L3)) a natural number n and an $\mathfrak{L}\mathfrak{F}$-arrow (a left-invertible embedding) $\psi : \mathcal{D} \hookrightarrow \mathcal{A}_n$ such that $\|\phi_n^\infty \circ \psi - \alpha\| < 1$. Use the amalgamation property to find a finite-dimensional C^*-algebra $\mathcal{G}$ and left-invertible embeddings $\phi' : \mathcal{E} \hookrightarrow \mathcal{G}$ and $\psi' : \mathcal{A}_n \hookrightarrow \mathcal{G}$ such that $\phi' \circ \phi = \psi' \circ \psi$ (see Diagram (8.1)). The Fraïssé condition (F) implies the existence of $m \geq n$ and a left-invertible embedding $\delta : \mathcal{G} \hookrightarrow \mathcal{A}_m$ such that $\delta \circ \psi' = \phi_n^m$. Let $\beta' = \phi_m^\infty \circ \delta \circ \phi'$. It is clearly left-invertible.

(8.1)



For every d in $\mathcal{D}$ we have

$$\beta' \circ \phi(d) = \phi_m^\infty \circ \delta \circ \phi' \circ \phi(d) = \phi_m^\infty \circ \delta \circ \psi' \circ \psi(d) = \phi_m^\infty \circ \phi_n^m \circ \psi(d) = \phi_n^\infty \circ \psi(d).$$

Therefore $\|\beta' \circ \phi - \alpha\| < 1$. Conjugating β' with a unitary in $\widetilde{\mathcal{A}}_{\mathfrak{F}}$ gives the required left-invertible embedding β (Lemma 2.2).

For the uniqueness, suppose $\mathcal{B}$ is the limit of a left-invertible sequence $(\mathcal{B}_n, \psi_n^m)$ of finite-dimensional C^*-algebras, satisfying the assumption of the theorem. Using this assumption and then (L3) we can show that $(\mathcal{B}_n, \psi_n^m)$ satisfies the Fraïssé condition (F) and therefore $\mathcal{B}$ is the Fraïssé limit of $\mathfrak{F}$. Uniqueness of the Fraïssé limit implies that $\mathcal{B}$ is isomorphic to $\mathcal{A}_{\mathfrak{F}}$. □

9. Unital categories

The proof of Theorem 7.2 also shows that the category of all finite-dimensional C^*-algebras (or any $\oplus$-stable category) and *unital* left-invertible embeddings has the (proper) amalgamation property. However, this category fails to have the joint embedding property (note that 0 is no longer an object of the category), since for example one cannot jointly embed M_2 and M_3 into a finite-dimensional C^*-algebra with unital left-invertible maps.

9.1. **The category $\widetilde{\mathfrak{F}}$.** Let $\widetilde{\mathfrak{F}}$ denote the category of all finite-dimensional C^*-algebras isomorphic to $\mathbb{C} \oplus \mathcal{D}$, for a finite-dimensional C^*-algebra $\mathcal{D}$, and unital left-invertible embeddings. This category is no longer $\oplus$-stable, however, the same proof as the one of Theorem 7.2 when the maps are unital, shows that $\ddagger\widetilde{\mathfrak{F}}$ has the proper amalgamation property. Therefore $\widetilde{\mathfrak{F}}$ is a Fraïssé category, since $\mathbb{C}$ is the initial object of this category and therefore the joint embedding property is a consequence of the amalgamation property. The Fraïssé limit $\mathcal{A}_{\widetilde{\mathfrak{F}}}$ of this category is a separable AF-algebra with the universality property that any unital AF-algebra which can be obtained as the limit of a left-invertible unital sequence of finite-dimensional C^*-algebras isomorphic to $\mathbb{C} \oplus \mathcal{D}$, can be embedded via a left-invertible unital embedding into $\mathcal{A}_{\widetilde{\mathfrak{F}}}$. The unital analogue of Theorem 8.1 states the following.

Corollary 9.1. *For every unital separable AF-algebra $\mathcal{B}$ there is a surjective homomorphism from $\mathcal{A}_{\widetilde{\mathfrak{F}}}$ onto $\mathcal{B}$.*

Proof. Suppose $\mathcal{B}$ is an arbitrary unital AF-algebra. Using Proposition 3.7 we can find a unital AF-algebra $\mathcal{A} \supseteq B$ which is the limit of a left-invertible unital sequence of finite-dimensional C^*-algebras, such that $\mathcal{B}$ is a quotient of $\mathcal{A}$. Thus $\mathbb{C} \oplus \mathcal{A}$ is the limit of a unital left-invertible sequence of finite-dimensional C^*-algebras of the form $\mathbb{C} \oplus \mathcal{D}$, for finite-dimensional $\mathcal{D}$. By the universality of $\mathcal{A}_{\widetilde{\mathfrak{F}}}$, there is a left-invertible unital embedding from $\mathbb{C} \oplus \mathcal{A}$ into $\mathcal{A}_{\widetilde{\mathfrak{F}}}$. Since $\mathcal{B}$ is a quotient of $\mathcal{A}$, there is a surjective homomorphism from $\mathbb{C} \oplus \mathcal{A}$ onto $\mathcal{B}$. Combining the two surjections gives us a surjective homomorphism from $\mathcal{A}_{\widetilde{\mathfrak{F}}}$ onto $\mathcal{B}$. □

Remark 9.2. Small adjustments in the proof of Lemma 7.4 show that $\mathcal{A}_{\widetilde{\mathfrak{F}}}$ has the Cantor property (in the sense of Definition 4.3). In fact, it is easy to check that $\mathcal{A}_{\widetilde{\mathfrak{F}}}$ is isomorphic to $\widetilde{\mathcal{A}_{\mathfrak{F}}}$, the unitization of $\mathcal{A}_{\mathfrak{F}}$. This, in particular, implies that $\mathcal{A}_{\mathfrak{F}}$ is not unital. Since if it was unital, then $\widetilde{\mathcal{A}_{\mathfrak{F}}}$ (and hence $\mathcal{A}_{\widetilde{\mathfrak{F}}}$) would be isomorphic to $\mathcal{A}_{\mathfrak{F}} \oplus \mathbb{C}$, but this is not possible since $\mathcal{A}_{\widetilde{\mathfrak{F}}}$ has the Cantor property and therefore has no minimal projections.

Definition 9.3. We say $\mathcal{D}$ is a *unital-retract* of the C^*-algebra $\mathcal{A}$ if there is a left-invertible unital embedding from $\mathcal{D}$ into $\mathcal{A}$.

9.2. **The category $\widetilde{\mathfrak{K}}_{\mathcal{A}}$.** If $\mathcal{A}$ is a unital AF-algebra with Cantor property (Definition 4.3), then let $\widetilde{\mathfrak{K}}_{\mathcal{A}}$ denote the category whose objects are finite-dimensional unital-retracts of $\mathcal{A}$ and morphisms are unital left-invertible embeddings. This category is not $\oplus$-stable, since it does not satisfy condition (1) of Definition 7.1. However, $\ddagger\widetilde{\mathfrak{K}}_{\mathcal{A}}$ still has proper amalgamations.

Proposition 9.4. *$\ddagger\widetilde{\mathfrak{K}}_{\mathcal{A}}$ has the proper amalgamation property.*

Proof. The proof is exactly the same as the proof of Lemma 7.2 where the maps are assumed to be unital. We only need to check that $\mathcal{D}\oplus\mathcal{E}_1\oplus\mathcal{F}_1$ is a unital-retract of $\mathcal{A}$. By Lemma 3.5, for some m both $\mathcal{E}\cong\mathcal{D}\oplus\mathcal{E}_1$ and $\mathcal{F}\cong\mathcal{D}\oplus\mathcal{F}_1$ are unital-retracts of $\mathcal{A}_m$. An easy argument using Proposition 3.2 shows that $\mathcal{D}\oplus\mathcal{E}_1\oplus\mathcal{F}_1$ is also a unital-retract of $\mathcal{A}_m$ and therefore a unital retract of $\mathcal{A}$. □

Also $\widetilde{\mathfrak{K}}_{\mathcal{A}}$ has a weakly initial object (by the next lemma). Therefore it is a Fraïssé category. Recall that an object is *weakly initial* in $\mathfrak{K}$ if it has at least one $\mathfrak{K}$-arrow to any other object of $\mathfrak{K}$.

Lemma 9.5. *Suppose $\mathcal{A}$ is a unital AF-algebra with Cantor property . The category $\widetilde{\mathfrak{K}}_{\mathcal{A}}$ has a weakly initial object, i.e., there is a finite-dimensional unital-retract of $\mathcal{A}$ which can be mapped into any other finite-dimensional unital-retract of $\mathcal{A}$ via a left-invertible unital embedding.*

Proof. Let $M_{k_1}\oplus\cdots\oplus M_{k_l}$ be an arbitrary $\widetilde{\mathfrak{K}}_{\mathcal{A}}$-object. Suppose that $\{k'_1,\dots,k'_t\}$ is the largest subset of $\{k_1,\dots,k_l\}$ such that k'_i cannot be written as $\sum_{\substack{j\le n\\ j\neq i}} x_j k'_j$ for any natural set numbers $\{x_j : j\le n \text{ and } j\neq i\}$, for any $i\le t$. Since $\{k'_1,\dots,k'_t\}$ is the largest such subset, $\mathcal{D}=M_{k'_1}\oplus\cdots\oplus M_{k'_t}$ is a unital-retract of $M_{k_1}\oplus\cdots\oplus M_{k_l}$ an therefore a unital-retract of $\mathcal{A}$. Suppose $\mathcal{F}$ is an arbitrary $\widetilde{\mathfrak{K}}_{\mathcal{A}}$-object. Let $(\mathcal{A}_n,\phi_n^m)$ be a $\widetilde{\mathfrak{K}}_{\mathcal{A}}$-sequence with limit $\mathcal{A}$ such that $\mathcal{A}_1\cong\mathcal{F}$. Then $\mathcal{D}$ is a unital-retract of some $\mathcal{A}_m$, so $\mathcal{A}_m=\dot{\mathcal{D}}\oplus\mathcal{E}$, for some $\mathcal{E}$ and $\dot{\mathcal{D}}\cong\mathcal{D}$.

Fix $i\le t$. Since ϕ_1^m is a unital embedding, there is a subalgebra of $\mathcal{F}$ isomorphic to $M_{n_1}\oplus\cdots\oplus M_{n_s}$ such that $\sum_{j=1}^s y_j n_j = k'_i$, for some $\{y_1,\dots,y_s\}\subseteq\mathbb{N}$. We claim that exactly one n_j is equal to k'_i and the rest are zero. If not, then for every $j\le s$ we have $0<n_j<k'_i$. Since ϕ_1^m is left-invertible, for every $j\le s$ a copy of M_{n_j} appears as a summand of $\mathcal{A}_m$. Also because there is a unital embedding from $\mathcal{D}$ into $\mathcal{A}_m$, for some $\{x_1,\dots,x_r\}\subseteq\mathbb{N}$ we have $n_j=\sum_{\substack{j'\le r\\ j'\neq i}} x_{j'}k'_{j'}$ for every $j\le s$. But then

$$k'_i=\sum_{j=1}^{s}\sum_{\substack{j'\le n\\ j'\neq i}} x_{j'}y_j k'_{j'},$$

which is a contradiction with the choice of k'_i. This means that $\mathcal{F}=\mathcal{F}_0\oplus\mathcal{F}_1$ such that $\mathcal{F}_0\cong\mathcal{D}$ and there is a unital homomorphism from $\mathcal{D}$ onto $\dot{\mathcal{F}}_1$. Therefore $\mathcal{D}$ is a unital retract of $\mathcal{F}$. □

Corollary 9.6. *The category $\widetilde{\mathfrak{K}}_{\mathcal{A}}$ is a Fraïssé category and $\ddagger\widetilde{\mathfrak{K}}_{\mathcal{A}}$ has the proper amalgamation property. The Fraïssé limit of $\widetilde{\mathfrak{K}}_{\mathcal{A}}$ is $\mathcal{A}$.*

Proof. The proof of the fact that $\mathcal{A}$ is the Fraïssé limit of $\widetilde{\mathfrak{K}}_{\mathcal{A}}$ is same as Theorem 7.7, where all the maps are unital. □

10. Surjectively universal countable dimension groups

A countable partially ordered abelian group $\langle G, G^+\rangle$ is a *dimension group* if it is isomorphic to the inductive limit of a sequence

$$\mathbb{Z}^{r_1} \xrightarrow{\alpha_1^2} \mathbb{Z}^{r_2} \xrightarrow{\alpha_2^3} \mathbb{Z}^{r_3} \xrightarrow{\alpha_3^4} \dots$$

for some natural numbers r_n, where α_i^j are positive group homomorphisms and $\mathbb{Z}^r$ is equipped with the ordering given by

$$(\mathbb{Z}^r)^+ = \{(x_1, x_2, \dots, x_r) \in \mathbb{Z}^r : x_i \geq 0 \text{ for } i = 1, \dots, r\}.$$

A partially ordered abelian group that is isomorphic to $\langle \mathbb{Z}^r, (\mathbb{Z}^r)^+\rangle$, for a non-negative integer r, is usually called a *simplicial group*. A *scale* S on the dimension group $\langle G, G^+\rangle$ is a generating, upward directed and hereditary subset of G^+ (see [3, IV.3]).

Notation. If $\langle G, S\rangle$ is a scaled dimension group as above, we can recursively pick order-units

$$\bar{u}_n = (u_{n,1}, u_{n,2}, \dots, u_{n,r_n}) \in (\mathbb{Z}^{r_n})^+$$

of $\mathbb{Z}^{r_n}$ such that $\alpha_n^{n+1}(\bar{u}_n) \leq \bar{u}_{n+1}$ and $S = \bigcup_n \alpha_n^\infty[[\bar{0}, \bar{u}_n]]$. Then we say the scaled dimension group $\langle G, S\rangle$ is the limit of the sequence $(\mathbb{Z}^{r_n}, \bar{u}_n, \alpha_n^m)$. If $(\bar{u}_n)$ can be chosen such that $\alpha_n^{n+1}(\bar{u}_n) = \bar{u}_{n+1}$ for every $n \in \mathbb{N}$, then G has an order-unit $u = \lim_n \alpha_n^\infty(\bar{u}_n)$. In this case we denote this dimension group with order-unit by $\langle G, u\rangle$.

An isomorphism between scaled dimension groups is a positive group isomorphism which sends the scale of the domain to the scale of the codomain. Given a separable AF-algebra $\mathcal{A}$, its K_0-group $\langle K_0(\mathcal{A}), K_0(\mathcal{A})^+\rangle$ is a (countable) dimension group and conversely any dimension group is isomorphic to K_0-group of a separable AF-algebra. The *dimension range* of $\mathcal{A}$,

$$\mathcal{D}(\mathcal{A}) = \{[p] : p \text{ is a projection of } \mathcal{A}\} \subseteq K_0(\mathcal{A})^+$$

is a scale for $\langle K_0(\mathcal{A}), K_0(\mathcal{A})^+\rangle$, and therefore $\langle K_0(\mathcal{A}), \mathcal{D}(\mathcal{A})\rangle$ is a scaled dimension group. Conversely, every scaled dimension group is isomorphic to $\langle K_0(\mathcal{A}), \mathcal{D}(\mathcal{A})\rangle$ for a separable AF-algebra $\mathcal{A}$. Elliott's classification of separable AF-algebras ([5]) states that $\langle K_0(\mathcal{A}), \mathcal{D}(\mathcal{A})\rangle$ is a complete isomorphism invariant for the separable AF-algebra $\mathcal{A}$.

Theorem 10.1 (Elliott [5])**.** *Two separable AF-algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if their scaled dimension groups are isomorphic. If $\mathcal{A}$ and $\mathcal{B}$ are unital, then they are isomorphic if and only if $\langle K_0(\mathcal{A}), [1_\mathcal{A}]\rangle \cong \langle K_0(\mathcal{B}), [1_\mathcal{B}]\rangle$, as partially ordered abelian groups with order-units.*

10.1. **Surjectively universal dimension groups.** The universality property of $\langle K_0(\mathcal{A}_\mathfrak{F}), \mathcal{D}(\mathcal{A}_\mathfrak{F})\rangle$ can be obtained by applying K_0-functor to Theorem 8.1.

Corollary 10.2. *The scaled (countable) dimension group $\langle K_0(\mathcal{A}_\mathfrak{F}), \mathcal{D}(\mathcal{A}_\mathfrak{F})\rangle$ maps onto any countable scaled dimension group.*

By applying K_0-functor to Corollary 8.4, we immediately obtain the following result.

Corollary 10.3. *$\langle K_0(\mathcal{A}_\mathfrak{F}), \mathcal{D}(\mathcal{A}_\mathfrak{F})\rangle$ is the unique scaled dimension group which is the limit of a sequence $(\mathbb{Z}^{r_n}, \bar{u}_n, \alpha_n^m)$ (as in Notation above) satisfying the following conditions:*

(1) for every $n \in \mathbb{N}$ and $1 \leq i \leq r_n$ there are $m \geq n$ and $1 \leq j, j' \leq r_m$ such that $j \neq j'$, $u_{n,i} = u_{m,j} = u_{m,j'}$ and $\pi_j \circ \alpha_n^m(u_{n,i}) = u_{m,j}$ and $\pi_{j'} \circ \alpha_n^m(u_{n,i}) = u_{m,j'}$, where π_j is the canonical projection from $\mathbb{Z}^{r_m}$ onto its j-th coordinate.

(2) for every $n, n' \in \mathbb{N}$, $1 \leq i' \leq r_{n'}$ and $\{x_1, \dots, x_{r_n}\} \subseteq \mathbb{N} \cup \{0\}$ such that $\sum_{i=1}^{r_n} x_i u_{n,i} \leq u_{n',i'}$ there are $m \geq n$ and $1 \leq j \leq r_m$ such that $u_{n',i'} \leq u_{m,j}$ and $\pi_j \circ \alpha_n^m(u_{n,i}) = x_i . u_{n,i}$ for every $i \in \{1, \dots, r_n\}$.

(3) For every $k \in \mathbb{N}$ there are natural numbers n and $1 \leq i \leq r_n$ such that $u_{n,i} = k$.

Corollary 10.4. *The (countable) dimension group with order-unit $\langle K_0(\mathcal{A}_{\widetilde{\mathfrak{F}}}), [1_{\mathcal{A}_{\widetilde{\mathfrak{F}}}}]\rangle$ maps onto (there is a surjective normalized positive group homomorphism) any countable dimension group with order-unit.*

A similar characterization of the dimension group with order-unit $\langle K_0(\mathcal{A}_{\widetilde{\mathfrak{F}}}), [1_{\mathcal{A}_{\widetilde{\mathfrak{F}}}}]\rangle$ holds where α_n^m are order-unit preserving and in condition (2) of Corollary 10.3 the inequality $\sum_{i=1}^{r_n} x_i u_{n,i} \leq u_{n',i'}$ is replaced with equality.

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