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WEAK SOLUTIONS OF THE ROBIN PROBLEM FOR THE OSEEN SYSTEM

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ABSTRACT. We study the Robin problem for the Oseen system in the Sobolev space $W^{1,q}(\Omega;\mathbb{R}^m)\times L^q(\Omega)$ on a bounded domain $\Omega\subset\mathbb{R}^m$ with Lipschitz boundary for $m=2$ or $m=3$. We prove the unique solvability of the problem for $3/2<q<3$ and $\partial\Omega$ Lipschitz, and for $1<q<\infty$ and $\partial\Omega$ of class $\mathcal{C}^1$. Then we study the problem on exterior domains. First we study the problem for the homogeneous Oseen system with $(\mathbf{u},p)\in W^{1,q}_{\rm loc}(\overline{\Omega};\mathbb{R}^m)\times L^q_{\rm loc}(\overline{\Omega})$ and the additional condition $\mathbf{u}(\mathbf{x})\to 0$, $p(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$. Then we study the Robin problem for the non-homogeneous Oseen system in homogeneous Sobolev spaces $D^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$. Denote by $\tilde{W}^{1,q}(\Omega;\mathbb{R}^m)$ the closure of $\mathcal{C}^\infty_c(\mathbb{R}^m;\mathbb{R}^m)$ in $D^{1,q}(\Omega,\mathbb{R}^m)$. If $\Omega\subset\mathbb{R}^3$ is an exterior domain with Lipschitz boundary and $3/2<q<3$ then there exists a unique solution of the Robin problem in $\tilde{W}^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$. We characterize all solutions of the problem in $D^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$.

1. INTRODUCTION

This paper is devoted to the Robin problem for the stationary Oseen system. The Dirichlet problem for the Oseen system is studied very often - see for example [4], [5], [6], [7], [10], [15], [17], [19], [33]. But there are only a few papers concerning the Robin problem for the Oseen system. A. Russo and A. Tartaglione studied in [32] an L^q-solution of the Robin problem for the Oseen system

$$\nabla p-\Delta\mathbf{u}+\lambda\partial_1\mathbf{u}=0\quad \nabla\cdot\mathbf{u}=0\quad\text{in }\Omega,\qquad T(\mathbf{u},p)\mathbf{n}+h\mathbf{u}=\mathbf{g}\qquad\text{on }\partial\Omega \tag{1.1}$$

where h is a sufficiently large positive number. Here $\Omega\subset\mathbb{R}^3$ is an exterior domain with connected Lipschitz boundary, and $q=2$ or $\partial\Omega$ is of class $\mathcal{C}^1$, $1<q<\infty$. The boundary condition $\mathbf{g}\in L^q(\partial\Omega;\mathbb{R}^3)$ is satisfied in the sense of non-tangential limit. The author studied in [27] an L^q-solutions of the Robin problem for the Oseen system (1.1),

$$T(\mathbf{u},p)\mathbf{n}^\Omega-\frac{\lambda}{2}n_1\mathbf{u}+h\mathbf{u}=\mathbf{g}\qquad\text{on }\partial\Omega \tag{1.2}$$

for bounded and unbounded domains in R^m with compact Lipschitz boundary and $m=2$ or $m=3$. Here h is a non-negative bounded function. It is supposed that $q=2$ or $\partial\Omega$ is of class $\mathcal{C}^1$.

This paper is devoted to the Robin problem for the Oseen system on bounded and unbounded domains with compact Lipschitz boundary in $\mathbb{R}^m$ with $m=2$ or $m=3$. We formulate a weak solution of the problem in Sobolev spaces $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$

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for Ω bounded by an analogical way as it is usual for elliptic equations (see [31]). Since $\mathcal{C}^\infty(\mathbb{R}^m,\mathbb{R}^m)\times\mathcal{C}^\infty(\mathbb{R}^m)$ is a dense subset of $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$, it is the only possible way how to extend the Robin problem to this space and ensure that classical solutions of the problem are weak solutions of the same problem. We prove the unique solvability of the Robin problem under the assumption that $3/2<q<3$ for $\partial\Omega$ Lipschitz, and $1<q<\infty$ for $\partial\Omega$ of class $\mathcal{C}^1$. Then we study weak solutions of the Robin problem in homogeneous Sobolev spaces for Ω unbounded.

2. Sobolev spaces

If X is a Banach space we denote by X' its dual space.

If $\Omega\subset\mathbb{R}^m$ is an open set and $1<q<\infty$, we define the Sobolev space $W^{1,q}(\Omega):=\{u\in L^q(\Omega);\partial_j u\in L^q(\Omega)$ for $j=1,\dots,m\}$. Further we denote by $\mathring{W}^{1,q}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ (the space of infinitely differentiable functions with compact support in Ω) in $W^{1,q}(\Omega)$. It is usual to denote $W^{-1,q}(\Omega):=[\mathring{W}^{1,q'}(\Omega)]'$ where $q'=q/(q-1)$. If Ω is unbounded we denote by $W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ (by $L^q_{\mathrm{loc}}(\overline{\Omega})$) the space of all functions u such that $u\in W^{1,q}(G)$ (that $u\in L^q(G)$) for all bounded open subsets G of Ω, respectively.

Suppose now that Ω has compact Lipschitz boundary. If $0<s<1$ we define $W^{s,q}(\partial\Omega):=\{u\in L^q(\partial\Omega);\|u\|_{W^{s,q}(\partial\Omega)}<\infty\}$ where

$$\|u\|_{W^{s,q}(\partial\Omega)}:=\left[\|u\|^q_{L^q(\partial\Omega)}+\int_{\partial\Omega}\int_{\partial\Omega}\frac{|u(x)-u(y)|^q}{|x-y|^{m-1+qs}}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)\right]^{1/q}.$$

Further we denote $W^{-s,q}(\partial\Omega):=[W^{s,q'}(\partial\Omega)]'$. According to [23, Theorem 6.8.13] there exists a unique continuous linear mapping $\gamma_\Omega:W^{1,q}(\Omega)\to W^{1-1/q,q}(\partial\Omega)$ called the trace such that $\gamma_\Omega u=u$ for all $u\in\mathcal{C}_c^\infty(\mathbb{R}^m)$. So, if $u\in W^{1,q'}(\Omega)$ then $\gamma_\Omega u\in W^{1-1/q',q'}(\partial\Omega)=W^{1/q,q'}(\partial\Omega)$. Therefore $W^{-1/q,q}(\partial\Omega)\hookrightarrow[W^{1,q'}(\Omega)]'$ if we define

$$\langle f,u\rangle:=\langle f,\gamma_\Omega u\rangle,\qquad f\in W^{-1/q,q}(\partial\Omega),u\in W^{1,q'}(\Omega).$$

3. Weak solution of the problem (1.1), (1.2)

Let $\Omega\subset\mathbb{R}^m$ be an open set with compact Lipschitz boundary. We are going to study the Robin problem for the Oseen system (1.1) with the boundary condition

$$T_\lambda(\mathbf{u},p)\mathbf{n}+h\mathbf{u}=\mathbf{g}\qquad\text{on }\partial\Omega.\tag{3.1}$$

Here $\mathbf{n}=\mathbf{n}^\Omega$ is the outward unit normal vector of Ω, $\mathbf{u}=(u_1,\dots,u_m)$ is a velocity field, p is a pressure, $\nabla\cdot\mathbf{u}=\partial_1u_1+\cdots+\partial_mu_m$, $\hat\nabla\mathbf{u}=[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]/2$,

$$T(\mathbf{u},p)\equiv2\hat\nabla\mathbf{u}-pI,\qquad T_\lambda(\mathbf{u},p)\mathbf{n}=T(\mathbf{u},p)\mathbf{n}-\frac{\lambda}{2}n_1\mathbf{u}\tag{3.2}$$

and $T(\mathbf{u},p)$ is the stress tensor. (I means the identity operator represented by the unit matrix.) If $h\equiv0$ we say about the Neumann problem.

If $(\mathbf{u},p)\in\mathcal{C}^2(\overline{\Omega};\mathbb{R}^m)\times\mathcal{C}^1(\overline{\Omega})$ is a classical solution of the problem (1.1), (3.1) and $\mathbf{\Phi}\in\mathcal{C}_c^\infty(\mathbb{R}^m,\mathbb{R}^m)$, then the Green formula gives

$$\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma=\int_\Omega[2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi}-p(\nabla\cdot\mathbf{\Phi})+\frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u}-\mathbf{u}\cdot\partial_1\mathbf{\Phi})]\,\mathrm{d}\mathbf{x}+\int_{\partial\Omega}h\mathbf{u}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma.\tag{3.3}$$

(Compare [37, p. 14].) This formula motivates definition of a weak solution of the Robin problem (1.1), (3.1).

Suppose first that Ω is bounded. Let $\mathbf{g} \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ with $1 < q < \infty$, $q' = q/(q-1)$. *We say that* $(\mathbf{u}, p) \in W^{1,q}(\Omega, R^m) \times L^q(\Omega)$ *is a weak solution of the problem* (1.1), (3.1) *if* $\nabla \cdot \mathbf{u} = 0$ *in* Ω *and*

$$\int_\Omega [2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p(\nabla \cdot \mathbf{\Phi}) + \tfrac{\lambda}{2}(\mathbf{\Phi} \cdot \partial_1 \mathbf{u} - \mathbf{u} \cdot \partial_1 \mathbf{\Phi})]\, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h \gamma_\Omega \mathbf{u} \cdot \gamma_\Omega \mathbf{\Phi}\, \mathrm{d}\sigma = \langle \mathbf{g}, \gamma_\mathbf{\Omega} \mathbf{\Phi} \rangle \tag{3.4}$$

for all $\mathbf{\Phi} \in W^{1,q'}(\Omega, \mathbb{R}^m)$ *(equivalently for all* $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$*).*

Let now Ω be unbounded. Let $\mathbf{g} \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ with $1 < q < \infty$. *We say that* $(\mathbf{u}, p) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, R^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ *is a weak solution of the problem* (1.1), (3.1) *if* $\nabla \cdot \mathbf{u} = 0$ *in* Ω *and* (3.4) *holds for all* $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$.

Remark that if $(\mathbf{u}, p) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, R^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ is a weak solution of the problem (1.1), (3.1), then the Green formula gives that $(\mathbf{u}, p)$ is a solution of (1.1) in Ω in the sense of distributions.

Lemma 3.1. *Let* $\Omega \subset \mathbb{R}^m$ *be open,* $\lambda \in \mathbb{R}^1$. *If* $(\mathbf{u}, p)$ *is a solution of the Oseen system* (1.1) *in* Ω *in the sense of distributions, then* $\mathbf{u} \in \mathcal{C}^\infty(\Omega, \mathbb{R}^m)$, $p \in \mathcal{C}^\infty(\Omega)$ *and therefore* $(\mathbf{u}, p)$ *is a classical solution of* (1.1) *in* Ω.

Proof. $p \in \mathcal{C}^\infty(\Omega)$ by [22, p. 30]. Since $-\Delta\mathbf{u} + \lambda\partial_1\mathbf{u} \in \mathcal{C}^\infty(\Omega, \mathbb{R}^m)$, [12, Chapter II, §8, Proposition 5] gives $\mathbf{u} \in \mathcal{C}^\infty(\Omega, \mathbb{R}^m)$. □

4. Oseen fundamental tensor

We are going to look for a solution of the problem (1.1), (3.1) in the form of an appropriate combination of boundary layer potentials. For this reason we need to define a fundamental solution of the Oseen system and boundary layer potentials.

If $O_{jk}(\mathbf{x})$, $Z_j(\mathbf{x})$ are tempered distributions, $j = 1, \ldots, m$, and $k = 1, \ldots, m+1$, then $O = \{O_{jk}\}$, $Z = \{Z_j\}$ is called a fundamental tensor for the Oseen equation (1.1) in $\mathbb{R}^m$ if

$$-\Delta O_{jk} + \lambda\partial_1 O_{jk} + \partial_j Z_k = \delta_{jk}\delta_0,$$
$$\partial_1 O_{1k} + \cdots + \partial_m O_{mk} = \delta_{k(m+1)}\delta_0.$$

According to [27, Corollary 1] there exists a unique fundamental tensor O^λ, Q^λ of the Oseen equation (1.1) such that $|O^\lambda(\mathbf{x})|, |Q^\lambda(\mathbf{x})| = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to \infty$. If $k \le m$ then

$$O^\lambda_{k,(m+1)}(\mathbf{x}) = Q^\lambda_k(\mathbf{x}) = Q_k(\mathbf{x}) = \frac{x_k}{\sigma_m |\mathbf{x}|^m}$$

where σ_m is the surface of the unit sphere in $\mathbb{R}^m$,

$$O^0_{jk}(\mathbf{x}) = \begin{cases} \frac{1}{2\sigma_m}\left[\delta_{jk}\frac{|\mathbf{x}|^{2-m}}{m-2} + \frac{x_j x_k}{|\mathbf{x}|^m}\right], & m > 2, \\ \frac{1}{4\pi}\left[\delta_{jk}\ \ln\frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^2}\right], & m = 2, \end{cases}$$

$$Q^\lambda_{m+1}(\mathbf{x}) = \delta_0(\mathbf{x}) - \lambda\frac{x_1}{\sigma_m |\mathbf{x}|^m}\ .$$

The explicit formula of O^λ_{ji} for $j, k \le m$ and $\lambda \ne 0$ can be found in [17]. This formula is very complicated and we only gather properties of the fundamental tensor. We have $O^\lambda_{jk} = O^\lambda_{kj} \in \mathcal{C}^\infty(R^m \setminus \{0\})$. If β is a multi-index, then we have

$$|\partial^\beta O^\lambda_{jk}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m-|\beta|)/2}) \quad \text{as } |\mathbf{x}| \to \infty. \tag{4.1}$$

If $\mathbf{z} \ne [c, 0, \ldots, 0]$ then

$$\lim_{r\to\infty} |O^\lambda(r\mathbf{z})| r^{(m-1)/2} = 0. \tag{4.2}$$

If $r>0$ and $q>1+1/m$ then we have

$$|\nabla O^\lambda_{jk}| \in L^q(R^m\setminus B(0;r)), \tag{4.3}$$

where $B(\mathbf{x};r)=\{\mathbf{y}\in\mathbb{R}^m; |\mathbf{x}-\mathbf{y}|<r\}$. If $m=3$ then

$$|\partial^\alpha(O^\lambda(\mathbf{x})-O^0(\mathbf{x}))| = O(|x|^{-|\alpha|}) \quad \text{as } |\mathbf{x}|\to 0. \tag{4.4}$$

If $m=2$ then

$$|O^\lambda(\mathbf{x})-O^0(\mathbf{x})| = O(1) \quad \text{as } |\mathbf{x}|\to 0, \tag{4.5}$$

$$|\nabla(O^\lambda(\mathbf{x})-O^0(\mathbf{x}))| = O(\ln|\mathbf{x}|) \quad \text{as } |\mathbf{x}|\to 0, \tag{4.6}$$

$$|\partial^\alpha(O^\lambda(\mathbf{x})-O^0(\mathbf{x}))| = O(|x|^{-|\alpha|+1}) \quad \text{as } |\mathbf{x}|\to 0 \quad \text{for } |\alpha|\geq 2. \tag{4.7}$$

Easy calculation yields

$$O^\lambda_{jk}(-\mathbf{x}) = O^{-\lambda}_{jk}(\mathbf{x}), \qquad 1\leq j,k\leq m. \tag{4.8}$$

5. Potentials

Denote $Q(\mathbf{x}) := (Q_1(\mathbf{x}),\dots,Q_m(\mathbf{x}))$, $\breve{O}^\lambda := (O^\lambda_{ij})_{i,j\leq m}$. For $\mathbf{\Psi}\in L^q(\partial\Omega,R^m)$ with $1<q<\infty$ define the velocity part of the Oseen single layer potential with density $\mathbf{\Psi}$ by

$$O^\lambda_\Omega\mathbf{\Psi}(\mathbf{x}) = \int_{\partial\Omega}\breve{O}^\lambda(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})$$

and the corresponding pressure part by

$$Q_\Omega\mathbf{\Psi}(\mathbf{x}) = \int_{\partial\Omega} Q(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}).$$

More generally, if $\mathbf{\Psi}=(\Psi_1,\dots,\Psi_m)$, where Ψ_j are distributions supported on $\partial\Omega$, then we define

$$[O^\lambda_\Omega\mathbf{\Psi}(\mathbf{x})]_i := \sum_{j=1}^m\langle\Psi_j, O^\lambda_{ij}(\mathbf{x}-\cdot)\rangle, \quad Q_\Omega\mathbf{\Psi}(\mathbf{x}) := \sum_{j=1}^m\langle\Psi_j, Q_j((\mathbf{x}-\cdot)\rangle, \quad \mathbf{x}\in\mathbb{R}^m\setminus\partial\Omega.$$

Clearly $(O^\lambda_\Omega\mathbf{\Psi}, Q_\Omega\mathbf{\Psi})$ is a solution of the Oseen system (1.1) in $R^m\setminus\partial\Omega$.

Define $K^{\Omega,\lambda}(\cdot,\mathbf{y}) = T_\lambda(\breve{O}^\lambda(\cdot-\mathbf{y}), Q(\cdot-\mathbf{y}))\mathbf{n}^\Omega(\mathbf{y})$ for $\mathbf{y}\in\partial\Omega$, $\mathbf{x}\in\mathbb{R}^m\setminus\{\mathbf{y}\}$, i.e.

$$\begin{aligned} K^{\Omega,\lambda}_{j,k}(\mathbf{x},\mathbf{y}) &= \mathbf{n}^\Omega(\mathbf{y})\cdot\nabla_{\mathbf{y}}O^\lambda_{jk}(\mathbf{x}-\mathbf{y}) + \sum_{i=1}^m n^\Omega_i(\mathbf{y})\frac{\partial}{\partial y_k}O^\lambda_{ji}(\mathbf{x}-\mathbf{y}) \\ &\quad + n^\Omega_k(\mathbf{y})Q^\lambda_j(\mathbf{x}-\mathbf{y}) + \frac{\lambda n^\Omega_1(\mathbf{y})}{2}O^\lambda_{jk}(\mathbf{x}-\mathbf{y}) \end{aligned}$$

for $j,k\leq m$. Denote

$$\begin{aligned} \Pi^{\Omega,\lambda}_k(\mathbf{x},\mathbf{y}) &= \mathbf{n}^\Omega(\mathbf{y})\cdot\nabla_{\mathbf{y}}Q^\lambda_k(\mathbf{x}-\mathbf{y}) + \sum_{i=1}^m n^\Omega_i(\mathbf{y})\frac{\partial}{\partial y_k}Q^\lambda_i(\mathbf{x}-\mathbf{y}) \\ &\quad + n^\Omega_k(\mathbf{y})Q^\lambda_{m+1}(\mathbf{x}-\mathbf{y}) + \frac{\lambda n^\Omega_1(\mathbf{y})}{2}Q^\lambda_k(\mathbf{x}-\mathbf{y}) \end{aligned}$$

for $k\leq m$. For $\mathbf{\Psi}\in L^q(\partial\Omega,R^m)$ we define the velocity part of the Oseen double layer potential with density $\mathbf{\Psi}$ by

$$(D^\lambda_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega}K^{\Omega,\lambda}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}), \quad \mathbf{x}\in\mathbb{R}^m\setminus\partial\Omega$$

and the corresponding pressure part by

$$(\Pi^\lambda_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \Pi^{\Omega,\lambda}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^m \setminus \partial\Omega.$$

For $\mathbf{x} \in \partial\Omega$ we denote

$$(K_{\Omega,\lambda}\mathbf{\Psi})(\mathbf{x}) = \lim_{\delta\downarrow 0} \int_{\partial\Omega\setminus B(\mathbf{x},\delta)} K^{\Omega,\lambda}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}),$$

$$(K'_{\Omega,\lambda}\mathbf{\Psi})(\mathbf{x}) = \lim_{\delta\downarrow 0} \int_{\partial\Omega\setminus B(\mathbf{x},\delta)} K^{\Omega,\lambda}(\mathbf{y},\mathbf{x})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}).$$

Lemma 5.1. *Let $\lambda \neq 0$, $1 < q < \infty$. If $\mathbf{f} \in W^{-1,q}(\mathbb{R}^m;\mathbb{R}^m)$ has compact support, and $m = 2$ or $m = 3$, then $\breve{O}^\lambda * \mathbf{f} \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^m)$, $Q * \mathbf{f} \in L^q_{\mathrm{loc}}(\mathbb{R}^m)$.*

Proof. According to [2, Theorem 4.11] there exist $\mathbf{u} \in W^{1,q}_{loc}(\mathbb{R}^m;\mathbb{R}^m)$, $p \in L^q_{\mathrm{loc}}(\mathbb{R}^m)$ such that

$$\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^m.$$

$(\breve{O}^\lambda * \mathbf{f}, Q * \mathbf{f})$ is a solution of the same system by the definition of the fundamental solution. Lemma 3.1 gives that $(Q * \mathbf{f} - p) \in \mathcal{C}^\infty(\mathbb{R}^m)$, $(\breve{O}^\lambda * \mathbf{f} - \mathbf{u}) \in \mathcal{C}^\infty(\mathbb{R}^m;\mathbb{R}^m)$. □

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \neq 0$ and $1 < q < \infty$. Then $O^\lambda_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to W^{1,q}(\Omega;\mathbb{R}^m)$, $Q_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to L^q(\Omega)$ are bounded linear operators.*

Proof. $Q_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to L^q(\Omega)$ is a bounded linear operator by [24, Theorem 4.4].

Denote $q' = q/(q-1)$. If $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ then $\mathbf{f} \in [W^{1,q'}(\Omega;\mathbb{R}^m)]' \subset W^{-1,q}(\mathbb{R}^m;\mathbb{R}^m)$. Since $\mathbf{f}$ has compact support, Lemma 5.1 forces that $O^\lambda_\Omega \mathbf{f} = \breve{O}^\lambda * \mathbf{f} \in W^{1,q}(\Omega;\mathbb{R}^m)$. Since $O^\lambda_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to W^{1,q}(\Omega;\mathbb{R}^m)$ is a closed linear operator, it is bounded by the Closed graph theorem. □

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \neq 0$ and $1 < q < \infty$. If $\mathbf{f} \in L^q(\partial\Omega;\mathbb{R}^m)$ denote by $\mathcal{O}^\lambda_\Omega \mathbf{f}$ the restriction of $O^\lambda_\Omega \mathbf{f}$ onto $\partial\Omega$. Then $\mathcal{O}^\lambda_\Omega : L^q(\partial\Omega;\mathbb{R}^m) \to W^{1,q}(\partial\Omega;\mathbb{R}^m)$ is bounded. The operator $\mathcal{O}^\lambda_\Omega$ can be extended by a unique way to a bounded linear operator $\mathcal{O}^\lambda_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$. If $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$, then $O^\lambda_\Omega \mathbf{f} \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ and $\mathcal{O}^\lambda_\Omega \mathbf{f}$ is the trace of $O^\lambda_\Omega \mathbf{f}$.*

Proof. $\mathcal{O}^\lambda_\Omega : L^q(\partial\Omega;\mathbb{R}^m) \to W^{1,q}(\partial\Omega;\mathbb{R}^m)$ is bounded by [30, Corollary 4.2.4] and [21, Lemma 3.7]. Since $L^q(\partial\Omega;\mathbb{R}^m)$ is a dense subspace of $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$, a possible continuous extension $\mathcal{O}^\lambda_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ is unique.

Put $q' = q/(q-1)$. Since $\mathcal{O}^{-\lambda}_\Omega : L^{q'}(\partial\Omega,\mathbb{R}^m) \to W^{1,q'}(\partial\Omega,\mathbb{R}^m)$ is bounded, the adjoint operator $[\mathcal{O}^{-\lambda}_\Omega]' : W^{-1,q}(\partial\Omega,\mathbb{R}^m) \to L^q(\partial\Omega,\mathbb{R}^m)$ is bounded, too. The symmetry of $\breve{O}^\lambda$ and the relation (4.8) give that $[\mathcal{O}^{-\lambda}_\Omega]' = \mathcal{O}^\lambda_\Omega$. Thus $\mathcal{O}^\lambda_\Omega : W^{-1,q}(\partial\Omega;\mathbb{R}^m) \to L^q(\partial\Omega;\mathbb{R}^m)$ is bounded. We now use the real interpolation. One has

$$(L^q(\partial\Omega), W^{1,q}(\partial\Omega))_{1-1/q,q} = W^{1-1/q,q}(\partial\Omega)$$

$$(W^{-1,q}(\partial\Omega), L^q(\partial\Omega))_{1-1/q,q} = W^{-1/q,q}(\partial\Omega)$$

by [1, Theorem 7.48] and [34, Lemma 41.3]. Thus $\mathcal{O}^\lambda_\Omega : W^{-1/q,q}(\partial\Omega;\mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ is bounded by [8, Proposition 1.6].

Let $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$. Choose $\mathbf{f}_k \in L^q(\partial\Omega;\mathbb{R}^m)$ such that $\mathbf{f}_k \to \mathbf{f}$ in $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ as $k \to \infty$. Let $r > 0$ be such that $\partial\Omega \subset B(0;r)$. Put $\omega = \Omega \cap B(0;r)$ and define $\mathbf{f} = 0$, $\mathbf{f}_k = 0$ on $\partial\omega \setminus \partial\Omega$. Then $O^\lambda_\Omega \mathbf{f} = O^\lambda_\omega \mathbf{f} \in W^{1,q}(\omega)$ and $O^\lambda_\Omega \mathbf{f}_k \to O^\lambda_\Omega \mathbf{f}$ in $W^{1,q}(\omega)$ as $k \to \infty$ by Lemma 5.2. Moreover, $\mathcal{O}^\lambda_\Omega \mathbf{f}_k(\mathbf{x})$ is the non-tangential limit of $O^\lambda_\Omega \mathbf{f}_k$ for almost all $\mathbf{x} \in \partial\Omega$ by Lemma 12.3. Since the non-tangential limit is equal to the trace for functions from $W^{1,q}_{\rm loc}(\Omega)$ by Lemma 12.2, we infer that $\gamma_\Omega O^\lambda_\Omega \mathbf{f}_k = \mathcal{O}^\lambda_\Omega \mathbf{f}_k$. Since $\gamma_\omega : W^{1,q}(\omega) \to W^{1-1/q,q}(\partial\omega)$ is continuous, we obtain $\gamma_\Omega O^\lambda_\Omega \mathbf{f} = \mathcal{O}^\lambda_\Omega \mathbf{f}$. □

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \neq 0$, $0 \le s \le 1$ and $1 < q < \infty$. Then $K_{\Omega,\lambda}$ is a bounded linear operator on $W^{s,q}(\partial\Omega,\mathbb{R}^m)$, and $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact on $W^{s,q}(\partial\Omega,\mathbb{R}^m)$. The operator $K'_{\Omega,\lambda}$ can be extended as a continuous operator on $W^{-s,q}(\partial\Omega,\mathbb{R}^m)$.*

Proof. $K_{\Omega,\lambda}$ is a bounded operator on $L^q(\partial\Omega,\mathbb{R}^m)$ and on $W^{1,q}(\partial\Omega,\mathbb{R}^m)$, and $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact on $L^q(\partial\Omega,\mathbb{R}^m)$ and on $W^{1,q}(\partial\Omega,\mathbb{R}^m)$ by [24, Corollary 3.3], [24, Proposition 3.5], [21, Proposition 3.1] and [21, Proposition 3.2]. Let now $0 < s < 1$. We use the real interpolation. Since

$$(L^q(\partial\Omega,\mathbb{R}^m), W^{1,q}(\partial\Omega,\mathbb{R}^m))_{s,q} = W^{s,q}(\partial\Omega,\mathbb{R}^m)$$

by [36, §7.3.1, Theorem], [34, Lemma 22.3] gives that $K_{\Omega,\lambda}$ is a bounded operator on $W^{s,q}(\partial\Omega,\mathbb{R}^m)$. Moreover, $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact on $W^{s,q}(\partial\Omega,\mathbb{R}^m)$ by [11, Theorem 1.1].

Put $q' = q/(q-1)$. Since $K'_{\Omega,\lambda}$ is the adjoint operator of $K_{\Omega,\lambda}$, and $K_{\Omega,\lambda}$ is bounded on $W^{s,q'}(\partial\Omega,\mathbb{R}^m)$, $K'_{\Omega,\lambda}$ is bounded on $W^{-s,q}(\partial\Omega,\mathbb{R}^m)$. □

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \neq 0$, $1 < q < \infty$ and $h \in L^\infty(\partial\Omega)$. If $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$, then $(\mathbf{u},p) := (O^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f}) \in W^{1,q}_{\rm loc}(\overline{\Omega};\mathbb{R}^m) \times L^q_{\rm loc}(\overline{\Omega})$ is a weak solution of (1.1), (3.1) with $\mathbf{g} = \frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f} + h\mathcal{O}^\lambda_\Omega \mathbf{f}$.*

Proof. According to Lemma 5.3 we can suppose $h \equiv 0$. If $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$, then $(O^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f}) \in W^{1,q}_{\rm loc}(\overline{\Omega};\mathbb{R}^m) \times L^q_{\rm loc}(\overline{\Omega})$ by Lemma 5.3 and [24, Theorem 4.4].

Suppose first that $\mathbf{f} \in L^q(\partial\Omega;\mathbb{R}^m)$ and Ω is bounded. (See the Appendix for the definition of the non-tangential cone $\Gamma_a(\mathbf{x})$, a non-tangential maximal function $M_a(v)$ and a non-tangential limit v_Ω.) According to Lemma 12.3 and Lemma 12.4 there exist non-tangential limits of $O^\lambda_\Omega \mathbf{f}$, $\nabla O^\lambda_\Omega \mathbf{f}$ and $Q_\Omega \mathbf{f}$ at almost all points of $\partial\Omega$ and

$$[T_\lambda(O^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f})\mathbf{n}]_\Omega = \frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f} \quad \text{a.e. on } \partial\Omega,$$

$$\|M_a(Q_\Omega \mathbf{f})\|_{L^q(\partial\Omega)} + \|M_a(\nabla O^\lambda_\Omega \mathbf{f})\|_{L^q(\partial\Omega)} + \|M_a(O^\lambda_\Omega \mathbf{f})\|_{L^q(\partial\Omega)} \le C\|\mathbf{f}\|_{L^q(\partial\Omega)}.$$

According to [38, Theorem 1.12] there is a sequence of domains Ω_j with boundaries of class C^∞ such that

- $\overline{\Omega_j} \subset \Omega$.
- There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(\mathbf{y}) \in \Gamma_a(\mathbf{y})$ for each j and each $\mathbf{y} \in \partial\Omega$ and $\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})|; \mathbf{y} \in \partial\Omega\} \to 0$ as $j \to \infty$.

- There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\sigma = \int_{\Lambda_j(E)} 1 \, d\sigma$, and so that $\omega_j \to 1$ point-wise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$.
- The normal vectors to Ω_j, $\mathbf{n}(\Lambda_j(\mathbf{y}))$, converge point-wise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$, to $\mathbf{n}(\mathbf{y})$.

Fix $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$. According to Lebesgue's lemma and Green's formula we obtain

$$\int_{\partial\Omega} \mathbf{\Phi}\cdot\left[\frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f}\right] d\sigma = \lim_{j\to\infty} \int_{\partial\Omega_j} \mathbf{\Phi}\cdot T_\lambda(\mathbf{u},p)\mathbf{n}\, d\sigma = \lim_{j\to\infty}\int_{\Omega_j} [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi})$$

$$+\frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u} - \mathbf{u}\cdot\partial_1\mathbf{\Phi})]\, d\mathbf{x} = \int_\Omega [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u} - \mathbf{u}\cdot\partial_1\mathbf{\Phi})]\, d\mathbf{x}.$$

Suppose now that Ω is bounded and $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$. Choose $\mathbf{f}_k \in L^q(\partial\Omega;\mathbb{R}^m)$ such that $\mathbf{f}_k \to \mathbf{f}$ in $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ as $k\to\infty$. Denote $(\mathbf{u}_k, p_k) := (O_\Omega^\lambda \mathbf{f}_k, Q_\Omega \mathbf{f}_k)$. Fix $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$. According to Lemma 5.4 and Lemma 5.2

$$\left\langle \frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f}, \mathbf{\Phi}\right\rangle = \lim_{k\to\infty}\int_{\partial\Omega} \mathbf{\Phi}\cdot\left[\frac{1}{2}\mathbf{f}_k - K'_{\Omega,\lambda}\mathbf{f}_k\right] d\sigma$$

$$= \lim_{k\to\infty}\int_\Omega [2\hat\nabla\mathbf{u}_k\cdot\hat\nabla\mathbf{\Phi} - p_k(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u}_k - \mathbf{u}_k\cdot\partial_1\mathbf{\Phi})]\, d\mathbf{x}$$

$$= \int_\Omega [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u} - \mathbf{u}\cdot\partial_1\mathbf{\Phi})]\, d\mathbf{x}.$$

Suppose now that Ω is unbounded and $\mathbf{f} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$. Let us fix $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m,\mathbb{R}^m)$. Choose $r \in (0,\infty)$ such that $\mathbf{\Phi}$ is supported in $B(0;r)$. Denote $\omega := \Omega \cap B(0;r)$. Define $\mathbf{f} = 0$ on $\partial\omega \setminus \partial\Omega$. Clearly $K'_{\omega,\lambda}\mathbf{f} = K'_{\Omega,\lambda}\mathbf{f}$ on $\partial\Omega$. Since $\mathbf{\Phi} = 0$ on $\partial\omega\setminus\partial\Omega$, one has

$$\left\langle \frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f}, \mathbf{\Phi}\right\rangle = \left\langle \frac{1}{2}\mathbf{f} - K'_{\omega,\lambda}\mathbf{f}, \mathbf{\Phi}\right\rangle = \int_\omega [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u}$$

$$-\mathbf{u}\cdot\partial_1\mathbf{\Phi})]\, d\mathbf{x} = \int_\Omega [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u} - \mathbf{u}\cdot\partial_1\mathbf{\Phi})]\, d\mathbf{x}.$$

□

Lemma 5.6. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let $\lambda \in \mathbb{R}^1$. If $1 < p \le 2 \le q < \infty$ then $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are Fredholm operators with index 0 in $L^q(\partial\Omega,\mathbb{R}^m)$ and in $W^{1,p}(\partial\Omega,\mathbb{R}^m)$.*

Proof. For $\lambda = 0$ see [30, Corollary 9.1.2]. Since $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact on $L^q(\partial\Omega,\mathbb{R}^m)$ and on $W^{1,q}(\partial\Omega,\mathbb{R}^m)$ by Lemma 5.4, we obtain the Lemma. □

6. Liouville's theorem for the Oseen system

Lemma 6.1. *Let $\lambda \in \mathbb{R}^1\setminus\{0\}$, p be a distribution in $\mathbb{R}^m$ and $u_1,\dots,u_m$ be tempered distributions in $\mathbb{R}^m$. If $-\Delta\mathbf{u} + \lambda\partial_1\mathbf{u} + \nabla p = 0$, $\nabla\cdot\mathbf{u} = 0$ in $\mathbb{R}^m$ in the sense of distributions, then $u_1,\dots,u_m$ and p are polynomials.*

Proof. See [27, Lemma 1] if p is a tempered distribution. Let now p be general. Since $\partial^\alpha u_j$ is a tempered distribution for arbitrary multi-index α, we infer that $\partial_j p = \Delta u_j - \lambda \partial_1 u_j$ is a tempered distribution. Since $-\Delta \partial_j \mathbf{u} + \lambda \partial_1 \partial_j \mathbf{u} + \nabla \partial_j p = 0$, $\nabla \cdot \partial_j \mathbf{u} = 0$ in $\mathbb{R}^m$, we deduce that $\partial_j u_1, \dots, \partial_j u_m$ and $\partial_j p$ are polynomials. Therefore $u_1, \dots, u_m$ and p are polynomials. □

7. Integral representation

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lispchitz boundary, $\lambda \in \mathbb{R}^1$, $1 < q < \infty$, $h \equiv 0$. If $(\mathbf{u}, p) \in W^{1,q}(\Omega; \mathbb{R}^m) \times L^q(\Omega)$ is a weak solution of the Neumann problem (1.1), (3.1) then*

$$\mathbf{u} = O^\lambda_\Omega \mathbf{g} + \mathcal{D}^\lambda_\Omega \gamma_\Omega \mathbf{u}, \qquad p = Q_\Omega \mathbf{g} + \Pi^\lambda_\Omega \gamma_\Omega \mathbf{u}. \tag{7.1}$$

Proof. Fix $\mathbf{x} \in \Omega$. Fix $r > 0$ such that $\overline{B(\mathbf{x}; r)} \subset \Omega$. Put $U := B(\mathbf{x}; r)$, $\omega := \Omega \setminus \overline{U}$, $\mathbf{F} := T_\lambda(\mathbf{u}, p)\mathbf{n}^U$ on ∂U, $\mathbf{G} := \mathbf{g}$ on $\partial\Omega$, $\mathbf{G} := -\mathbf{F}$ on ∂U. Since $(\mathbf{u}, p)$ is a classical solution of the problem

$$\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = 0,\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } U, \quad T_\lambda(\mathbf{u}, p)\mathbf{n}^U = \mathbf{F} \quad \text{on } \partial U, \tag{7.2}$$

one has

$$\mathbf{u}(\mathbf{x}) = O^\lambda_U \mathbf{F}(\mathbf{x}) + \mathcal{D}^\lambda_U \mathbf{u}(\mathbf{x}), \qquad p = Q_U \mathbf{F} + \Pi^\lambda_U \mathbf{u} \tag{7.3}$$

by [21, Proposition 3.4]. Since $(\mathbf{u}, p)$ is a weak solution of the problem (7.2), $(\mathbf{u}, p)$ is a weak solution of the problem

$$\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = 0,\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \omega, \quad T_\lambda(\mathbf{u}, p)\mathbf{n}^\omega = \mathbf{G} \quad \text{on } \partial\omega.$$

Fix $k \in \{1, \dots, m\}$ and put $\mathbf{\Phi}(\mathbf{y}) := (O^\lambda_{1k}(\mathbf{x}-\mathbf{y}), \dots, O^\lambda_{mk}(\mathbf{x}-\mathbf{y}))$, $\varphi(\mathbf{y}) := Q_k(\mathbf{x}-\mathbf{y})$, $\tilde{\mathbf{\Phi}}(\mathbf{y}) := (O^\lambda_{1,m+1}(\mathbf{x}-\mathbf{y}), \dots, O^\lambda_{m,m+1}(\mathbf{x}-\mathbf{y}))$, $\tilde\varphi(\mathbf{y}) := Q_{m+1}(\mathbf{x}-\mathbf{y})$. Remember that

$$-\Delta \mathbf{\Phi} - \lambda \partial_1 \mathbf{\Phi} + \nabla \varphi = 0, \quad \nabla \cdot \mathbf{\Phi} = 0 \quad \text{in } \omega,$$
$$-\Delta \tilde{\mathbf{\Phi}} - \lambda \partial_1 \tilde{\mathbf{\Phi}} + \nabla \tilde\varphi = 0, \quad \nabla \cdot \tilde{\mathbf{\Phi}} = 0 \quad \text{in } \omega.$$

According to the Green formula

$$[O^\lambda_\omega \mathbf{G}(\mathbf{x})]_k = \langle \mathbf{G}, \mathbf{\Phi} \rangle = \int_\omega [2\hat\nabla \mathbf{u} \cdot \hat\nabla \mathbf{\Phi} + \frac{\lambda}{2}(\mathbf{\Phi} \cdot \partial_1 \mathbf{u} - \mathbf{u} \partial_1 \mathbf{\Phi})]\, \mathrm{d}\mathbf{y} = -[\mathcal{D}^\lambda_\omega \mathbf{u}(\mathbf{x})]_k,$$

$$Q_\omega \mathbf{G}(\mathbf{x}) = \langle \mathbf{G}, \tilde{\mathbf{\Phi}} \rangle = \int_\omega [2\hat\nabla \mathbf{u} \cdot \hat\nabla \tilde{\mathbf{\Phi}} + \frac{\lambda}{2}(\tilde{\mathbf{\Phi}} \cdot \partial_1 \mathbf{u} - \mathbf{u} \partial_1 \tilde{\mathbf{\Phi}})]\, \mathrm{d}\mathbf{y} = -\Pi^\lambda_\omega \mathbf{u}(\mathbf{x}).$$

Adding with (7.3) we obtain (7.1). □

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded open set with compact Lispchitz boundary, $\lambda \in \mathbb{R}^1 \setminus \{0\}$, $1 < q < \infty$, $h \equiv 0$. Suppose that $(\mathbf{u}, p) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ is a weak solution of the Neumann problem (1.1), (3.1). Suppose that $\mathbf{u}$ is bounded at infinity or $\partial_j \mathbf{u} \in L^q(\Omega; \mathbb{R}^m)$ for $j = 1, \dots, m$. Then there exist $\mathbf{u}_\infty \in \mathbb{R}^3$ and $p_\infty \in \mathbb{R}^1$ such that $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$, $p(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$ and*

$$\mathbf{u} = O^\lambda_\Omega \mathbf{g} + \mathcal{D}^\lambda_\Omega \gamma_\Omega \mathbf{u} + \mathbf{u}_\infty, \qquad p = Q_\Omega \mathbf{g} + \Pi^\lambda_\Omega \gamma_\Omega \mathbf{u} + p_\infty. \tag{7.4}$$

Proof. Choose $r \in (0, \infty)$ such that $\partial\Omega \subset B(0; r)$ and put $G := \Omega \cap B(0; r)$. Define $\mathbf{g} := T_\lambda(\mathbf{u}, p)\mathbf{n}^G$ on $\partial B(0; r)$. Then

$$\mathbf{u} = O^\lambda_G \mathbf{g} + \mathcal{D}^\lambda_G \gamma_G \mathbf{u}, \qquad p = Q_G \mathbf{g} + \Pi^\lambda_G \gamma_G \mathbf{u} \quad \text{in } G \tag{7.5}$$

by Proposition 7.1. So, if we define

$$\mathbf{v} = \begin{cases} \mathbf{u} - O^\lambda_\Omega \mathbf{g} - \mathcal{D}^\lambda_\Omega \gamma_\Omega \mathbf{u}, & \text{in } \Omega, \\ O^\lambda_{B(0;r)} \mathbf{g} + \mathcal{D}^\lambda_{B(0;r)} \gamma_{B(0;r)} \mathbf{u}, & \text{in } B(0;r), \end{cases} \tag{7.6}$$

$$\tau = \begin{cases} p - Q_\Omega \mathbf{g} - \Pi^\lambda_\Omega \gamma_\Omega \mathbf{u}, & \text{in } \Omega, \\ Q_{B(0;r)} \mathbf{g} + \Pi^\lambda_{B(0;r)} \gamma_{B(0;r)} \mathbf{u}, & \text{in } B(0;r), \end{cases} \tag{7.7}$$

then $\mathbf{v}$, τ are well defined on $\Omega \cap B(0;r)$. Clearly, $(\mathbf{v}, \tau)$ is a solution of (1.1) in $\mathbb{R}^m$. Moreover, $v_1, \dots, v_m$ are tempered distributions. (See [28, Lemma 1.25.9].) Lemma 6.1 gives that $\tau, v_1, \dots, v_m$ are polynomials.

We now show that $\mathbf{v}$ is constant. If $\mathbf{u}$ is bounded at infinity, then $\mathbf{v}$ is bounded at infinity and therefore $\mathbf{v} \equiv \mathbf{u}_\infty$ for some constant $\mathbf{u}_\infty$. Let now $\partial_j u_k \in L^q(\Omega)$. Since $\partial_j O^\lambda_\Omega \mathbf{g}(\mathbf{x}) + \partial_j \mathcal{D}^\lambda_\Omega \gamma_\Omega \mathbf{u}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ and $\partial_j v_k = \partial_j u_k - \partial_j O^\lambda_\Omega g_k - \partial_j \mathcal{D}^\lambda_\Omega \gamma_\Omega u_k$ is a polynomial, we deduce that $\partial_j v_k \equiv 0$. Therefore there exists $\mathbf{u}_\infty \in \mathbb{R}^m$ such that $\mathbf{v} \equiv \mathbf{u}_\infty$.

The equation (1.1) gives $\nabla \tau \equiv 0$. So, there exists $p_\infty \in \mathbb{R}^1$ such that $\tau \equiv p_\infty$. Thus (7.4) holds. Properties of the fundamental solution give that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. □

8. Solvability of the problem (1.1), (3.1)

Let $\Omega \subset \mathbb{R}^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$, and $\lambda \neq 0$. Let $G(1), \dots, G(k)$ be all bounded components of $\mathbb{R}^m \setminus \overline{\Omega}$. Fix open balls $B(j)$ such that $\overline{B}(j) \subset G(j)$. It is a tradition to look for a solution of the Robin problem in the form of a single layer potential. Unfortunately, it is not possible for domains with holes. We look for a solution of the Robin problem (1.1), (3.1) by virtue of a modified Oseen single layer potential. Choose $\mathbf{\Psi}_j \in W^{1,\infty}(\partial G(j), \mathbb{R}^m)$ such that

$$\int_{\partial G(j)} \mathbf{\Psi}_j \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma \neq 0. \tag{8.1}$$

For $\mathbf{f} \in W^{-1/q,q}(\partial\Omega, R^m)$ with $1 < q < \infty$ define the modified Oseen single layer potential by

$$\tilde{O}^\lambda_\Omega \mathbf{f} := O^\lambda_\Omega \mathbf{f} + \sum_{j=1}^k (D^\lambda_{B(j)} \mathbf{n}^{B(j)}) \langle \mathbf{f}, \mathbf{\Psi}_j \rangle, \tag{8.2}$$

$$\tilde{Q}^\lambda_\Omega \mathbf{f} := Q_\Omega \mathbf{f} + \sum_{j=1}^k (\Pi^\lambda_{B(j)} \mathbf{n}^{B(j)}) \langle \mathbf{f}, \mathbf{\Psi}_j \rangle. \tag{8.3}$$

(If Ω is a bounded domain with connected boundary then $\tilde{O}^\lambda_\Omega \mathbf{f} = O^\lambda_\Omega \mathbf{f}$, $\tilde{Q}^\lambda_\Omega \mathbf{f} = Q_\Omega \mathbf{f}$.) Put

$$\tau^\lambda_{h,\Omega} \mathbf{f} := \frac{1}{2} \mathbf{f} - K'_{\Omega,\lambda} \mathbf{f} + \sum_{j=1}^k [T_\lambda(D^\lambda_{B(j)} \mathbf{n}^{B(j)}, \Pi^\lambda_{B(j)} \mathbf{n}^{B(j)}) \mathbf{n}^\Omega] \langle \mathbf{f}, \mathbf{\Psi}_j \rangle + h \tilde{O}^\lambda_\Omega \mathbf{f}.$$

Then $(\mathbf{u}, p) = (\tilde{O}^\lambda_\Omega \mathbf{f}, \tilde{Q}^\lambda_\Omega \mathbf{f})$ is a weak solution of the problem (1.1), (3.1) in the space $W^{1,q}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ if and only if $\tau^\lambda_{h,\Omega} \mathbf{f} = \mathbf{g}$. (See Lemma 5.5.)

Proposition 8.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let $\lambda \in \mathbb{R}^1$, $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$. Suppose that $3/2 < q < 3$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Then $\frac{1}{2}I \pm K'_{\Omega,\lambda}$ and $\tau^\lambda_{h,\Omega}$ are bounded Fredholm operators with index 0 in $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$, and $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are bounded Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$.*

Proof. First we show that $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are bounded Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$. If $\partial\Omega$ is of class $\mathcal{C}^1$, then $\frac{1}{2}I \pm K_{\Omega,0}$ are Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$ by [24, p. 232]. Since $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact in $W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$ by Lemma 5.4, $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$. Let now $3/2 < q < 3$. Since $1/2 < q - 1 < 2$ there exist $p, r' \in (1, 2)$ such that $q - 1 = p/r'$. Denote $r = r'/(r' - 1)$. Now we use the real interpolation. Put $\theta = 1 - 1/q$. Since $(1-\theta)/r + \theta/p = (1/q)[1/r + (q-1)/p] = 1/q$, one has

$$[L^r(\partial\Omega), W^{1,p}(\partial\Omega)]_{\theta,q} = W^{1-1/q,q}(\partial\Omega).$$

(Compare [14, Corollary 6.8].) Remark that $r \in (2, \infty)$. Lemma 5.6 gives that $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are bounded Fredholm operators with index 0 in $W^{1,p}(\partial\Omega, \mathbb{R}^m)$ and in $L^r(\partial\Omega, \mathbb{R}^m)$. So, $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are bounded Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$ by [28, Proposition 1.10.4].

Denote $q' = q/(q-1)$. We have proved that $\frac{1}{2}I \pm K_{\Omega,\lambda}$ are bounded Fredholm operators with index 0 in $W^{1-1/q',q'}(\partial\Omega, \mathbb{R}^m)$. The duality argument gives that $\frac{1}{2}I \pm K'_{\Omega,\lambda}$ are bounded Fredholm operators with index 0 in $[W^{1-1/q',q'}(\partial\Omega, \mathbb{R}^m)]' = W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$.

The operator $\tau^\lambda_{h,\Omega} - [\frac{1}{2}I - K'_{\Omega,\lambda}]$ is compact in $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$ by Lemma 5.3. Hence $\tau^\lambda_{h,\Omega}$ is a Fredholm operator with index 0 in $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$. □

Lemma 8.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}^1 \setminus \{0\}$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\mathbf{g} \in W^{-1/2,2}(\partial\Omega; \mathbb{R}^m)$. Suppose that $(\mathbf{u}, p)$ is a weak solution of the Robin problem (1.1), (3.1) in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}^m) \times L^2_{\mathrm{loc}}(\overline{\Omega})$. If Ω is unbounded suppose moreover $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. If $\gamma_\Omega \mathbf{u} = 0$ on $\partial\Omega$ then $\mathbf{u} \equiv 0$. If $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$ and $p \equiv 0$.*

Proof. Suppose that $\mathbf{g} \equiv 0$ or $\gamma_\Omega \mathbf{u} = 0$ on $\partial\Omega$. If Ω is bounded then

$$0 = \langle \mathbf{g}, \gamma_\Omega \mathbf{u} \rangle = 2 \int_\Omega |\hat{\nabla} \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h |\mathbf{u}|^2 \, \mathrm{d}\sigma. \tag{8.4}$$

Suppose now that Ω is unbounded. Define $h = 0$ on $\mathbb{R}^m \setminus \partial\Omega$. Choose $r_0 > 0$ such that $\partial\Omega \subset B(0; r_0)$. For $r > r_0$ denote $\Omega(r) = \Omega \cap B(0; r)$, $\mathbf{g}_r = \mathbf{g}$ on $\partial\Omega$, $\mathbf{g}_r = T_\lambda(\mathbf{u}, p)\mathbf{n}^{\Omega(r)}$ on $\partial\Omega(r) \setminus \partial\Omega$. Then $(\mathbf{u}, p)$ is a weak solution of

$$\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = 0 \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega(r),$$
$$T_\lambda(\mathbf{u}, p)\mathbf{n} + h\mathbf{u} = \mathbf{g}_r \qquad \text{on } \partial\Omega(r).$$

Thus

$$\langle \mathbf{g}_r, \gamma_{\Omega(r)} \mathbf{u} \rangle = 2 \int_{\Omega(r)} |\hat{\nabla} \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h |\mathbf{u}|^2 \, \mathrm{d}\sigma. \tag{8.5}$$

Proposition 7.2 gives $\mathbf{u} = O^\lambda_\Omega(\mathbf{g} - h\gamma_\Omega \mathbf{u}) + \mathcal{D}^\lambda_\Omega \gamma_\Omega \mathbf{u}$, $p = Q_\Omega(\mathbf{g} - h\gamma_\Omega \mathbf{u}) + \Pi^\lambda_\Omega \gamma_\Omega \mathbf{u}$. According to (4.1) there exists a constant C such that

$$|\mathbf{u}(r\mathbf{y})||T_\lambda(\mathbf{u}(r\mathbf{y}), p(r\mathbf{y})| \leq C r^{1-m} \quad \text{for } |\mathbf{y}| = 1.$$

If $\mathbf{y} \neq [c, 0, \dots, 0]$ then $|\mathbf{u}(r\mathbf{y})||T_\lambda(\mathbf{u}(r\mathbf{y}), p(r\mathbf{y})|r^{m-1} \to 0$ as $r \to \infty$ by (4.2). Since $\langle \mathbf{g}, \gamma_\Omega \mathbf{u}\rangle = 0$, we get from (8.5) using Lebesgues lemma

$$2 \int_\Omega |\hat\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h|\mathbf{u}|^2 \, \mathrm{d}\sigma = \lim_{r\to\infty} 2 \int_{\Omega(r)} |\hat\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h|\mathbf{u}|^2 \, \mathrm{d}\sigma$$

$$= \lim_{r\to\infty} \langle \mathbf{g}_r, \gamma_{\Omega(r)} \mathbf{u}\rangle = \lim_{r\to\infty} \int_{\partial B(0;1)} r^{m-1} \mathbf{u}(r\mathbf{y}) \cdot T_\lambda(\mathbf{u}(r\mathbf{y}), p(r\mathbf{y}))\mathbf{n}(\mathbf{y}) \, \mathrm{d}\sigma(\mathbf{y}) = 0.$$

Since (8.4) holds, we infer $\hat\nabla \mathbf{u} = 0$ in Ω. So, [26, Lemma 3.1] gives that $\mathbf{u}$ is linear. Thus $\Delta \mathbf{u} \equiv 0$, $\nabla p = -\lambda \partial_1 \mathbf{u}$ and $p \in \mathcal{C}^\infty(\overline{\Omega})$. If $\gamma_\Omega \mathbf{u} = 0$ on $\partial\Omega$, then $\mathbf{u} \equiv 0$ by the maximum principle for the Laplace equation. If $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$, $p \equiv 0$ by [27, Theorem 1]. □

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let $\lambda \in \mathbb{R}^1 \setminus \{0\}$, $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that $3/2 < q < 3$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Then $\tau^\lambda_{h,\Omega}$ is an isomorphism on $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$. Let $\mathbf{g} \in W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$. Put*

$$\mathbf{u} := \tilde O^\lambda_\Omega [\tau^\lambda_{h,\Omega}]^{-1} \mathbf{g}, \qquad p := \tilde Q^\lambda_\Omega [\tau^\lambda_{h,\Omega}]^{-1} \mathbf{g}. \tag{8.6}$$

If Ω is bounded then $(\mathbf{u}, p)$ is a unique weak solution of the Robin problem (1.1), (3.1) in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$. If Ω is unbounded then $(\mathbf{u}, p)$ is a unique weak solution of the Robin problem (1.1), (3.1),

$$\mathbf{u}(\mathbf{x}) \to 0, \quad p(\mathbf{x}) \to 0 \qquad \text{as } |\mathbf{x}| \to \infty \tag{8.7}$$

in $W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$.

Proof. Let us remark that

$$W^{1-1/q(2),q(2)}(\partial\Omega) \hookrightarrow W^{1-1/q(1),q(1)}(\partial\Omega), \; W^{-1/q(2),q(2)}(\partial\Omega) \hookrightarrow W^{-1/q(1),q(1)}(\partial\Omega)$$

for $1 < q(1) < q(2) < \infty$ by [35, Theorem 1.97].

Let now $(\tilde{\mathbf{u}}, \tilde p)$ be a weak solution of the Robin problem (1.1), (3.1) with $\mathbf{g} \equiv 0$ in $W^{1,q}_{loc}(\overline{\Omega}, \mathbb{R}^m) \times L^q_{loc}(\overline{\Omega})$. If Ω is unbounded suppose moreover $\tilde{\mathbf{u}}(\mathbf{x}) \to 0$, $\tilde p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Since $T_\lambda(\tilde{\mathbf{u}}, \tilde p)\mathbf{n}^\Omega = -h\gamma_\Omega \tilde{\mathbf{u}}$, Proposition 7.1 and Proposition 7.2 give

$$\tilde{\mathbf{u}} = -O^\lambda_\Omega(h\gamma_\Omega \tilde{\mathbf{u}}) + \mathcal{D}^\lambda_\Omega \gamma_\Omega \tilde{\mathbf{u}}, \qquad \tilde p = -Q_\Omega(h\gamma_\Omega \tilde{\mathbf{u}}) + \Pi^\lambda_\Omega \gamma_\Omega \tilde{\mathbf{u}} \quad \text{in } \Omega. \tag{8.8}$$

According to Lemma 5.3, Lemma 12.2 and Lemma 12.4

$$\gamma_\Omega \tilde{\mathbf{u}} = -\mathcal{O}^\lambda_\Omega(h\gamma_\Omega \tilde{\mathbf{u}}) + \frac{1}{2}\gamma_\Omega \tilde{\mathbf{u}} + K_{\Omega,\lambda}\gamma_\Omega \tilde{\mathbf{u}} \quad \text{on } \partial\Omega.$$

Define

$$L\mathbf{f} := \frac{1}{2}\mathbf{f} - K_{\Omega,\lambda}\mathbf{f} + \mathcal{O}^\lambda_\Omega h\mathbf{f}.$$

We have proved that $L(\gamma_\Omega \tilde{\mathbf{u}}) = 0$. Moreover, $\gamma_\Omega \tilde{\mathbf{u}} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$. The operator L is a Fredholm operator with index 0 in $W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$, in $W^{1/2,2}(\partial\Omega; \mathbb{R}^m)$ and in $W^{1,2}(\partial\Omega; \mathbb{R}^m)$ by Proposition 8.1, Lemma 5.6 and Lemma 5.3. Thus $\gamma_\Omega \tilde{\mathbf{u}} \in W^{1,2}(\partial\Omega; \mathbb{R}^m)$ by [28, Lemma 1.8.4]. So, $(\tilde{\mathbf{u}}, \tilde p) \in W^{1,2}_{loc}(\overline{\Omega}, \mathbb{R}^m) \times L^2_{loc}(\overline{\Omega})$ by (8.8), Lemma 5.3, Lemma 5.2, [30, Theorem 10.5.1], [21, Proposition 3.2] and [28, Lemma 1.28.1]. Lemma 8.2 gives that $\tilde{\mathbf{u}} \equiv 0$, $\tilde p \equiv 0$.

Let $\mathbf{f} \in W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$ and $\tau^\lambda_{h,\Omega}\mathbf{f} = 0$. Then $(\tilde{O}^\lambda_\Omega \mathbf{f}, \tilde{Q}^\lambda_\Omega \mathbf{f})$ is a weak solution of the problem (1.1), (3.1) with $\mathbf{g} \equiv 0$ in $W^{1,q}_{loc}(\overline{\Omega}, \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$. We have proved that $\tilde{O}^\lambda_\Omega \mathbf{f} = 0$ and $\tilde{Q}^\lambda_\Omega \mathbf{f} = 0$ in Ω. The Green formula gives

$$\int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot \mathcal{O}^\lambda_\Omega \mathbf{f}\, \mathrm{d}\sigma = \int_{G(i)} \nabla \cdot O^\lambda_\Omega \mathbf{f}\, \mathrm{d}\mathbf{x} = 0, \qquad i = 1, \ldots, k,$$

$$\int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot D^\lambda_{B(j)} \mathbf{n}^{B(j)}\, \mathrm{d}\sigma = \int_{G(i)} \nabla \cdot D^\lambda_{B(j)} \mathbf{n}^{B(j)}\, \mathrm{d}\mathbf{x} = 0, \qquad i \neq j.$$

Thus

$$0 = \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot \tilde{\mathcal{O}}^\lambda_\Omega \mathbf{f}\, \mathrm{d}\sigma = \langle \mathbf{f}, \mathbf{\Psi}_i \rangle \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot D^\lambda_{B(i)} \mathbf{n}^{B(i)}\, \mathrm{d}\mathbf{x}. \tag{8.9}$$

Using Lemma 12.4 and Lemma 12.5 for $D^\lambda_{B(i)} \mathbf{n}^{B(i)}$ on $B(i)$ and on $G(i) \setminus \overline{B(i)}$ we obtain

$$\int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left(\frac{1}{2} \mathbf{n}^{B(i)} + K_{B(i),\lambda} \mathbf{n}^{B(i)} \right) \mathrm{d}\sigma = 0,$$

$$\int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left(-\frac{1}{2} \mathbf{n}^{B(i)} + K_{B(i),\lambda} \mathbf{n}^{B(i)} \right) \mathrm{d}\sigma + \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot D^\lambda_{B(i)} \mathbf{n}^{B(i)}\, \mathrm{d}\mathbf{x} = 0.$$

Hence

$$\int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot D^\lambda_{B(i)} \mathbf{n}^{B(i)}\, \mathrm{d}\mathbf{x} = \int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \mathbf{n}^{B(i)}\, \mathrm{d}\sigma \neq 0.$$

According to (8.9) we have

$$\langle \mathbf{f}, \mathbf{\Psi}_i \rangle = 0, \qquad i = 1, \ldots, k. \tag{8.10}$$

Thus $O^\lambda_\Omega \mathbf{f} = \tilde{O}^\lambda_\Omega \mathbf{f} = 0$ and $Q_\Omega \mathbf{f} = \tilde{Q}^\lambda_\Omega \mathbf{f} = 0$ in Ω. So,

$$\frac{1}{2} \mathbf{f} - K'_{\Omega,\lambda} \mathbf{f} = 0 \tag{8.11}$$

by Lemma 5.5 and $\mathcal{O}^\lambda_\Omega \mathbf{f} = 0$ on $\partial\Omega$ by Lemma 5.3. Using Lemma 8.2 for $(O^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f})$ on $\omega = \mathbb{R}^m \setminus \overline{\Omega}$ we obtain $O^\lambda_\Omega \mathbf{f} = 0$ in ω. Since $\nabla Q_\Omega \mathbf{f} = \Delta O^\lambda_\Omega \mathbf{f} - \lambda O^\lambda_\Omega \mathbf{f} = 0$ in ω, the function $Q_\Omega \mathbf{f}$ is locally constant in ω. So, if S is a component of $\partial\omega$, then there exists a constant c_S such that

$$\frac{1}{2} \mathbf{f} - K'_{\omega,\lambda} \mathbf{f} = c_S \mathbf{n}^\omega \quad \text{on } S$$

by Lemma 5.5. Since $K'_{\Omega,\lambda} = -K'_{\omega,\lambda}$ we get by adding (8.11) that $\mathbf{f} = -c_S \mathbf{n}^\Omega$ on S. (8.10) and (8.1) give that $c_{\partial G(i)} = 0$. So, if Ω is unbounded then $\mathbf{f} \equiv 0$. Let now Ω be bounded. Denote by C the boundary of the unbounded component of ω. Then $\mathbf{f} = 0$ on $\partial\Omega \setminus C$, $\mathbf{f} = c\mathbf{n}^\Omega$ on C. According to [26] we have $Q_\Omega \mathbf{f} = c$ in Ω. Therefore

$$\frac{1}{2} \mathbf{f} - K'_{\Omega,\lambda} \mathbf{f} = T_\lambda(O^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f}) \mathbf{n}^\Omega = -c\mathbf{n}^\Omega.$$

(8.11) gives that $c = 0$ and thus $\mathbf{f} \equiv 0$.

$\tau^\lambda_{h,\Omega}$ is a Fredholm operator with index 0 in $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$ by Proposition 8.1. Since the kernel of $\tau^\lambda_{h,\Omega}$ is trivial, $\tau^\lambda_{h,\Omega}$ is an isomorphism in $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$. So, $(\mathbf{u}, p)$ given by (8.6) is a weak solution of the problem (1.1), (3.1) in $W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$. If Ω is unbounded then (8.7) holds. $\square$

9. Non-homogeneous system on bounded domains

Suppose that $\Omega \subset \mathbb{R}^m$ is a bounded open set with Lipschitz boundary. If $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^m)$, $p \in \mathcal{C}^1(\overline{\Omega})$ and

$$\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega,$$

$$T_\lambda(\mathbf{u}, p)\mathbf{n} + h\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega$$

and $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$, then the Green formula gives

$$\int_\Omega \mathbf{f} \cdot \mathbf{\Phi} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma = \int_\Omega [2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p(\nabla \cdot \mathbf{\Phi})$$

$$+ \frac{\lambda}{2}(\mathbf{\Phi} \cdot \partial_1 \mathbf{u} - \mathbf{u} \cdot \partial_1 \mathbf{\Phi})] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\mathbf{u} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma.$$

(Compare [37, p. 14].) If we define

$$\langle \mathbf{F}, \mathbf{\Phi} \rangle := \int_\Omega \mathbf{f} \cdot \mathbf{\Phi} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma,$$

then

$$\int_\Omega [2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p(\nabla \cdot \mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi} \cdot \partial_1 \mathbf{u} - \mathbf{u} \cdot \partial_1 \mathbf{\Phi})] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\mathbf{u} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma = \langle \mathbf{F}, \mathbf{\Phi} \rangle.$$

We can formally write that

$$[\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u}]|_\Omega + [T_\lambda(\mathbf{u}, p)\mathbf{n} + h\mathbf{u}]|_{\partial\Omega} = \mathbf{F}. \tag{9.1}$$

We can interpret (9.1) and $\nabla \cdot \mathbf{u} = 0$ as

$$\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = \mathbf{F}|_\Omega \quad \text{in} \quad \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{9.2a}$$

$$T_\lambda(\mathbf{u}, p)\mathbf{n} + h\mathbf{u} = \mathbf{F}|_{\partial\Omega} \qquad \text{on } \partial\Omega. \tag{9.2b}$$

This motivates the following definition of a weak solution of (9.2):

Let $1 < q < \infty$, $q' = q/(q-1)$, $\mathbf{F} \in [W^{1,q'}(\Omega, \mathbb{R}^m)]'$. *We say that* $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ *is a weak solution of the problem* (9.2) *(or equivalently of the problem* (9.1)*) if* $\nabla \cdot \mathbf{u} = 0$ *and*

$$\int_\Omega [2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p(\nabla \cdot \mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi} \cdot \partial_1 \mathbf{u} - \mathbf{u} \cdot \partial_1 \mathbf{\Phi})] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\gamma_\Omega \mathbf{u} \cdot \gamma_\Omega \mathbf{\Phi} \, \mathrm{d}\sigma = \langle \mathbf{F}, \mathbf{\Phi} \rangle$$

for all $\mathbf{\Phi} \in W^{1,q'}(\Omega, \mathbb{R}^m)$ *(equivalently for all* $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$*).*

Since $\mathcal{C}^2(\overline{\Omega}; \mathbb{R}^m) \times \mathcal{C}^1(\overline{\Omega})$ is a dense subset of $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$, every definition of the Robin problem for the Oseen system in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ agreeing to classical solutions must be equivalent with our definition.

If $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ is a weak solution of the problem (9.2) and $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^m)$ then the Green formula gives

$$\langle \mathbf{F}, \mathbf{\Phi} \rangle = -\int_\Omega [\mathbf{u} \cdot (\Delta \mathbf{\Phi} + \lambda \partial_1 \mathbf{\Phi}) + p \nabla \cdot \mathbf{\Phi}] \, \mathrm{d}\mathbf{x},$$

i.e. $\nabla p - \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} = \mathbf{F}$ in Ω in the sense of distributions.

Theorem 9.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, and $m = 2$ or $m = 3$. Suppose that $\lambda \in \mathbb{R}^1 \setminus \{0\}$, $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Assume that $3/2 < q < 3$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Denote $q' = q/(q-1)$. If $\mathbf{F} \in [W^{1,q'}(\Omega, \mathbb{R}^m)]'$ then there exists a unique weak solution $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of the Robin problem (9.2). Moreover,*

$$\|\mathbf{u}\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \leq C\|\mathbf{F}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}$$

where C does not depend on $\mathbf{F}$.

Proof. The uniqueness of a solution of the problem follows from Theorem 8.3.

Denote $X := \{(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega); \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}$. Define a mapping $Z : X \to [W^{1,q}(\Omega, \mathbb{R}^m)]'$ by

$$\langle Z(\mathbf{u}, p), \mathbf{\Phi}\rangle := \int_\Omega [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u} - \mathbf{u}\cdot\partial_1\mathbf{\Phi})]\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\gamma_\Omega\mathbf{u}\cdot\gamma_\Omega\mathbf{\Phi}\,\mathrm{d}\sigma.$$

Then Z is continuous.

Let now $\mathbf{F} \in [W^{1,q'}(\Omega, \mathbb{R}^m)]'$. For $\mathbf{\Phi} \in W^{1,q'}(\mathbb{R}^m, \mathbb{R}^m)$ we denote $\langle\tilde{\mathbf{F}}, \mathbf{\Phi}\rangle := \langle\mathbf{F}, \mathbf{\Phi}|_\Omega\rangle$. Then $\tilde{\mathbf{F}} \in [W^{1,q'}(\mathbb{R}^m, \mathbb{R}^m)]' = W^{-1,q}(\mathbb{R}^m, \mathbb{R}^m)$ has compact support. Lemma 5.1 gives that $(\breve{O}^\lambda * \tilde{\mathbf{F}}, Q * \tilde{\mathbf{F}}) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$. Properties of the fundamental solution give that $\nabla\cdot(\breve{O}^\lambda * \tilde{\mathbf{F}}) = 0$, $\nabla(Q * \tilde{\mathbf{F}}) - \Delta(\breve{O}^\lambda * \tilde{\mathbf{F}}) + \lambda\partial_1(\breve{O}^\lambda * \tilde{\mathbf{F}}) = \tilde{\mathbf{F}}$. Denote $\mathbf{G} := Z(\breve{O}^\lambda * \tilde{\mathbf{F}}, Q * \tilde{\mathbf{F}})$. Then $(\breve{O}^\lambda * \tilde{\mathbf{F}}, Q * \tilde{\mathbf{F}})$ is a weak solution of the problem

$$\nabla\tilde{p} - \Delta\tilde{\mathbf{u}} + \lambda\partial_1\tilde{\mathbf{u}} = \mathbf{G}|_\Omega, \quad \nabla\cdot\tilde{\mathbf{u}} = 0 \qquad \text{in } \Omega,$$
$$T_\lambda(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}^\Omega + h\mathbf{u} = \mathbf{G}|_{\partial\Omega} \quad \text{on } \partial\Omega.$$

So, $\mathbf{F} - \mathbf{G} \in [W^{1,q'}(\Omega, \mathbb{R}^m)]'$, $\mathbf{G} = \mathbf{F}$ in Ω. Lemma 12.1 gives that $\mathbf{F} - \mathbf{G} \in W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$. According to Theorem 8.3 there exists a weak solution $(\hat{\mathbf{u}}, \hat{p}) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of the problem

$$\nabla\hat{p} - \Delta\hat{\mathbf{u}} + \lambda\partial_1\hat{\mathbf{u}} = 0, \quad \nabla\cdot\hat{\mathbf{u}} = 0 \qquad \text{in } \Omega,$$
$$T_\lambda(\hat{\mathbf{u}}, \hat{p})\mathbf{n}^\Omega + h\mathbf{u} = \mathbf{F} - \mathbf{G} \quad \text{on } \partial\Omega.$$

Thus $(\mathbf{u}, p) := (\breve{O}^\lambda * \tilde{\mathbf{F}}, Q * \tilde{\mathbf{F}}) + (\hat{\mathbf{u}}, \hat{p}) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ is a weak solution of the Robin problem (9.2).

Z is an injective continuous linear mapping X onto $[W^{1,q'}(\Omega, \mathbb{R}^m)]'$. So, it is an isomorphism. $\square$

10. Function spaces on unbounded domains

It is well known that it is not reasonable to study boundary value problems for unbounded domains in Sobolev spaces. So, boundary value problems are studied in homogeneous Sobolev spaces (see for example [13], [16], [17], [18], [19],[20]) or in weighted Sobolev spaces (see for example [3], [4], [5], [15]).

Let $\Omega \subset \mathbb{R}^m$ be a domain (i.e. an open connected set). For $1 < q < \infty$ we define the homogeneous Sobolev space $D^{k,q}(\Omega) := \{u \in L^1_{\mathrm{loc}}(\Omega); \partial^\beta u \in L^q(\Omega)\ \forall|\beta| = k\}$. Then $D^{k,q}(\Omega) \subset W^{k,q}_{\mathrm{loc}}(\Omega)$. Fix a bounded open set G such that $\overline{G} \subset \Omega$. Then $D^{k,q}(\Omega)$ is a Banach space with the norm

$$\|u\|_{D^{k,q}(\Omega),G} := \|u\|_{L^q(G)} + \|\ |\nabla^k u|\ \|_{L^q(\Omega)}.$$

Moreover, different choices of G give equivalent norms. (See [25, §1.5.3, Corollary 2].) See more about homogeneous Sobolev spaces in the books [17] and [25].

Members of $[D^{1,q}(\Omega)]'$ are not distributions in general. That is a reason why we define some smaller space. Denote by $\tilde W^{1,q}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\mathbb{R}^m)$ in $D^{1,q}(\Omega)$. Clearly, $[\tilde W^{1,q}(\Omega)]'$ is formed by distributions.

Lemma 10.1. *Suppose that $\Omega = \mathbb{R}^m$ or Ω is a domain with compact Lipschitz boundary, and $1 < q < \infty$. Then $D^{1,q}(\Omega) \subset W^{1,q}_{loc}(\overline{\Omega})$. If Ω is bounded then $D^{1,q}(\Omega) = W^{1,q}(\Omega)$ and the corresponding norms are equivalent.*

(See [25, §1.5.2–§1.5.4] or [28, Proposition 1.25.2].)

Lemma 10.2. *Let $\Omega \subset \mathbb{R}^m$ be a domain such that $\overline{\Omega} \neq \mathbb{R}^m$. Let $1 < q < \infty$. Denote by $\mathring{D}^{1,q}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $D^{1,q}(\Omega)$. Then $\|\nabla\varphi\|_{L^q(\Omega)}$ is an equivalent norm in $\mathring{D}^{1,q}(\Omega)$.*

(See [28, Lemma 1.25.4].)

Lemma 10.3. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact Lipchitz boundary and $1 < q < \infty$. Denote by $P_0(\mathbb{R}^m)$ the space of constant functions on $\mathbb{R}^m$.*

- *If $q \geq m$ then $\tilde W^{1,q}(\Omega) = D^{1,q}(\Omega)$.*
- *If $q < m$ then $D^{1,q}(\Omega) = \tilde W^{1,q}(\Omega) \oplus P_0(\mathbb{R}^m)$, and $\tilde W^{1,q}(\Omega)$ is formed by $u \in D^{1,q}(\Omega)$ such that*

$$\lim_{r\to\infty} \int_{\partial B(0;1)} |u(r\mathbf{x})| \, \mathrm{d}\sigma(\mathbf{x}) = 0. \tag{10.1}$$

If we denote

$$\|u\|_{\tilde W^{1,q}(\Omega)} := \|\nabla u\|_{L^q(\Omega)},$$

then $\|u\|_{\tilde W^{1,q}(\Omega)}$ is a norm on $\tilde W^{1,q}(\Omega)$ equivalent to the norm induced from $D^{1,q}(\Omega)$.

Proof. If $q \geq m$ then $\tilde W^{1,q}(\Omega) = D^{1,q}(\Omega)$ by [28, Lemma 3.38.11].

Let $q < m$. Then $D^{1,q}(\Omega) = \tilde W^{1,q}(\Omega) \oplus P_0(\mathbb{R}^m)$ by [28, Lemma 3.38.11]. So, $\tilde W^{1,q}(\Omega)$ is isomorphic with the factor space $D^{1,q}(\Omega)/P_0(\mathbb{R}^m)$. Since $\|\nabla u\|_{L^q(\Omega)}$ is an equivalent norm on $D^{1,q}(\Omega)/P_0(\mathbb{R}^m)$ by [28, Lemma 1.25.5], it is an equivalent norm on $\tilde W^{1,q}(\Omega)$. Fix $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ such that $\varphi = 1$ on a neighborhood of $\mathbb{R}^m \setminus \Omega$. For $u \in D^{1,q}(\Omega)$ define $v_u = (1-\varphi)u$ in Ω, $v_u = 0$ elsewhere. Then $u \in \tilde W^{1,q}(\Omega)$ if and only if $v_u \in \mathring{D}^{1,q}(\Omega)$. [28, Proposition 3.38.5] gives that $u \in \tilde W^{1,q}(\Omega)$ if and only if (10.1) holds. $\square$

11. Non-homogeneous system on unbounded domains

Suppose that $\Omega \subset \mathbb{R}^m$ is an unbounded domain with compact Lipschitz boundary. Let $F_1, \dots, F_m$ be distributions in $\mathbb{R}^m$, $\mathbf{F} = (F_1, \dots, F_m)$, $\mathbf{u} \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^m)$ and $p \in L^q_{\mathrm{loc}}(\overline{\Omega})$ with $1 < q < \infty$. *We say that $(\mathbf{u}, p)$ is a weak solution of the problem (9.2) if $\nabla\cdot\mathbf{u} = 0$ and*

$$\int_\Omega [2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi}\cdot\partial_1\mathbf{u} - \mathbf{u}\cdot\partial_1\mathbf{\Phi})] \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\gamma_\Omega\mathbf{u}\cdot\gamma_\Omega\mathbf{\Phi} \,\mathrm{d}\sigma = \langle\mathbf{F},\mathbf{\Phi}\rangle$$

for all $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$.

Theorem 11.1. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact Lipschitz boundary, and $m = 2$ or $m = 3$. Suppose that $\lambda \in \mathbb{R}^1 \setminus \{0\}$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Let $q \in (2, 3)$ for $m = 2$, and $q \in (3/2, \infty)$ for $m = 3$. Suppose that $3/2 < q < 3$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Put $q' = q/(q-1)$. Fix $\mathbf{F} \in [\tilde{W}^{1,q'}(\Omega, \mathbb{R}^m)]'$. Then there exists a weak solution $(\mathbf{u}, p) \in \tilde{W}^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of the Robin problem (9.2).*

- *For $k = 1, \ldots, m$ denote $\mathbf{b}^k = (\delta_{1k}, \ldots, \delta_{mk})$. Then there exists a unique weak solution $(\mathbf{w}^k, \tau^k) \in D^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of (1.1), (3.1) with $\mathbf{g} \equiv 0$ such that $\mathbf{w}^k(\mathbf{x}) \to \mathbf{b}^k$ as $|\mathbf{x}| \to \infty$. Moreover, $(\mathbf{w}^1, \tau^1), \ldots, (\mathbf{w}^m, \tau^m)$ are linearly independent, and the general form of a weak solution $(\mathbf{w}, \tau)$ of the Robin problem (9.2) in $D^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ is*

$$\mathbf{w} = \mathbf{u} + \sum_{k=1}^{m} c_k \mathbf{w}^k, \quad \tau = p + \sum_{k=1}^{m} c_k \tau^k, \quad c_j \in \mathbb{R}^1. \tag{11.1}$$

  (Remember that $D^{1,q}(\Omega, \mathbb{R}^m) = \tilde{W}^{1,q}(\Omega, \mathbb{R}^m)$ for $q \geq m$.)
- *If $m = 3$ and $3/2 < q < 3$ (i.e. if $q < m$) then $(\mathbf{u}, p)$ is a unique solution of (9.2) in $\tilde{W}^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$.*

Proof. First we show that there exists a solution $(\mathbf{v}, \rho) \in \tilde{W}^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of (9.2a). Fix $\mathbf{z} \in \mathbb{R}^m$ and $r \in (0, \infty)$ such that $B(\mathbf{z}; 3r) \cap \Omega = \emptyset$. Put $\omega := \mathbb{R}^m \setminus \overline{B(\mathbf{z}; r)}$. For $\mathbf{w} \in \mathring{D}^{1,q'}(\Omega; \mathbb{R}^m)$ define $E\mathbf{w} = \mathbf{w}$ in Ω, $E\mathbf{w} = 0$ elsewhere. Then $X := \{E\mathbf{w}; \mathbf{w} \in \mathring{D}^{1,q'}(\Omega; \mathbb{R}^m)\}$ is a closed subspace of $\mathring{D}^{1,q'}(\omega; \mathbb{R}^m)$. Define $\tilde{\mathbf{F}}(E\mathbf{w}) := \mathbf{F}(\mathbf{w})$ for $\mathbf{w} \in \mathring{D}^{1,q'}(\Omega, \mathbb{R}^m)$. Then $\tilde{\mathbf{F}}$ is a bounded linear operator on X. Remark that $\tilde{\mathbf{F}} = \mathbf{F}$ in Ω in the sense of distributions. According to Hahn-Banach theorem there exists $\mathcal{F} \in [\mathring{D}^{1,q'}(\omega, \mathbb{R}^m)]'$ such that $\mathcal{F} = \tilde{\mathbf{F}}$ on X. Lemma 10.2, [17, Theorem VII.7.2 and Remark VII.7.3] give that there exists $(\mathbf{w}, \rho) \in D^{1,q}(\mathbb{R}^m, \mathbb{R}^m) \times L^q(\mathbb{R}^m)$ such that $-\Delta \mathbf{w} + \lambda \partial_1 \mathbf{w} + \nabla \rho = \mathcal{F}$, $\nabla \cdot \mathbf{w} = 0$ in $\mathbb{R}^m$. According to Lemma 10.3 there exist $\mathbf{c} \in \mathbb{R}^m$ and $\mathbf{v} \in \tilde{W}^{1,q}(\Omega, \mathbb{R}^m)$ such that $\mathbf{w} = \mathbf{v} + \mathbf{c}$. Clearly, $(\mathbf{v}, \rho)$ is a solution of (9.2a).

For $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$ define

$$\langle \mathbf{G}, \mathbf{\Phi} \rangle := \int_\Omega [2\hat{\nabla}\mathbf{v} \cdot \hat{\nabla}\mathbf{\Phi} - \rho(\nabla \cdot \mathbf{\Phi}) + \frac{\lambda}{2}(\mathbf{\Phi} \cdot \partial_1 \mathbf{v} - \mathbf{v} \cdot \partial_1 \mathbf{\Phi})] \, d\mathbf{x} + \int_{\partial\Omega} h\mathbf{v} \cdot \mathbf{\Phi} \, d\sigma.$$

Then $\mathbf{G} = \mathbf{F}$ in Ω. Choose $r \in (0, \infty)$ such that $\partial\Omega \subset B(0; r)$. Then $\mathbf{g} := \mathbf{G} - \mathbf{F} \in [W^{1,q'}(\Omega \cap B(0; r); \mathbb{R}^m)]'$. Since $\mathbf{g}$ is supported on $\partial\Omega$, Lemma 12.1 gives that $\mathbf{g} \in W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$. Put $\mathbf{f} := [\tau^\lambda_{h,\Omega}]^{-1}\mathbf{g}$, $\tilde{\mathbf{u}} := \tilde{O}^\lambda_\Omega \mathbf{f}$, $\tilde{p} := \tilde{Q}^\lambda_\Omega \mathbf{f}$. Theorem 8.3 gives that $(\tilde{\mathbf{u}}, \tilde{p}) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ is a weak solution of (1.1), (3.1), (8.7). Moreover, $(\tilde{\mathbf{u}}, \tilde{p}) \in \tilde{W}^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ by (4.1), (4.3) and Lemma 10.3. Thus $(\mathbf{u}, p) := (\mathbf{v} + \tilde{\mathbf{u}}, \rho + \tilde{p}) \in \tilde{W}^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ is a weak solution of the problem (9.2).

Put $\mathbf{f}^k := [\tau^\lambda_{h,\Omega}]^{-1}(h + n_1/2)\mathbf{b}^k$, $\mathbf{w}^k := \mathbf{b}^k - \tilde{O}^\lambda_\Omega \mathbf{f}^k$, $\tau^k := -\tilde{Q}^\lambda_\Omega \mathbf{f}^k$. Theorem 8.3 gives that $(\mathbf{w}^k, \tau^k) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^m) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ is a unique weak solution of (1.1), (3.1) with $\mathbf{g} \equiv 0$ such that $\mathbf{w}^k(\mathbf{x}) \to \mathbf{b}^k$ as $|\mathbf{x}| \to \infty$. Moreover, $(\mathbf{w}^k, \tau^k) \in D^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ by (4.1) and (4.3). Since $\mathbf{b}^1, \ldots, \mathbf{b}^m$ are linearly independent, we deduce that $(\mathbf{w}^1, \tau^1), \ldots, (\mathbf{w}^m, \tau^m)$ are linearly independent. If $(\mathbf{w}, \tau)$ is given by (11.1), then $(\mathbf{w}, \tau)$ is a weak solution of the Robin problem (9.2) in $D^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\partial\Omega)$.

Let $(\mathbf{w}, \tau)$ be a weak solution of the Robin problem (9.2) in $D^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$. Then $(\mathbf{w} - \mathbf{u}, \tau - p)$ is a weak solution of (1.1), (3.1) with $\mathbf{g} \equiv 0$ in $D^{1,q}(\Omega, \mathbb{R}^m) \times$

$L^q(\Omega)$. According to Proposition 7.2 there exist $\tau_\infty \in \mathbb{R}^1$ and $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$ such that $(\mathbf{w}(\mathbf{x}) - \mathbf{u}(\mathbf{x})) \to \mathbf{c}$, $(\tau(\mathbf{x}) - p(\mathbf{x})) \to \tau_\infty$ as $|\mathbf{x}| \to \infty$. Since $(\tau - p) \in L^q(\Omega)$ we deduce that $\tau_\infty = 0$. Put

$$\tilde{\mathbf{w}} := \mathbf{w} - \mathbf{u} - \sum_{k=1}^m c_k \mathbf{w}^k, \quad \tilde{\tau} := \tau - p - \sum_{k=1}^m c_k \tau^k.$$

Then $(\tilde{\mathbf{w}}, \tilde{\tau})$ is a weak solution of (1.1), (3.1), (8.7) with $\mathbf{g} \equiv 0$ in $D^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$. Theorem 8.3 gives that $(\tilde{\mathbf{w}}, \tilde{\tau}) \equiv 0$. So, $(\mathbf{w}, \tau)$ is given by (11.1).

Let now $m = 3$ and $3/2 < q < 3$ (i.e. if $q < m$). Suppose that $(\mathbf{w}, \tau)$ is a weak solution of the Robin problem (9.2) in $\tilde{W}^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$. Then there exist $c_1, \dots, c_m$ such that (11.1) holds. Since $\mathbf{u}, \mathbf{w} \in \tilde{W}^{1,q}(\Omega, \mathbb{R}^m)$, Lemma 10.3 gives

$$0 = \lim_{r \to \infty} \int_{\partial B(0;1)} |\mathbf{u}(r\mathbf{x})| \, \mathrm{d}\sigma(\mathbf{x}) = |\mathbf{c}|.$$

Thus $c_1 = \dots = c_m = 0$ and $\mathbf{w} = \mathbf{u}$, $\tau = p$. $\square$

12. Appendix

12.1. **Function spaces.**

Lemma 12.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $1 < q < \infty$, $q' = q/(q-1)$. Then $\{f \in [W^{1,q'}(\Omega)]'; spt\ f \subset \partial\Omega\} = W^{-1/q,q}(\partial\Omega)$.*

Proof. $\gamma_\Omega : W^{1,q'}(\Omega) \to W^{1/q,q'}(\partial\Omega)$ is bounded. On the other hand, there exists a bounded extension operator $E : W^{1/q,q'}(\partial\Omega) \to W^{1,q'}(\Omega)$ by [23, Theorem 6.9.2]. Hence $\{f \in [W^{1,q'}(\Omega)]'; spt\ f \subset \partial\Omega\} = [W^{1/q,q'}(\partial\Omega)]' = W^{-1/q,q}(\partial\Omega)$. $\square$

12.2. **Non-tangential limit.** Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. If $\mathbf{x} \in \partial\Omega$, $a > 0$ denote the non-tangential approach regions of opening a at the point $\mathbf{x}$ by

$$\Gamma_a(\mathbf{x}) := \{\mathbf{y} \in \Omega; |\mathbf{x} - \mathbf{y}| < (1 + a) \operatorname{dist}(\mathbf{y}, \partial\Omega)\}.$$

If v is a function defined in Ω then

$$v_\Omega(\mathbf{x}) := \lim_{\Gamma_a(\mathbf{x}) \ni \mathbf{y} \to \mathbf{x}} v(\mathbf{y})$$

is the non-tangential limit of $\mathbf{v}$ at $\mathbf{x}$. We denote the non-tangential maximal function of v on $\partial\Omega$ by

$$M_a(v)(\mathbf{x}) = \sup\{|v(\mathbf{y})|; \mathbf{y} \in \Gamma_a(\mathbf{x})\}.$$

Lemma 12.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, and $1 < q < \infty$. Let $u \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$. If there exists the non-tangential limit of u at almost all points of $\partial\Omega$, then the non-tangential limit of u is equal to the trace of u.*

(See [9, Corollary 5.7].)

Lemma 12.3. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \in \mathbb{R}^1$ and $1 < q < \infty$. If $\mathbf{f} \in L^q(\partial\Omega; \mathbb{R}^m)$ then $\mathcal{O}^\lambda_\Omega \mathbf{f}(\mathbf{x})$ is the non-tangential limit of $O^\lambda_\Omega \mathbf{f}$ at $\mathbf{x}$ for almost all $\mathbf{x} \in \partial\Omega$ and*

$$\|M_a(O^\lambda_\Omega \mathbf{f})\|_{L^q(\partial\Omega)} \le C \|\mathbf{f}\|_{L^q(\partial\Omega)}.$$

Proof. It is a consequence of [27, Proposition 1] and (4.4), (4.5). $\square$

Lemma 12.4. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \in \mathbb{R}^1$ and $1 < q < \infty$. Then there exists a constant C such that if $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^m)$ then*

$$\|M_a(Q_\Omega \mathbf{f})\|_{L^q(\partial\Omega)} + \|M_a(\nabla O_\Omega^\lambda \mathbf{f})\|_{L^q(\partial\Omega)} + \|M_a(D_\Omega^\lambda \mathbf{f})\|_{L^q(\partial\Omega)} \leq C\|\mathbf{f}\|_{L^q(\partial\Omega)},$$

there exist non-tangential limits of $Q_\Omega \mathbf{f}$, $\nabla O_\Omega^\lambda \mathbf{f}$, $D_\Omega^\lambda \mathbf{f}$ at almost all points of $\partial\Omega$ and

$$[T_\lambda(O_\Omega^\lambda \mathbf{f}, Q_\Omega \mathbf{f})\mathbf{n}]_\Omega = \frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f} \quad \text{a.e. on } \partial\Omega,$$

$$[D_\Omega^\lambda \mathbf{f}]_\Omega = \frac{1}{2}\mathbf{f} + K_{\Omega,\lambda}\mathbf{f} \quad \text{a.e. on } \partial\Omega.$$

Proof. For $\lambda = 0$ see [30, Proposition 4.2.3 and Corollary 4.3.2] and [24, Proposition 3.2]. Let now $\lambda \neq 0$. Since there exists a constant c such that $|\nabla(O^\lambda(\mathbf{x}) - O^0(\mathbf{x}))| \leq c|\mathbf{x}|^{3/2-m}$, the lemma is a consequence of [27, Proposition 1] and Lemma 12.3. □

Lemma 12.5. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\lambda \in \mathbb{R}^1$ and $1 < q < \infty$. If $\mathbf{f} \in L^q(\partial\Omega; \mathbb{R}^m)$ then*

$$\int_{\partial\Omega} \mathbf{n}^\Omega \cdot [D_\Omega^\lambda \mathbf{f}]_\Omega \, \mathrm{d}\sigma = \int_{\partial\Omega} \mathbf{n}^\Omega \cdot \left(\frac{1}{2}\mathbf{f} + K_{\Omega,\lambda}\mathbf{f}\right) \mathrm{d}\sigma = 0.$$

Proof. The lemma is a consequence of Lemma 12.4 and [29, Proposition 2.4]. □

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