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Combinatorial homotopy theory for operads

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Abstract. We introduce an explicit combinatorial characterization of the minimal model $\mathcal{O}_\infty$ of the coloured operad $\mathcal{O}$ encoding non-symmetric operads. In our description of $\mathcal{O}_\infty$, the spaces of operations are defined in terms of hypergraph polytopes and the composition structure generalizes the one of the A_∞-operad. As further generalizations of this construction, we present a combinatorial description of the W-construction applied on $\mathcal{O}$, as well as of the minimal model of the coloured operad $\mathcal{C}$ encoding non-symmetric cyclic operads.

Contents

Introduction

Sullivan's classical construction of minimal models of rational homotopy theory has been made available to operad theory by Markl, in his paper [17], together with the subsequent papers of Hinich [13], Spitzweck [25], Vogt [30], Berger-Moerdijk [2], Cisinski-Moerdijk [6] and Robertson [24], in which model structures of various categories of operads have been investigated. In [17, Theorem 3.1], Markl introduced the notion of a minimal model (i.e. resolution) of a monochrome dg operad and he proved that any such operad $\mathcal{P}$, with $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \Bbbk$, with $\Bbbk$ being a field of characteristic zero, admits a minimal model, which is unique up to isomorphism. In [18, Definition 2], Markl generalized the notion of a minimal model to coloured dg operads. We recall his definition below.

Definition 1. Let $\mathcal{P} = (\mathcal{P}, d_{\mathcal{P}})$ be a C-coloured dg operad. A minimal model of $\mathcal{P}$ is a C-coloured dg operad $\mathfrak{M}_{\mathcal{P}} = (\mathcal{T}_C(E), d_{\mathfrak{M}})$, where $\mathcal{T}_C(E)$ is the free C-coloured operad on a C-coloured collection E, together with a map $\alpha_{\mathcal{P}} : \mathfrak{M}_{\mathcal{P}} \to \mathcal{P}$ of dg coloured operads, such that

1. $\alpha_{\mathcal{P}} : \mathfrak{M}_{\mathcal{P}} \to \mathcal{P}$ is a quasi-isomorphism, and
2. $d_{\mathfrak{M}}(E)$ consists of decomposable elements of $\mathcal{T}_C(E)$, i.e. $d_{\mathfrak{M}}(E) \subseteq \mathcal{T}_C(E)^{(\geq 2)}$, where $\mathcal{T}_C(E)^{(\geq 2)} \subseteq \mathcal{T}_C(E)$ is determined by planar trees with at least two vertices.

For Koszul quadratic operads, Markl's notion of minimal model coincides with the cobar construction on the quadratic dual of an operad, given by Ginzburg and Kapranov in [12], and, in particular, provides the structure encoding higher operations of most classical strongly homotopy algebras, such as A_∞-, L_∞- and C_∞-algebras. A detailed description of these algebras can be found in [20, Section 3.10]. In recent applications of homotopy theory of algebras over operads, especially in theoretical physics, an explicit description of the structure maps of minimal models remains essential; see [14] for an up to date review on how higher homotopy structures naturally govern field theories. Such a description is often obtained by a direct calculation of a particular model, which tends to be a rather involved task and calls for new methods and conceptual approaches for understanding the homotopy properties of algebraic structures.

In this paper, we introduce an explicit combinatorial characterization of the minimal model of the coloured operad $\mathcal{O}$ encoding non-symmetric operads, introduced by Van der Laan in his work [29] on extending Koszul duality theory of Ginzburg-Kapranov [12] to coloured operads. The novelty of our characterization is its interpretation in terms of hypergraph polytopes, introduced by Došen and Petrić

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in [9] and further developed by Curien, Ivanović and the author in [7], whose hypergraphs arise in a certain way from rooted trees – we refer to them as operadic polytopes. In particular, each operadic polytope is a truncated simplex displaying the homotopy replacing the associativity relations for the partial composition operations pertinent to the corresponding rooted tree. In this way, our operad structure generalizes the structure of Stasheff's topological A_∞-operad [27]: the family of associahedra corresponds to the suboperad determined by linear rooted trees. We then introduce a combinatorial description of the cubical subdivision of operadic polytopes, obtaining in this way the Boardman-Vogt-Berger-Moerdijk resolution of $\mathcal{O}$, i.e. its W-construction, introduced in [4]. Finally, by modifying the underlying formalism of trees, we obtain the minimal model of the coloured operad $\mathcal{C}$ encoding non-symmetric cyclic operads, whose algebras yield a notion of strongly homotopy cyclic operads for which the associativity relations for the partial composition operations are coherently relaxed up to homotopy, while the relations involving the action of cyclic permutations are kept strict.

We hope that our explicit construction of operadic polytopes, together with the fact that they admit the structure of an infinity operad, will be of interest in the context of recent developments around Koszulity in operadic categories of Batanin and Markl [1]. From a different, but closely related point of view, we believe that it provides a valuable addition to Ward's recent work [31], proving that the operad encoding modular operads is Koszul and indicating that such a proof can be given in terms of cellular chains on a family of polytopes that generalizes graph associahedra.

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Notation and conventions.

Operads. We work with $\mathbb{N}$-coloured reduced operads in the symmetric monoidal category dgVect of dg vector spaces over a field $\Bbbk$ of characteristic 0. In dgVect, the monoidal structure is given by the classical tensor product $\otimes$, and the switching map $\tau : V\otimes W \to W\otimes V$ is defined by $\tau(v\otimes w) := (-1)^{|v||w|} w\otimes v$, where v and w are homogeneous elements of degrees $|v|$ and $|w|$, respectively. We use the classical Koszul sign convention. We work with homological grading; the differential is a map of degree -1. We denote with $\mathcal{T}_C(K)$ the free C-coloured (symmetric) operad on a C-coloured (symmetric) collection K. A detailed construction of $\mathcal{T}_C(K)$ is given in [4, Section 3]. Our main references for the general theory of operads and related notions are [20] and [15].

Ordinals. We denote with $[n]$ the set $\{1,\dots,n\}$, and with Σ_n the symmetric group on $[n]$.

Trees. A rooted tree $\mathcal{T}$ is a finite connected contractible graph on a non-empty vertex set, together with a distinguished external edge $r(\mathcal{T})$, called the root of $\mathcal{T}$. We shall denote with $v(\mathcal{T})$, $e(\mathcal{T})$ and $l(\mathcal{T})$ the sets of vertices, (internal) edges and external edges (or leaves) of $\mathcal{T}$, respectively. We shall write $E(\mathcal{T})$ for the union $e(\mathcal{T})\cup l(\mathcal{T})$ of all the edges of $\mathcal{T}$. We shall write $i(\mathcal{T})$ for the set $l(\mathcal{T})\backslash\{r(\mathcal{T})\}$, and we shall refer to the elements of $i(\mathcal{T})$ as the inputs (or the input leaves) of $\mathcal{T}$. The set of inputs $i(v)$ and the root $r(v)$ of a vertex $v\in v(\mathcal{T})$ are defined in the standard way through the source and target maps obtained by reading $\mathcal{T}$ from the input leaves to the root. The notation for all these various sets defining a rooted tree will often also be used for their respective cardinalities. We shall denote with $\rho(\mathcal{T})$ the unique vertex of $\mathcal{T}$ whose root is $r(\mathcal{T})$, and we shall refer to it as the root vertex of $\mathcal{T}$. Throughout the paper, *edge* will always mean an *internal edge*.

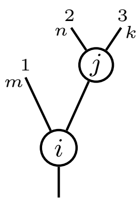
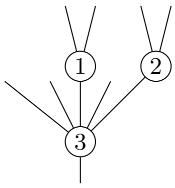
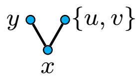
A rooted tree $\mathcal{T}$ is called planar if each vertex of $\mathcal{T}$ comes equipped with an ordering of its inputs. In this case, the inputs of $\mathcal{T}$ admit a canonical labeling from left to right, relative to the planar embedding of $\mathcal{T}$, by 1 through n, for $n = |i(\mathcal{T})|$. Planar rooted trees are isomorphic if there exists an isomorphism of the correspondig graphs that preserves the root and the planar structure. We denote with $\texttt{Tree}(n)$ the set of planar rooted trees with n inputs.

There are two principal constructions on planar rooted trees: grafting and substitution. For trees $\mathcal{T}_1\in\texttt{Tree}(n)$ and $\mathcal{T}_2\in\texttt{Tree}(m)$ and an index $1\le i\le n$, the grafting of $\mathcal{T}_2$ to $\mathcal{T}_1$ along the input i is the tree $\mathcal{T}_1\circ_i\mathcal{T}_2$, obtained by identifying the root of $\mathcal{T}_2$ with the i-th input of $\mathcal{T}_1$. If $v\in V(\mathcal{T}_1)$ is such that $|i(v)| = m$, the substitution of the vertex v of $\mathcal{T}_1$ by $\mathcal{T}_2$ is the tree $\mathcal{T}_1\bullet_v\mathcal{T}_2$, obtained by replacing the vertex v by the tree $\mathcal{T}_2$, identifying the m inputs of v with the m inputs of $\mathcal{T}_2$, using the respective planar structures. The trees $\mathcal{T}_1\circ_i\mathcal{T}_2$ and $\mathcal{T}_1\bullet_v\mathcal{T}_2$ can be rigorously defined either in terms of disjoint unions of sets of vertices, edges and leaves of $\mathcal{T}_1$ and $\mathcal{T}_2$, or by preassuming the appropriate disjointness of sets and taking the ordinary union instead; we take the latter convention. Moreover, we shall assume that all the edges and leaves that need to be identified in these two constructions are a priori the same.

A corolla is a rooted tree with only one vertex. Each planar rooted tree $\mathcal{T}$ is either a planar corolla, or there exist planar rooted trees $\mathcal{T}_1, \dots, \mathcal{T}_p$, a corolla t_n with n inputs, for $n \geq p$, and a monotone injection $(p,n) : [p] \to [n]$, such that $\mathcal{T}$ is obtained by identifying the roots of $\mathcal{T}_i$'s with the inputs $(p,n)(i)$ of t_n. In the latter case, we write $\mathcal{T} = t_n(\mathcal{T}_1, \dots, \mathcal{T}_p)$, implicitly bookkeeping the data of the correspondence (p,n). Note that this recursive definition allows for an inductive reasoning.

A subtree of a planar rooted tree $\mathcal{T}$ is a connected subgraph $\mathcal{S}$ of $\mathcal{T}$ which is itself a planar rooted tree, such that $r(\mathcal{S}) = r(v)$, for some $v \in v(\mathcal{T})$, and such that, if a vertex v of $\mathcal{T}$ is present in $\mathcal{S}$, then all the inputs and the root of v in $\mathcal{T}$ must also be present in $\mathcal{S}$; it is assumed that the source and the target maps of $\mathcal{S}$ are the appropriate restrictions of the ones of $\mathcal{T}$, and that the planar structure of $\mathcal{S}$ is inherited from $\mathcal{T}$. In this way, each subtree of $\mathcal{T}$ is completely determined by a subset of vertices of $\mathcal{T}$, and therefore also by a subset of internal edges of $\mathcal{T}$ (by taking all the vertices adjacent with those edges). We can, therefore, speak about the subtree $\mathcal{T}(X)$ of $\mathcal{T}$ determined by a subset X of vertices (resp. of edges) of $\mathcal{T}$. For an edge $e \in e(\mathcal{T}) \cup \{r(\mathcal{T})\}$, the subtree of $\mathcal{T}$ rooted at e is the subtree of $\mathcal{T}$ determined by all the vertices of $\mathcal{T}$ that are descendants of the vertex v whose root is e, including v itself.

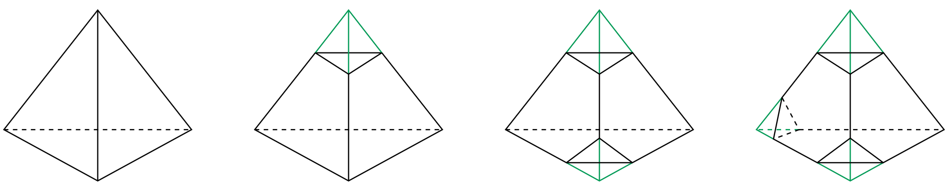
In this paper, we shall work with three different kinds of rooted trees. In order to help the reader navigate between them, in the following table we briefly summarize their characterizations and the corresponding notational conventions.

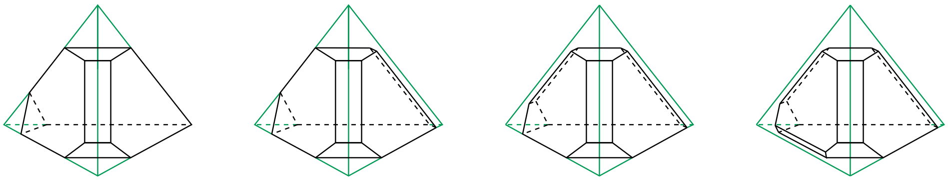
Composite trees	Operadic trees	Constructs
planar rooted trees with $\mathbb{N}$-coloured edges, whose vertices encode the operadic $\circ_i$ operations	planar rooted trees with monochrome edges and totally ordered vertex sets	non-planar trees labeling the faces of hypergraph polytopes
$T =$ 	$\mathcal{T} =$ 	$C =$ 

A planar unrooted tree is a finite connected contractible graph on a non-empty vertex set, each of whose vertices comes equipped with a cyclic ordering of all the adjacent edges. Planar unrooted trees are isomorphic if there exists an isomorphism of the corresponding graphs that preserves the cyclic orderings of the sets of edges adjacent to vertices. By forgetting the data of the root of a planar rooted tree $\mathcal{T}$, one canonically obtains a planar unrooted tree that we shall denote by $\mathcal{U}(\mathcal{T})$.

1. Hypergraph polytopes

A hypergraph polytope is a polytope that may be characterized as a truncated simplex, whereby the truncations are only performed on the faces of the original simplex and not on the faces already obtained as a result of a truncation. In particular, in each dimension, the family of hypergraph polytopes consists of an interval of simple polytopes starting with a simplex and ending with a permutohedron. As an illustration, here is a sequence of truncations of the 3-dimensional simplex that leads to a polytope called *hemiassociaherdon*:





The hemiassociahedron is not as well-known as certain other notable members of the family of hypergraph polytopes, like simplices, hypercubes, associahedra, cyclohedra and permutohedra, but, like all those polytopes, it also has a role in characterizing infinity structures: it displays a particular homotopy of strongly homotopy operads. The hemiassociahedron will be *our favourite polytope* in this article.

The attribute *hypergraph* in the designation *hypergraph polytopes* is meant to indicate the particular style of combinatorial description of the polytopes from this family: the face lattice of each hypergraph polytope can be derived from the data of a hypergraph whose hyperedges encode the truncations of the simplex that lead to the polytope in question. This particular characterization of truncated simplices has been introduced by Došen and Petrić in [9] and further developed by Curien, Ivanović and the author in [7]. Truncated simplices were originally investigated by Feichtner and Sturmfels in [11] and by Postnikov in [22], by means of different – and predating – combinatorial tools: *nested sets* and *tubings*, while, in [23], they first appeared under the name of *nestohedra*.

This section is a recollection on the combinatorial description of the familly of hypergraph polytopes and is entiriely based on [9] and [7]. In particular, we shall consider hypergraph polytopes as abstract polytopes only, disregarding their geometric characterization as a bounded intersection of a finite set of half-spaces, which is also given in the two references. We refer to [21] for the definition of an abstract polytope and related notions.

1.1. **Hypergraph terminology.** A hypergraph is a generalization of a graph for which an edge can relate an arbitrary number of vertices. Formally, a hypergraph $\mathbf{H}$ is given by a set H of *vertices* and a subset $\mathbf{H} \subseteq \mathcal{P}(H)\backslash\emptyset$ of *hyperedges*, such that $\bigcup \mathbf{H} = H$. Note the abuse of notation here: we used the bold letter $\mathbf{H}$ to denote both the hypergraph itself and its set of hyperedges. We justify this identification by requiring all our hypergraphs to be *atomic*, meaning that $\{x\} \in \mathbf{H}$ for all $x \in H$. Additionally, we shall assume that all our hypergraphs are *non-empty*, meaning that $H \neq \emptyset$, *finite*, meaning that H is finite and *connected*, meaning that there are no non-trivial partitions $H = H_1 \cup H_2$, such that $\mathbf{H} = \{X \in \mathbf{H} \,|\, X \subseteq H_1\} \cup \{Y \in \mathbf{H} \,|\, Y \subseteq H_2\}$. There is one more property of hypergraphs that we shall encounter (but not a priori ask for) in the construction of hypergraph polytopes: the property of being *saturated*. We say that a hypergraph $\mathbf{H}$ is saturated when, for every $X, Y \in \mathbf{H}$ such that $X \cap Y \neq \emptyset$, we have that $X \cup Y \in \mathbf{H}$. Every hypergraph can be saturated by adding the missing (unions of) hyperedges. Let us introduce the notation

$$\mathbf{H}_X := \{Z \in \mathbf{H} \,|\, Z \subseteq X\},$$

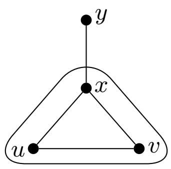
for a hypergraph $\mathbf{H}$ and $X \subseteq H$. The *saturation* of $\mathbf{H}$ is then formally defined as the hypergraph

$$\mathit{Sat}(\mathbf{H}) := \{X \,|\, \emptyset \subsetneq X \subseteq H \text{ and } \mathbf{H}_X \text{ is connected}\}.$$

Example 1. The hypergraph

$$\mathbf{H} = \{\{x\}, \{y\}, \{u\}, \{v\}, \{x, y\}, \{x, u\}, \{x, v\}, \{u, v\}, \{x, u, v\}\}$$

can be represented pictorially as follows:



Here, the hyperedge $\{x, u, v\}$ is represented by the circled-out area aroud the vertices x, u and v. The hypergraph $\mathbf{H}$ is not saturated. The saturation of $\mathbf{H}$ is the hypergraph

$$\mathit{Sat}(\mathbf{H}) = \{\{x\}, \{y\}, \{u\}, \{v\}, \{x, y\}, \{x, u\}, \{x, v\}, \{u, v\}, \{x, y, u\}, \{x, y, v\}, \{x, u, v\}, \{x, y, u, v\}\}.$$

△

We additionally import the following notational conventions and terminology from [7]. For a hypergraph $\mathbf{H}$ and $X \subseteq H$, we set

$$\mathbf{H}\backslash X := \mathbf{H}_{H\backslash X}.$$

Observe that for each (not necessarily connected) finite hypergraph there exists a partition $H = H_1 \cup \ldots \cup H_m$, such that each hypergraph $\mathbf{H}_{H_i}$ is connected and $\mathbf{H} = \bigcup(\mathbf{H}_{H_i})$. The $\mathbf{H}_{H_i}$'s are called the *connected components* of $\mathbf{H}$. We shall write $\mathbf{H}_i$ for $\mathbf{H}_{H_i}$. We shall use the notation

$$\mathbf{H}\backslash X \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_n \qquad (\text{resp. } \mathbf{H}\backslash X \rightsquigarrow \{\mathbf{H}_i \mid 1 \leq i \leq n\})$$

to indicate that $\mathbf{H}_1, \ldots, \mathbf{H}_n$ are the (resp. $\{\mathbf{H}_i \mid 1 \leq i \leq n\}$ is the set of) connected components of $\mathbf{H}\backslash X$.

1.2. **The abstract polytope of a hypergraph.** We next define the abstract polytope

$$\mathcal{A}(\mathbf{H}) = (A(\mathbf{H}) \cup \{\emptyset\}, \leq_{\mathbf{H}})$$

associated to a hypergraph $\mathbf{H}$. We shall recall the representation of $\mathcal{A}(\mathbf{H})$ given in [7], which coincides, up to isomorphism, with the one of [9]. The advantage of the representation of $\mathcal{A}(\mathbf{H})$ given in [7] lies in the *tree notation* for all the faces of hypergraph polytopes that encodes face inclusion as edge contraction – this combinatorial decription reveals the operad structure on a particular subfamily of the family of hypergraph polytopes and was essential for the main purpose of this paper.

The elements of the set $A(\mathbf{H})$, to which we refer as the *constructs* of $\mathbf{H}$, are the non-planar, vertex-decorated rooted trees defined recursively as follows. Let $\emptyset \neq Y \subseteq H$ be a non-empty subset of vertices of $\mathbf{H}$.

- If $Y = H$, then the tree with a single vertex decorated with H and without any inputs, is a construct of $\mathbf{H}$; we denote it by H.
- If $Y \subsetneq H$, if $\mathbf{H}\backslash Y \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_n$, and if $C_1, \ldots, C_n$ are constructs of $\mathbf{H}_1, \ldots, \mathbf{H}_n$, respectively, then the tree whose root vertex is decorated by Y and that has n inputs, on which the respective C_i 's are grafted, is a construct of $\mathbf{H}$; we denote it by $Y\{C_1, \ldots, C_n\}$.

We write $C : \mathbf{H}$ to indicate that C is a construct of $\mathbf{H}$. A *construction* is a construct whose vertices are all decorated with singletons.

The partial order $\leq_{\mathbf{H}}$ on $A(\mathbf{H}) \cup \{\emptyset\}$ is defined by the following three rules.

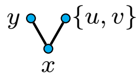
- For all $C : \mathbf{H}$, $\emptyset \leq_{\mathbf{H}} C$.
- If $\mathbf{H}\backslash Y \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_n$, $\mathbf{H}_1\backslash X \rightsquigarrow \mathbf{H}_{11}, \ldots, \mathbf{H}_{1m}$, $C_{1j} : \mathbf{H}_{1j}$ for $1 \leq j \leq m$, and $C_i : \mathbf{H}_i$ for $2 \leq i \leq n$, then
$$Y\{X\{C_{11}, \ldots, C_{1m}\}, C_2, \ldots, C_n\} \leq_{\mathbf{H}} (Y \cup X)\{C_{11}, \ldots, C_{1m}, C_2, \ldots, C_n\}.$$
- If $\mathbf{H}\backslash Y \rightsquigarrow \mathbf{H}_1, \ldots, \mathbf{H}_n$, $C_i : \mathbf{H}_i$ for $2 \leq i \leq n$, and $C_1 \leq_{\mathbf{H}_1} C_1'$, then
$$Y\{C_1, C_2, \ldots, C_n\} \leq_{\mathbf{H}} Y\{C_1', C_2, \ldots, C_n\}.$$

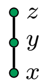
Therefore, given a construct $C : \mathbf{H}$, one can obtain a larger construct by contracting an edge of C and merging the decorations of the vertices adjacent with that edge. Note that the partial order $\leq_{\mathbf{H}}$ is well-defined, in the sense that, if $C_1 : \mathbf{H}$ and if $C_1 \leq_{\mathbf{H}} C_2$ is inferred, then $C_2 : \mathbf{H}$ can be inferred.

The faces of $\mathcal{A}(\mathbf{H})$ are ranked by integers ranging from -1 to $|H|-1$. The face $\emptyset$ is the unique face of rank -1, whereas the rank of a construct $C : \mathbf{H}$ is $|H| - |v(C)|$. In particular, constructions are faces of rank 0, whereas the construct $H : \mathbf{H}$ is the unique face of rank $|H|-1$. We take the usual convention to name the faces of rank 0 *vertices*, the faces of rank 1 *edges* and the faces of rank $|H|-2$ *facets*. The ranks of the faces of $\mathcal{A}(\mathbf{H})$ correspond to their actual dimension when realized in Euclidean space. We refer to [9, Section 9] and [7, Section 3] for a geometric realization of $\mathcal{A}(\mathbf{H})$. In the next section, in conformity with this geometric realization, we shall provide examples of hypergraph polytopes.

The fact that the poset $\mathcal{A}(\mathbf{H})$ is indeed an abstract polytope of rank $|H| - 1$ follows by translating the definition of $\mathcal{A}(\mathbf{H})$, using the order isomorphism [7, Proposition 2], to the formalism of hypergraph polytopes presented in [9], as a consequence of [9, Section 8], where the axioms of abstract polytopes are verified for the latter presentation of $\mathcal{A}(\mathbf{H})$.

Convention 1. In order to facilitate the notation for constructs, we shall represent their singleton vertices without the braces. For example, instead of $\{x\}\{\{y\}, \{u, v\}\}$ and $\{x\}\{\{y\}\{\{z\}\}\}$, we shall write $x\{y, \{u, v\}\}$ and $x\{y\{z\}\}$, respectively. Also, we shall freely confuse the vertices of constructs with the sets decorating them, since they are a fortiori all distinct. In particular, we shall denote the vertices of constructs with capital letters specifying those sets. Finally, in order to provide more intuition for the partial order defined in terms of edge contraction, and later also for the composition of constructs underlying our infinity operad structure, we shall use the graphical representation for constructs indicated in the Introduction. For example,

the constructs $x\{y,\{u,v\}\}$ and $x\{y\{z\}\}$ will be drawn as 

and 

,

respectively. Notice that for constructs we do not draw the root.

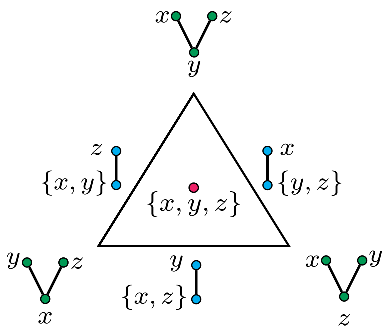
1.3. **Examples.** This section contains examples of various hypergraph polytopes; in [9, Appendix B] and [7, Section 2.4, Section 2.6], the reader can find more of them. Given that our hypergraph vocabulary is now settled, before we give the individual examples, let us first provide the intuition on the very first characterization of hypergraph polytopes that we have mentioned: the geometric description in terms of truncated simplices.

If $\mathbf{H}$ is a hypergraph with the vertex set $H = \{x_1,\ldots,x_{n+1}\}$, then $\mathbf{H}$ encodes the truncation instructions to be applied to the $(|H|-1)$-dimensional simplex, as follows. Start by labeling the facets of the $(|H|-1)$-dimensional simplex by the vertices of $\mathbf{H}$. Then, for each hyperedge $X \in Sat(\mathbf{H})\backslash(\{H\} \cup \{\{x_i\}\,|\,x_i \in H\})$, truncate the face of the simplex defined as the intersection of the facets contained in X. This intuition is formalized as the geometric realization of hypergraph polytopes in [9, Section 8] and [7, Section 3].

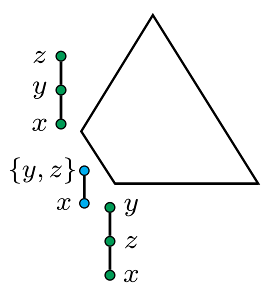
1.3.1. *Simplex.* The hypergraph encoding the n-dimensional simplex is the hypergraph with $n+1$ vertices and no non-trivial hyperedges:

$$\mathbf{S}_{n+1} = \{\{x_1\},\ldots,\{x_{n+1}\},\{x_1,\ldots,x_{n+1}\}\}.$$

In dimension 2, the poset of constructs of the hypergraph $\mathbf{S}_3 = \{\{x\},\{y\},\{z\},\{x,y,z\}\}$ can be realized as a triangle:



Let us now illustrate how truncations arise by adding non-trivial hyperedges to the "bare" simplex hypergraph $\mathbf{S}_3$. Consider the hypergraph $\mathbf{S}_3 \cup \{\{y,z\}\}$. For this hypergraph, the vertex $x\{y,z\}$ is no longer well-defined, since $(\mathbf{S}_3 \cup \{\{y,z\}\})\backslash\{x\}$ no longer contains two connected components, but only one. In the polytope associated to $\mathbf{S}_3 \cup \{\{y,z\}\}$, the vertex $x\{y,z\}$ gets replaced by two new vertices: $x\{y\{z\}\}$ and $x\{z\{y\}\}$, and the edge $x\{\{y,z\}\}$ between them, which can be realized by truncating $x\{y,z\}$ in the above realization of the triangle:

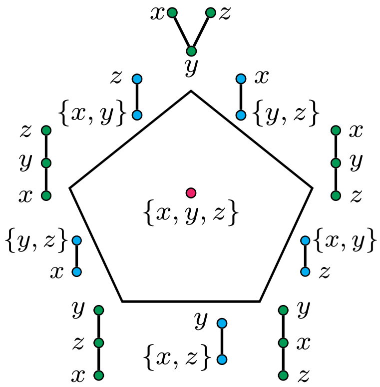


By additionally adding the hyperedge $\{x,y\}$ to $\mathbf{S}_3 \cup \{\{y,z\}\}$, the vertex $z\{x,y\}$ will also be truncated. This leads us to our second example.

1.3.2. *Associahedron.* The hypergraph encoding the n-dimensional associahedron is the linear graph with $n+1$ vertices:

$$\mathbf{A}_{n+1} = \{\{x_1\}, \dots, \{x_{n+1}\}, \{x_1, x_2\}, \dots, \{x_n, x_{n+1}\}\}.$$

In dimension 2, the poset of constructs of the hypergraph $\mathbf{A}_3 = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}\}$ (i.e. of the hypergraph $\mathbf{S}_3 \cup \{\{x, y\}, \{y, z\}\}$) encodes the face lattice of a pentagon as follows:



Starting from the construct representation of the n-dimensional associahedron, one can retrieve Stasheff's original representation in terms of (partial) parenthesisations of a word $a_1 \cdots a_{n+2}$ on $n+2$ letters, or, equivalently, of planar rooted trees with $n+2$ leaves, as follows. The idea is to consider each vertex x_i of $\mathbf{A}_{n+1}$ as the mutiplication of letters a_i and a_{i+1}, as suggested in the following expression:

$$a_1 \cdot_{x_1} a_2 \cdot_{x_2} a_3 \cdot_{x_3} \;\cdots\; \cdot_{x_{n+1}} a_{n+2}.$$

A given construct should then be read from the leaves to the root, interpreting each vertex as an instruction for inserting a pair of parentheses around the group of (possibly already partially parenthesised) letters spanned by all the multiplications determined by the vertex. For example, in dimension 2, and taking

$$a \cdot_x b \cdot_y c \cdot_z d$$

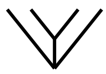
as the layout for building the parentheses, the constructs

$$x\{y\{z\}\}, \quad \{x, y\}\{z\}, \quad \text{and} \quad y\{x, z\}$$

correspond to parenthesised words

$$(a(b(cd))), \quad (ab(cd)), \quad \text{and} \quad ((ab)(cd)),$$

respectively. In the other direction, the construct corresponding to a planar rooted tree with $n+2$ leaves is recovered as follows. First, label the $n+1$ intervals between the leaves of a given tree by $x_1, \dots, x_{n+1}$. Then, considering x_i's as balls, let them fall, and decorate each vertex of the tree by the set of balls which end up falling to that vertex. Finally, remove the input leaves of the starting tree. For example, in dimension 2 again, and writing x, y, z for x_1, x_2, x_3, respectively, the planar rooted trees

 and 

correspond to constructs

$$x\{z\{y\}\} \qquad \text{and} \qquad \{x, z\}\{y\},$$

respectively.

1.3.3. *Permutohedron.* The hypergraph encoding the n-dimensional permutoheron is the complete graph with $n+1$ vertices:

$$\mathbf{P}_{n+1} = \{\{x_1\}, \dots, \{x_{n+1}\} \cup \{\{x_i, x_j\} \mid i, j \in \{1, \dots, n+1\} \text{ and } i \neq j\}.$$

In dimension 2, the hypergraph $\mathbf{P}_3 = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{z, x\}\}$ (i.e. the hypergraph $\mathbf{A}_3 \cup \{\{z, x\}\}$) encodes the following set of constructs:

Rank	Faces
2	$\{x, y, z\}$
1	$\{x, y\}\{z\}$, $\{y, z\}\{x\}$, $\{x, z\}\{y\}$, $x\{\{y, z\}\}$, $z\{\{x, y\}\}$, $y\{\{x, z\}\}$
0	$x\{y\{z\}\}$, $x\{z\{y\}\}$, $y\{x\{z\}\}$, $y\{z\{x\}\}$, $z\{x\{y\}\}$, $z\{y\{x\}\}$
−1	$\emptyset$

The corresponding realization is obtained by truncating the top vertex of the 2-dimensional associahedron from §1.3.2.

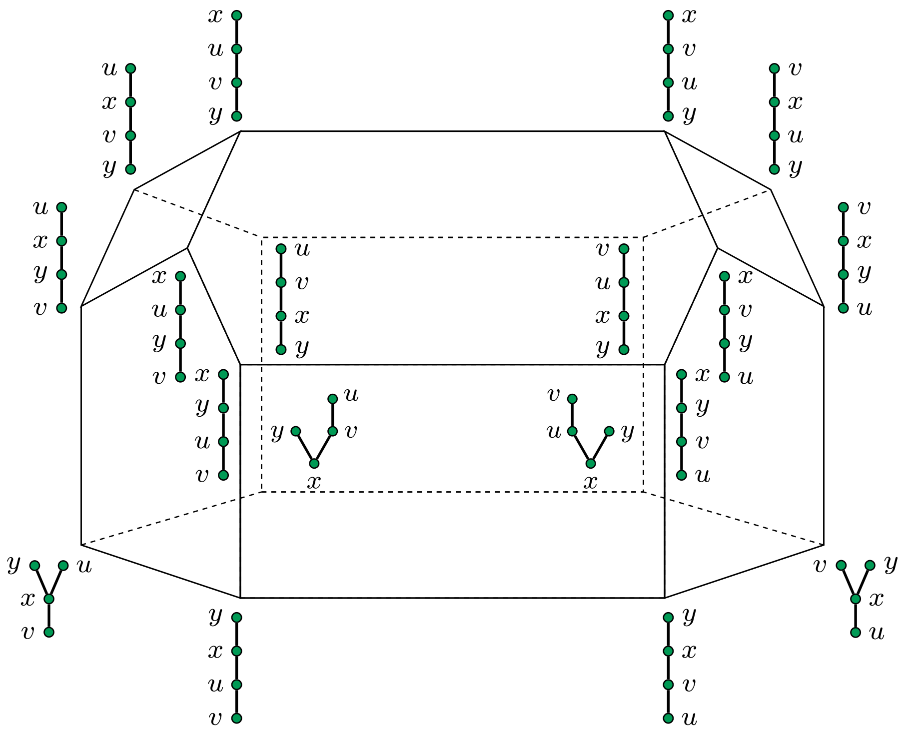
1.3.4. *Hemiassociahedron.* We finish this section with the description of the 3-dimensional hemiassociahedron, whose construction in terms of simplex truncation we illustrated in the introduction to this section. The hypergraph encoding the 3-dimensional hemiassociahedron is the hypergraph $\mathbf{H}$ from Example 1:

$$\mathbf{H} = \{\{x\}, \{y\}, \{u\}, \{v\}, \{x, y\}, \{x, u\}, \{x, v\}, \{u, v\}, \{x, u, v\}\}.$$

As an exercise, the reader may now label the facets of the 3-dimensional simplex in such a way that the sequence of truncations from page 3 can be read in terms of non-trivial hyperedges of $Sat(\mathbf{H})$. The following table, listing the constructs of rank 2 of $\mathbf{H}$, might come in handy:

Rank	Faces	
2	Hexagons	$y\{\{x, u, v\}\}$, $\{y, u, v\}\{x\}$, $\{x, u, v\}\{y\}$
	Pentagons	$\{x, y, v\}\{u\}$, $v\{\{x, y, u\}\}$, $\{x, y, u\}\{v\}$, $u\{\{x, y, v\}\}$
	Squares	$\{x, y\}\{\{u, v\}\}$, $\{u, v\}\{\{x, y\}\}$, $\{y, v\}\{\{x, u\}\}$, $\{y, u\}\{\{x, v\}\}$

together with the following realization, in which we labeled the vertices of the hemiassociahedron:



Notice that we did not specify the hypergraph for the general case of an n-dimensional hemiassociahedron. Indeed, the question of the generalization of the 3-dimensional hemiassociahedron to an arbitrary finite dimension has more than one possible answer. For example, we might define the hypergraph encoding the n-dimensional hemiassociahedron by $\mathbf{H}_{n+1} = \mathbf{P}_n \cup \{\{y\}, \{x_i, y\}\}$, where y is different from all the vertices of the permutohedron hypergraph $\mathbf{P}_n$ and x_i is one of the vertices of $\mathbf{P}_n$, but other possibilities exist as well. In the framework of strongly homotopy structures, an appropriate generalization should be such that the resulting family of hemiassociahedra is closed under the composition product of the structure in question. Finding such a generalization seems like an interesting task.

2. The combinatorial homotopy theory for operads

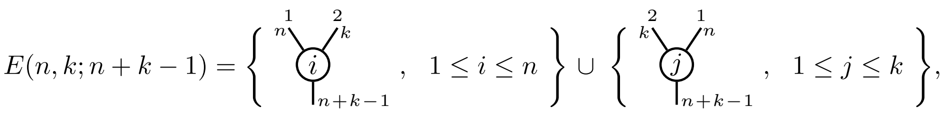
This section contains the combinatorial description of the minimal model $\mathcal{O}_\infty$ of the coloured operad $\mathcal{O}$ encoding non-symmetric non-unital reduced operads. The operad $\mathcal{O}$ is the quadratic operad whose generators and relations presentation is given by the "non-symmetric portion" of [8, Definition 5], where the coloured operad encoding symmetric operads is defined. This definition describes $\mathcal{O}$ in terms of *composite trees* (i.e. binary trees whose vertices encode the $\circ_i$ operations) and grafting. Under the name PsOpd, and by specifying its spaces of operations, the operad $\mathcal{O}$ is defined earlier in [29, Definition 4.1], where it is proven to be self-dual Koszul, and where $\mathcal{O}_\infty$ is subsequently defined as the cobar construction $\Omega\,\mathrm{PsOpd}^{\mathrm{i}}$ of the cooperad $\mathrm{PsOpd}^{\mathrm{i}}$. This alternative characterizatiom describes $\mathcal{O}$ in terms of *operadic trees* (i.e., trees with vertices indexed bijectively by $[k]$) and substitution. The same style of the definition of $\mathcal{O}$ can also be found in [4, Example 1.5.6], where arity 0 and units are additionally allowed.

We start this section by recalling and relating in §2.1 the two equivalent definitions of the operad $\mathcal{O}$. The combinatorial description of the minimal model $\mathcal{O}_\infty$ that we construct in §2.2 will be directly tied to the characterization of $\mathcal{O}$ given by [29, Definition 4.1]. To each operadic tree $\mathcal{T}$ with k vertices, we shall associate in a particular way a hypergraph $\mathbf{H}_\mathcal{T}$ with $k-1$ vertices, in such a way that the faces of the abstract polytope $\mathcal{A}(\mathbf{H}_\mathcal{T})$ become the operations of $\mathcal{O}_\infty$ that replace (or split) $\mathcal{T}$, and that the order relation of $\mathcal{A}(\mathbf{H}_\mathcal{T})$ determines the differential of $\mathcal{O}_\infty$. As further generalizations of our construction, in §2.3, we introduce a cubical subdivision of operadic polytopes, obtaining in this way precisely the combinatorial Boardman-Vogt-Berger-Moerdijk resolution of $\mathcal{O}$, i.e. the W-construction for coloured operads, introduced in [4], applied on $\mathcal{O}$. Finally, in §2.4, by switching from operadic trees to *cyclic operadic trees*, we obtain the combinatorial description of the minimal model of the coloured operad $\mathcal{C}$ encoding non-symmetric non-unital reduced cyclic operads.

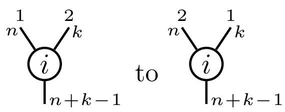
2.1. **Operads as algebras over the colored operad $\mathcal{O}$.** In this section, we recall from [8, Definition 5] and [29, Definition 4.1] the two definitions of the coloured operad $\mathcal{O}$ encoding non-symmetric operads. We relate these two characterizations through a correspondence between the underlying formalisms of composite and operadic trees.

2.1.1. *$\mathcal{O}$ in terms of composite trees.* Below is the "non-symmetric portion" of [8, Definition 5], obtained from [8, Definition 5] by leaving out the generators encoding the action of the symmetric groups. Note that the resulting operad $\mathcal{O}$ (called $\mathcal{O}_{ns}$ in [8]) itself remains a symmetric operad. As a final remark before the definition, we note that $\mathcal{O}$ will incipiently be defined as a coloured operad in the category of sets, and that it is turned into a dg operad by the strong symmetric monoidal functor sending a set X to the direct sum $\bigoplus_{x\in X}\mathbb{k}$.

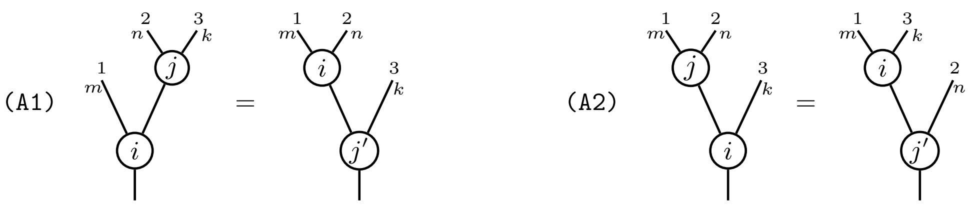
Definition 2. The coloured operad $\mathcal{O}$ is the $\mathbb{N}$-coloured operad defined by $\mathcal{O}=\mathcal{T}_\mathbb{N}(E)/(R)$, where the set of generators E is given by binary operations



for $n,k\geq 1$, equipped with the action of the transposition (21) that sends



, and the set R of relations by equalities



where, in (A1), $j'=j+i-1$, and, in (A2), it is assumed that $i<j$ and $j'=j+k-1$.

Remark 1. $\mathcal{O}$ is a dg operad: for each $k\geq 2$, the vector space $\mathcal{T}_\mathbb{N}(E)(n_1,\ldots,n_k;n)/R(n_1,\ldots,n_k;n)$ is concentrated in degree zero, and the differential is trivial. Therefore, the homology of

$$\mathcal{O}(n_1,\ldots,n_k;n)=\bigoplus_{m\geq 0}(\mathcal{T}_\mathbb{N}(E)(n_1,\ldots,n_k;n)/R(n_1,\ldots,n_k;n))_m$$

is trivial for $m\neq 0$, while, for $m=0$, we have $H_0(\mathcal{O}(n_1,\ldots,n_k;n),0)\cong\mathcal{O}(n_1,\ldots,n_k;n)$.

Let $(End(A), d_{End})$ be the dg endomorphism operad on a dg $\mathbb{N}$-module $A = \{(A(n), d_{A(n)})\}_{n\geq 1}$, i.e. the $\mathbb{N}$-coloured dg operad defined by

$$End(A)(n_1, \ldots, n_k; n) := \mathrm{Hom}(A(n_1) \otimes \cdots \otimes A(n_k); A(n)), \qquad k \geq 1,$$

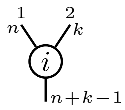
where $\mathrm{Hom}_p(A(n_1) \otimes \cdots \otimes A(n_k); A(n))$ is the vector space of homogeneous degree p linear maps $f : A(n_1) \otimes \cdots \otimes A(n_k) \to A(n)$, with the partial composition operations (resp. the action of the symmetric groups) induced by substitution (resp. permutation respecting the Koszul sign rule) of the tensor factors, and with the differential d_{End} defined on $f \in \mathrm{Hom}_p(A(n_1) \otimes \cdots \otimes A(n_k); A(n))$ by

$$d_{End}(f) := d_{A(n)} \circ f - (-1)^p \sum_{i=1}^{k} f \circ_i d_{A(n_i)}.$$

(Note that when the formula defining d_{End} is applied to elements, additional signs appear due to the Koszul sign rule.)

Lemma 1. *Algebras over $\mathcal{O}$ are (non-unital, non-symmetric, reduced) dg operads.*

Proof. By definition, an $\mathcal{O}$-algebra is a degree 0 homomorphism of $\mathbb{N}$-coloured dg operads $\chi : (\mathcal{O}, 0) \to (End(A), d_{End})$. Therefore, an $\mathcal{O}$-algebra is a dg $\mathbb{N}$-module $(A, d) = \{(A(n), d_{A(n)})\}_{n\geq 1}$ endowed with operations



$$: A(n) \otimes A(k) \to A(n+k-1)$$

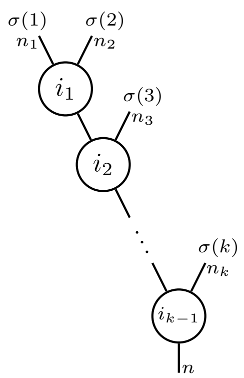
satisfying the obvious associativity axioms, whereby the equality $\chi \circ 0 = d_{End} \circ \chi$ satisfied by χ guarantees that those operations are compatible with d. ■

2.1.2. $\mathcal{O}$ *in terms of operadic trees.* We next recall from [29, Definition 4.1] and [4, Example 1.5.6] the characterization of $\mathcal{O}$ in terms of operadic trees.

Lemma 2. *The vector space $\mathcal{O}(n_1, \ldots, n_k; n)$ is spanned by* operadic trees*, i.e. equivalence classes of pairs $(\mathcal{T}, \sigma)$, where $\mathcal{T} \in \mathtt{Tree}(n)$ has k vertices and $\sigma : [k] \to v(\mathcal{T})$ is a bijection such that the vertex $\sigma(i)$ has n_i inputs, under the equivalence relation defined by:*

$$(\mathcal{T}_1, \sigma_1) \sim (\mathcal{T}_2, \sigma_2) \text{ iff there exists an isomorphim } \varphi : \mathcal{T}_1 \to \mathcal{T}_2, \text{ such that } \varphi \circ \sigma_1 = \sigma_2{}^1.$$

Proof. By [8, Proposition 3], $\mathcal{O}$ is spanned by binary planar $\mathbb{N}$-coloured left combs whose vertex decorations, read from top to bottom, are nondecreasing, together with a labeling of the leaves with a permutation on the number of them. Formally, this basis is the set of normal forms of the confluent and terminating rewriting system obtained by orienting the relations `(A1)` and `(A2)` from left to right. To each left comb

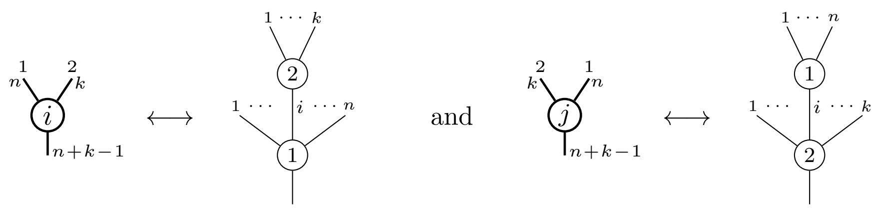
$T =$ 

such that $i_1 \leq \cdots \leq i_{k-1}$, we associate an operadic tree $\omega(T)$, as follows: denote with t_i the planar corolla with n_i inputs, decorated with $\sigma(i)$, and define

$$\omega(T) = (\cdots((t_1 \circ_{i_1} t_2) \circ_{i_2} t_3) \cdots) \circ_{i_{k-1}} t_k,$$

where $\circ_{i_j}$, $1 \leq j \leq k-1$, denotes the grafting operation on rooted trees (that preserves vertex decorations). In particular, the correspondence between the generators of $\mathcal{O}$ and operadic trees with two vertices is given by

[1]In the equality $\varphi \circ \sigma_1 = \sigma_2$, we abuse the notation by writing φ for what is actually the vertex component of φ. We shall continue with this practice whenever specifying compatibilities involving tree isomorphisms.



Notice that the fact that the vertex decorations of T are nondecreasing means that $\omega(T)$ is defined by grafting the corollas in the *left-recursive way*, i.e. from bottom to top and from left to right. This property is used for the definition of the inverse of α: a composite tree is recovered by traversing an operadic tree in the left-recursive manner, as we illustrate in Example 2 that follows. ■

The partial composition operations $\circ_i$ of $\mathcal{O}$ translate to the basis given by Lemma 2 as follows: for $(\mathcal{T}_1,\sigma_1)\in\mathcal{O}(n_1,\dots,n_k;n)$ and $(\mathcal{T}_2,\sigma_2)\in\mathcal{O}(m_1,\dots,m_l;n_i)$, we have

$$(\mathcal{T}_1,\sigma_1)\circ_i(\mathcal{T}_2,\sigma_2)=(\mathcal{T}_1\bullet_i\mathcal{T}_2,\sigma_1\bullet_i\sigma_2),$$

where $\mathcal{T}_1\bullet_i\mathcal{T}_2$ is the planar rooted tree obtained by replacing the vertex $\sigma_1(i)$ (i.e. the vertex indexed by i) of $\mathcal{T}_1$ by the tree $\mathcal{T}_2$, identifying the n_i inputs of $\sigma_1(i)$ in $\mathcal{T}_1$ with the n_i input edges of $\mathcal{T}_2$ using the respective planar structures, and $\sigma_1\bullet_i\sigma_2$ is defined by

$$(\sigma_1\bullet_i\sigma_2)(j)=\begin{cases}\sigma_1(j), & j<i\\ \sigma_2(j-i+1), & i\le j\le i+l-1\\ \sigma_1(j-l+1), & i+l\le j\le k+l-1.\end{cases}$$

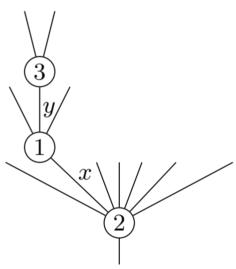
Indeed, it can be shown that

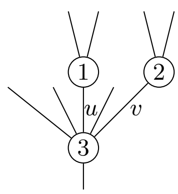
$$(\mathcal{T}_1\bullet_i\mathcal{T}_2,\sigma_1\bullet_i\sigma_2)=\omega(nf(\omega^{-1}(\mathcal{T}_1,\sigma_1)\circ_i\omega^{-1}(\mathcal{T}_2,\sigma_2))),$$

where ω is the bijection from the proof of Lemma 2, and nf is the normal form function of the rewriting system generated by orienting the relations `(A1)` and `(A2)` from left to right. The action of the symmetric group is defined by $(\mathcal{T},\sigma)^\kappa=(\mathcal{T},\sigma\circ\kappa)$.

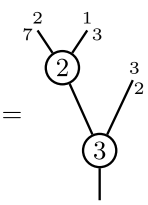
The following example illustrates the correspondence between the partial composition operation of $\mathcal{O}$ in terms of composite trees and grafting, and operadic trees and substitution.

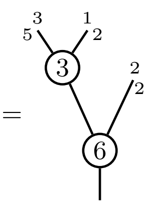
Example 2. For operadic trees

$(\mathcal{T}_1,\sigma_1)=$ 

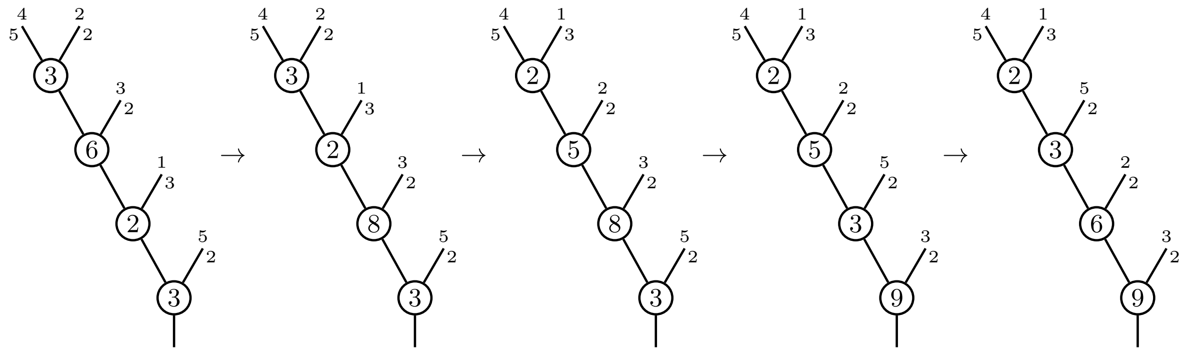
and $(\mathcal{T}_2,\sigma_2)=$ 

we have

$\omega^{-1}(\mathcal{T}_1,\sigma_1)=$ 

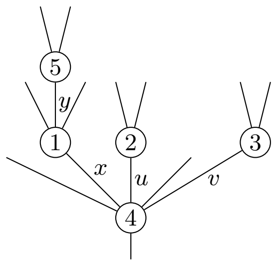
and $\omega^{-1}(\mathcal{T}_2,\sigma_2)=$ 

The normalizing sequence for $\omega^{-1}(\mathcal{T}_1,\sigma_1)\circ_2\omega^{-1}(\mathcal{T}_2,\sigma_2)$ is given by



The operadic tree corresponding to the last composite tree in the sequence is

$$(\mathcal{T},\sigma) =$$



and we indeed have that $(\mathcal{T}_1,\sigma_1)\circ_2(\mathcal{T}_2,\sigma_2) = (\mathcal{T},\sigma)$. △

Observe that, although $\mathcal{O}$ is an operad with the free action of the symmetric group, the relation (A2) contains a non-trivial permutation of the inputs, making it a *non-regular* operad. This means that $\mathcal{O}$ cannot be characterized starting from a non-symmetric operad, by tensoring the space of operations with the regular representation of $\Bbbk[\Sigma_n]$, and by tensoring the partial composition operation with the composition map of the symmetric operad Ass.

Indeed, such a characterization would require that the restriction of the structure of $\mathcal{O}$ to *left-recursive* operadic trees, i.e. operadic trees with a canonical order of vertices that we define below, is closed under the operadic composition of $\mathcal{O}$, which fails to be true.

A left-recursive operadic tree is an operadic tree $(\mathcal{T},\sigma)$, for which $\sigma:[k]\to v(\mathcal{T})$ is the following canonical indexing of the vertices of $\mathcal{T}$:

- if $\mathcal{T}$ is a corolla t_n, then $\sigma:\{1\}\to v(t_n)$ is trivially defined by $\sigma(1)=\rho(t_n)$;
- if $\mathcal{T}=t_m(\mathcal{T}_1,\dots,\mathcal{T}_p)$ and if $\leq_i$ is the linear order on $v(\mathcal{T}_i)$ determined by the left-recursive structure of $\mathcal{T}_i$, then σ is derived from the following linear order on $v(\mathcal{T})$:

$$u\leq v \quad\Longleftrightarrow\quad \begin{cases} u=\rho(t_m)\\ u,v\in v(\mathcal{T}_i) \quad\text{and}\quad u\leq_i v\\ u\in v(\mathcal{T}_i),\, v\in v(\mathcal{T}_j) \quad\text{and}\quad \mathcal{T}_i<\mathcal{T}_j,\end{cases}$$

  where $\mathcal{T}_i<\mathcal{T}_j$ means that $\mathcal{T}_i$ comes before $\mathcal{T}_j$ with respect to the order of inputs of $\rho(t_m)$.

Hence, in a left-recursive operadic tree, the vertices are indexed from bottom to top and from left to right by 1 through k. Observe that this indexing is invariant under planar isomorphisms. In what follows, when refering to a left-recursive operadic tree $(\mathcal{T},\sigma)$, given that σ is canonically determined, we shall write simply $\mathcal{T}$.

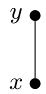
The reader may now want to compose the operadic trees $\mathcal{T}_1$ and $\mathcal{T}_2$ from Example 2, considered as left-recursive trees, to see that the result will not be a left-recursive operadic tree. Nevertheless, note that the composition $(\mathcal{T}_1,\sigma_1)\circ_2(\mathcal{T}_2,\sigma_2)$ from that example can be calculated by the substitution operation on $\mathcal{T}_1$ and $\mathcal{T}_2$ considered as left-recursive operadic trees, followed by the reindexing of the vertices of the resulting (non-left-recursive) tree in a uniquely determined way.

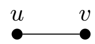
Convention 2. The data of an operadic tree $\mathcal{T}$ involves *non-skeletal* and *skeletal* identifications of its the edges and vertices: the non-skeletal data is given by the *names* of edges and vertices as elements of $e(\mathcal{T})\cup i(\mathcal{T})$ and $v(\mathcal{T})$, respectively, and the skeletal data is the index of an edge (resp. vertex) given by the planar structure (resp. by the left-recursive indexing). We shall freely mix these two ways of specifying edges and vertices and use whatever is more suitable for the purpose at hand. In particular, note that the non-skeletal description of edges eases the portrayal of operadic composition operation, as it bypasses the reindexing involved in the skeletal setting.

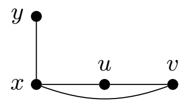
2.2. **The combinatorial $\mathcal{O}_\infty$ operad.** In this section, we define the combinatorial $\mathcal{O}_\infty$ operad as the dg operad defined on the faces of *operadic polytopes*, i.e. hypergraph polytopes whose hypergraphs are the edge-graphs of operadic trees, with the differential determined by the partial order on those faces. We start by formalizing the latter type of hypergraphs.

2.2.1. *The edge-graph of a planar rooted tree.* The *edge-graph* of a planar rooted tree $\mathcal{T}$ is the hypergraph $\mathbf{H}_\mathcal{T}$ defined as follows: the vertices of $\mathbf{H}_\mathcal{T}$ are the (internal) edges of $\mathcal{T}$ and two vertices are connected by an edge in $\mathbf{H}_\mathcal{T}$ whenever, as edges of $\mathcal{T}$, they share a common vertex. Notice that the names (and possible indexing) of the vertices of $\mathcal{T}$, as well as the leaves of $\mathcal{T}$, play no role in the definition of the edge-graph of $\mathcal{T}$.

Example 3. The edge-graphs of operadic trees $(\mathcal{T}_1, \sigma_1)$, $(\mathcal{T}_2, \sigma_2)$ and $(\mathcal{T}, \sigma)$ from Example 2 are

$\mathbf{H}_{\mathcal{T}_1} =$ 

$\mathbf{H}_{\mathcal{T}_2} =$ 

and $\mathbf{H}_{\mathcal{T}} =$ 

respectively. Observe that the edge-graph $\mathbf{H}_\mathcal{T}$ of the tree $\mathcal{T}$ is precisely the hypergraph of the hemiassociahedron (cf. §1.3.4), making $\mathcal{T}$ *our favourite operadic tree.* △

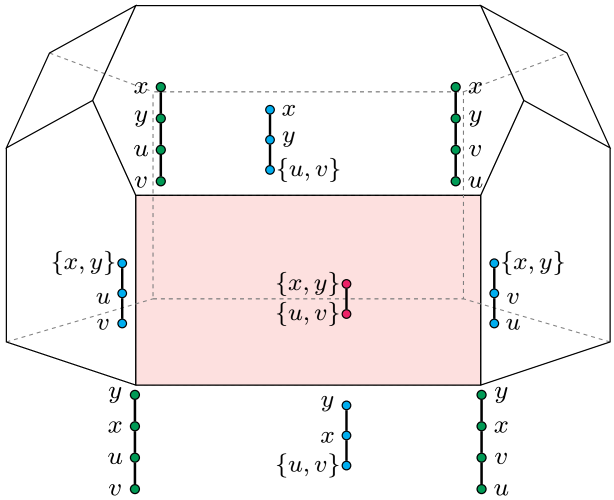
Observe, in Example 3, the additional data given by the relative position of vertices of $\mathbf{H}_{\mathcal{T}_1}$ (one above the other) and $\mathbf{H}_{\mathcal{T}_2}$ (one next to the other). This data is implicitly present in the edge-graphs of planar rooted trees: since edge-graphs inherit their structure from planar rooted trees, their vertices can naturally be arranged in levels, both vertically (from bottom to top) and horizontally (from left to right). This observation is essential for the interpretation of the edges of operadic polytopes in terms of homotopies replacing the relations `(A1)` and `(A2)` defining the operad $\mathcal{O}$. The latter interpretation has been defined in detail in [7, Section 4]. Let us recall here the idea.

Recall from §1.2 that the vertices of a hypergraph polytope are encoded by the constructions of the corresponding hypergraph, i.e. by the constructs whose vertices are decorated by singletons only. In addition, the edges of a hypergraph polytope are encoded by the constructs whose vertices are all singletons, except one, which is a two-element set. Let $\mathcal{T}$ be an operadic tree and let C be a construct encoding an edge of the operadic polytope $\mathbf{H}_\mathcal{T}$; suppose that $\{x, y\}$ is the unique two-element set vertex of C. We show how $\mathbf{H}_\mathcal{T}$, together with its bipartition of vertical and horizontal edges, determines the type of C in terms of homotopies replacing the relations `(A1)` and `(A2)`, as well as the direction of the corresponding edge corresponding to the orientation of `(A1)` and `(A2)` from left to right[2]. In order to state the criterion, we shall use the fact that, among all the paths between two vertices of $\mathbf{H}_\mathcal{T}$, there exists a unique one of minimal length; this fact is proven in [7, Lemma 11]. The criterion is the following:

> If the shortest path between x and y in $\mathbf{H}_\mathcal{T}$ is made of vertical edges only, then the edge encoded by C corresponds to the homotopy replacing the relation `(A1)`, and is oriented towards the vertex encoded by the construction in which the vertex x appears above the vertex y if and only if the vertical level of x is inferior to the vertical level of y in $\mathbf{H}_\mathcal{T}$. Otherwise, the edge encoded by C corresponds to the homotopy replacing the relation `(A2)`, and is oriented towards the vertex encoded by the construction in which the vertex x appears above the vertex y if and only if the horizontal level of x is inferior to the horizontal level of y in $\mathbf{H}_\mathcal{T}$.

Example 4. Let us derive the edge information for the facet of the hemiassociahedron given by the marked square in the realization below:

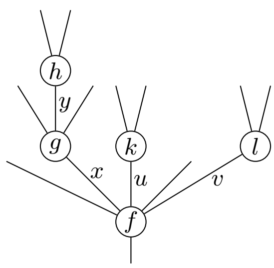
[2]Strictly speaking, in [7], the authors worked in the non-skeletal operadic setting and with the opposite orientation of `(A1)`. In the non-skeletal environment, the colours of $\mathcal{O}$ are arbitrary finite sets and the vertices of composite trees are decorated by the elements of those sets. This in particular means that the non-skeletal variant of the relation `(A2)` does not admit a natural orientation, as opposed to the skeletal one.



According to the criterion, the edges $\{u,v\}\{y\{x\}\}$ and $\{u,v\}\{x\{y\}\}$ encode the homotopies for (A2), whereas the edges $v\{u\{\{x,y\}\}\}$ and $u\{v\{\{x,y\}\}\}$ encode the homotopies for (A1). From the point of view of *categorified operads* [10], corresponding to strongly homotopy operads for which the operations given by operadic trees with more than three vertices vanish, the construct $\{u,v\}\{\{x,y\}\}$, encoding the entire square, is the homotopy identity for the naturality relation

$$\begin{array}{ccc} (((f\circ_x g)\circ_y h)\circ_u k)\circ_v l & \xleftarrow{\texttt{(A2)}} & (((f\circ_x g)\circ_y h)\circ_v l)\circ_u k \\ \uparrow{\scriptstyle \texttt{(A1)}} & & \uparrow{\scriptstyle \texttt{(A1)}} \\ ((f\circ_x(g\circ_y h))\circ_u k)\circ_v l & \xleftarrow{\texttt{(A2)}} & ((f\circ_x(g\circ_y h))\circ_v l)\circ_u k \end{array}$$

pertaining to the operation



△

The following two lemmas are straightforward consequences of the definition of the edge-graph of an operadic tree.

Lemma 3. *The subtrees of an operadic tree $(\mathcal{T},\sigma)$ that have at least two vertices, considered as left-recursive operadic trees, are in a one-to-one correspondence with the connected subsets of $H_{\mathcal{T}}$, i.e. non-empty subsets X of vertices of $\mathbf{H}_{\mathcal{T}}$ such that the hypergraph $(\mathbf{H}_{\mathcal{T}})_X$ is connected.*

Remark 2. Thanks to Lemma 3, for an operadic tree $(\mathcal{T},\sigma)$ and $\emptyset \neq X \subseteq e(\mathcal{T})$, we can index the connected components of $\mathbf{H}_{\mathcal{T}}\backslash X$ by the corresponding left-recursive subtrees of $\mathcal{T}$, by writing

$$\mathbf{H}_{\mathcal{T}}\backslash X \rightsquigarrow \mathbf{H}_{\mathcal{T}_1},\ldots,\mathbf{H}_{\mathcal{T}_n}.$$

However, one must be careful with the induced decomposition on the level of trees! Observe that the subtrees $\mathcal{T}_1,\ldots,\mathcal{T}_n$ of $\mathcal{T}$ do not in general make a decomposition of $\mathcal{T}$, in the sense that the removal of the edges from the set X may result in a number of subtrees of $\mathcal{T}$ reduced to a corolla.

Lemma 4. *Suppose that $(\mathcal{T},\sigma)=(\mathcal{T}_1,\sigma_1)\circ_i(\mathcal{T}_2,\sigma_2)$, and that, for a subset $\emptyset \neq X \subseteq e(\mathcal{T}_1)$ of edges of $\mathcal{T}_1$, we have $\mathbf{H}_{\mathcal{T}_1}\backslash X \rightsquigarrow \mathbf{H}_{(\mathcal{T}_1)_1},\ldots,\mathbf{H}_{(\mathcal{T}_1)_p}$. If there exists an index $1\leq j\leq p$, such that the subtree $(\mathcal{T}_1)_j$ of $\mathcal{T}_1$ contains the vertex v indexed by i in $\mathcal{T}_1$, and if l is the index that the vertex v gets in the left-recursive ordering of the vertices of $(\mathcal{T}_1)_j$, then*

$$\mathbf{H}_{\mathcal{T}_1\bullet_i\mathcal{T}_2}\backslash X \rightsquigarrow \{(\mathbf{H}_1)_k \,|\, 1\leq k\leq p, k\neq j\}\cup\{\mathbf{H}_{(\mathcal{T}_1)_j\bullet_l\mathcal{T}_2}\}.$$

Otherwise, we have that

$$\mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2} \backslash X \rightsquigarrow \{(\mathbf{H}_1)_k \,|\, 1 \leq k \leq p\} \cup \{\mathbf{H}_{\mathcal{T}_2}\}.$$

An isomorphism of planar rooted trees induces an isomorphism on the corresponding hypergraphs and their constructs in the natural way: the form of the hypergraph matters, not the names of the hyperedges. For an isomorphism $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ of planar rooted trees and constructs $C_1 : \mathbf{H}_{\mathcal{T}_1}$ and $C_2 : \mathbf{H}_{\mathcal{T}_2}$, we shall write $C_1 \sim_\varphi C_2$ to denote that C_1 and C_2 are isomorphic via φ. In addition, for a hypergraph $\mathbf{H}$, the polytope $\mathcal{A}(\mathbf{H})$ will be considered modulo renaming of the vertices of $\mathbf{H}$.

2.2.2. $\mathcal{O}_\infty$ *as an operad of vector spaces.* Define, for $k \geq 2$, $n_1, \ldots, n_k \geq 1$, and $n = (\sum_{i=1}^k n_i) - k + 1$, $\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n)$ to be the vector space spanned by triples $(\mathcal{T}, \sigma, C)$, such that $(\mathcal{T}, \sigma)$ is an operadic tree representing an operation of $\mathcal{O}(n_1, \ldots, n_k; n)$ and $C : \mathbf{H}_\mathcal{T}$, subject to the equivalence relation generated by:

> $(\mathcal{T}_1, \sigma_1, C_1) \sim (\mathcal{T}_2, \sigma_2, C_2)$ iff there exists an isomorphism $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$, such that $\varphi \circ \sigma_1 = \sigma_2$ and $C_1 \sim_\varphi C_2$.

Hence, for a fixed operadic tree $(\mathcal{T}, \sigma) \in \mathcal{O}(n_1, \ldots, n_k; n)$, the subspace of $\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n)$ determined by $(\mathcal{T}, \sigma)$ is spanned by all the (isomorphism classes of) constructs of the hypergraph $\mathbf{H}_\mathcal{T}$:

$$\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n) := \mathsf{Span}_\Bbbk \Bigg(\bigoplus_{(\mathcal{T},\sigma) \in \mathcal{O}(n_1, \ldots, n_k; n)} A(\mathbf{H}_\mathcal{T}) \Bigg).$$

Note that for $n \neq (\sum_{i=1}^k n_i) - k + 1$, we set $\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n)$ to be the zero vector space. The $\mathbb{N}$-coloured collection

$$\{\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n) \mid n_1, \ldots, n_k \geq 1\}$$

admits the following operad structure. The composition operation

$$\circ_i : \mathcal{O}_\infty(n_1, \ldots, n_k; n) \otimes \mathcal{O}_\infty(m_1, \ldots, m_l; n_i) \to \mathcal{O}_\infty(n; n_1, \ldots, n_{i-1}, m_1, \ldots, m_l, n_{i+1}, \ldots n_k)$$

is defined by

$$(\mathcal{T}_1, \sigma_1, C_1) \circ_i (\mathcal{T}_2, \sigma_2, C_2) = (\mathcal{T}_1 \bullet_i \mathcal{T}_2, \sigma_1 \bullet_i \sigma_2, C_1 \bullet_i C_2),$$

where the composition on the level of operadic trees is determined by the composition product of the operad $\mathcal{O}$, and the construct $C_1 \bullet_i C_2 : \mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2}$ is defined as follows.

- If $C_1 = H_{\mathcal{T}_1}$, then
$$C_1 \bullet_i C_2 := H_{\mathcal{T}_1}\{C_2\}.$$
- Suppose that $C_1 = X\{C_{11}, \ldots, C_{1p}\}$, where $\mathbf{H}_{\mathcal{T}_1} \backslash X \rightsquigarrow (\mathbf{H}_{\mathcal{T}_1})_1, \ldots, (\mathbf{H}_{\mathcal{T}_1})_p$ and $C_{1q} : (\mathbf{H}_{\mathcal{T}_1})_q$. If there exists an index $1 \leq j \leq p$, such that the subtree $(\mathcal{T}_1)_j$ of $\mathcal{T}_1$ contains the vertex v indexed by i in $\mathcal{T}_1$, we define
$$C_1 \bullet_i C_2 := X\{C_{11}, \ldots, C_{1j} \bullet_l C_2, \ldots, C_{1p}\},$$
where l is the left-recursive index of the vertex v in $\mathcal{T}_1$. Otherwise, we define
$$C_1 \bullet_i C_2 := X\{C_{11}, \ldots, \ldots, C_{1p}, C_2\}.$$

The action of the symmetric group is defined by $(\mathcal{T}, \sigma, C)^\kappa = (\mathcal{T}, \sigma \circ \kappa, C)$.

Lemma 5. *The composition operation of* $\mathcal{O}_\infty$ *is well-defined.*

Proof. We prove that $C_1 \bullet_i C_2$ is indeed a construct of $\mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2}$. As for the first case defining $C_1 \bullet_i C_2$, by Lemma 3, since $\mathcal{T}_2$ is a subtree of $\mathcal{T}_1 \bullet_i \mathcal{T}_2$, we have that $\mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2} \backslash H_{\mathcal{T}_1} \rightsquigarrow \mathbf{H}_{\mathcal{T}_2}$. Therefore, since $C_2 : \mathbf{H}_{\mathcal{T}_2}$, we indeed have that $H_{\mathcal{T}_1}\{C_2\} : \mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2}$. The legitimacy of the second case defining $C_1 \bullet_i C_2$ is a direct consequence of Lemma 4. ■

The following lemma provides a non-inductive characterization of $C_1 \bullet_i C_2$.

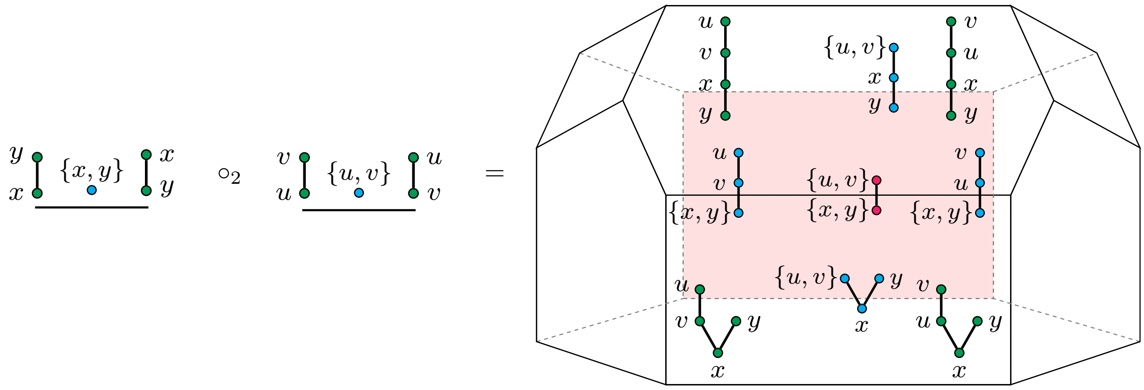
Lemma 6. $C_1 \bullet_i C_2$ *is the unique construct of the hypergraph* $\mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2}$*, such that* $v(C_1 \bullet_i C_2) = v(C_1) \cup v(C_2)$*,* $\rho(C_1 \bullet_i C_2) = \rho(C_1)$*, and such that there exists an edge of* $C_1 \bullet_i C_2$ *whose removal results precisely in* C_1 *and* C_2*.*

Remark 3. Note that, if $(\mathcal{T}_1, \sigma_1, C_1) \circ_i (\mathcal{T}_2, \sigma_2, C_2) = (\mathcal{T}_1 \bullet_i \mathcal{T}_2, \sigma_1 \bullet_i \sigma_2, C_1 \bullet_i C_2)$, then $C_1 \bullet_i C_2 \leq H_{\mathcal{T}_1}\{H_{\mathcal{T}_2}\}$ in $\mathcal{A}(\mathbf{H}_{\mathcal{T}_1 \bullet_i \mathcal{T}_2})$.

Example 5. The picture below displays all the 9 instances of the partial composition

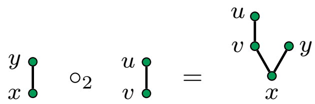
$$\circ_2 : \mathcal{O}_\infty(3,7,2;10) \otimes \mathcal{O}_\infty(2,2,5;7) \to \mathcal{O}_\infty(3,2,2,5,2;10)$$

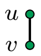
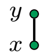
determined by operadic trees $(\mathcal{T}_1, \sigma_1)$ and $(\mathcal{T}_2, \sigma_2)$ from Example 2. The resulting 9 constructs are the faces of the square $\{x,y\}\{\{u,v\}\}$ of the 3-dimensional hemiassociahedron.



Observe that the rank of the composition is the sum of the corresponding ranks.

Let us provide the details of the construction of the composition



By definition, we consider the left-recursive subtrees of $\mathcal{T}_1$ obtained by removing the edge x and we search for the one containing the vertex that used to be indexed by 1 in $\mathcal{T}_1$. Since this subtree is reduced to a corolla, the resulting construct will have  grafted to the root vertex of . △

The proof that the operad $\mathcal{O}_\infty$ is free as an operad of vector spaces will rely on the operation of collapsing an edge in a rooted tree. We recall the relevant definitions and results below.

Definition 3. Let $\mathcal{T} \in \texttt{Tree}(n)$ and let e be an (internal) edge of $\mathcal{T}$. We define $\mathcal{T}\backslash e \in \texttt{Tree}(n)$ to be the rooted tree obtained by collapsing the edge e *downwards*, i.e. in such a way that the vertex that remains after e is collapsed is the target vertex of e, i.e. the root vertex $\rho(\mathcal{T}(\{e\}))$ of the subtree of $\mathcal{T}$ determined by e; after the collapse, the inputs of $\rho(\mathcal{T}(\{e\}))$ will be all the inputs of $\mathcal{T}(\{e\})$ and they will be ordered as in $\mathcal{T}(\{e\})$. The remaining of the structure of $\mathcal{T}$ remains the same in $\mathcal{T}\backslash e$.

As for the edge collapses of operadic trees $(\mathcal{T}, \sigma)$, we take the convention to consider both $\mathcal{T}\backslash e$ and $\mathcal{T}(\{e\})$ as left-recursive operadic trees.

The following lemma is a straightforward consequence of Definition 3.

Lemma 7. *For $(\mathcal{T}, \sigma) \in \mathcal{O}(n_1, \ldots, n_k; n)$ and $e \in e(\mathcal{T})$, there exists a unique permutation $\sigma_e \in \Sigma_k$, such that the equality $(\mathcal{T}, \sigma) = (\mathcal{T}\backslash e \circ_{\rho(\mathcal{T}(\{e\}))} \mathcal{T}(\{e\}))^{\sigma_e}$ holds in the operad $\mathcal{O}$.*

Remark 4 (Coherence of edge collapses)**.** Note that, if $e_1, e_2 \in e(\mathcal{T})$, then $(\mathcal{T}\backslash e_1)\backslash e_2 = (\mathcal{T}\backslash e_2)\backslash e_1$. This equality ensures that, having fixed a set of edges of a tree, the order of collapsing the edges from that set has no effect on the resulting tree.

Note that, if a fixed set of edges determines a subtree $\mathcal{S}$ of $\mathcal{T}$, then the root vertex $\rho(\mathcal{S})$ of $\mathcal{S}$ remains a vertex in the tree $\mathcal{T}\backslash\mathcal{S}$, obtained by collapsing all the edges of $\mathcal{S}$.

The following result is a consequence of Lemma 7 and Remark 4.

Lemma 8. *For $(\mathcal{T}, \sigma) \in \mathcal{O}(n_1, \ldots, n_k; n)$ and a subtree $\mathcal{S}$ of $\mathcal{T}$, considered as a left-recursive operadic tree, there exists a unique permutation $\sigma_{\mathcal{S}} \in \Sigma_k$, such that the equality $(\mathcal{T}, \sigma) = (\mathcal{T}\backslash\mathcal{S} \circ_{\rho(\mathcal{S})} \mathcal{S})^{\sigma_{\mathcal{S}}}$ holds in the operad $\mathcal{O}$.*

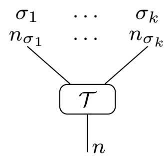
We now have all the prerequisites for proving that the $\mathcal{O}_\infty$ operad is free. The idea is simple and it has already been indicated in §2.2.1: constructs of arbitrary hypergraphs are non-planar trees, but if a hypergraph is the edge-graph of some operadic tree, then the hypergraph itelf, as well as its constructs, inherit a canonical planar embedding from that tree. This observation gives us a way to represent each triple $(\mathcal{T}, \sigma, C)$, where $(\mathcal{T}, \sigma)$ is an operadic tree and $C : \mathbf{H}_{\mathcal{T}}$ as a planar tree of a free operad.

Theorem 1. *As an operad of vector spaces, $\mathcal{O}_\infty$ is the free $\mathbb{N}$-coloured operad generated by the equivalence classes of left-recursive operadic trees:*

$$\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}\left(\bigoplus_{k\geq 2} \; \bigoplus_{n_1,\ldots,n_k \geq 1} \; \bigoplus_{\mathcal{T}\in \mathcal{O}(n_1,\ldots,n_k;n)} \mathbb{k} \right).$$

Proof. We define an isomorphism α between $\mathcal{O}_\infty$ and the operad of $\mathbb{N}$-coloured planar rooted trees whose vertices are decorated by left-recursive operadic trees and whose leaves are labeled by a permutation on the number of them, by induction on the number of vertices of the construct C of a given operation $(\mathcal{T},\sigma,C) \in \mathcal{O}_\infty(n_1,\ldots,n_k;n)$. Denote, for $1 \leq i \leq k$, with σ_i the index given by σ to the i-th vertex in the left-recursive ordering of $\mathcal{T}$.

- If C is the maximal construct $e(\mathcal{T})$, then

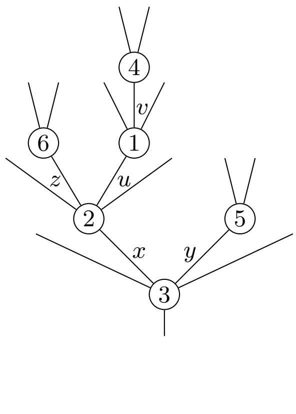
$\alpha(\mathcal{T},\sigma,e(\mathcal{T})) =$ 

  Note that each input of $\alpha(\mathcal{T},\sigma,e(\mathcal{T}))$ is uniquely determined by any of the following two data: its position in the planar structure (the number), or, the name of the vertex indexed by i in $\mathcal{T}$ (the name).
- Suppose that $C = X\{C_1,\ldots,C_p\}$, where $\mathbf{H}_{\mathcal{T}}\backslash X \rightsquigarrow \mathbf{H}_{\mathcal{T}_1},\ldots,\mathbf{H}_{\mathcal{T}_p}$ and $C_i : \mathbf{H}_{\mathcal{T}_i}$. Let $\mathcal{T}_X$ be the left-recursive operadic tree obtained from $(\mathcal{T},\sigma)$ by collapsing all the edges from $e(\mathcal{T})\backslash X$ (see Remark 4). Observe that the collapse of the edges $e(\mathcal{T})\backslash X$ that defines $\mathcal{T}_X$ is, in fact, the collapse of the subtrees $\mathcal{T}_i$, $1 \leq i \leq p$, of $\mathcal{T}$. By the definition of a collapse, each $\mathcal{T}_i$ will collapse to the vertex $\rho(\mathcal{T}_i)$ of $\mathcal{T}_X$; in particular, since the $\mathcal{T}_i$'s are mutually disjoint subtrees of $\mathcal{T}$, all the vertices $\rho(\mathcal{T}_i)$ will be mutually distinct. We define

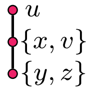
$$\alpha(\mathcal{T},\sigma,C) = \Big((\cdots((\alpha(\mathcal{T}_X,X) \circ_{\rho(\mathcal{T}_1)} \alpha(\mathcal{T}_1,C_1)) \circ_{\rho(\mathcal{T}_2)} \alpha(\mathcal{T}_2,C_2))\cdots) \circ_{\rho(\mathcal{T}_p)} \alpha(\mathcal{T}_p,C_p) \Big)^{\sigma_C},$$

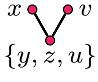
  where we the inputs of $\alpha(\mathcal{T}_X,X)$ involved in grafting are represented by their names, in order to avoid the reindexing, and where σ_C is the permutation determined uniquely thanks to (iterated application of) Lemma 8. Note that each tree $\mathcal{T}_i$ above is considered as left-recursive.

For example, for the operadic tree

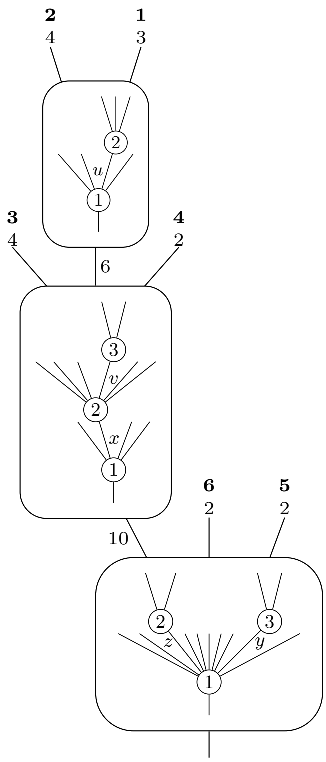
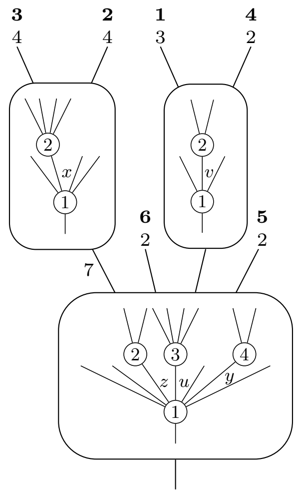
$(\mathcal{T},\sigma) =$ 

and constructs

$C_1 =$ 

and $C_2 =$ 

of the hypergraph $\mathbf{H}_{\mathcal{T}}$ associated to $\mathcal{T}$, we have that

$\alpha(\mathcal{T}, \sigma, C_1) =$  and $\alpha(\mathcal{T}, \sigma, C_2) =$ 

where the edge and leaf colours given by natural numbers are represented using regular font, and the indexing of the leaves is represented using bold font. Observe that, modulo leaves, $\alpha(\mathcal{T}, C)$ has the same shape as C.

The inverse of α is defined by composing the left-recursive operadic trees that decorate the nodes of an element (T, σ) of the free operad, in the way dictated by the edges of that element, followed by reindexing the vertices of the resulting tree as specified by σ, and by extracting the corresponding construct in the following way: first, remove all the leaves of T, and then, for each vertex of T, replace the operadic tree that decorates that vertex by the maximal constructs of its associated hypergraph. Lemma 8 ensures that this correspondence is indeed an isomorphism. ■

Having in mind the free operad description of $\mathcal{O}_\infty$, we adopt the following notational convention about constructs.

Convention 3. If we wish to incorporate the specification of the planar embedding of a construct $C : \mathbf{H}_{\mathcal{T}}$ into the notation for C, we shall write $C = X(C_1, \ldots, C_p)$ instead of $C = X\{C_1, \ldots, C_p\}$, if $C_1, \ldots, C_p$ appear in that order in the tree $\alpha(\mathcal{T}, \sigma, C)$.

2.2.3. $\mathcal{O}_\infty$ *as a differential graded operad.* In order to equip the $\mathcal{O}_\infty$ operad with a grading and a differential, we shall use the free operad structure of $\mathcal{O}_\infty$ and count the edges and leaves that lie in a particular position relative to some other edge or a leaf, in the way formalized by the following definition.

Definition 4. Let $\mathcal{T}$ be a planar rooted tree, and let $e \in e(\mathcal{T})$ and $l \in i(\mathcal{T})$ be an internal edge and an input leaf of $\mathcal{T}$, respectively.

- The internal edges *below* e (resp. l) in $\mathcal{T}$ are the internal edges of $\mathcal{T}$ that lie on the unique path from the vertex $\rho(\mathcal{T}(\{e\}))$ (resp. $\rho(\mathcal{T}(\{l\}))$) to $\rho(\mathcal{T})$.
- The edges and leaves *on the left* (resp. *on the right*) from e are the edges and leaves of $\mathcal{T}$ which are strictly on the left (resp. right) from the unique path from the first (resp. last) leaf of the subtree of $\mathcal{T}$ rooted at e, to $r(\mathcal{T})$. The edges and leaves on the left (resp. on the right) from l are the edges and leaves of $\mathcal{T}$ which are strictly on the left (resp. right) from the unique path from l to $r(\mathcal{T})$.

For $e \in e(\mathcal{T})$ and $l \in i(\mathcal{T})$, denote with $E_{\leq e}(\mathcal{T})$ the sum of the number of all edges and leaves on the left from e and the number of all edges below e in $\mathcal{T}$, and with $E_{l>}(\mathcal{T})$ the number of all edges and leaves on the right from l in $\mathcal{T}$.

We grade the vector space $\mathcal{O}_\infty(n_1, n_2, \ldots, n_k; n)$ by setting

$$|(\mathcal{T}, \sigma, C)| = e(\mathcal{T}) - v(C) = k - 1 - v(C).$$

Note that the grading agrees with the rank of the construct $C : \mathbf{H}_{\mathcal{T}}$ in $\mathcal{A}(\mathbf{H}_{\mathcal{T}})$; in particular, $0 \leq |(\mathcal{T}, C)| \leq k-2$. Observe also that the partial composition operation of $\mathcal{O}_\infty$ respects the grading, in the sense that

$$|(\mathcal{T}_1,\sigma_1,C_1)\circ_i(\mathcal{T}_2,\sigma_2,C_2)| = |(\mathcal{T}_1,\sigma_1,C_1)| + |(\mathcal{T}_2,\sigma_2,C_2)|.$$

If $(\mathcal{T},\sigma)$ is clear from the context, we shall often write $|C|$ for what is actually $|(\mathcal{T},\sigma,C)|$.

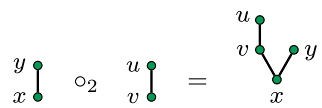
In the graded version of the $\mathcal{O}_\infty$ operad, signs show up in the definition of the partial composition: we adapt the definition of $\circ_i$ by setting

$$(\mathcal{T}_1,\sigma_1,C_1)\circ_i(\mathcal{T}_2,\sigma_2,C_2) = (-1)^{\varepsilon}(\mathcal{T}_1 \bullet_i \mathcal{T}_2, \sigma_1 \bullet_i \sigma_2, C_1 \bullet_i C_2),$$

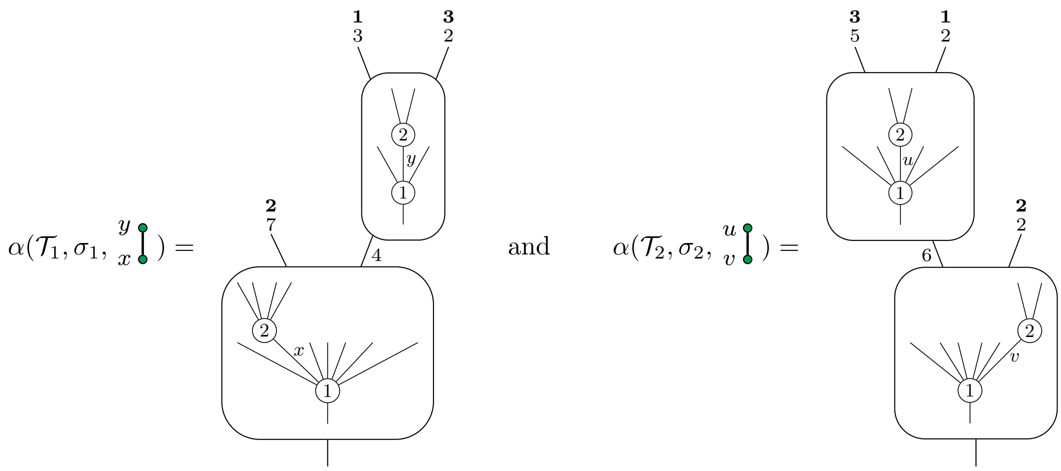
where ε is the number of edges and leaves of $\alpha(\mathcal{T}_1,\sigma_1,C_1)$ on the right of the leaf indexed by i, multiplied by the number of all edges and leaves of $\alpha(\mathcal{T}_2,\sigma_2,C_2)$, minus the root:

$$\varepsilon = E_{i>}(\alpha(\mathcal{T}_1,\sigma_1,C_1))\cdot(E(\alpha(\mathcal{T}_2,\sigma_2,C_2))-1).$$

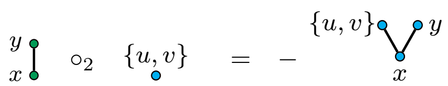
Example 6. In the graded setting, the composition



from Example 5 gets multiplied by $+$. Indeed,



and, therefore, $\varepsilon = 3\cdot 4 = 12$. On the other hand, we have



since $\alpha(\mathcal{T}_2,\sigma_2,\{u,v\})$ has one edge less than $\alpha(\mathcal{T}_2,\sigma_2,\{u\})$, which gives $\varepsilon = 3\cdot 3 = 9$. △

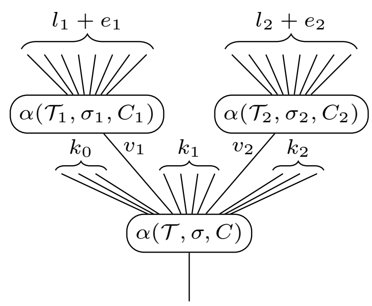
In the following lemma, we prove that Theorem 1 extends to the graded context, in which signs show up in the computation of composition in a free operad (see [15, Section 5.8.7]).

Lemma 9. *With the composition product adapted to the graded context, $\mathcal{O}_\infty$ is the free $\mathbb{N}$-coloured graded operad with respect to the set of generators given in Theorem 1.*

Proof. By Theorem 1, it remains to be shown that, for an operation $(\mathcal{T},\sigma,C)\in\mathcal{O}_\infty(n_1,\dots,n_k;n)$ and two distinct vertices $v_1,v_2\in v(\mathcal{T})$, such that the index of v_1 in the left-recursive decoration of $\mathcal{T}$ is less than the index of v_2, we have that

$$((\mathcal{T},\sigma,C)\circ_{v_1}(\mathcal{T}_1,\sigma_1,C_1))\circ_{v_2}(\mathcal{T}_2,\sigma_2,C_2) = (-1)^{|C_1|\cdot|C_2|}((\mathcal{T},\sigma,C)\circ_{v_2}(\mathcal{T}_2,\sigma_2,C_2))\circ_{v_1}(\mathcal{T}_1,\sigma_1,C_1).$$

In the free operad description of $\mathcal{O}_\infty$, the two compositions of the above equality are represented by the planar tree



where, for $i = 1, 2$, $\alpha(\mathcal{T}_i, \sigma_i, C_i)$ has l_i inputs and e_i edges, and where k_0 (resp. k_1, k_2) is the number of leaves and edges of $\alpha(\mathcal{T}, \sigma, C)$ on the left from v_1 (resp. between v_1 and v_2, on the right from v_2). The sign of the composition on the left-hand side is then determined by $\varepsilon_1 = (k_1+k_2+1)(l_1+e_1)+k_2(l_2+e_2)$, while, for the right-hand side, we have $\varepsilon_2 = k_2(l_2+e_2)+(k_1+k_2+l_2+e_2+1)(l_1+e_1)$. Additionally, for $i = 1, 2$, we have that $|C_i| = l_i - e_i - 2$. A straightforward calculation shows that $\varepsilon_1 =_{\text{mod}2} \varepsilon_2 + |C_1| \cdot |C_2|$, which proves the claim. ∎

In order to equip the graded operad $\mathcal{O}_\infty$ with a differential, we now formalize the action of splitting the vertices of constructs of edge-graphs of operadic trees. Let $(\mathcal{T}, \sigma, C) \in \mathcal{O}_\infty(n_1, \ldots, n_k; n)$ and let $V \in v(C)$ be such that $|V| \geq 2$. Let $\mathcal{T}_V$ be the left-recursive operadic tree obtained from $\mathcal{T}$ by contracting all the edges of $\mathcal{T}$ except the ones contained in V. Let X and Y be non-empty disjoint sets such that $X \cup Y = V$ and such that $X\{Y\} : \mathbf{H}_{\mathcal{T}_V}$. We define the construct $C[X\{Y\}/V]$ of $\mathbf{H}_{\mathcal{T}}$ by induction on the number of vertices of C, as follows.

- If $C = e(\mathcal{T})$, we set
$$C[X\{Y\}/V] := X\{Y\}.$$
- Suppose that $C = Z\{C_1, \ldots, C_p\}$, where $\mathbf{H}_{\mathcal{T}} \backslash Z \rightsquigarrow \mathbf{H}_{\mathcal{T}_1}, \ldots, \mathbf{H}_{\mathcal{T}_p}$ and $C_i : \mathbf{H}_{\mathcal{T}_i}$.
  - If there exists an index $1 \leq i \leq p$, such that $V \in v(C_i)$, we define
$$C[X\{Y\}/V] := Z\{C_1, \ldots, C_{i-1}, C_i[X\{Y\}/V], C_{i+1}, \ldots, C_p\}.$$
  - Suppose that $V = Z$ and let $\{i_1, \ldots, i_q\} \cup \{j_1, \ldots, j_r\}$ be the partition of the set $\{1, \ldots, p\}$ such that the trees $\mathcal{T}_{i_s}$, for $1 \leq s \leq q$, contain an edge sharing a vertex with some edge of Y, while the trees $\mathcal{T}_{i_t}$, for $1 \leq t \leq q$, have no edges sharing a vertex with the edges of Y. We define
$$C[X\{Y\}/V] := X\{Y\{C_{i1}, \ldots C_{iq}\}, C_{j1}, \ldots, C_{jr}\}.$$
  If, exceptionally, $\{i_1, \ldots, i_q\} = \emptyset$ (resp. $\{j_1, \ldots, j_r\} = \emptyset$), we have that $C[X\{Y\}/V] := X\{Y, C_1, \ldots, C_p\}$ (resp. $C[X\{Y\}/V] := X\{Y\{C_1, \ldots, C_p\}\}$).

Therefore, intuitively, $C[X\{Y\}/V]$ is obtained from C by splitting the vertex V into the edge $X\{Y\}$.

Lemma 10. *The non-planar rooted tree $C[X\{Y\}/V]$ is indeed a construct of $\mathbf{H}_{\mathcal{T}}$.*

Proof. By induction on the number of vertices of C. If C has a single vertex $V = e(\mathcal{T})$, then $\mathbf{H}_{\mathcal{T}_V} = \mathbf{H}_{\mathcal{T}}$, and, therefore, for any decomposition $X \cup Y = e(\mathcal{T})$, such that $X\{Y\} : \mathbf{H}_{\mathcal{T}_V}$, we trivially also have that $X\{Y\} : \mathbf{H}_{\mathcal{T}}$.

Suppose that $C = Z(C_1, \ldots, C_p)$, where $\mathbf{H}_{\mathcal{T}} \backslash Z \rightsquigarrow \mathbf{H}_{\mathcal{T}_1}, \ldots, \mathbf{H}_{\mathcal{T}_p}$ and $C_i : \mathbf{H}_{\mathcal{T}_i}$.

- If there exists $1 \leq i \leq p$, such that $V \in v(C_i)$, we conclude by the induction hypothesis for C_i.
- If $V = Z$ and if $\{i_1, \ldots, i_q\} \cup \{j_1, \ldots, j_r\}$ is the partition as above, then, since $X\{Y\} : \mathcal{T}_V$, it must be the case that the set of edges $Y \cup \bigcup_{i \in \{i_1, \ldots, i_q\}} e(\mathcal{T}_i)$ determines a single subtree $\mathcal{S}$ of $\mathcal{T}$. Therefore,
$$\mathbf{H}_{\mathcal{T}} \backslash X \rightsquigarrow \{\mathbf{H}_{\mathcal{S}}\} \cup \{\mathbf{H}_{\mathcal{T}_j} \mid j \in \{j_1, \ldots, j_r\}\}.$$
  Then, if $\{i_1, \ldots, i_q\} \neq \emptyset$, the conclusion follows since $Y\{C_{i1}, \ldots C_{iq}\} : \mathbf{H}_{\mathcal{S}}$. If $\{i_1, \ldots, i_q\} = \emptyset$, the conclusion follows since Y is trivially the maximal construct of $\mathbf{H}_{\mathcal{S}}$.

∎

The differential $d_{\mathcal{O}_\infty}$ of $\mathcal{O}_\infty$ is defined in terms of splitting the vertices of constructs, as follows: for $(\mathcal{T},\sigma,C)\in\mathcal{O}_\infty(n_1,\ldots,n_k;n)$, we set

$$d_{\mathcal{O}_\infty}(\mathcal{T},\sigma,C):=\sum_{\substack{V\in v(C)\\ |V|\geq 2}}\ \sum_{\substack{(X,Y)\\ X\cup Y=V\\ X\{Y\}:\mathbf{H}_{\mathcal{T}_V}}}(-1)^{\delta}(\mathcal{T},\sigma,C[X\{Y\}/V]),$$

where δ is the number of edges and leaves in $\alpha(\mathcal{T},\sigma,C[X\{Y\}/V])$ which are on the left from or below the edge determined by $X\{Y\}$:

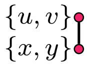
$$\delta=E_{\leq X\{Y\}}(\alpha(\mathcal{T},\sigma,C[X\{Y\}/V])).$$

Observe that, for $(\mathcal{T},C)\in\mathcal{O}_\infty(n_1,\ldots,n_k;n)$, we have $|(\mathcal{T},C)|=k-1-v(C)$ and $|d_{\mathcal{O}_\infty}(\mathcal{T},C)|=k-1-(v(C)+1)$, which shows that $d_{\mathcal{O}_\infty}$ is indeed a map od degree -1. In particular, for generators $(\mathcal{T},e(\mathcal{T}))$ we have:

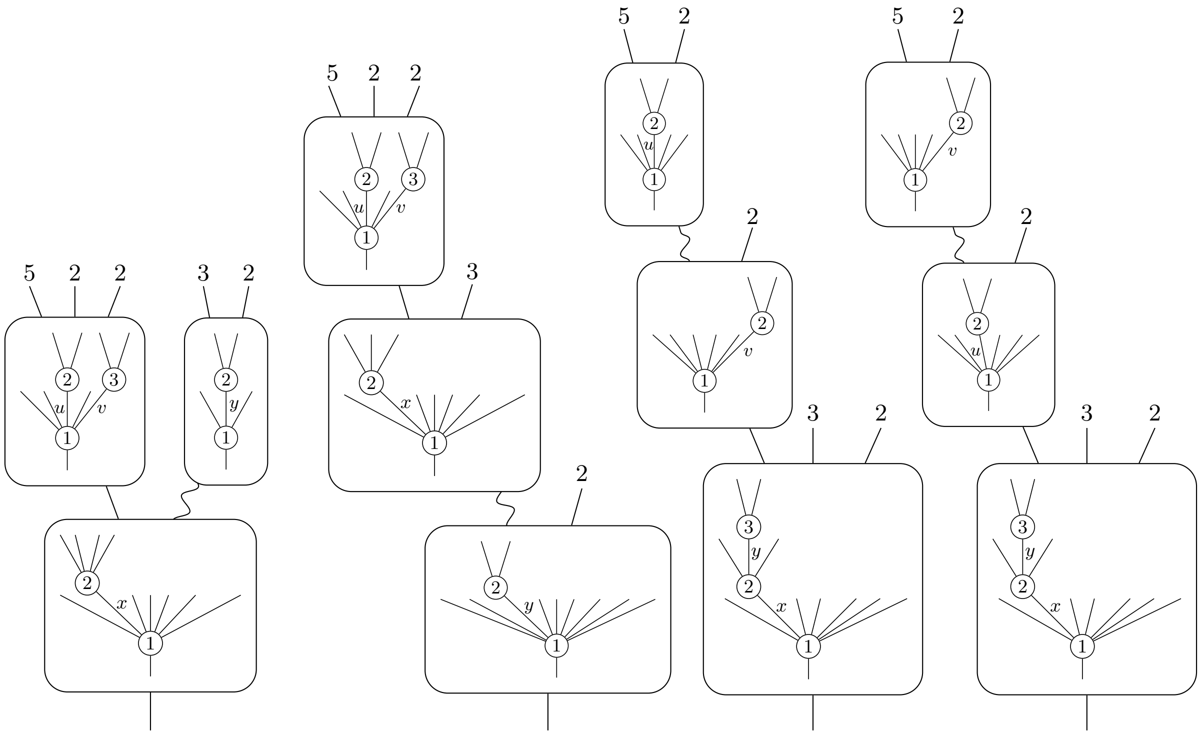
$$d_{\mathcal{O}_\infty}(\mathcal{T},e(\mathcal{T}))=\sum_{\substack{(X,Y)\\ X\cup Y=e(\mathcal{T})\\ X\{Y\}:\mathbf{H}_{\mathcal{T}}}}(-1)^{\delta}(\mathcal{T},X\{Y\})=\sum_{\substack{(X,Y)\\ X\cup Y=e(\mathcal{T})\\ X\{Y\}:\mathbf{H}_{\mathcal{T}}}}(-1)^{\delta}(\mathcal{T}_X,X)\circ_{\rho(\mathcal{T}_Y)}(\mathcal{T}_Y,Y),$$

which shows that $d_{\mathcal{O}_\infty}$ is minimal.

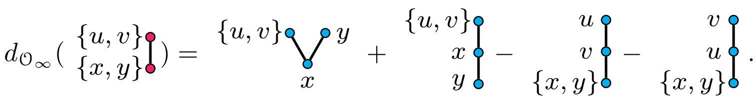
Convention 4. The differential $d_{\mathcal{O}_\infty}$ of $\mathcal{O}_\infty$ acts on an operation $(\mathcal{T},\sigma,C)$ by splitting the vertices of C, whereas the underlying operadic tree $(\mathcal{T},\sigma)$ remains unchanged. When expressing this action, assuming that $(\mathcal{T},\sigma)$ is clear from the context, we shall often write simply $d_{\mathcal{O}_\infty}(C)$.

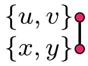
Example 7. We calculate the differential of $(\mathcal{T},\sigma,$ 

$)$, where $(\mathcal{T},\sigma)$ is our favourite operadic tree (see Example 3 and Example 5). By splitting the vertices $\{x,y\}$ and $\{u,v\}$, we get the following four planar trees in the free operad representation of $\mathcal{O}_\infty$:



in which the curly lines represent the edges that arise from the splitting. By counting the edges and the leaves on the left from and below those edges, we get 4, 0, 1 and 1, respectively. Therefore,



Observe that the geometric interpretation of this differential is the boundary of the square of the 3-dimensional hemiassociahedron encoded by 

(see Example 5). △

In the following technical lemma, we prove that $(\mathcal{O}_\infty, d_{\mathcal{O}_\infty})$ is indeed a dg operad. For the sake of readability, we shall write $(\mathcal{T}, C)$ for what is actually $(\mathcal{T}, \sigma, C)$.

Lemma 11. *The map d has the following properties:*

1. $d^2_{\mathcal{O}_\infty} = 0$, *and*
2. *the composition structure of $\mathcal{O}_\infty$ is compatible with $d_{\mathcal{O}_\infty}$, i.e.*

$$d_{\mathcal{O}_\infty}((\mathcal{T}_1, C_1) \circ_i (\mathcal{T}_2, C_2)) = d_{\mathcal{O}_\infty}(\mathcal{T}_1, C_1) \circ_i (\mathcal{T}_2, C_2) + (-1)^{|C_1|}(\mathcal{T}_1, C_1) \circ_i d_{\mathcal{O}_\infty}(\mathcal{T}_2, C_2).$$

Proof. 1. The proof that $d_{\mathcal{O}_\infty}$ squares to zero goes by case analysis relative to the configuration of vertices of C that got split in constructing the two occurences of the same summand $(\mathcal{T}, C')$ in $d^2_{\mathcal{O}_\infty}(\mathcal{T}, C)$, by showing that the corresponding summands have the opposite sign. This is an easy analysis of the relative edge positions: in all the cases, there will exist exactly one edge that is counted in calculating the sign of one of the two instances of $(\mathcal{T}, C')$, but not in calculating the sign of the other one.

For example, suppose that C' is obtained by splitting two different vertices of C, i.e. that

$$C' = C[X_1\{Y_1\}/V_1][X_2\{Y_2\}/V_2] = C[X_2\{Y_2\}/V_2][X_1\{Y_1\}/V_1]$$

for some $V_1, V_2 \in v(C)$. Suppose, moreover, that V_1 is above V_2 in C, and let $V_2\{U\}$ be the first edge on the path from V_2 to V_1. Suppose finally that, after splitting V_2, the edge $V_2\{U\}$ splits into $X_2\{Y_2\{U\}\}$, i.e. that the vertex Y_2 stays on the path from X_2 to X_1. Under these assumptions on the shape of C', let p_1 (resp. p_2) be the number of internal edges between V_1 and V_2 (resp. below V_2) in $\alpha(\mathcal{T}, C)$, and let l_i, for $i = 1, 2$, be the number of edges and leaves on the left from the edge $X_i\{Y_i\}$ in $\alpha(\mathcal{T}, C[X_i\{Y_i\}/V_i])$. Finally, let l be the number of edges and leaves on the left from the edge $V_2\{U\}$ in $\alpha(\mathcal{T}, C[X_1\{Y_1\}/V_1])$. The signs of the operations $C[X_1\{Y_1\}/V_1][X_2\{Y_2\}/V_2]$ and $C[X_2\{Y_2\}/V_2][X_1\{Y_1\}/V_1]$ are then induced from the sums

$$\underbrace{l_1 + p_1 + p_2 + l}_{X_1\{Y_1\}/V_1} + \underbrace{l_2 + p_2}_{X_2\{Y_2\}/V_2} \quad \text{and} \quad \underbrace{l_2 + p_2}_{X_2\{Y_2\}/V_2} + \underbrace{l_1 + p_1 + p_2 + l + 1}_{X_1\{Y_1\}/V_1},$$

respectively, and the conclusion follows since they differ by 1.

2. Denote $\mathcal{T} = \mathcal{T}_1 \bullet_i \mathcal{T}_2$ and $C = C_1 \bullet_i C_2$. For the left-hand side of the equality, we have

$$d_{\mathcal{O}_\infty}((\mathcal{T}_1, C_1) \circ_i (\mathcal{T}_2, C_2)) = \sum_{\substack{V \in v(C) \\ |V| \geq 2}} \sum_{\substack{(X,Y) \\ X \cup Y = V \\ X\{Y\} : \mathbf{H}_{\mathcal{T}_V}}} (-1)^{\delta+\varepsilon}(\mathcal{T}, C[X\{Y\}/V])$$

where $\delta = E_{\leq X\{Y\}}(\alpha(\mathcal{T}, C[X(Y)/V]))$ and $\varepsilon = E_{i>}(\alpha(\mathcal{T}_1, C_1)) \cdot (E(\alpha(\mathcal{T}_2, C_2)) - 1)$. We prove the stated equality by case analysis with respect to the origin of the vertex V relative to $v(C_1)$ and $v(C_2)$, and, if $V \in v(C_1)$, the position of the edge $X\{Y\}$ relative to the leaf i of $\alpha(\mathcal{T}_1, C_1)$.

Suppose that $V \in v(C_1)$.

- If $X\{Y\} \in E_{\leq i}(\alpha(\mathcal{T}_1, C_1[X(Y)/V]))$, then
  - $E_{\leq X\{Y\}}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V])) = \delta$, and
  - $E_{i>}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V])) \cdot (E(\alpha(\mathcal{T}_2, C_2)) - 1) = \varepsilon$.
- If $X\{Y\} \in E_{i>}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V]))$, then
  - $E_{\leq X\{Y\}}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V])) = \delta - (E(\alpha(\mathcal{T}_2, C_2)) - 1)$, and
  - $E_{i>}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V])) \cdot (E(\alpha(\mathcal{T}_2, C_2)) - 1) = \varepsilon + E(\alpha(\mathcal{T}_2, C_2)) - 1$.

In both cases, we have that

$$E_{\leq X\{Y\}}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V])) + E_{i>}(\alpha(\mathcal{T}_1, C_1[X\{Y\}/V])) \cdot (E(\alpha(\mathcal{T}_2, C_2)) - 1) = \delta + \varepsilon,$$

which means that $(-1)^{\delta+\varepsilon}(\mathcal{T}, C[X\{Y\}/V])$ appears as a summand in $d_{\mathcal{O}_\infty}(\mathcal{T}_1, C_1) \circ_i (\mathcal{T}_2, C_2)$.

If $V \in V(C_2)$, then

- $E_{\leq X\{Y\}}(\alpha(\mathcal{T}_2, C_2[X\{Y\}/V])) = \delta - E_{\leq i}(\alpha(\mathcal{T}_1, C_1)) - 1$, and
- $E_{i>}(\alpha(\mathcal{T}_1, C_1)) \cdot (E(\alpha(\mathcal{T}_2, C_2[X\{Y\}/V])) - 1) = \varepsilon - E_{i>}(\alpha(\mathcal{T}_1, C_1))$.

Observe that

$$E_{i>}(\alpha(\mathcal{T}_1,C_1))\cup E_{\leq i}(\alpha(\mathcal{T}_1,C_1))\cup\{i\}=E(\alpha(\mathcal{T}_1,C_1))\backslash\{\rho(\mathcal{T}_1)\}. \tag{2.1}$$

Therefore, if $|(\mathcal{T}_1,C_1)|$ is even (resp. odd), then the cardinality of (2.1) is even (resp. odd), and, hence,

$$\delta-E_{\leq i}(\alpha(\mathcal{T}_1,C_1))-1+\varepsilon-E_{i>}(\alpha(\mathcal{T}_1,C_1))=_{\text{mod }2}\delta+\varepsilon$$

$$(\text{resp. }\ \delta-E_{\leq i}(\alpha(\mathcal{T}_1,C_1))-1+\varepsilon-E_{i>}(\alpha(\mathcal{T}_1,C_1))=_{\text{mod }2}\delta+\varepsilon+1\ ),$$

meaning that $(-1)^{\delta+\varepsilon}(\mathcal{T},C[X\{Y\}/V])$ appears as a summand in $(-1)^{|(\mathcal{T}_1,C_1)|}(\mathcal{T}_1,C_1)\circ_i d_{\mathcal{O}_\infty}(\mathcal{T}_2,C_2)$.

The opposite direction is treated by an analogous analysis. ■

2.2.4. *The homology of* $(\mathcal{O}_\infty,d_{\mathcal{O}_\infty})$. We first prove that

$$\mathcal{O}^{k-2}_\infty(n_1,\dots,n_k;n)\xrightarrow{d^{k-2}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)}\ \cdots\ \xrightarrow{d^{1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)}\mathcal{O}^{0}_\infty(n_1,\dots,n_k;n)$$

is an exact sequence.

Lemma 12. *For* $0<m\leq k-2$*, we have that* $\operatorname{Ker} d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)=\operatorname{Im} d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$.

Proof. Since $d_{\mathcal{O}_\infty}$ squares to zero, we have that $\operatorname{Im} d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)\subseteq\operatorname{Ker} d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$. For the proof of the opposite inclusion, let $L\in\operatorname{Ker}\ d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$. Since $|L|\neq 0$, it must be the case that, in $d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(L)$, the summands cancel each other out pairwise. Therefore, we may assume, without loss of generality, that the linear combination L is made of triples whose first two components are all given by the same operadic tree $\mathcal{T}=(\mathcal{T},\sigma)$, i.e. that

$$L=k_1(\mathcal{T},C_1)+\cdots+k_p(\mathcal{T},C_p),$$

where $k_i\in\{+1,-1\}$. Then, for each summand $(-1)^\delta(\mathcal{T},C_i[X\{Y\}/V])$ in $d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(L)$, there exists a summand $(-1)^{\delta+1}(\mathcal{T},C_j[U\{V\}/W])$ in $d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(L)$, such that

$$C_i[X\{Y\}/Z]=C_j[U\{V\}/W].$$

The latter equality means that $k_i(\mathcal{T},C_i)$ and $k_j(\mathcal{T},C_j)$ appear as summands in

$$d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(k_{ij}(\mathcal{T},C)),$$

where C is obtained by collapsing the edge $U\{V\}$ of C_i (or, equivalently, by collapsing the edge $X\{Y\}$ of C_j), for some $k_{ij}\in\{+1,-1\}$. It is easy to see that, if $d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(\mathcal{T},C)$ contains a summand $(\mathcal{T},C[P\{Q\}/R])$ different than $k_i(\mathcal{T},C_i)$ and $k_j(\mathcal{T},C_j)$, then $(\mathcal{T},C[P\{Q\}/R])$ also appears in L. For example, if

$$(\mathcal{T},C[P\{Q\}/R][P_1\{P_2\}/P])$$

is a summand of $d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(\mathcal{T},C[P\{Q\}/R])$, then the construct $C[P\{Q\}/R][P_1\{P_2\}/P]$ can also be obtained as a result of splitting first the vertex R of C either into P_1 and $P_2\cup Q$, or into P_2 and $P_1\cup Q$ by $d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$, followed by splitting the vertex decorated by the union of the appropriate sets by $d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$.

The preimage of L with respect to $d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$ is obtained by reconstructing, for each cancellable pair in $d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)(L)$, the appropriate construct and its coefficient in $\mathcal{O}^{m+1}_\infty(n_1,\dots,n_k;n)$ in this way (throwing away the duplicates), and by taking the sum of the obtained constructs. ■

For $m=0$, we have that $\operatorname{Ker} d^0_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)=\mathcal{O}^0_\infty(n_1,\dots,n_k;n)$, since constructions have no vertices that could be split. As for the image of $d^1_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)$, we have

$$\operatorname{Im} d^1_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)=\mathsf{Span}_{\Bbbk}\left(\bigoplus_{(\mathcal{T},\sigma)\in\mathcal{O}(n_1,\dots,n_k;n)}\ \bigoplus_{C:\mathbf{H}_{\mathcal{T}},|C|=1}\pm C[x\{y\}/\{x,y\}]\pm C[y\{x\}/\{x,y\}]\right),$$

where $\{x,y\}$ is the unique two-element vertex of C and the signs are determined by the position of the vertices x and y in $\mathbf{H}_{\mathcal{T}}$ (considered as the edge-graph with levels), using the criterion from 2.2.1: if the shortest path between x and y is made of vertical edges only, then one of the vertices x and y is above the other and the construction which respects this position gets multiplied by $-$, and the other one by $+$; otherwise, both constructions get the sign $+$. By collapsing $\operatorname{Im} d_1$ to zero, for each $\mathcal{T}\in\mathcal{O}(n_1,\dots,n_k;n)$, all the constructions of $\mathbf{H}_{\mathcal{T}}$ get glued into a single equivalence class; in particular, different trees from $\mathcal{O}(n_1,\dots,n_k;n)$ give rise to different classes. Therefore,

$$\operatorname{Ker} d^m_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)/\operatorname{Im} d^{m+1}_{\mathcal{O}_\infty}(n_1,\dots,n_k;n)=\begin{cases}\{0\}, & m\neq 0\\ \mathcal{O}(n_1,\dots,n_k;n), & m=0.\end{cases}$$

which entails that $H(\mathcal{O}_\infty, d_{\mathcal{O}_\infty}) \cong H(\mathcal{O}, 0)$. The witnessing quasi-isomorphism $\alpha_{\mathcal{O}} : \mathcal{O}_\infty \to \mathcal{O}$ is simply the first projection on degree zero, and the zero map elsewhere.

Having established the freeness of $\mathcal{O}_\infty$, the minimality of $d_{\mathcal{O}_\infty}$ and the quasi-isomorphism with $\mathcal{O}$, we can now finally conclude.

Theorem 2. *The operad $\mathcal{O}_\infty$ is the minimal model for the operad $\mathcal{O}$.*

2.2.5. *Stasheff's associahedra as a suboperad of $\mathcal{O}_\infty$.* The earliest example of an explicit description of the minimal model of a dg operad, predating even the notion of minimal model itself, is given by the dg A_∞-operad, the minimal model of the non-symmetric operad As for associative algebras, often described in terms of Stasheff's associahedra [27].

Recall that the non-symmetric associative operad As, encoding the category of non-unital associative algebras, is defined by $As(n) = \mathsf{Span}_{\mathbb{k}}(\{t_n\})$, for $n \geq 2$, where t_n the isomorphism class of a planar corolla with n inputs. The one-dimensional space $As(n)$ is concentrated in degree is zero and the differential is trivial.

In the standard dg framework, the A_∞-operad is the quasi-free dg operad $A_\infty = \mathcal{T}(\bigoplus_{n\geq 2} \mathsf{Span}_{\mathbb{k}}(\{t_n\}))$, where $|t_n| = n-2$, with the differential given by

$$d(t_n) = \sum_{\substack{n=p+q+r\\ k=p+r+1\\ k,q\geq 2}} (-1)^p \; t_k \circ_{p+1} t_q.$$

The dg A_∞-operad is the minimal model for As. Indeed, the map $\alpha_{As} : A_\infty \to As$, defined as the identity on t_2 and as the zero map elsewhere, induces a homology isomorphism $H_\bullet(A_\infty, d) \cong As$, whereas d is clearly defined in terms of decomposable elements of A_∞.

Let K_n, for $n \geq 2$, denote the $(n-2)$-dimensional associahedron, i.e. a CW complex whose cells of dimension k are in bijection with rooted planar trees having n leaves and $n-k-1$ vertices. The sequence of associahedra $\mathcal{K} = \{K_n\}_{n\geq 2}$ is naturally endowed with the structure of a non-symmetric topological operad: the composition

$$\circ_i : K_r \times K_s \to K_{r+s-1}$$

is defined as follows: for faces $k_1 \in K_r$ and $k_2 \in K_s$, $\circ_i(k_1, k_2)$ is the face of K_{r+s-1} obtained by grafting the tree encoding the face k_2 to the leaf i of the tree encoding the face k_1. The topological operad $\mathcal{K}$ is turned into the dg A_∞-operad by taking cellular chain complexes on $\mathcal{K}$; we refer to [15, Proposition 9.2.4.] for the details of this transition.

Let $\mathcal{A}_\infty$ be the suboperad of the $\mathcal{O}_\infty$ operad determined by linear operadic trees, i.e. operadic trees $\mathcal{T}$ with univalent vertices such that a vertex i is always adjacent to the vertex $i-1$ (and not to a vertex $i-j$, for some $j > 1$). Observe that the univalency requirement ensures that linear operadic trees are closed under the operation of substitution of trees.

Theorem 3. *The operad $\mathcal{A}_\infty$ is the minimal model for the operad As.*

Proof. The restriction to linear operadic trees that defines $\mathcal{A}_\infty$ collapses the set of colours $\mathbb{N}$ of the operad $\mathcal{O}$ to the singleton set $\{1\}$, making therefore $\mathcal{A}_\infty$ a monochrome operad. The conclusion follows from the correspondence between the construct description of associahedra and the standard description underlying the definition of the A_∞-operad, established in §1.3.2. ■

2.3. **The combinatorial Boardman-Vogt-Berger-Moerdijk resolution of $\mathcal{O}$.** In [4], Berger and Moerdijk constructed a cofibrant resolution for coloured operads in arbitrary monoidal model categories, by generalizing the Boardman-Vogt W-construction for topological operads [5]. In this section, by introducing a cubical subdivision of the faces of operadic polytopes, we define the operad $\mathcal{O}_\infty^{\circ}$, which is precisely the W-construction applied on $\mathcal{O}$.

In order to provide the combinatorial description of the W-construction of $\mathcal{O}$, we are going to generalize the notion of a construct of the edge-graph $\mathbf{H}_{\mathcal{T}}$ of an operadic tree $\mathcal{T}$, to the notion of a circled construct of $\mathbf{H}_{\mathcal{T}}$. A circled construct should be thought of as a two-level construct, i.e. a construct whose vertices are constructs themselves; the idea is that circles determine those "higher" vertices, in the same way as in the definition of the monad of trees. The circles that we add to a construct of $\mathbf{H}_{\mathcal{T}}$ arise from decompositions of $\mathcal{T}$, and themselves determine a decomposition of that construct – this construction is dual to the one defining the isomorphism α in the proof of Theorem 1.

Definition 5. Let $(\mathcal{T}, \sigma) \in \mathcal{O}(n_1, \ldots, n_k; n)$. The set of *circled constructs* of the hypergraph $\mathbf{H}_{\mathcal{T}}$ is generated by the following two rules.

- For each (ordinary) construct $C : \mathbf{H}_{\mathcal{T}}$, the construct C together with a single circle that entirely surrounds it, is a circled construct of $\mathbf{H}_{\mathcal{T}}$.
- If $(\mathcal{T},\sigma) = (\mathcal{T}_1,\sigma_1)\bullet_i(\mathcal{T}_2,\sigma_2)$ and if C_1 and C_2 are circled constructs of $\mathbf{H}_{\mathcal{T}_1}$ and $\mathbf{H}_{\mathcal{T}_2}$, respectively, then the construct $C_1 \bullet_i C_2 : \mathbf{H}_{\mathcal{T}}$, determined uniquely by the composition $\circ_i$ of $\mathcal{O}_\infty$, together with all the circles of C_1 and C_2 (and no other circle), is a circled construct of $\mathbf{H}_{\mathcal{T}}$.

In what follows, we shall write $C^\circ : \mathbf{H}_{\mathcal{T}}$ to denote that C° is a circled construct of $\mathbf{H}_{\mathcal{T}}$; if $C^\circ : \mathbf{H}_{\mathcal{T}}$, we shall denote with C the ordinary construct of $\mathbf{H}_{\mathcal{T}}$ obtained by forgetting the circles of C°. We shall write C°_p to indicate that the circled construct C° has p circles. Denote with $A^\circ(\mathbf{H}_{\mathcal{T}})$ the set of all circled constructs of $\mathbf{H}_{\mathcal{T}}$.

Remark 5. Note that the set of edges of a circled construct $C^\circ : \mathbf{H}_{\mathcal{T}}$ can be decomposed into two disjoint subsets: the subset of *circled edges*, i.e. of the edges that lie within a circle, and the subset of *connecting edges*, i.e. of the edges that connect two adjacent circles. The disjointness is ensured by the fact that Definition 5 disallowes nested circles.

Remark 6. Circled constructs are generalizations of *circled trees*, in the terminology of [15, Appendix C.2.3.].

Let, for $k \geq 2$, $n_1,\dots,n_k \geq 1$, and $n = (\sum_{i=1}^k n_i)-k+1$, $\mathcal{O}^\circ_\infty(n_1,\dots,n_k;n)$ be the vector space spanned by triples $(\mathcal{T},\sigma,C^\circ)$, where $(\mathcal{T},\sigma)$ is an operadic tree representing an operation of $\mathcal{O}(n_1,\dots,n_k;n)$ and $C^\circ : \mathbf{H}_{\mathcal{T}}$, subject to the equivalence relation generated by:

> $(\mathcal{T}_1,\sigma_1,C^\circ_1) \sim (\mathcal{T}_2,\sigma_2,C^\circ_2)$ iff there exists an isomorphism $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$, such that $\varphi\circ\sigma_1 = \sigma_2$, $C_1 \sim_\varphi C_2$, and such that, modulo the renaming φ, the circles of C°_1 are exactly the circles of C°_2.

Therefore,

$$\mathcal{O}^\circ_\infty(n_1,\dots,n_k;n) = \mathsf{Span}_{\Bbbk}\left(\bigoplus_{(\mathcal{T},\sigma)\in\mathcal{O}(n_1,\dots,n_k;n)} A^\circ(\mathbf{H}_{\mathcal{T}})\right).$$

The space $\mathcal{O}^\circ_\infty(n_1,\dots,n_k;n)$ is graded by $|(\mathcal{T},C^\circ_p)| = |v(C)| - p$.

The $\mathbb{N}$-coloured graded collection

$$\{\mathcal{O}^\circ_\infty(n_1,\dots,n_k;n)\,|\,n_1,\dots,n_k \geq 1\}$$

admits the following operad structure: the composition operation

$$\circ_i : \mathcal{O}^\circ_\infty(n_1,\dots,n_k;n)\otimes\mathcal{O}^\circ_\infty(m_1,\dots,m_l;n_i) \to \mathcal{O}^\circ_\infty(n_1,\dots,n_{i-1},m_1,\dots,m_l,n_{i+1},\dots,n_k;n)$$

is defined by

$$(\mathcal{T}_1,\sigma_1,C^\circ_1)\circ_i(\mathcal{T}_2,\sigma_2,C^\circ_2) = (-1)^\varepsilon(\mathcal{T}_1\bullet_i\mathcal{T}_2,\sigma_1\bullet_i\sigma_2,C^\circ_1\bullet_i C^\circ_2),$$

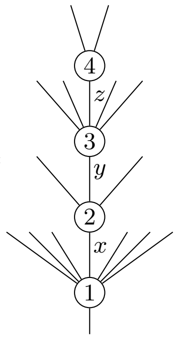
where $C^\circ_1\bullet_i C^\circ_2$ is defined by the second rule of Definition 5, and the sign $(-1)^\varepsilon$ is determined as follows. Observe that, for an operation $(\mathcal{T},\sigma,C^\circ)$, the circles of C° carry over to $\alpha(\mathcal{T},\sigma,C)$, where α is the isomorphism from the proof of Theorem 1. This decomposes the set of edges of $\alpha(\mathcal{T},\sigma,C)$ into the set of circled edges and the set of connecting edges. Then, ε is the number of connecting edges and leaves of $\alpha(\mathcal{T}_1,\sigma_1,C_1)$ on the right from the leaf indexed by i, multiplied by the number of all connecting edges and leaves of $\alpha(\mathcal{T}_2,\sigma_2,C_2)$, minus the root. Therefore, the sign is calculated in the analogous way as for the partial composition of $\mathcal{O}_\infty$, save that the vertices of operations of $\mathcal{O}_\infty$ are identified with the circles of operations of $\mathcal{O}^\circ_\infty$.

Finally, the derivative d° of $\mathcal{O}^\circ_\infty$ will be the difference $d^\circ_1 - d^\circ_0$ of two derivatives. The derivative d°_1 acts on $(\mathcal{T},\sigma,C^\circ)$ by turning circled edges of C° into connecting edges, by splitting the circles in two. The associated signs are determined as follows. Let us fix a summand in $d^\circ_1(\mathcal{T},\sigma,C^\circ)$. Suppose that C' is the construct surrounded by the circle of C° that we split in two, let C'_1 and C'_2 be the constructs surrounded by the two resulting circles in the summand, and let X and Y be the unions of the sets decorating the vertices of C'_1 and C'_2 (i.e. X and Y are the sets obtained by collapsing all the edges of C'_1 and C'_2), respectively. Suppose, without loss of generality, that the circle surrounding C'_1 is below the circle surrounding C'_2. The sign of the resulting summand is given by $(-1)^{\delta+e(C'_1)+e(C'_2)}$, where $(-1)^\delta$ is the sign of the summand of $d_{\mathcal{O}_\infty}(\mathcal{T},\sigma,C[(X\cup Y)/X\{Y\}])$ whose new edge is determined by $X\{Y\}$. Here, $C[(X\cup Y)/X\{Y\}]$ denotes the construct obtained from C by collapsing the edge $X\{Y\}$. Observe that the component d°_1 does not affect the overall configuration of edges of C as a plain construct. The component d°_0 collapses circled edges; each resulting summand $(\mathcal{T},\sigma,C[(U\cup V)/U\{V\}]^\circ)$ will be multiplied by the sign that $(\mathcal{T},\sigma,C)$ gets as a summand of $d_{\mathcal{O}_\infty}(\mathcal{T},\sigma,C[(U\cup V)/U\{V\}])$. The component d°_0 does not affect circles. One might think of d_1° as a coercion of the derivative $d_{\mathcal{O}_\infty}$ to $\mathcal{O}_\infty^\circ$, by the identification of the vertices of the operations of $\mathcal{O}$ with the circles of the operations of $\mathcal{O}_\infty^\circ$, since both d_1° and $d_{\mathcal{O}_\infty}$ split the corresponding entity. On the other hand, d_0° can be seen as the inverse of $d_{\mathcal{O}_\infty}$: d_0° collapses the edges, whereas $d_{\mathcal{O}_\infty}$ creates new edges by splitting the vertices. Therefore, intuitively, d_1° acts *globally*, and d_0° *locally*. The sum $e(C_1') + e(C_2')$ in the sign $(-1)^{\delta+e(C_1')+e(C_2')}$ pertaining to d_1° has to be added in order to ensure that d° squares to zero. More precisely, it is needed for the pairs of summands in $(d^\circ)^2(\mathcal{T},\sigma,C^\circ)$ that should be cancelled out and which arise by applying d_0° and d_1° in the opposite order. As a consequence of Lemma 11, d° agrees with the composition of $\mathcal{O}_\infty^\circ$.

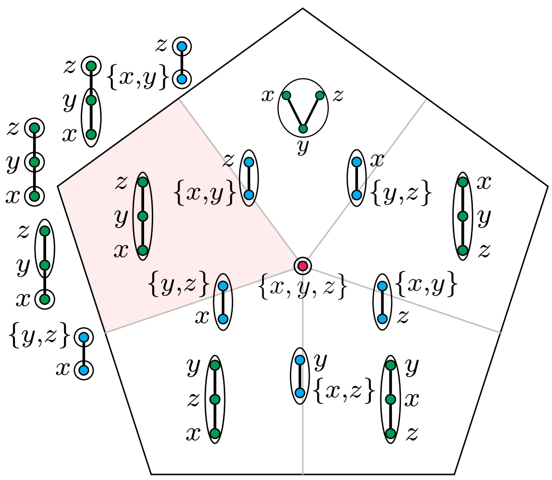
In the following two examples, we describe two cubical decompositions of operadic polytopes, given by the appropriate posets of circled constructs, where, like it was the case for ordinary constructs, the partial order is induced by the action of the differential d°.

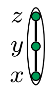
Example 8. For the linear tree

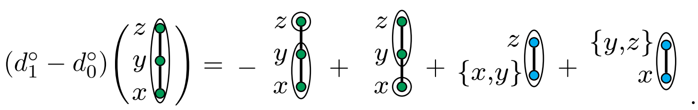
$(\mathcal{T},\sigma)$ =

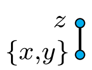
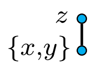


by taking all the circled constructs of $\mathbf{H}_{\mathcal{T}}$, we recover a familiar picture of the cubical decomposition of the pentagon from [15, Appendix C.2.3.], in which we fully labeled only the faces of the shaded square:



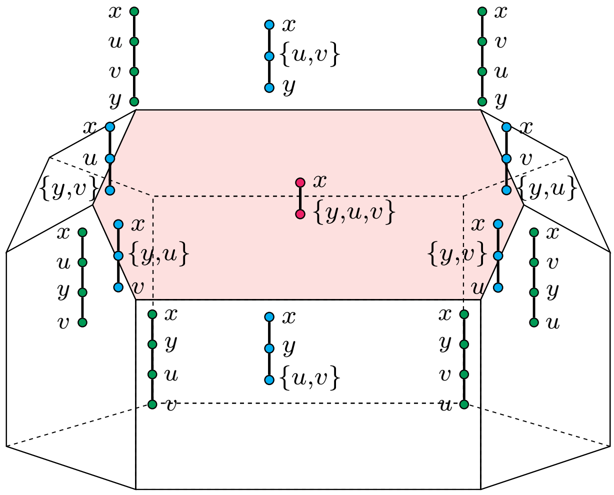
By calculating the derivative of the circled construction $C^\circ =$ , we get the boundary of the square:



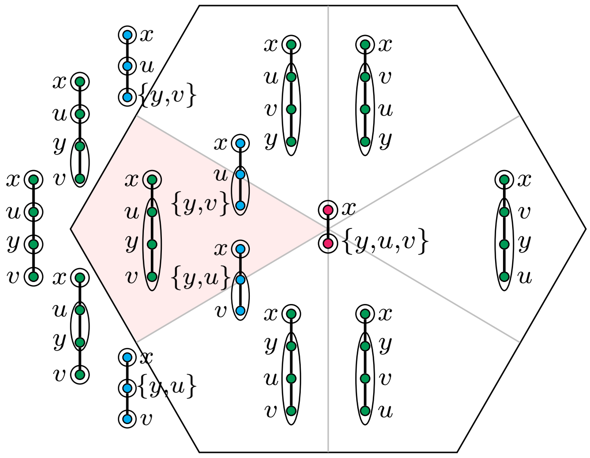
Here is how we calculated some of the signs above. The first summand on the right-hand side is obtained by applying d_1° on C°, by turning the circled edge $y\{z\}$ into a connecting edge. The sign rule says that this summand will be multiplied by $(-1)^{\delta+1+0}$, where $(-1)^\delta$ is the sign that the ordinary construct  gets as a summand of $d_{\mathcal{O}_\infty}(\mathcal{T},\sigma,\ \{x,y,z\}\bullet)$. Since $(-1)^\delta = 1$, the final sign is $-$. The third summand on the right-hand side is obtained by applying d_0° on C°, by collapsing the circled edge $x\{y\}$. By the sign rule, it will be multiplied by the sign that C gets as a summand of $d_{\mathcal{O}_\infty}(\mathcal{T},\sigma,$ $)$, which is $-$, which, together with the $-$ in front of d_0°, results in $+$. The reader may readily check that applying the differential d° on the sum on the right-hand side of the above equality results in 0. △

Since the basis of $\mathcal{O}_\infty^\circ(n_1,\ldots,n_k;n)$ is given by the faces of cubical subdivisions of arbitrary operadic polytopes, and not just the associahedra, we give below an example of the cubical subdivision of one of the hexagonal faces of the hemiassociahedron (see 1.3.4).

Example 9. The cubical subdivision of the shaded hexagonal face



of the hemiassociahedron is given by



where, once again, we fully labeled only the shaded square. △

Theorem 4. *The operad $\mathcal{O}_\infty^\circ$ is the W-construction $W(H,\mathcal{O})$ for the dg operad $\mathcal{O}$ and the interval $H = N_*(\Delta^1)$ of normalized chains on the 1-simplex Δ^1.*

Proof. Recall from [3, Section 8.5.2.] that the operad $W(H,\mathcal{O})$ is constructed like the free $\mathbb{N}$-coloured operad over $\mathcal{O}$, but with the additional assignment of a "length in H" to each edge of the corresponding trees. More precisely, since $H = N_*(\Delta^1)$ is the complex concentrated in degrees 0 and 1 defined by:

$$H_0 = \mathsf{Span}_{\Bbbk}(\{\lambda_0\}) \oplus \mathsf{Span}_{\Bbbk}(\{\lambda_1\}), \quad H_1 = \mathsf{Span}_{\Bbbk}(\{\lambda\}), \quad \text{and } d_1(\lambda) = \lambda_1 - \lambda_0,$$

and since the lengths of the edges of a planar rooted tree T are formally defined by the tensor product $H(T) := \bigotimes_{e\in e(T)} H$, the space $W(H,\mathcal{O})(n_1,\ldots,n_k;n)$ is spanned by the equivalence classes of quadruples $(\mathcal{T},\sigma,C,h)$, where $(\mathcal{T},\sigma,C) \in \mathcal{O}_\infty(n_1,\ldots,n_k;n)$ and $h \in H(\alpha(\mathcal{T},C))$, where α is the isomorphism from Theorem 1, under the equivalence relation generated by

$$(\mathcal{T},\sigma,C[X\{Y\}/V],s_{X\{Y\}}(h)) \sim (\mathcal{T},\sigma,C,h),$$

for $X\{Y\} : \mathcal{T}_V$ and $s_e : H(\alpha(\mathcal{T},C)) \to H(\alpha(\mathcal{T},C[X\{Y\}/V]))$ defined by setting the length of $X\{Y\}$ to be λ_0. The partial composition operation of $W(H,\mathcal{O})$ is defined by grafting trees and assigning the length λ_1 to the new edge. Finally, the differential ∂_W of $W(H,\mathcal{O})$ is the sum $\partial_W = \partial_{\mathcal{O}} + \partial_H$ of the *internal differential* $\partial_{\mathcal{O}}$, which, being induced from the differential of the operad $\mathcal{O}$, is trivially 0, and the *external differential* ∂_H, which is itself a sum of the operators ∂_H^1 and ∂_H^2, induced by the degree zero elements of H, which act by assigning the corresponding length to the edges.

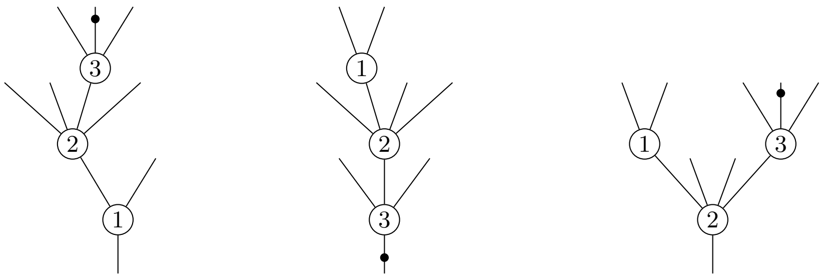
The isomorphism between $W(H, \mathcal{O})$ and $\mathcal{O}_\infty^\circ$ is reflected by the fact that the elements of the basis of $W(H, \mathcal{O})(n_1, \ldots, n_k; n)$ determined by an operadic tree $\mathcal{T}$ are in one-to-one correspondence with the circled constructs of $\mathbf{H}_{\mathcal{T}}$. Indeed, a circled construct $C^\circ : \mathbf{H}_{\mathcal{T}}$ determines $h \in H(\alpha(\mathcal{T}, \sigma, C))$ as follows: if X is a vertex of the underlying construct $C : \mathbf{H}_{\mathcal{T}}$, then, in $\alpha(\mathcal{T}, \sigma, C)$, the edges determined by the set X are "already collapsed" and hence their length is taken to be λ_0; the connecting edges of C° will get the length λ_1 and the circled edges will get the length λ. With this identification, it becomes obvious that the external differential ∂_H is precisely $d_1^\circ - d_0^\circ$. ∎

2.4. **Combinatorial homotopy theory for cyclic operads.** In [16, Section 1.6.3], Lukács defined the $\mathbb{N}$-coloured operad $\mathcal{C}$ governing non-symmetric cyclic operads. His definition is based on *cyclic operadic trees*, i.e. operadic trees additionally equipped with bijections labeling clockwise all their leaves in a cyclic way with respect to the planar embedding, and the appropriate modification of the substitution operation. In this section, we show that, by switching from operadic to cyclic operadic trees, while preserving the faces of operadic polytopes as components of operations, one obtains the minimal model $\mathcal{C}_\infty$ for the operad $\mathcal{C}$.

2.4.1. *Cyclic operads as algebras over the colored operad* $\mathcal{C}$. Consider the equivalence classes of *cyclic operadic trees*, i.e. of triples $(\mathcal{T}, \sigma, \tau)$, where $(\mathcal{T}, \sigma)$ is an operadic tree and $\tau : \{0, 1, \ldots, n\} \to l(\mathcal{T})$ is a bijection labeling clockwise all the leaves of $\mathcal{T}$ in a cyclic way with respect to the planar embedding of $\mathcal{T}$, under the equivalence relation generated by:

> $(\mathcal{T}_1, \sigma_1, \tau_1) \sim (\mathcal{T}_2, \sigma_2, \tau_2)$ iff there exists an isomorphism $\varphi : \mathcal{U}(\mathcal{T}_1) \to \mathcal{U}(\mathcal{T}_2)$ of planar unrooted trees $\mathcal{U}(\mathcal{T}_1)$ and $\mathcal{U}(\mathcal{T}_2)$, obtained from $\mathcal{T}_1$ and $\mathcal{T}_2$ by forgetting the respective roots, such that $\sigma_1 = \varphi \circ \sigma_2$ and $\tau_1 = \varphi \circ \tau_2$.

Example 10. The following three cyclic operadic trees, in which the marked leaf is the 0-th in the cyclic order, are equivalent:



△

Observe that each equivalence class of a cyclic operadic tree $(\mathcal{T}, \sigma, \tau)$ can be represented by a triple $(\underline{\mathcal{T}}, \underline{\sigma}, \underline{\tau})$, for which $\underline{\tau}(0)$ is the root of $\underline{\mathcal{T}}$. In Example 10, this canonical representative is the tree in the middle.

Let, for $k \geq 1$, $n_1, \ldots, n_k \geq 1$ and $n = (\sum_{i=1}^k n_i) - k + 1$, $\mathcal{C}(n_1, \ldots, n_k; n)$ be the vector space spanned by the equivalence classes of cyclic operadic trees $(\mathcal{T}, \sigma, \tau)$, where $(\mathcal{T}, \sigma)$ is an operadic tree in $\mathcal{O}(n_1, \ldots, n_k, n)$. Note that this definition allows cyclic operadic trees with only one vertex (and no internal edges).

The operad structure on the collection $\mathcal{C}$ is defined in terms of substitution of trees that takes into account the cyclic orderings of leaves: when identifying the leaves of a vertex of the first tree with the leaves of the second tree, the root of that vertex is considered as 0-th in the planar order. More precisely, for $(\mathcal{T}_1, \sigma_1, \tau_1) \in \mathcal{C}(n_1, \ldots, n_k; n)$ and $(\mathcal{T}_2, \sigma_2, \tau_2) \in \mathcal{C}(m_1, \ldots, m_l; n_i)$, we have

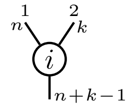
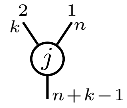
$$(\mathcal{T}_1, \sigma_1, \tau_1) \circ_i (\mathcal{T}_2, \sigma_2, \tau_2) = (\mathcal{T}_1 \bullet_i \underline{\mathcal{T}_2}, \sigma_1 \bullet_i \underline{\sigma_2}, \tau_1),$$

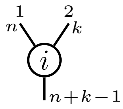
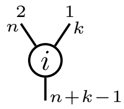
where the $\bullet_i$ operation is the one defined for operadic trees in §2.1.2, and where $(\underline{\mathcal{T}_2}, \underline{\sigma_2}, \underline{\tau_2})$ is the canonical representative of the class determined by $(\mathcal{T}_2, \sigma_2, \tau_2)$. The action of the symmetric group is defined by $(\mathcal{T}, \sigma, \tau)^\kappa = (\mathcal{T}, \sigma \circ \kappa, \tau)$.

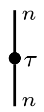
Lemma 13. *Algebras over* $\mathcal{C}$ *are (non-unital, non-symmetric, reduced) dg cyclic operads.*

Proof. This has been proven in [16, Proposition 1.6.4]. An alternative proof can be obtained by translating Lukács' definition of $\mathcal{C}$ into an equivalent definition that describes $\mathcal{C}$ in terms of generators (i.e. composite trees) and relations. Such a definition is obtained by extending the set of generators E and the set of relations R from Definition 2 by adding the unary generators encoding cyclic permutations and the relations governing them, as follows.

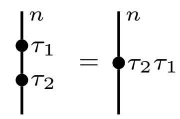
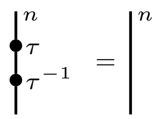
Indeed, relying on Lemma 2, it can be shown that $\mathcal{C} \simeq \mathcal{T}_{\mathbb{N}}(\hat{E})/(\hat{R})$, where the set of generators $\hat{E}$ is given by binary operations

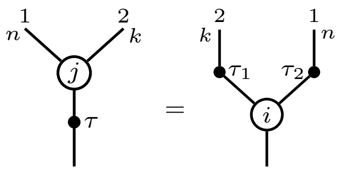
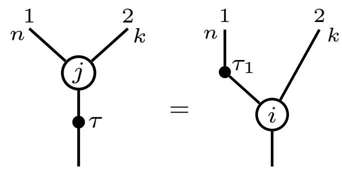
$\hat{E}(n,k;n+k-1) = \Big\{$  $,\quad 1 \le i \le n \Big\} \cup \Big\{$  $,\quad 1 \le j \le k \Big\},$

for $n, k \geq 1$, equipped with the action of the transposition (21) that sends  to , and unary operations

$\hat{E}(n;n) = \Big\{$  $,\quad \tau \in \mathbb{Z}^{op}_{n+1} \backslash \{id\} \Big\}$

for $n \geq 1$, realized as cyclic permutations (i.e. bijections preserving the cyclic order) of the cyclically ordered set $(0, 1, \ldots, n)$, and the set of relations $\hat{R}$ is given by the relation (A1) from Definition 2, together with the following equalities, concerning the action of cyclic permutations:

(C1)  (C2) 

(C3)  (C4) 

where, in (C3), it is assumed that $j \le \tau(0) \le k+j-1$, i.e. that $\tau(0) = k - i + j$ for some $1 \le i \le k$, and τ_1 is determined by $\tau_1(i) = 0$ and τ_2 by $\tau_2(0) = j$, and, in (C4), it is assumed that $1 \le \tau(0) \le j-1$ (resp. $k+j \le \tau(0) \le n+k-1$), and τ_1 is determined by $\tau_1(0) = \tau(0)$ (resp. $\tau_1(0) = \tau(0) - k + 1$). Note that the relation (A2) from Definition 2 does not appear in the above description of $\mathcal{C}$, since it can be proven by a sequence of equalities witnessed by the relations (A1), (C1), (C2), (C3) and (C4).

A $\mathcal{C}$-algebra is then a dg $\mathbb{N}$-module $(A, d) = \{(A(n), d_{A(n)})\}_{n \geq 1}$, equipped with the obvious unary and binary operations. Observe that the unary generators determine a right $\mathbb{Z}_{n+1}$ action on each $A(n)$, making $\{A(n)\}_{n\geq 1}$ a cyclic $\mathbb{N}$-module, the relations (A1) and (A2) ensure the associativity of the operadic composition, while the relations (C3) and (C4) establish the compatibility of the action of cyclic permutations with the operadic composition. That this is indeed a cyclic operad follows according to [19, Proposition 42.]. ■

2.4.2. *The combinatorial $\mathcal{C}_\infty$ operad.* The $\mathcal{C}_\infty$ operad is the dg operad whose structure is induced by the one of the $\mathcal{O}_\infty$ operad, by replacing operadic trees with cyclic operadic trees, while preserving the faces of operadic polytopes as components of operations. Here, the operadic polytope associated to a cyclic operadic tree $(\mathcal{T}, \sigma, \tau)$ remains simply the hypergraph polytope for the edge-graph $\mathbf{H}_{\mathcal{T}}$ of the tree $\mathcal{T}$. In particular, if $\mathcal{T}$ has only one vertex, its associated edge-graph is the empty hypergraph $\mathbf{H}_\emptyset$ (i.e. the unique hypergraph whose set of vertices is empty), whose set of constructs is the singleton containing the empty construct. Observe that this modification is well defined, since equivalent cyclic operadic trees have the same associated edge-graph.

More precisely, for $k \geq 1$, $n_1, \ldots, n_k \geq 1$ and $n = (\sum_{i=1}^k n_i) - k + 1$, we define $\mathcal{C}_\infty(n_1, \ldots, n_k; n)$ to be the vector space spanned by the equivalence classes of quadruples $(\mathcal{T}, \sigma, \tau, C)$, where $(\mathcal{T}, \sigma, \tau)$ is a cyclic operadic tree in $\mathcal{C}(n_1, \ldots, n_k; n)$ and $C : \mathbf{H}_{\mathcal{T}}$. For $(\mathcal{T}_1, \sigma_1, \tau_1, C_1) \in \mathcal{C}_\infty(n_1, \ldots, n_k; n)$ and $(\mathcal{T}_2, \sigma_2, \tau_2, C_2) \in \mathcal{C}_\infty(m_1, \ldots, m_l; n_i)$, we define the partial composition operation $\circ_i$ by

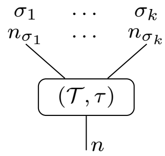
$$(\mathcal{T}_1, \sigma_1, \tau_1, C_1) \circ_i (\mathcal{T}_2, \sigma_2, \tau_2, C_2) := (\mathcal{T}_1 \bullet_i \underline{\mathcal{T}_2}, \sigma_1 \bullet_i \underline{\sigma_2}, C_1 \bullet_i C_2, \tau_1),$$

where $(\underline{\mathcal{T}_2}, \underline{\sigma_2}, \underline{\tau_2})$ is the canonical representative of the class determined by $(\mathcal{T}_2, \sigma_2, \tau_2)$, and $C_1 \bullet_i C_2$ is defined exactly as in §2.2.2. This determines an operad on vector spaces which is easily proven to be free over the equivalence classes represented of left-recursive cyclic operadic trees (i.e. cyclic operadic trees whose underlying operadic trees are left-recursive):

$$\mathcal{C}_\infty \simeq \mathcal{T}_{\mathbb{N}}\Bigg(\bigoplus_{k\geq 1}\ \bigoplus_{n_1,\dots,n_k\geq 1}\ \bigoplus_{(\mathcal{T},\tau)\in\mathcal{C}(n_1,\dots,n_k;n)} \Bbbk\Bigg).$$

Indeed, the corresponding isomorphism α is defined by introducing the decomposition of an operation $(\mathcal{T},\sigma,\tau,C)\in\mathcal{C}_\infty(n_1,\dots,n_k;n)$ by the construction analogous to the one from the proof of Theorem 1, which, in addition, takes into account the data given by τ, as follows. Denote, for $1\leq i\leq k$, with σ_i the index given by σ to the i-th vertex in the left-recursive ordering of $\mathcal{T}$.

- If C is the maximal construct $e(\mathcal{T})$, then

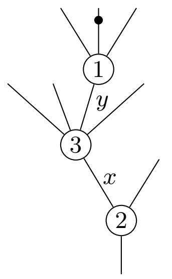
$\alpha(\mathcal{T},\sigma,\tau,e(\mathcal{T}))=$ 

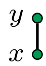
- Suppose that $C = X\{C_1,\dots,C_p\}$, where $\mathbf{H}_{\mathcal{T}}\backslash X \rightsquigarrow \mathbf{H}_{\mathcal{T}_1},\dots,\mathbf{H}_{\mathcal{T}_p}$ and $C_i : \mathbf{H}_{\mathcal{T}_i}$. Let $(\mathcal{T}_X,\tau)$ be the left-recursive cyclic operadic tree obtained from $(\mathcal{T},\sigma,\tau)$ by collapsing all the edges from $e(\mathcal{T})\backslash X$. Note that this construction preserves τ. Once again, the collapse of the edges $e(\mathcal{T})\backslash X$ that defines $\mathcal{T}_X$ is, in fact, the collapse of the subtrees $\mathcal{T}_i$, $1\leq i\leq p$, of $\mathcal{T}$. Denote with τ_i the indexing of the leaves of $\mathcal{T}_i$ that sends 0 to the root of $\mathcal{T}_i$. We define

$$\alpha(\mathcal{T},\sigma,\tau,C)=\Big((\cdots(\alpha(\mathcal{T}_X,\tau,X)\circ_{\rho(\mathcal{T}_1)}\alpha(\mathcal{T}_1,\tau_1,C_1))\cdots)\circ_{\rho(\mathcal{T}_p)}\alpha(\mathcal{T}_p,\tau_p,C_p)\Big)^{\sigma_C},$$

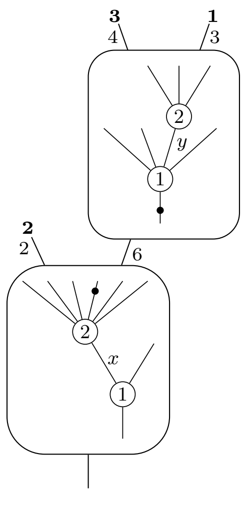
  where σ_C is the permutation determined uniquely thanks to Lemma 8.

Example 11. The free operad representation of the operation $(\mathcal{T},\sigma,\tau,C)$, where

$(\mathcal{T},\sigma,\tau)=$ 

and $C=$ 

is given by



△

The space $\mathcal{C}_\infty(n_1,\dots,n_k;n)$ is then graded by setting $|(\mathcal{T},\sigma,\tau,C)| := e(\mathcal{T})-v(C)$. Analogously as we did for the operad $\mathcal{O}_\infty$, the free operad structure of $\mathcal{C}_\infty$ is used for the introduction of signs for the corresponding dg extension. In particular, the differential $d_{\mathcal{C}_\infty}$ of $\mathcal{C}_\infty$ acts like the differential $d_{\mathcal{O}_\infty}$ of

$\mathcal{O}_\infty$, i.e. it splits the vertices of constructs and does not change the underlying cyclic operadic tree; exceptionally, $d_{\mathcal{C}_\infty}$ maps the empty construct to 0.

We can, therefore, conclude.

Theorem 5. *The operad $\mathcal{C}_\infty$ is the minimal model for the operad $\mathcal{C}$.*

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