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Jiří Neustupa

Minsuk Yang

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Jiří Neustupa & Minsuk Yang

Abstract

We show that $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$ is a regular point of a suitable weak solution $(\mathbf{u}, \mathbf{b}, p)$ of the MHD equations in $\Omega \times (0,T)$ (where Ω is a domain in $\mathbb{R}^3$ and $T > 0$) if the limit inferior (for $t \to t_0-$) of the sum of the L^3–norms of $\mathbf{u}$ and $\mathbf{b}$ over an arbitrarily small ball $B_\rho(\mathbf{x}_0)$ is less than infinity.

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1 Introduction

1.1 The system of MHD equations, a weak and a suitable weak solution

The motion of a viscous incompressible electrically conducting fluid in domain $\Omega \subset \mathbb{R}^3$ in the time interval $(0,T)$ (where $T > 0$), in the absence of an acting external body force and magnetic or electric field, is described by the system of magneto-hydro-dynamical equations (which is abbreviated to MHD equations)

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b} \;=\; -\nabla p + \nu \Delta \mathbf{u}, \tag{1.1}$$

$$\partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} \;=\; \xi \Delta \mathbf{b}. \tag{1.2}$$

$$\operatorname{div} \mathbf{u} \;=\; \operatorname{div} \mathbf{b} \;=\; 0 \tag{1.3}$$

in $Q_T := \Omega \times (0,T)$. The unknowns are the velocity $\mathbf{u}$ of the fluid, the magnetic field $\mathbf{b}$ and the pressure p. The coefficients ν and ξ (which are supposed to be positive constants) represent the kinematic viscosity and the magnetic diffusivity, respectively. In order to formulate a consistent initial–boundary value problem, we complete equations (1.1)–(1.3) by the initial conditions

$$\mathbf{u}(\,.\,,0) = \mathbf{u}_0, \qquad \mathbf{b}(\,.\,,0) = \mathbf{b}_0 \tag{1.4}$$

and by appropriate boundary conditions. Usually considered boundary conditions are the so called Navier-type conditions

$$\text{a)} \quad \mathbf{b} \cdot \mathbf{n} = 0, \qquad \text{b)} \quad \mathbf{curl}\, \mathbf{b} \times \mathbf{n} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T) \tag{1.5}$$

(where $\mathbf{n}$ denotes the outer normal vector field on $\partial\Omega$) for the magnetic field and either the no-slip condition

$$\mathbf{u} \;=\; \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T) \tag{1.6}$$

or the Navier-type conditions

$$\text{a)} \quad \mathbf{u}\cdot\mathbf{n} = 0, \qquad \text{b)} \quad \mathbf{curl}\,\mathbf{u}\times\mathbf{n} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T) \tag{1.7}$$

for the velocity. (See e.g. [18], [19], [21], [37], [38] and [39].)

Independent of the boundary conditions, we call the pair $(\mathbf{u},\mathbf{b}) \in \big[L^\infty(0,T;\, \mathbf{L}^2(\Omega)) \cap L^2(0,T;\, \mathbf{W}^{1,2}(\Omega))\big]^2$ a *weak solution* to the system (1.1)–(1.3), with the initial conditions (1.4)), if $\mathbf{u}$ and $\mathbf{b}$ are divergence-free (in the sense of distributions) in Q_T and the integral identities

$$\int_0^T\!\!\int_\Omega \big[-\mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi} - \mathbf{b}\cdot\nabla\mathbf{b}\cdot\boldsymbol{\phi} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_\Omega \mathbf{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\mathbf{x}, \tag{1.8}$$

$$\int_0^T\!\!\int_\Omega \big[-\mathbf{b}\cdot\partial_t\boldsymbol{\phi} + \mathbf{u}\cdot\nabla\mathbf{b}\cdot\boldsymbol{\phi} - \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi} + \xi\nabla\mathbf{b} : \nabla\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_\Omega \mathbf{b}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\mathbf{x}, \tag{1.9}$$

hold for all infinitely differentiable divergence-free vector functions $\boldsymbol{\phi}$ in Q_T, with a compact support in $\Omega\times[0,T)$. A distribution p in Q_T is said to be an *associate pressure* if $\mathbf{u}$, $\mathbf{b}$ and p satisfy the equations (1.1)–(1.3) in the sense of distributions in Q_T. If $(\mathbf{u},\mathbf{b})$ is a weak solution and the pressure p is a locally integrable function in Q_T, such that the product $p\mathbf{u}$ is integrable in Q_T, and $\mathbf{u}$, $\mathbf{b}$, p satisfy the so called *the localized energy inequality*

$$\begin{aligned} &\int_{Q_T} 2\big(\nu|\nabla\mathbf{u}|^2 + \xi|\nabla\mathbf{b}|^2\big)\,\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\le \int_{Q_T} \big[|\mathbf{u}|^2\,(\partial_t\psi + \nu\Delta\psi) + |\mathbf{b}|^2\,(\partial_t\psi + \xi\Delta\psi) \\ &\qquad + \big(|\mathbf{u}|^2 + |\mathbf{b}|^2 + 2p\big)\,(\mathbf{u}\cdot\nabla\psi) - 2(\mathbf{u}\cdot\mathbf{b})\,(\mathbf{b}\cdot\nabla\psi)\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \end{aligned} \tag{1.10}$$

for every non-negative infinitely differentiable scalar function ψ in Q_T, compactly supported in Q_T, then we call $(\mathbf{u},\mathbf{b},p)$ a *suitable weak solution* to the system (1.1)–(1.3).

A sketch of the construction of a suitable weak solution (which is analogous to the Navier–Stokes equations) can be found in the paper [11] by Ch. He and Z. Xin. Note that while the existence of a weak solution is guaranteed in any domain $\Omega\subset\mathbb{R}^3$ (which can be proven by the same method as for the Navier–Stokes equations), the existence of a suitable weak solution is a subtler problem, because, as it has already been said above, it requires the existence of the associated pressure as in function in Q_T with an appropriate rate of integrability. For the Navier–Stokes equations, the existence of such a pressure is known if Ω is a "smooth" bounded or exterior domain in $\mathbb{R}^3$, or a half-space or the whole $\mathbb{R}^3$, see e.g. H. Sohr and W. von Wahl [36]. The situation in the theory of the MHD equations is similar: although one can define the notion of a suitable weak solution regardless the shape and smoothness of Ω, the existence of the suitable weak solution is known only if domain Ω is "smooth" and of one of the aforementioned types.

1.2 Previous results on regularity of weak solutions to the MHD equations

Recall that the point $(\mathbf{x}_0, t_0) \in Q_T$ is said to be a *regular point* of the a solution $(\mathbf{u}.\mathbf{b})$ if there exists a neighbourhood $U \subset Q_T$ of this point such that both $\mathbf{u}$ and $\mathbf{b}$ are essentially bounded in U. Other points of Q_T are called *singular points* of solution $(\mathbf{u}, \mathbf{b})$.

By analogy with the Navier–Stokes equations, the question of regularity of weak (and particularly also suitable weak) solutions to the MHD equations (1.1)–(1.3) (i.e. the question whether singular points can develop in a weak solution if all the given data are sufficiently smooth and integrable) is generally open.

Ch. He and Z. Xin [11] (2005) derived a series of local regularity criteria (i.e. criteria for the regularity at a point $(\mathbf{x}_0, t_0) \in Q_T$) for a suitable weak solution in terms of the quantities

$$\sup_{t_0-\rho^2 \le t < t_0} \frac{1}{\rho} \int_{B_r(\mathbf{x}_0)} |\mathbf{u}|^2 \, \mathrm{d}\mathbf{x}, \qquad \frac{1}{\rho^{5-a}} \int_{t_0-\rho^2}^{t_0} \int_{B_\rho(\mathbf{x}_0)} |\mathbf{u}|^a \, \mathrm{d}\mathbf{x} \, \mathrm{d}t,$$
$$\frac{1}{\rho} \int_{t_0-\rho^2}^{t_0} \int_{B_\rho(\mathbf{x}_0)} |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t, \qquad \frac{1}{\rho} \int_{t_0-\rho^2}^{t_0} \int_{B_\rho(\mathbf{x}_0)} |\mathbf{curl}\, \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

for the velocity field $\mathbf{u}$ and the analogous quantities for the magnetic field $\mathbf{b}$ or the pressure p. Some generalizations or slight improvements can be found in the papers [17] (2009) by K. Kang, J. Lee and [41] (2013) by W. Wang, Z. Zhang. Note that one can deduce from some criteria that the 1-dimensional Hausdorff measure of the set of hypothetic singular points of the suitable weak solution in Q_T is zero.

It follows from paper [24] (2007) by A. Mahalov, B. Nicolaenko and T. Shilkin that if $(\mathbf{u}, \mathbf{b})$ is a weak solution, the velocity $\mathbf{u}$ satisfies Serrin's condition (the authors call it the Ladyzhenskaya–Prodi–Serrin condition) $\mathbf{u} \in L^r(t_0 - \rho^2, t_0;\, \mathbf{L}^s(B_\rho(\mathbf{x}_0)))$ (where $\rho > 0$, $s > 3$ and $2/r + 3/s = 1$) then $\mathbf{u}$ and $\mathbf{b}$, together with all their spatial derivatives (of all orders) are Hölder–continuous in $B_{\rho/2}(\mathbf{x}_0) \times (t_0 - \frac{1}{4}\rho^2, t_0)$. The authors also consider the critical case $r = \infty$, $s = 3$, but here, they need both $\mathbf{u}$ and $\mathbf{b}$ to be in $L^\infty(t_0 - \rho^2, t_0;\, \mathbf{L}^3(B_\rho(\mathbf{x}_0)))$.

In paper [24], the authors have also excluded the possibility of existence of a collapsing self-similar weak solution with the generating profile in $\mathbf{L}^3(\mathbb{R}^3)$. (The same result for the Navier–Stokes equations was already known before, see [25].)

J. Wu [42] (2004) considered a weak solution $(\mathbf{u}, \mathbf{b})$ in $\mathbb{R}^3 \times (0, T)$ and showed that if both the velocity $\mathbf{u}$ and the magnetic field $\mathbf{b}$ lie in the space $L^r(0, T;\, \mathbf{L}^s(\mathbb{R}^3))$ for some exponents $r > 3$, $s \ge 2$ such that $2/r + 3/s = 1$, then in fact $\mathbf{u}$ and $\mathbf{b}$ belong to $L^\infty(0, T;\, \mathbf{L}^s(\mathbb{R}^3))$, whereas $\nabla \mathbf{u}$

and $\nabla \mathbf{b}$ belong to $L^\infty(0,T;\, L^2(\mathbb{R}^3)^{3\times 3})$. The authors have also proved some finer results for axially symmetric solutions.

Ch. He and Z. Xin [10] (2005) proved that if $\Omega = \mathbb{R}^3$, $(\mathbf{u}, \mathbf{b})$ is a weak solution and the velocity $\mathbf{u}$ satisfies $\mathbf{u} \in L^r(0,T;\, \mathbf{L}^s(\mathbb{R}^3))$ with $2/r + 3/s \leq 1$; $s > 3$, or $\mathbf{u} \in C([0,T];\, \mathbf{L}^3(\mathbb{R}^3))$, or $\nabla \mathbf{u} \in L^r(0,T;\, L^s(\mathbb{R}^3)^{3\times 3})$ with $2/r + 3/s = 2$, $1 < r \leq 2$ then $(\mathbf{u}, \mathbf{b})$ is smooth in $\mathbb{R}^3 \times (0,T)$. Analogous results can also be found in paper [47] (2005) by Y. Zhou. It is remarkable that no conditions are imposed on the magnetic field $\mathbf{b}$. Some refinements of these results are proven in the papers [4] (2008, by Q. Chen, C. Miao and Z. Zhang, sufficient conditions for regularity formulated by means of norms of Besov spaces) and [50] (2010, by Y. Zhou and S. Gala, sufficient conditions for regularity formulated in a certain multiplier space). By analogy with the result of P. Constantin and C. Fefferman [5], concerning the Navier–Stokes equations, it is also shown in [10] that $(\mathbf{u}, \mathbf{b})$ is smooth if $\mathbf{curl}\, \mathbf{u}$ does not change the direction “too quickly”.

The regularity of the weak solution $(\mathbf{u}, \mathbf{b})$ in the limiting case, when both $\mathbf{u}$ and $\mathbf{b}$ are supposed to be in $L^\infty(0,T;\, \mathbf{L}^3(\mathbb{R}^3))$, was proven in paper [40] (2012) by W. Wang and Z. Zhang. The proof is based on the blow-up analysis using the backward uniqueness and unique continuation theorems for parabolic equations, developed by L. Escauriaza, G. Seregin and V. Šverák in [6] (2003).

Some logarithmically improved regularity criteria for the MHD equations can be found in the papers [7] (2011, by J. Fan, S. Jiang, G. Nakamura and Y. Zhou) and [49] (2012, by Y. Zhou and J. Fan).

Regularity of the weak solution $(\mathbf{u}, \mathbf{b})$ in dependence of conditions imposed only on some components of $\mathbf{u}$, $\mathbf{b}$, or $\nabla \mathbf{u}$ and $\nabla \mathbf{b}$, was studied by E. Ji, J. Lee in [12] (2010), by C. Cao, J. Wu in [3] (2010; importance of the directional derivative of $\mathbf{u}$), X. Jia and Y. Zhou in [13] (2012), [14] (2014), [15] (2015) and [16] (2016), by L. Ni, Z. Guo and Y. Zhou in [32] (2012), by H. Lin, L. Du in [22] (2013), by K. Yamazaki in [43] (2014) and [44] (2014), by Z. Zhang in [45] (2015) and [46] (2015) and by Ch. Qian in [33] (2018).

The regularity of a suitable weak solution $(\mathbf{u}, \mathbf{b}, p)$ under some conditions imposed on the pressure is proven in the papers [48] (2006, by Y. Zhou), [17] (2009, by K. Kang and J. Lee), [3] (2010, by C. Cao and J. Wu), [49] (2012, by Y. Zhou and J. Fan), [22] (2013, by H. Lin and L. Du) and [14] (2014, by X. Jia and Y. Zhou).

The regularity of solution $(\mathbf{u}, \mathbf{b})$ up to the boundary was studied by K. Kang, J.-M. Kim in [18] and [19], by J.-M. Kim in [21] (2017), by V. Vyalov and T. Shilkin in [37] (2011), by V. Vyalov in [38] (2012) and [39] (2014).

1.3 The result of this paper

We assume that Ω is either the whole space $\mathbb{R}^3$, or a bounded or exterior domain in $\mathbb{R}^3$ with the boundary of class $C^{2+(h)}$ for some $h > 0$, or a half-space in $\mathbb{R}^3$ and $(\mathbf{u}, \mathbf{b}, p)$ is a suitable weak solution to the system (1.1)–(1.3), such that $\mathbf{u}$ satisfies either the no-slip boundary condition (1.6) (if $\Omega \neq \mathbb{R}^3$) or the Navier-type boundary conditions (1.7) or Navier's boundary conditions

$$\text{a)} \quad \mathbf{u}\cdot\mathbf{n} = 0, \qquad \text{b)} \quad \big[\mathbb{T}_{\mathrm{d}}(\mathbf{u})\cdot\mathbf{n}\big]_\tau + \gamma\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega\times(0,T)) \tag{1.11}$$

(for Ω bounded). In (1.11), $\mathbb{T}_{\mathrm{d}}(\mathbf{u}) := 2\nu\,(\nabla\mathbf{u})_{\mathrm{sym}}$ denotes the dynamic stress tensor induced by the velocity field $\mathbf{u}$, subscript τ denotes the tangential component and γ (a non-negative constant) denotes the coefficient of friction between the fluid and the boundary of Ω. As to the magnetic field $\mathbf{b}$, we only assume that it satisfies the boundary condition (1.5) a).

The reason, why we do not need any other assumption regarding the boundary condition (or conditions) satisfied by the $\mathbf{b}$, is that we shall apply certain results on the local interior regularity of pressure from [31] in the next section and the pressure appears only in the momentum equation (1.1), which is an evolution equation for velocity $\mathbf{u}$. Thus, p is a global quantity, whose local properties in the neighbourhood of any point $\mathbf{x}_0 \in \Omega$ are more influenced by the boundary conditions satisfied by $\mathbf{u}$ than by the conditions satisfied by $\mathbf{b}$ on $\partial\Omega\times(0,T)$.

The main results are formulated in the next theorems:

Theorem 1. *Let Ω be a domain in $\mathbb{R}^3$, satisfying the aforementioned assumptions, and $(\mathbf{u}, \mathbf{b}, p)$ be a suitable weak solution to the system* (1.1)–(1.3) *in Q_T with the boundary conditions satisfied by $\mathbf{u}$ and $\mathbf{b}$ in relation with the type of Ω, as is specified above. There exists $\varepsilon > 0$ such that if $(\mathbf{x}_0, t_0) \in Q_T$ and the condition*

$$\liminf_{t\to t_0-} \big(\|\mathbf{u}(\,.\,,t)\|_{3;\,B_\rho(\mathbf{x}_0)} + \|\mathbf{b}(\,.\,,t)\|_{3;\,B_\rho(\mathbf{x}_0)}\big) \;<\; \infty, \tag{1.12}$$

holds for some $\rho > 0$, then $(\mathbf{x}_0, t_0)$ is a regular point of the solution $(\mathbf{u}, \mathbf{b}, p)$.

(Here, $\|\,.\,\|_{3;\,B_\rho(\mathbf{x}_0)}$ denotes the L_3–norm over the ball $B_\rho(\mathbf{x}_0)$.)

The theorem generalizes the result from [24], where the authors required

$$\operatorname*{ess\,sup}_{t_0-\rho^2<t<t_0} \big(\|\mathbf{u}(\,.\,,t)\|_{3;\,B_\rho(\mathbf{x}_0)} + \|\mathbf{b}(\,.\,,t)\|_{3;\,B_\rho(\mathbf{x}_0)}\big) \;<\; \infty$$

instead of our condition (1.12).

Note that the theorem is also valid in the special case $\mathbf{b} \equiv \mathbf{0}$, and it states that the necessary condition for the development of a singularity in a suitable weak solution $(\mathbf{u}, p)$ to the Navier–Stokes equations at the point $(\mathbf{x}_0, t_0) \in Q_T$ is that the limit (for $t \to t_0-$) of $\|\mathbf{u}(\,.\,,t)\|_{3;\,B_\rho(\mathbf{x}_0)}$ (for all $\rho > 0$

arbitrarily small) is equal to infinity. This statement generalizes the result of G. Seregin from paper [34], where the author considered $\Omega = \mathbb{R}^3$ and the main theorem says that t_0 is the so called epoch of irregularity only if

$$\lim_{t\to t_0-} \|\mathbf{u}(\,.\,,t)\|_{3;\,\mathbb{R}^3} \;=\; \infty.$$

(Recall that t_0 is called an *epoch of irregularity* of solution $\mathbf{u}$ if $\mathbf{u}$ is smooth on the interval $(t_0-\delta, t_0)$ for some $\delta > 0$ and it has a singular point on the time level $t = t_0$, which means that it "blows up" for $t \to t_0-$.)

2 Proof of Theorem 1 – part I

Notation. *We use the notation $A \lesssim B$ if there exists a generic positive constant C such that $|A| \leq C|B|$. In order to stress the dependence of constant C on parameter M, we write $A \lesssim_M B$.*

2.1 Localization to the neighbourhood of $\mathbf{x}_0$ and the definition of the functions $\widehat{\mathbf{u}}$, $\widehat{\mathbf{b}}$, $\widehat{p}$

There exist positive numbers ρ_1 and ρ_2 such that $0 < \rho_1 < \rho_2$ and $A_{\rho_1,\rho_2}(\mathbf{x}_0) \times (0,T)$ contains no singular points, where we denote the annulus in $\mathbb{R}^3$ by

$$A_{\rho_1,\rho_2}(\mathbf{x}_0) \;:=\; \{\mathbf{x} \in \mathbb{R}^3;\; \rho_1 < |\mathbf{x}-\mathbf{x}_0| < \rho_2\}.$$

This follows from the fact that the 1–dimensional Hausdorff measure of singular points of a solution $(\mathbf{u}, \mathbf{b}, p)$ is zero, using the same arguments as in [26] or [27]. Without loss of generality, we can choose ρ_2 so small that $\rho_2 \leq \rho$. Thus, $\mathbf{u}$ are $\mathbf{b}$ are essentially bounded in $A_{\rho_1,\rho_2}(\mathbf{x}_0) \times (\delta, T-\delta)$ for each $\delta > 0$. (We assume that δ is so small that $3\delta < t_0 < T - 3\delta$.)

Applying the results of [24], one can deduce that $\mathbf{u}$, $\mathbf{b}$, and all their spatial derivatives (of all orders), are Hölder–continuous in $A_{\rho_3,\rho_4}(\mathbf{x}_0) \times (2\delta, T-2\delta)$ for all ρ_3 and ρ_4 such that $\rho_1 < \rho_3 < \rho_4 < \rho_2$. As $\partial_t\mathbf{b}$ is expressed by equation (1.2) $\mathbf{u}$, $\mathbf{b}$ and their spatial derivatives, we observe that $\partial_t\mathbf{b}$, together with all its spatial derivatives, is Hölder–continuous in $A_{\rho_3,\rho_4}(\mathbf{x}_0) \times (2\delta, T-2\delta)$, too. An information on the regularity of the functions p and $\partial_t\mathbf{u}$ in $A_{\rho_3,\rho_4}(\mathbf{x}_0) \times (2\delta, T-2\delta)$ follows from [31]. Function $\mathbf{b}$ is only supposed to satisfy the boundary condition (1.5) a) in [31]. As p is a "global quantity" and ∇p appears only in the balance of momentum equation (1.1), the interior regularity of p mainly depends on boundary conditions, satisfies by function $\mathbf{u}$ on $\partial\Omega \times (0,T)$. (See also paragraph 1.3.) Concretely, it follows from [31] that

(a) if $\Omega = \mathbb{R}^3$ and $\rho_3 < \rho_5 < \rho_6 < \rho_4$ then $\partial_t\mathbf{u}$, ∇p and all their spatial derivatives (of all orders) are in are essentially bounded in $A_{\rho_5,\rho_6}(\mathbf{x}_0) \times (3\delta, T-3\delta)$,

(b) if Ω is a bounded or exterior domain in $\mathbb{R}^3$ with the boundary at least of the class $C^{2+(h)}$ for some $h > 0$ or a half-space in $\mathbb{R}^3$ and $\mathbf{u}$ satisfies the boundary condition (1.6) then $\partial_t\mathbf{u}$, ∇p and all their spatial derivatives (of all orders) are in $L^{\mu}(3\delta, T-3\delta;\, \mathbf{L}^{\infty}(A_{\rho_5,\rho_6}(\mathbf{x}_0)))$ for any $\mu \in (1,2)$,

(c) if Ω is a bounded domain in $\mathbb{R}^3$ with the boundary at least of the class $C^{2+(h)}$ for some $h > 0$ and $\mathbf{u}$ satisfies the boundary conditions (1.7) then one can make the same conclusion on $\partial_t\mathbf{u}$, ∇p as in item (a),

(d) if Ω is a bounded domain in $\mathbb{R}^3$ with the boundary at least of the class $C^{2+(h)}$ for some $h > 0$ and $\mathbf{u}$ satisfies the boundary conditions (1.11) then $\partial_t\mathbf{u}$, ∇p and all their spatial derivatives (of all orders) are in $L^4(3\delta, T-3\delta;\, \mathbf{L}^{\infty}(A_{\rho_5,\rho_6}(\mathbf{x}_0)))$.

Analogous results for the Navier–Stokes equations can be found in [30], [28], [29] and [35].

Let ρ_7 and ρ_8 be positive numbers, satisfying $\rho_5 < \rho_7 < \rho_8 < \rho_6$. Let η be an infinitely differentiable cut–off function in $[0,\infty)$ such that

$$\eta(\sigma) \begin{cases} = 1 & \text{for } 0 \le \sigma \le \rho_7, \\ \in (0,1) & \text{for } \rho_5 < \sigma < \rho_8, \\ = 0 & \text{for } \rho_8 \le \sigma. \end{cases}$$

In order to obtain functions supported only in the closure of $B_{\rho_4}(\mathbf{x}_0) \times (0,T)$, we multiply $\mathbf{u}$ and $\mathbf{b}$ by η. Obviously, $\operatorname{div}(\eta\mathbf{u}) = \nabla\eta\cdot\mathbf{u}$ and $\operatorname{div}(\eta\mathbf{b}) = \nabla\eta\cdot\mathbf{b}$. In order to obtain divergence-free functions, we put

$$\widehat{\mathbf{u}} := \eta\mathbf{u} - \mathbf{u}_{\mathrm{corr}} \qquad \text{and} \qquad \widehat{\mathbf{b}} := \eta\mathbf{b} - \mathbf{b}_{\mathrm{corr}},$$

where the correcting terms $\mathbf{u}_{\mathrm{corr}}$ and $\mathbf{b}_{\mathrm{corr}}$ satisfy $\operatorname{div}\mathbf{u}_{\mathrm{corr}} = \nabla\eta\cdot\mathbf{u}$ and $\operatorname{div}\mathbf{b}_{\mathrm{corr}} = \nabla\eta\cdot\mathbf{b}$. The existence of appropriate functions $\mathbf{u}_{\mathrm{corr}}$ and $\mathbf{b}_{\mathrm{corr}}$ follows e.g. from [8, Theorem III.3.2] or [1, Theorem 2.4]. Due to these theorems, there exists a linear mapping

$$\mathfrak{B} : W_0^{m,2}(A_{\rho_5,\rho_6}(\mathbf{x}_0)) \to \mathbf{W}_0^{m+1,2}(A_{\rho_5,\rho_6}(\mathbf{x}_0))$$

for all $m \in \{0\}\cup\mathbb{N}$ such that for all $f \in W_0^{m,2}(A_{\rho_5,\rho_6}(\mathbf{x}_0))$, satisfying $\int_{A_{\rho_5,\rho_6}(\mathbf{x}_0)} f \,\mathrm{d}\mathbf{x} = 0$,

1. $\operatorname{div}\mathfrak{B}f = f$ a.e. in $A_{\rho_5,\rho_6}(\mathbf{x}_0)$,
2. $\|\nabla^{m+1}\mathfrak{B}f\|_{2;\,A_{\rho_5,\rho_6}(\mathbf{x}_0)} \lesssim \|\nabla^m f\|_{2;\,A_{\rho_5,\rho_6}(\mathbf{x}_0)}$.

Mapping $\mathfrak{B}$ is often called the Bogovskij operator. Let us denote by $S_\rho(\mathbf{x}_0)$ the sphere with the center at point $\mathbf{x}_0$ and radius ρ. Then $\partial A_{\rho_5,\rho_6}(\mathbf{x}_0) = S_{\rho_5}(\mathbf{x}_0)\cup S_{\rho_6}(\mathbf{x}_0)$. Since $\eta = 1$ on $S_{\rho_5}(\mathbf{x}_0)$, $\eta = 0$ on $S_{\rho_6}(\mathbf{x}_0)$ and $\operatorname{div}\mathbf{u} = 0$ in $A_{\rho_5,\rho_6}(\mathbf{x}_0)$, we have

$$\int_{A_{\rho_5,\rho_6}(\mathbf{x}_0)} \nabla\eta\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; \int_{S_{\rho_5}(\mathbf{x}_0)} \eta\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S + \int_{S_{\rho_6}(\mathbf{x}_0)} \eta\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S - \int_{A_{\rho_5,\rho_6}(\mathbf{x}_0)} \eta\operatorname{div}\mathbf{u}\,\mathrm{d}\mathbf{x}$$

$$= \int_{S_{\rho_5}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S \;=\; -\int_{B_{\rho_5}(\mathbf{x}_0)} \operatorname{div}\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; 0,$$

we may put

$$\mathbf{u}_{\rm corr}(\,.\,,t) := \mathfrak{B}\big[\nabla\eta\cdot\mathbf{u}(\,.\,,t)\big] \qquad \text{and} \qquad \mathbf{b}_{\rm corr}(\,.\,,t) := \mathfrak{B}\big[\nabla\eta\cdot\mathbf{b}(\,.\,,t)\big].$$

As $\nabla\eta\cdot\mathbf{u}(\,.\,,t)\in W^{m,2}_0(A_{\rho_5,\rho_6}(\mathbf{x}_0))$ for any $m\in\{0\}\cup\mathbb{N}$, we obtain $\mathbf{u}_{\rm corr}(\,.\,,t)\in \mathbf{W}^{m,2}_0(A_{\rho_5,\rho_6}(\mathbf{x}_0))$ for any $m\in\{0\}\cup\mathbb{N}$. Since all spatial derivatives of $\nabla\eta\cdot\mathbf{u}$ are essentially bounded in $A_{\rho_5,\rho_6}(\mathbf{x}_0)\times(3\delta,T-3\delta)$, we deduce that for any $m\in\{0\}\cup\mathbb{N}$,

$$\mathbf{u}_{\rm corr}\in L^\infty(3\delta,T-3\delta;\,\mathbf{W}^{m,2}_0(A_{\rho_5,\rho_6}(\mathbf{x}_0))).$$

The function $\mathbf{b}_{\rm corr}$ satisfies the same inclusion.

Extending $\mathbf{u}_{\rm corr}$ and $\mathbf{b}_{\rm corr}$ by zero outside $A_{\rho_5,\rho_6}(\mathbf{x}_0)$, and extending also $\eta\mathbf{u}$ and $\eta\mathbf{b}$ by zero outside Ω, we observe that the functions $\widehat{\mathbf{u}}:=\eta\mathbf{u}-\mathbf{u}_{\rm corr}$ and $\widehat{\mathbf{b}}:=\eta\mathbf{b}-\mathbf{b}_{\rm corr}$ are divergence-free in $\mathbb{R}^3\times(3\delta,T-3\delta)$, they coincide with $\mathbf{u}$ and $\mathbf{b}$, respectively, in $B_{\rho_7}(\mathbf{x}_0)\times(\delta,T-\delta)$, they are equal to zero in $(\mathbb{R}^3\smallsetminus B_{\rho_8}(\mathbf{x}_0)\times(3\delta,T-3\delta)$ and all their spatial derivatives are essentially bounded in $A_{\rho_5,\rho_6}(\mathbf{x}_0)\times(3\delta,T-3\delta)$.

One can deduce that $\widehat{\mathbf{u}}$, $\widehat{\mathbf{b}}$ and $\widehat{p}:=\eta p$ is a suitable weak solution to the MHD equations

$$\partial_t\widehat{\mathbf{u}}+\widehat{\mathbf{u}}\cdot\nabla\widehat{\mathbf{u}}-\widehat{\mathbf{b}}\cdot\nabla\widehat{\mathbf{b}} \;=\; -\nabla\widehat{p}+\nu\Delta\widehat{\mathbf{u}}+\mathbf{f}, \tag{2.1}$$

$$\partial_t\widehat{\mathbf{b}}+\widehat{\mathbf{u}}\cdot\nabla\widehat{\mathbf{b}}-\widehat{\mathbf{b}}\cdot\nabla\widehat{\mathbf{u}} \;=\; \xi\Delta\widehat{\mathbf{b}}+\mathbf{g}, \tag{2.2}$$

$$\operatorname{div}\widehat{\mathbf{u}} \;=\; \operatorname{div}\widehat{\mathbf{b}} \;=\; 0 \tag{2.3}$$

in $\mathbb{R}^3\times(3\delta,T-3\delta)$, where

$$\begin{aligned}
\mathbf{f} \;=\;& -\partial_t\mathbf{u}_{\rm corr}-\eta(1-\eta)\,\mathbf{u}\cdot\nabla\mathbf{u}+\eta(1-\eta)\,\mathbf{b}\cdot\nabla\mathbf{b}+\eta\,(\mathbf{u}\cdot\nabla\eta)\,\mathbf{u}-\eta\,(\mathbf{b}\cdot\nabla\eta)\,\mathbf{b}\\
&-\mathbf{u}_{\rm corr}\cdot\nabla(\eta\mathbf{u})-\eta\mathbf{u}\cdot\nabla\mathbf{u}_{\rm corr}+\mathbf{u}_{\rm corr}\cdot\nabla\mathbf{u}_{\rm corr}+\mathbf{b}_{\rm corr}\cdot\nabla(\eta\mathbf{b})+\eta\mathbf{b}\cdot\nabla\mathbf{b}_{\rm corr}\\
&-\mathbf{b}_{\rm corr}\cdot\nabla\mathbf{b}_{\rm corr}+p\,\nabla\eta-2\nu\,\nabla\eta\cdot\nabla\mathbf{u}-\nu\,(\Delta\eta)\,\mathbf{u},\\
\mathbf{g} \;=\;& -\partial_t\mathbf{b}_{\rm corr}-\eta(1-\eta)\,\mathbf{u}\cdot\nabla\mathbf{b}+\eta(1-\eta)\,\mathbf{b}\cdot\nabla\mathbf{u}+\eta\,(\mathbf{u}\cdot\nabla\eta)\,\mathbf{b}-\eta\,(\mathbf{b}\cdot\nabla\eta)\,\mathbf{u}\\
&-\mathbf{u}_{\rm corr}\cdot\nabla(\eta\mathbf{b})-\eta\mathbf{u}\cdot\nabla\mathbf{b}_{\rm corr}+\mathbf{u}_{\rm corr}\cdot\nabla\mathbf{b}_{\rm corr}+\mathbf{b}_{\rm corr}\cdot\nabla(\eta\mathbf{u})+\eta\mathbf{b}\cdot\nabla\mathbf{u}_{\rm corr}\\
&-\mathbf{b}_{\rm corr}\cdot\nabla\mathbf{u}_{\rm corr}-2\xi\,\nabla\eta\cdot\nabla\mathbf{b}-\xi\,(\Delta\eta)\,\mathbf{b}.
\end{aligned}$$

Obviously, $\mathbf{f}$ and $\mathbf{g}$ are supported in $A_{\rho_5,\rho_6}(\mathbf{x}_0)\times[3\delta,T-3\delta]$ and $\mathbf{f},\,\mathbf{g}\in L^\alpha(3\delta,T-3\delta;\,\mathbf{L}^\infty(\mathbb{R}^3))$ for any $\alpha\in(1,2)$ in case b) and $\alpha=\infty$ in cases a) and c). The same statements also hold on all spatial derivatives of $\mathbf{f}$ and $\mathbf{g}$.

2.2 Rescaling of $\widehat{\mathrm{u}}$, $\widehat{\mathrm{b}}$ and $\widehat{p}$ and the definition of the functions $\widehat{\mathrm{u}}^{(k)}$, $\widehat{\mathrm{b}}^{(k)}$ and $\widehat{p}^{(k)}$

It follows from (1.11) that there exists an increasing sequence of time instants $\{t_k\}$ in $(0,t_0)$ such that $t_k \nearrow t_0$ and

$$\|\mathbf{u}(\,.\,,t_k)\|_{3;\,B_\rho(\mathbf{x}_0)} + \|\mathbf{b}(\,.\,,t_k)\|_{3;\,B_\rho(\mathbf{x}_0)} \;\leq\; C \;<\; \infty \qquad \text{for all } k\in\mathbb{N}.$$

Hence there exists $M>0$ such that the localized functions $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{b}}$ satisfy

$$\|\widehat{\mathbf{u}}(\,.\,,t_k)\|_{3;\,\mathbb{R}^3} + \|\widehat{\mathbf{b}}(\,.\,,t_k)\|_{3;\,\mathbb{R}^3} \;\leq\; M \qquad \text{for all } k\in\mathbb{N}. \tag{2.4}$$

We can rescale $\widehat{\mathbf{u}}$, $\widehat{\mathbf{b}}$ and the associated pressure $\widehat{p}$ according to the formulas

$$\mathbf{x} = \mathbf{x}_0 + \lambda_k\mathbf{y}, \quad t = t_0 + \lambda_k^2 s, \quad \lambda_k = \sqrt{\frac{t_0-t_k}{S}}, \tag{2.5}$$

$$\widehat{\mathbf{u}}^{(k)}(\mathbf{y},s) = \lambda_k\,\widehat{\mathbf{u}}(\mathbf{x},t), \quad \widehat{\mathbf{b}}^{(k)}(\mathbf{y},s) = \lambda_k\,\widehat{\mathbf{b}}(\mathbf{x},t), \quad \widehat{p}^{(k)}(\mathbf{y},s) = \lambda_k^2\,\widehat{p}(\mathbf{x},t). \tag{2.6}$$

Here, S is a positive number which will be specified later. We also define

$$\mathbf{f}^{(k)}(\mathbf{y},s) := \lambda_k^3\,\mathbf{f}(\mathbf{x},t), \qquad \text{and} \qquad \mathbf{g}^{(k)}(\mathbf{y},s) := \lambda_k^3\,\mathbf{g}(\mathbf{x},t). \tag{2.7}$$

The rescaled functions $\widehat{\mathbf{u}}^{(k)}$, $\widehat{\mathbf{b}}^{(k)}$ and $\widehat{p}^{(k)}$ satisfy the equations

$$\partial_s\widehat{\mathbf{u}}^{(k)} + \widehat{\mathbf{u}}^{(k)}\cdot\nabla\widehat{\mathbf{u}}^{(k)} - \widehat{\mathbf{b}}^{(k)}\cdot\nabla\widehat{\mathbf{b}}^{(k)} \;=\; -\nabla\widehat{p}^{(k)} + \nu\Delta\widehat{\mathbf{u}}^{(k)} + \mathbf{f}^{(k)}, \tag{2.8}$$

$$\partial_s\widehat{\mathbf{b}}^{(k)} + \widehat{\mathbf{u}}^{(k)}\cdot\nabla\widehat{\mathbf{b}}^{(k)} - \widehat{\mathbf{b}}^{(k)}\cdot\nabla\widehat{\mathbf{u}}^{(k)} \;=\; \xi\Delta\widehat{\mathbf{b}}^{(k)} + \mathbf{g}^{(k)}, \tag{2.9}$$

$$\operatorname{div}\widehat{\mathbf{u}}^{(k)} \;=\; \operatorname{div}\widehat{\mathbf{b}}^{(k)} \;=\; 0 \tag{2.10}$$

in $\mathbb{R}^3\times(-S,0)$, where the operators ∇, div and Δ now act on the spatial variable $\mathbf{y}$. Moreover, the functions $\widehat{\mathbf{u}}^{(k)}$ and $\widehat{\mathbf{b}}^{(k)}$ satisfy the initial conditions

$$\widehat{\mathbf{u}}^{(k)}(\mathbf{y},-S) = \lambda_k\,\widehat{\mathbf{u}}(\lambda_k\mathbf{y},t_k) \qquad \text{and} \qquad \widehat{\mathbf{b}}^{(k)}(\mathbf{y},-S) = \lambda_k\,\widehat{\mathbf{b}}(\lambda_k\mathbf{y},t_k). \tag{2.11}$$

Since the rescaling (2.5), (2.6) preserves the L^3–norm, we have

$$\|\widehat{\mathbf{u}}^{(k)}(\,.\,,-S)\|_{3;\,\mathbb{R}^3} + \|\widehat{\mathbf{b}}^{(k)}(\,.\,,-S)\|_{3;\,\mathbb{R}^3} \;\leq\; M \qquad \text{for all } k\in\mathbb{N}. \tag{2.12}$$

Note that although we treat (2.8)–(2.11) as an initial value problem in $\mathbb{R}^3\times(-S,0)$, all the functions $\widehat{\mathbf{u}}^{(k)}$, $\widehat{\mathbf{b}}^{(k)}$, $\widehat{p}^{(k)}$, $\mathbf{f}^{(k)}$ and $\mathbf{g}^{(k)}$ are in fact supported only in the closure of $A_{\rho_5/\lambda_k,\rho_6/\lambda_k}(\mathbf{0})\times(-S,0)$.

2.3 Splitting of $\widehat{\mathrm{u}}^{(k)}$ and $\widehat{\mathrm{b}}^{(k)}$ and the definition of the functions $\widetilde{\mathrm{u}}^{(k)}$, $\widetilde{\mathrm{b}}^{(k)}$, $\mathbf{w}_{\mathbf{u}}^{(k)}$, $\mathbf{w}_{\mathbf{b}}^{(k)}$

Functions $\widehat{\mathbf{u}}^{(k)}$ and $\widehat{\mathbf{b}}^{(k)}$ can be expressed as the sums

$$\widehat{\mathbf{u}}^{(k)} = \widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}, \qquad \text{and} \qquad \widehat{\mathbf{b}}^{(k)} = \widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)},$$

where $\widetilde{\mathbf{u}}^{(k)}$, $\widetilde{\mathbf{b}}^{(k)}$, $\mathbf{w}_{\mathbf{u}}^{(k)}$ and $\mathbf{w}_{\mathbf{b}}^{(k)}$ satisfy the equations

$$\begin{aligned} \partial_s\widetilde{\mathbf{u}}^{(k)} + (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) - (\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) & \\ = -\nabla\widehat{p}^{(k)} + \nu\Delta\widetilde{\mathbf{u}}^{(k)} + \mathbf{f}^{(k)}, & \qquad (2.13) \\ \partial_s\widetilde{\mathbf{b}}^{(k)} + (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) - (\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) & \\ = \xi\Delta\widetilde{\mathbf{b}}^{(k)} + \mathbf{g}^{(k)}, & \qquad (2.14) \\ \partial_s\mathbf{w}_{\mathbf{u}}^{(k)} = \nu\Delta\mathbf{w}_{\mathbf{u}}^{(k)}, & \qquad (2.15) \\ \partial_s\mathbf{w}_{\mathbf{b}}^{(k)} = \xi\Delta\mathbf{w}_{\mathbf{b}}^{(k)}, & \qquad (2.16) \\ \operatorname{div}\widetilde{\mathbf{u}}^{(k)} = \operatorname{div}\mathbf{w}_{\mathbf{u}}^{(k)} = \operatorname{div}\widetilde{\mathbf{b}}^{(k)} = \operatorname{div}\mathbf{w}_{\mathbf{b}}^{(k)} = 0 & \qquad (2.17) \end{aligned}$$

in $\mathbb{R}^3 \times (-S, 0)$ and the initial conditions

$$\widetilde{\mathbf{u}}^{(k)}(\,.\,, -S) = \widetilde{\mathbf{b}}^{(k)}(\,.\,, -S) = \mathbf{0}, \qquad (2.18)$$

$$\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,, -S) = \widehat{\mathbf{u}}^{(k)}(\,.\,, -S), \qquad (2.19)$$

$$\mathbf{w}_{\mathbf{b}}^{(k)}(\,.\,, -S) = \widehat{\mathbf{b}}^{(k)}(\,.\,, -S). \qquad (2.20)$$

Note that the functions $\mathbf{w}_{\mathbf{u}}^{(k)}$ and $\mathbf{w}_{\mathbf{b}}^{(k)}$ automatically satisfy the conditions $\operatorname{div}\mathbf{w}_{\mathbf{u}}^{(k)} = \operatorname{div}\mathbf{w}_{\mathbf{b}}^{(k)} = 0$ in $\mathbb{R}^3 \times (-S, 0)$, because their initial values $\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,, -S)$ and $\mathbf{w}_{\mathbf{b}}^{(k)}(\,.\,, -S)$ are divergence-free. Applying inequality (A) on p. 190 and the lemma on p. 196 in [9], we obtain the estimates

$$\int_{-S}^{0} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,, s)\|_5^5 \, \mathrm{d}s + \operatorname*{ess\,sup}_{-S<s<0} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,, s)\|_3 \le c_1, \qquad (2.21)$$

$$\int_{-S}^{0} \|\mathbf{w}_{\mathbf{b}}^{(k)}(\,.\,, s)\|_5^5 \, \mathrm{d}s + \operatorname*{ess\,sup}_{-S<s<0} \|\mathbf{w}_{\mathbf{b}}^{(k)}(\,.\,, s)\|_3 \le c_2, \qquad (2.22)$$

where c_1 and c_2 depend on M, but they are independent of S and k.

2.4 Decomposition of the pressure

In this subsection, we decompose the pressure in Lemma 1 by an analogous way in [20] and estimate each term consisting the pressure in Lemma 2. The detailed proofs of these lemmas are elaborated in Appendix.

Lemma 1. *Let $\mathbb{K}$ be the second order tensor with the entries k_{ij} given by*

$$k_{ij}(\mathbf{z}-\mathbf{y}) \;=\; \frac{\partial^2}{\partial z_i\,\partial z_j}\,\frac{1}{|\mathbf{z}-\mathbf{y}|}.$$

For $\mathbf{x}\in\mathbb{R}^3$ (which has nothing to do with $\mathbf{x}$ from (2.5)*) and $\mathbf{y}\in B_{3/2}(\mathbf{x})$, we have*

$$\widehat{p}^{(k)}(\mathbf{y},s) \;=\; \widehat{p}^{1(k)}_{\mathbf{x}}(\mathbf{y},s)+\widehat{p}^{2(k)}_{\mathbf{x}}(\mathbf{y},s)+c^{(k)}_{\mathbf{x}}(s)+d^{(k)}(s), \tag{2.23}$$

where

$$\begin{aligned}
\widehat{p}^{1(k)}_{\mathbf{x}}(\mathbf{y},s) \;&=\; \frac{1}{4\pi}\int_{B_2(\mathbf{x})}\mathbb{K}(\mathbf{z}-\mathbf{y}):\widehat{\mathbf{M}}^{(k)}(\mathbf{z},s)\;\mathrm{d}\mathbf{z}-\frac{1}{3}\big(|\widehat{\mathbf{u}}^{(k)}(\mathbf{y},s)|^2-|\widehat{\mathbf{b}}^{(k)}(\mathbf{y},s)|^2\big),\\
\widehat{p}^{2(k)}_{\mathbf{x}}(\mathbf{y},s) \;&=\; \frac{1}{4\pi}\int_{\mathbb{R}^3\smallsetminus B_2(\mathbf{x})}\big[\mathbb{K}(\mathbf{z}-\mathbf{y})-\mathbb{K}(\mathbf{z}-\mathbf{x})\big]:\widehat{\mathbf{M}}^{(k)}(\mathbf{z},s)\;\mathrm{d}\mathbf{z},\\
c^{(k)}_{\mathbf{x}}(s) \;&=\; \frac{1}{4\pi}\int_{\mathbb{R}^3\smallsetminus B_2(\mathbf{x})}\mathbb{K}(\mathbf{z}-\mathbf{x}):\widehat{\mathbf{M}}^{(k)}(\mathbf{z},s)\;\mathrm{d}\mathbf{z},\\
d^{(k)}(s) \;&=\; -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{z}-\mathbf{y}|}\,\mathrm{div}\,\mathbf{f}^{(k)}(\mathbf{z},s)\;\mathrm{d}\mathbf{z},
\end{aligned}$$

and

$$\widehat{\mathbf{M}}^{(k)}(\mathbf{z},s)=\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)\otimes\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)-\widehat{\mathbf{b}}^{(k)}(\mathbf{z},s)\otimes\widehat{\mathbf{b}}^{(k)}(\mathbf{z},s)$$

Note that a direct computation shows

$$k_{ij}(\mathbf{z}-\mathbf{y}) \;=\; \frac{\partial^2}{\partial z_i\,\partial z_j}\,\frac{1}{|\mathbf{z}-\mathbf{y}|} \;=\; -\frac{\partial}{\partial z_i}\,\frac{z_j-y_j}{|\mathbf{z}-\mathbf{y}|^3} \;=\; 3\,\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^5}-\frac{\delta_{ij}}{|\mathbf{z}-\mathbf{y}|^3}.$$

Lemma 2. *Let $1<q<\infty$ and*

$$\|f\|_{q;\,\mathrm{unif}} \;:=\; \sup_{\mathbf{x}\in\mathbb{R}^3}\;\|f\|_{q;\,B_1(\mathbf{x})}.$$

We have the following estimates for each parts of pressure.

$$\|\widehat{p}^{1(k)}_{\mathbf{x}}(\,.\,,s)\|_{3/2;\,B_{3/2}(\mathbf{x})} \;\lesssim_M\; \|\widetilde{\mathbf{u}}^{(k)}(\,.\,,s)\|^2_{3;\,B_2(\mathbf{x})}+\|\widetilde{\mathbf{b}}^{(k)}(\,.\,,s)\|^2_{3;\,B_2(\mathbf{x})}+1, \tag{2.24}$$

$$\sup_{\mathbf{y}\in B_{3/2}(\mathbf{x})}|\widehat{p}^{2(k)}_{\mathbf{x}}(\mathbf{y},s)| \;\lesssim_M\; \|\widetilde{\mathbf{u}}^{(k)}(\,.\,,s)\|^2_{2;\,\mathrm{unif}}+\|\widetilde{\mathbf{b}}^{(k)}(\,.\,,s)\|^2_{2;\,\mathrm{unif}}+1, \tag{2.25}$$

$$|c^{(k)}_{\mathbf{x}}(s)| \;\lesssim_M\; 1, \tag{2.26}$$

$$\big|d^{(k)}(s)\big| \;\lesssim\; \lambda_k^2\,\|\mathrm{div}\,\mathbf{f}(\,.\,,t_0+\lambda_k^2 s)\|_{\infty;\,\mathbb{R}^3}\,(\rho_6^2-\rho_5^2). \tag{2.27}$$

Moreover, $d^{(k)}\in L^{\mu}(-S,0)$ for any $\mu\in(1,2)$.

2.5 An estimate, following from the localized energy inequality

In this subsection, we apply the localized energy inequality and derive an important estimate, see Lemma 4. In order to express the estimate neatly, we at first define a few functionals and give their estimates in Lemma 3. The detailed proofs of these lemmas are elaborated in Appendix.

Definition 1 (Functionals). *For $-S \le s_0 \le s < 0$, we denote*

$$\begin{aligned}
\alpha_k(s) &:= \|\widetilde{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 + \|\widetilde{\mathbf{b}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2,\\
\beta_k(s_0,s) &:= \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{s_0}^{s} \big(\|\nabla\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^2 + \|\nabla\widetilde{\mathbf{b}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^2\big)\,\mathrm{d}\tau,\\
\gamma_k(s_0,s) &:= \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{s_0}^{s} \big(\|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{3;\,B_1(\mathbf{x})}^3 + \|\widetilde{\mathbf{b}}^{(k)}(\,.\,,\tau)\|_{3;\,B_1(\mathbf{x})}^3\big)\,\mathrm{d}\tau,\\
\delta_k(s_0,s) &:= \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{s_0}^{s} \big(\|\widehat{p}_{\mathbf{x}}^{1(k)}(\,.\,,\tau)\|_{3/2;\,B_1(\mathbf{x})}^{3/2} + \|\widehat{p}_{\mathbf{x}}^{2(k)}(\,.\,,\tau)\|_{3/2;\,B_1(\mathbf{x})}^{3/2}\big)\,\mathrm{d}\tau.
\end{aligned}$$

Lemma 3. *For a.a. $s \in (-S,0)$,*

$$\alpha_k(s) \lesssim \lambda_k^{-1} \tag{2.28}$$

and for all $s \in (-S,0)$

$$\gamma_k(s_0,s) \lesssim \int_{s_0}^{s} \alpha_k^{3/2}(\tau)\,\mathrm{d}\tau + \left(\int_{s_0}^{s} \alpha_k^3(\tau)\,\mathrm{d}\tau\right)^{1/4} \beta_k^{3/4}(s_0,s), \tag{2.29}$$

$$\delta_k(s_0,s) \lesssim \gamma_k(s_0,s) + \int_{s_0}^{s} \alpha_k^{3/2}(\tau)\,\mathrm{d}\tau + (s-s_0). \tag{2.30}$$

Lemma 4. *There exist positive constants A, B, C_M such that for a.a $s_0 \in [-S,0)$ and all $s \in (s_0,0)$,*

$$\begin{aligned}
&\alpha_k(s) + \beta_k(s_0,s)\\
&\le A\alpha_k(s_0) + B\mathcal{F}^{(k)}(s_0,s)\\
&\quad + C_M\bigg(\int_{s_0}^{s} (\alpha_k(\tau) + \alpha_k^3(\tau))\,\mathrm{d}\tau + (s-s_0)^{1/2} + \int_{s_0}^{s} \alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^5\,\mathrm{d}\tau\bigg),
\end{aligned} \tag{2.31}$$

where

$$\mathcal{F}^{(k)}(s_0,s) := \int_{s_0}^{s} \big({d_{\mathbf{x}}^{(k)}}^{3/2}(\tau) + \|\mathbf{f}^{(k)}(\,.\,,\tau)\|_\infty^{3/2} + \|\mathbf{g}^{(k)}(\,.\,,\tau)\|_\infty^{3/2}\big)\,\mathrm{d}\tau.$$

(The subscript M in C_M indicates the dependence on number M from inequality (2.4).)

Note that $\mathcal{F}^{(k)}(s_0,s) \to 0$ uniformly with respect to $s_0 \in [-S,0)$ and $s \in (s_0,0)$ as $k \to \infty$. Indeed, it follows from (2.7), (2.27) and the definition of $\mathcal{F}^{(k)}(s)$ that

$$\begin{aligned}
\mathcal{F}^{(k)}(s_0,s) &\lesssim \lambda_k^3 \int_{-S}^{0} \big(\|\mathbf{f}(\,.\,,t_0+\lambda_k^2\tau)\|_{1,\infty;\,\mathbb{R}^3}^{3/2} + \|\mathbf{g}(\,.\,,t_0+\lambda_k^2\tau)\|_{\infty;\,\mathbb{R}^3}^{3/2}\big)\,\mathrm{d}\tau\\
&= \lambda_k \int_{t_k}^{t_0} \big(\|\mathbf{f}(\,.\,,t)\|_{1,\infty;\,\mathbb{R}^3}^{3/2} + \|\mathbf{g}(\,.\,,t_0+\lambda_k^2\tau)\|_{\infty;\,\mathbb{R}^3}^{3/2}\big)\,\mathrm{d}t\\
&\lesssim \lambda_k.
\end{aligned} \tag{2.32}$$

2.6 Another estimate of $\alpha_k(s)$ and the choice of S

Recall that due to (2.18), $\alpha_k(-S) = 0$. Moreover, applying (2.28) and choosing $s_0 = -S$, we obtain from (2.31): for all $s \in (-S, 0)$,

$$\alpha_k(s) \leq \; B\mathcal{F}^{(k)}(-S,s) + C_M\biggl(\lambda_k^{-3}(s+S) + (s+S)^{1/2} + \lambda_k^{-1}\int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5 \; \mathrm{d}\tau\biggr). \tag{2.33}$$

Since the right side goes to 0 as $s \to -S$, $\alpha_k(s) \leq 1/10$ for s in some right neighbourhood of $-S$. We will further prove that if $S < 1$ is small enough and k is sufficiently large then $\alpha_k(s)$ satisfies the same estimate for s on the whole interval $(-S, 0)$. Let

$$s_1 \;:=\; \sup\{\sigma \in [-S,0] : \alpha_k(s) \leq 1/10 \text{ for } s \in [-S,\sigma)\}.$$

It follows from (2.33) that $-S < s_1 \leq 0$. Suppose that $s_1 < 0$ and $-S < s < s_1$. By analogy with (2.33), we obtain from (2.31):

$$\alpha_k(s) \;\lesssim_M\; \int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5\, \alpha_k(\tau)\; \mathrm{d}\tau + [(s+S)^{1/2} + \lambda_k].$$

By the generalized Gronwall inequality we have

$$\alpha_k(s) \;\lesssim_M\; \int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5\, [(\tau+S)^{1/2} + \lambda_k]\, \mathrm{e}^{H(s)-H(\tau)}\; \mathrm{d}\tau + [(s+S)^{1/2} + \lambda_k],$$

where $H(s) = \int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5 \; \mathrm{d}\tau$. Estimating the norm of $\mathbf{w}_{\mathbf{u}}^{(k)}$ by means of (2.21)), we obtain

$$\begin{aligned}
\alpha_k(s) \;&\lesssim_M\; \int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5\, [(\tau+S)^{1/2} + \lambda_k]\; \mathrm{d}\tau + [(s+S)^{1/2} + \lambda_k] \\
&\lesssim_M\; [(s+S)^{1/2} + \lambda_k]\biggl(\int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5 \; \mathrm{d}\tau + 1\biggr) \\
&\lesssim_M\; [S^{1/2} + \lambda_k]
\end{aligned} \tag{2.34}$$

for $s \in (-S, s_1)$. Suppose from now on that S is so small and k is so large (i.e. λ_k is so small) that $\alpha_k(s) \leq \frac{1}{20(1+A)}$. Inequality (2.31), considered with $s < s_0 < s_1$ and $s > s_1$, yields

$$\begin{aligned}
\alpha_k(s) \;\leq\; & \frac{A}{20(1+A)} + B\,\mathcal{F}^{(k)}(s_0,s) \\
& + C_M\left(\lambda_k^{-3}\,(s-s_0) + (s-s_0)^{1/2} + \int_{s_0}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5 \; \mathrm{d}\tau\right)
\end{aligned}$$

As the right hand side depends continuously on s_0, the same inequality also holds if s_0 is replaced by s_1. Then, as the right hand side is less than $\frac{1}{10}$ for s in some right neighbourhood of s_1, we obtain the contradiction with the assumption that $s_1 < 0$. Hence we may assume that

$$\alpha_k(s) \;\leq\; \frac{1}{10} \tag{2.35}$$

for all $s \in [-S, 0)$ and for all k. The importance of this inequality lies in the fact that, in contrast to (2.28), it provides an estimate of $\alpha_k(s)$ independent of k. Inequalities (2.31) and (2.35) imply that the estimate

$$\alpha_k(s) + \beta_k(-S, s) \;\le\; B\,\mathcal{F}^{(k)}(-S, s) + C_M\biggl((s+S)^{1/2} + \int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{5;\,\mathrm{unif}}^5\,\mathrm{d}\tau\biggr) \tag{2.36}$$

holds for all $s \in [-S, 0)$.

3 The limit transition for $k \to \infty$

3.1 The limit transition in the sequences $\{\mathbf{w}_{\mathbf{u}}^{(k)}\}$ and $\{\mathbf{w}_{\mathbf{b}}^{(k)}\}$

There exist functions $\widehat{\mathbf{u}}_0$, $\widehat{\mathbf{b}}_0 \in \mathbf{L}^3(\mathbb{R}^3)$ and subsequences of $\{\widehat{\mathbf{u}}^{(k)}(\,.\,,-S)\}$ and $\{\widehat{\mathbf{b}}^{(k)}(\,.\,,-S)\}$ (which we denote in the same way) such that $\widehat{\mathbf{u}}^{(k)}(\,.\,,-S) \rightharpoonup \widehat{\mathbf{u}}_0$ and $\widehat{\mathbf{b}}^{(k)}(\,.\,,-S) \rightharpoonup \widehat{\mathbf{b}}_0$ in $\mathbf{L}^3(\mathbb{R}^3)$. By analogy with [34], one can deduce that since

$$\begin{aligned} \sup_{-S<s<0} \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 + \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{-S}^{0} \|\nabla\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,B_1(\mathbf{x}_0)}^2\,\mathrm{d}s \;\le\; C(M) \;<\; \infty,\\ \sup_{-S<s<0} \|\mathbf{w}_{\mathbf{b}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 + \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{-S}^{0} \|\nabla\mathbf{w}_{\mathbf{b}}^{(k)}(\,.\,,s)\|_{2;\,B_1(\mathbf{x}_0)}^2\,\mathrm{d}s \;\le\; C(M) \;<\; \infty, \end{aligned} \tag{3.1}$$

there exist limit functions $\mathbf{w}_{\mathbf{u}}$ and $\mathbf{w}_{\mathbf{b}}$ such that $\mathbf{w}_{\mathbf{u}}^{(k)} \to \mathbf{w}_{\mathbf{u}}$ and $\mathbf{w}_{\mathbf{b}}^{(k)} \to \mathbf{w}_{\mathbf{b}}$ (together with all derivatives) uniformly on all sets of the type $B_R(\mathbf{0}) \times (s, 0]$ for any $R > 0$ and $s \in (-S, 0)$. The limit functions are divergence-free and represent strong solutions of equations (2.15) and (2.16) on the time interval $(-S, 0)$, satisfying the initial conditions $\mathbf{w}_{\mathbf{u}}(\,.\,,-S) = \widehat{\mathbf{u}}_0$ and $\mathbf{w}_{\mathbf{b}}(\,.\,,-S) = \widehat{\mathbf{b}}_0$. They also satisfy the same estimates as (3.1), i.e.

$$\begin{aligned} \sup_{-S<s<0} \|\mathbf{w}_{\mathbf{u}}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 + \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{-S}^{0} \|\nabla\mathbf{w}_{\mathbf{u}}(\,.\,,s)\|_{2;\,B_1(\mathbf{x}_0)}^2\,\mathrm{d}s \;\le\; C(M) \;<\; \infty,\\ \sup_{-S<s<0} \|\mathbf{w}_{\mathbf{b}}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 + \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{-S}^{0} \|\nabla\mathbf{w}_{\mathbf{b}}(\,.\,,s)\|_{2;\,B_1(\mathbf{x}_0)}^2\,\mathrm{d}s \;\le\; C(M) \;<\; \infty. \end{aligned} \tag{3.2}$$

Moreover, both the functions $\mathbf{w}_{\mathbf{u}}$ and $\mathbf{w}_{\mathbf{b}}$ are in $C\bigl([-S, 0];\, \mathbf{L}^3(\mathbb{R}^3)\bigr) \cap \mathbf{L}^5\bigl(\mathbb{R}^3 \times (-S, 0)\bigr)$, see e.g. [9].

3.2 Weak limits of the sequences $\{\widetilde{\mathbf{u}}^{(k)}\}$, $\{\widetilde{\mathbf{b}}^{(k)}\}$ and $\{\widehat{p}^{(k)}\}$

From (2.26), (2.29) and (2.36), we deduce that there exist subsequences of $\{\widetilde{\mathbf{u}}^{(k)}\}$, $\{\widetilde{\mathbf{b}}^{(k)}\}$ and $\{\widehat{p}^{(k)}\}$, which we again denote by $\{\widetilde{\mathbf{u}}^{(k)}\}$, $\{\widetilde{\mathbf{b}}^{(k)}\}$ and $\{\widehat{p}^{(k)}\}$, and limit functions $\widetilde{\mathbf{u}}$, $\widetilde{\mathbf{b}}$ and $\widehat{p}$ so that

$$\widetilde{\mathbf{u}}^{(k)} \longrightarrow \widetilde{\mathbf{u}}, \qquad \widetilde{\mathbf{b}}^{(k)} \longrightarrow \widetilde{\mathbf{b}} \qquad \text{weakly–}* \text{ in } L^\infty(-S, 0;\, \mathbf{L}^2(B_a(\mathbf{0}))), \tag{3.3}$$

$$\nabla\widetilde{\mathbf{u}}^{(k)} \longrightarrow \nabla\widetilde{\mathbf{u}}, \qquad \nabla\widetilde{\mathbf{b}}^{(k)} \longrightarrow \nabla\widetilde{\mathbf{b}} \qquad \text{weakly in } L^2\bigl(B_a(\mathbf{0}) \times (-S, 0)\bigr)^{3\times 3}, \tag{3.4}$$

$$\widehat{p}^{(k)} \longrightarrow \widehat{p} \qquad \text{weakly in } L^{3/2}\bigl(B_a(\mathbf{0}) \times (-S, 0)\bigr) \tag{3.5}$$

for $k \to \infty$ and for any $a > 0$. Using (2.29), we can also deduce that the subsequences can be chosen so that

$$\widetilde{\mathbf{u}}^{(k)} \longrightarrow \widetilde{\mathbf{u}}, \qquad \widetilde{\mathbf{b}}^{(k)} \longrightarrow \widetilde{\mathbf{b}} \qquad \text{weakly in } \mathbf{L}^3\big(B_a(\mathbf{0}) \times (-S, 0)\big). \tag{3.6}$$

3.3 A strong convergence of the sequences $\{\widetilde{\mathbf{u}}^{(k)}\}$ and $\{\widetilde{\mathbf{b}}^{(k)}\}$

In order to show that the convergence of $\{\widetilde{\mathbf{u}}^{(k)}\}$ and $\{\widetilde{\mathbf{b}}^{(k)}\}$ is strong in an appropriate function space, we need an information on the time derivatives $\partial_s\widetilde{\mathbf{u}}^{(k)}$ and $\partial_s\widetilde{\mathbf{b}}^{(k)}$. Consider a test function $\boldsymbol{\phi} \in \mathbf{C}_0^\infty\big(B_a(\mathbf{0})) \times (-S, 0)\big)$, multiply equation (2.13) by $\boldsymbol{\phi}$ and integrate over $B_a(\mathbf{0}) \times (-S, 0)$. We obtain

$$\int_{-S}^0 \int_{B_a(\mathbf{0})} \big[\partial_s\widetilde{\mathbf{u}}^{(k)} \cdot \boldsymbol{\phi} - (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\boldsymbol{\phi} \cdot (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) + (\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla\boldsymbol{\phi} \cdot (\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)})$$
$$+ p^{(k)} \operatorname{div} \boldsymbol{\phi} - \nu\nabla\widetilde{\mathbf{u}}^{(k)} : \nabla\boldsymbol{\phi}\big] \, \mathrm{d}\mathbf{y} \, \mathrm{d}s \; = \; \int_{-S}^0 \int_{B_a(\mathbf{0})} \mathbf{f}^{(k)} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{y} \, \mathrm{d}s$$

and

$$\bigg|\int_{-S}^0 \int_{B_a(\mathbf{0})} \partial_s\widetilde{\mathbf{u}}^{(k)} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{y} \, \mathrm{d}s\bigg|$$
$$\leq \int_{-S}^0 (\|\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}\|_{3;\, B_a(\mathbf{0})}^2 + \|\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}\|_{3;\, B_a(\mathbf{0})}^2) \, \|\nabla\boldsymbol{\phi}\|_{3;\, B_a(\mathbf{0})} \, \mathrm{d}s$$
$$+ \int_{-S}^0 \|p^{(k)}\|_{3/2;\, B_a(\mathbf{0})} \, \|\operatorname{div} \boldsymbol{\phi}\|_{3;\, B_a(\mathbf{0})} \, \mathrm{d}s + \nu \int_{-S}^0 \|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\, B_a(\mathbf{0})} \, \|\nabla\boldsymbol{\phi}\|_{2;\, B_a(\mathbf{0})} \, \mathrm{d}s$$
$$+ \int_{-S}^0 \|\mathbf{f}^{(k)}\|_{2;\, B_a(\mathbf{0})} \, \|\boldsymbol{\phi}\|_{2;\, B_a(\mathbf{0})} \, \mathrm{d}s.$$

Applying (2.26), (2.36), (2.29) and the continuous imbedding $\mathbf{W}^{1,2}(B_a(\mathbf{0})) \hookrightarrow \mathbf{L}^3(B_a(\mathbf{0}))$, we obtain

$$\bigg|\int_{-S}^0 \partial_s\widetilde{\mathbf{u}}^{(k)} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{y} \, \mathrm{d}s\bigg| \; \lesssim_a \; \bigg(\int_{-S}^0 \|\nabla\boldsymbol{\phi}\|_{3;\, B_a(\mathbf{0}))}^3 \, \mathrm{d}s\bigg)^{1/3}.$$

Consequently, $\{\partial_s\widetilde{\mathbf{u}}^{(k)}\}$ is uniformly bounded in the dual space to $L^3(-S, 0;\mathbf{W}_0^{1,3}(B_a(\mathbf{0}))$, which is $L^{3/2}(-S, 0;\, \mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0})))$. (Here, $\mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0}))$ denotes the dual space to $\mathbf{W}_0^{1,3}(B_a(\mathbf{0}))$.) Applying the version of the Lions–Aubin theorem (see [23, Théorème 5.1.]), and applying the same arguments also to the sequence $\{\widetilde{\mathbf{b}}^{(k)}\}$, one can deduce that

$$\widetilde{\mathbf{u}}^{(k)} \longrightarrow \widetilde{\mathbf{u}}, \qquad \widetilde{\mathbf{b}}^{(k)} \longrightarrow \widetilde{\mathbf{b}} \qquad \text{strongly in } \mathbf{L}^3\big(B_a(\mathbf{0}) \times (-S, 0)\big). \tag{3.7}$$

It also follows from the aforementioned results that the sequence $\{\widetilde{\mathbf{u}}^{(k)}\}$ is uniformly bounded in $W^{1,3/2}(-S, 0;\, \mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0}))$. Since $\mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0})) \hookrightarrow\hookrightarrow \mathbf{W}_0^{-1-\gamma}(B_a(\mathbf{0}))$ (for any $\gamma > 0$), we also have $W^{1,3/2}(-S,0;\, \mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0})) \hookrightarrow\hookrightarrow C\big((-S,0];\, \mathbf{W}_0^{-1-\gamma}(B_a(\mathbf{0})\big)$. Consequently, as the same conclusions also hold for functions $\widetilde{\mathbf{b}}^{(k)}$, we also get

$$\widetilde{\mathbf{u}}^{(k)} \longrightarrow \widetilde{\mathbf{u}}, \quad \widetilde{\mathbf{b}}^{(k)} \longrightarrow \widetilde{\mathbf{b}} \quad \text{strongly in } C\big((-S,0];\, \mathbf{W}_0^{-1-\gamma}(B_a(\mathbf{0})\big) \text{ for any } \gamma > 0. \tag{3.8}$$

3.4 The limits of $\widetilde{\mathbf{u}}(\,.\,,s)$ and $\widetilde{\mathbf{b}}(\,.\,,s)$ for $s \to -S+$

Note that as (3.3)–(3.7) hold for all $a > 0$, the limit functions $\widetilde{\mathbf{u}}$, $\widetilde{\mathbf{b}}$ and q are in fact defined a.e. in $\mathbb{R}^3 \times (-S,0)$. Thus, due to (3.5), one has $q \in L^{3/2}(-S,0;\, L^{3/2}_{loc}(\mathbb{R}^3))$. Moreover, one can easily derive from (2.32), (2.36), (3.2), (3.3) and (3.4) that for any $s \in (-S,0)$,

$$\begin{aligned} &\operatorname*{ess\,sup}_{-S<\tau<s} \big(\|\widetilde{\mathbf{u}}(\,.\,,\tau)\|^2_{2;\,\mathrm{unif}} + \|\widetilde{\mathbf{b}}(\,.\,,\tau)\|^2_{2;\,\mathrm{unif}}\big) \\ &\quad + \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{-S}^{s} \big(\|\nabla\widetilde{\mathbf{u}}(\,.\,,\tau)\|^2_{2;\,B_1(\mathbf{x})} + \|\nabla\widetilde{\mathbf{b}}(\,.\,,\tau)\|^2_{2;\,B_1(\mathbf{x})}\big)\, \mathrm{d}\tau \\ &\lesssim_M \ (s+S)^{1/2} + \int_{-S}^{s} \|\mathbf{w}_{\mathbf{u}}(\,.\,,\tau)\|^5_{5;\,\mathrm{unif}}\, \mathrm{d}\tau. \end{aligned} \tag{3.9}$$

This implies that

$$\|\widetilde{\mathbf{u}}(\,.\,,s)\|_{2;\,\mathrm{unif}} + \|\widetilde{\mathbf{b}}(\,.\,,s)\|_{2;\,\mathrm{unif}} \ \longrightarrow\ 0 \qquad \text{for } s \to -S+\,. \tag{3.10}$$

3.5 A weak continuity of $\widetilde{\mathbf{u}}(\,.\,,s)$ and $\widetilde{\mathbf{b}}(\,.\,,s)$ in dependence on s

Recall that

$$\|\partial_s\widetilde{\mathbf{u}}^{(k)}\|_{\mathcal{H}_a} + \|\partial_s\widetilde{\mathbf{b}}^{(k)}\|_{\mathcal{H}_a} \ \le\ C(a)$$

for any $a > 0$, where we have used the abbreviated notation

$$\mathcal{H}_a \ :=\ L^{3/2}(-S,0;\, \mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0}))).$$

The same inequality is also preserved in the limit for $k \to \infty$, which means that

$$\|\partial_s\widetilde{\mathbf{u}}\|_{\mathcal{H}_a} + \|\partial_s\widetilde{\mathbf{b}}\|_{\mathcal{H}_a} \ \le\ C(a). \tag{3.11}$$

Hence $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{b}}$ are continuous functions from $[-S,0]$ to $\mathbf{W}_0^{-1,3/2}(B_a(\mathbf{0}))$. From this and (3.9), one can easily deduce that given a function $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(B_a(\mathbf{0}))$, the mapping $s \mapsto \int_{B_a(\mathbf{0})} \widetilde{\mathbf{u}}(\mathbf{y},s)\cdot\boldsymbol{\varphi}(\mathbf{y})\, \mathrm{d}\mathbf{y}$ is a continuous function in $[-S,0]$ and the same statement also holds on $\widetilde{\mathbf{b}}$. It means, due to the density of $\mathbf{W}_0^{1,3}(B_a(\mathbf{0}))$ in $\mathbf{L}^2(B_a(\mathbf{0}))$ and the possibility to choose $a > 0$ arbitrarily large, that

$$\begin{aligned} &\int_{\mathbb{R}^3} \widetilde{\mathbf{u}}(\mathbf{y},\,.\,)\cdot\boldsymbol{\varphi}(\mathbf{y})\, \mathrm{d}\mathbf{y} \in C\big([-S,0]\big) \\ &\int_{\mathbb{R}^3} \widetilde{\mathbf{b}}(\mathbf{y},\,.\,)\cdot\boldsymbol{\varphi}(\mathbf{y})\, \mathrm{d}\mathbf{y} \in C\big([-S,0]\big) \end{aligned} \tag{3.12}$$

for any $\boldsymbol{\varphi} \in \mathbf{L}^2(\mathbb{R}^3)$ with a compact support in $\mathbb{R}^3$.

3.6 The functions $\mathbf{U}$ and $\mathbf{B}$

Put

$$\mathbf{U} := \widetilde{\mathbf{u}} + \mathbf{w}_{\mathbf{u}} \qquad \text{and} \qquad \mathbf{B} := \widetilde{\mathbf{b}} + \mathbf{w}_{\mathbf{b}}. \tag{3.13}$$

Due to the properties of the functions $\mathbf{w}_{\mathbf{u}}$ and $\mathbf{w}_{\mathbf{b}}$, listed at the beginning of this section, and (3.9)–(3.12), the functions $\mathbf{U}$ and $\mathbf{B}$ satisfy

$$\begin{aligned} & \operatorname*{ess\,sup}_{-S<\tau<0} \big(\|\mathbf{U}(\,.\,,\tau)\|^2_{2;\,\mathrm{unif}} + \|\mathbf{B}(\,.\,,\tau)\|^2_{2;\,\mathrm{unif}} \big) \\ & \quad + \sup_{\mathbf{x}\in\mathbb{R}^3} \int_{-S}^0 \big(\|\nabla\mathbf{U}(\,.\,,\tau)\|^2_{2;\,B_1(\mathbf{x})} + \|\nabla\mathbf{B}(\,.\,,\tau)\|^2_{2;\,B_1(\mathbf{x})} \big)\, \mathrm{d}\tau \\ & \leq\ C(M), \end{aligned} \tag{3.14}$$

and

$$\mathbf{U}(\,.\,,s) \to \mathbf{w}_{\mathbf{u}}(\,.\,,-S) \qquad \text{and} \qquad \mathbf{B}(\,.\,,s) \to \mathbf{w}_{\mathbf{b}}(\,.\,,-S) \tag{3.15}$$

in $\mathbf{L}^2_{loc}(\mathbb{R}^3)$ as $s \to -S+$ and

$$\begin{aligned} & \int_{\mathbb{R}^3} \mathbf{U}(\mathbf{y},\,.\,) \cdot \boldsymbol{\varphi}(\mathbf{y})\, \mathrm{d}\mathbf{y} \in C\big([-S,0]\big), \\ & \int_{\mathbb{R}^3} \mathbf{B}(\mathbf{y},\,.\,) \cdot \boldsymbol{\varphi}(\mathbf{y})\, \mathrm{d}\mathbf{y} \in C\big([-S,0]\big) \end{aligned} \tag{3.16}$$

for any $\boldsymbol{\varphi} \in \mathbf{L}^2(\mathbb{R}^3)$ with a compact support in $\mathbb{R}^3$.

3.7 Non-triviality of the function $|\mathbf{U}| + |\mathbf{B}|$

Suppose, for a while, that $(\mathbf{x}_0, t_0)$ is a singular point of the solution $\mathbf{u}$, $\mathbf{b}$ to the equations (1.1)–(1.3). (I.e. the solutions blows up in the neighborhood of the point $\mathbf{x}_0$ for $t \to t_0$.) Then there exist $\epsilon > 0$ and $\delta_* > 0$ such that

$$\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} \big(|\mathbf{u}|^3 + |\mathbf{b}|^3 \big)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t\ >\ \epsilon$$

for all $\delta \in (0, \delta_*)$, see [41]. Since $\mathbf{u}$ and $\mathbf{b}$ coincide with $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{b}}$ resprectively in $B_{\rho_7}(\mathbf{x}_0) \times (3\delta, t_0)$, the functions $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{b}}$ satisfy the same inequality

$$\frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{B_\delta(\mathbf{x}_0)} \big(|\widehat{\mathbf{u}}|^3 + |\widehat{\mathbf{b}}|^3 \big)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t\ >\ \epsilon \tag{3.17}$$

for all $\delta > 0$ sufficiently small. One can calculate from the transformation formulas (2.5), (2.6) and the inequality (3.17) that

$$\begin{aligned}
&\int_{-S/2}^{0}\int_{B_a(\mathbf{0})}\big(|\widehat{\mathbf{u}}^{(k)}|^3+|\widehat{\mathbf{b}}^{(k)}|^3\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \\
&= \frac{1}{\lambda_k^2}\int_{T-\lambda_k^2(S/2)}^{T}\int_{B_{a\lambda_k}(\mathbf{0})}\big(|\widehat{\mathbf{u}}|^3+|\widehat{\mathbf{b}}|^3\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \qquad (3.18)\\
&\geq \frac{S/2}{\lambda_k^2(S/2)}\int_{T-\lambda_k^2(S/2)}^{T}\int_{B_{\lambda_k\sqrt{S/2}}(\mathbf{0})}\big(|\widehat{\mathbf{u}}|^3+|\widehat{\mathbf{b}}|^3\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;>\; \frac{S}{2}\,\epsilon
\end{aligned}$$

if $a \geq \sqrt{S/2}$ and λ_k is so small (i.e. k is so large) that $\lambda_k\sqrt{S/2} < \delta_*$. Due to (3.7) and the uniform convergence of the sequence $\{\mathbf{w}_{\mathbf{u}}^{(k)}\}$ to $\mathbf{w}_{\mathbf{u}}$ on $B_a(\mathbf{0})\times(-S/2,0)$, the left hand side of (3.18) satisfies

$$\begin{aligned}
&\lim_{k\to\infty}\int_{-S/2}^{0}\int_{B_a(\mathbf{0})}\big(|\mathbf{u}^{(k)}|^3+|\widehat{\mathbf{b}}^{(k)}|^3\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \\
&= \int_{-S/2}^{0}\int_{B_a(\mathbf{0})}\big(|\widetilde{\mathbf{u}}+\mathbf{w}_{\mathbf{u}}|^3+|\widetilde{\mathbf{b}}+\mathbf{w}_{\mathbf{b}}|^3\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \\
&= \int_{-S/2}^{0}\int_{B_a(\mathbf{0})}\big(|\mathbf{U}|^3+|\mathbf{B}|^3\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}s.
\end{aligned}$$

Combining this with (3.18) we obtain the inequality

$$\int_{-S/2}^{0}\int_{B_a(\mathbf{0})}\big(|\mathbf{U}|^3+|\mathbf{B}|^3\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \;\geq\; \frac{S}{2}\,\epsilon, \qquad (3.19)$$

which shows that the sum $|\mathbf{U}|+|\mathbf{B}|$ is a non-trivial function.

4 Proof of Theorem 1 – completion

4.1 The finiteness of the norms $\|\widehat{\mathbf{u}}(\,.\,,t_0)\|_3$ and $\|\widehat{\mathbf{b}}(\,.\,,t_0)\|_3$

One can derive from (2.4) that the functions $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{b}}$ satisfy

$$\|\widehat{\mathbf{u}}(\,.\,,t_0)\|_3+\|\widehat{\mathbf{b}}(\,.\,,t_0)\|_3 \;\leq\; M. \qquad (4.1)$$

Indeed, since the sequences $\{\widehat{\mathbf{u}}(\,.\,,t_k)\}$ and $\{\widehat{\mathbf{b}}(\,.\,,t_k)\}$ are bounded, there exist subsequences, weakly convergent to some functions $\widehat{\mathbf{u}}_*$ and $\widehat{\mathbf{b}}_*$ in $\mathbf{L}^3(\mathbb{R}^3)$. Due to (1.5), the norms of the limit functions satisfy

$$\|\widehat{\mathbf{u}}_*\|_3+\|\widehat{\mathbf{b}}_*\|_3 \;\leq\; M. \qquad (4.2)$$

However, as the functions $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{b}}$ are weakly continuous from $(0,T]$ to $\mathbf{L}^2(\mathbb{R}^3)$, we have

$$\widehat{\mathbf{u}}_* = \widehat{\mathbf{u}}(\,.\,,t_0) \qquad \text{and} \qquad \widehat{\mathbf{b}}_* = \widehat{\mathbf{b}}(\,.\,,t_0).$$

This and (4.2) yield (4.1).

4.2 More on the functions U and B

Let $\boldsymbol{\phi}$ be an infinitely differentiable divergence-free vector function with a compact support in $\mathbb{R}^3 \times (-S, 0)$. Then

$$
\begin{aligned}
&\left| \int_{-S}^{0} \int_{\mathbb{R}^3} \mathbf{f}^{(k)}(\mathbf{y}, s) \cdot \boldsymbol{\phi}(\mathbf{y}, s) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \right| \\
&\leq \int_{-S}^{0} \big\| \mathbf{f}^{(k)}(\,.\,, s) \big\|_{\infty;\,\mathbb{R}^3} \int_{\mathbb{R}^3} |\boldsymbol{\phi}(\mathbf{y}, s)| \, \mathrm{d}\mathbf{y} \, \mathrm{d}s \\
&\leq \left(\int_{-S}^{0} \big\| \mathbf{f}^{(k)}(\,.\,, s) \big\|_{\infty;\,\mathbb{R}^3}^{3/2} \, \mathrm{d}s \right)^{2/3} \left[\int_{-S}^{0} \left(\int_{\mathbb{R}^3} |\boldsymbol{\phi}(\mathbf{y}, s)| \, \mathrm{d}\mathbf{y} \right)^{3} \mathrm{d}s \right]^{1/3} \\
&\lesssim \left(\int_{-S}^{0} \big\| \mathbf{f}^{(k)}(\,.\,, s) \big\|_{\infty;\,\mathbb{R}^3}^{3/2} \, \mathrm{d}s \right)^{2/3} \\
&= \left(\int_{t_k}^{t_0} \lambda_k^{9/2} \, \| \mathbf{f}(\,.\,, t) \|_{\infty;\,\mathbb{R}^3}^{3/2} \, \frac{\mathrm{d}t}{\lambda_k^2} \right)^{2/3}
\end{aligned}
\tag{4.3}
$$

and the last integral goes to zero as $k \to \infty$. Applying (3.3)–(3.8), (3.10), (3.15) and (4.3) (and the same estimates satisfied by functions $\mathbf{g}^{(k)}$), we deduce that $(\mathbf{U}, \mathbf{B}, \widehat{p})$ is a suitable weak solution to the system (2.1)–(2.3) (with $\mathbf{f} = \mathbf{0}$ and $\mathbf{g} = \mathbf{0}$) in $\mathbb{R}^3 \times (-S, 0)$, satisfying the initial conditions

$$
\mathbf{U}(\,.\,, -S) = \widehat{\mathbf{u}}(\,.\,, -S) \qquad \text{and} \qquad \mathbf{B}(\,.\,, -S) = \widehat{\mathbf{b}}(\,.\,, -S).
$$

If we denote the duality between the elements of $W_0^{-1-\gamma,3/2}(B_a(\mathbf{0}))$ and $\mathbf{W}_0^{1+\gamma,3}(B_a(\mathbf{0}))$ by $\langle\,.\,,\,.\,\rangle_{B_a(\mathbf{0})}$, then it follows from (3.8) that for each $\boldsymbol{\varphi} \in \mathbf{W}_0^{1+\gamma}(B_a(\mathbf{0}))$, we have

$$
\begin{aligned}
\big| \big\langle \mathbf{U}(\,.\,, 0), \boldsymbol{\varphi} \big\rangle_{B_a(\mathbf{0})} \big| &= \big| \lim_{k\to\infty} \big\langle \widehat{\mathbf{u}}^{(k)}(\,.\,, 0), \boldsymbol{\varphi} \big\rangle_{B_a(\mathbf{0})} \big| \\
&= \left| \lim_{k\to\infty} \int_{B_a(\mathbf{0})} \widehat{\mathbf{u}}^{(k)}(\mathbf{y}, 0) \cdot \boldsymbol{\varphi}(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&\leq \lim_{k\to\infty} \left(\int_{B_a(\mathbf{0})} |\widehat{\mathbf{u}}^{(k)}(\mathbf{y}, 0)|^3 \, \mathrm{d}\mathbf{y} \right)^{1/3} \left(\int_{B_a(\mathbf{0})} |\boldsymbol{\varphi}(\mathbf{y})|^{3/2} \, \mathrm{d}\mathbf{y} \right)^{2/3} \\
&= \lim_{k\to\infty} \left(\int_{B_{\lambda_k a}(\mathbf{0})} |\widehat{\mathbf{u}}(\mathbf{x}, 0)|^3 \, \mathrm{d}\mathbf{x} \right)^{1/3} \left(\int_{B_a(\mathbf{0})} |\boldsymbol{\varphi}(\mathbf{y})|^{3/2} \, \mathrm{d}\mathbf{y} \right)^{2/3}.
\end{aligned}
$$

Since $\lambda_k a \to 0$ as $k \to \infty$, the last limit equals zero and hence

$$
\big\langle \mathbf{U}(\,.\,, 0), \boldsymbol{\varphi} \big\rangle_{B_a(\mathbf{0})} = 0.
$$

This holds for all functions $\boldsymbol{\varphi} \in \mathbf{W}_0^{1+\gamma}(B_a(\mathbf{0}))$ and as the same arguments can also be applied to function $\mathbf{B}$, we obtain the equalities

$$
\mathbf{U}(\,.\,, 0) \;=\; \mathbf{B}(\,.\,, 0) \;=\; \mathbf{0}. \tag{4.4}
$$

These equalities hold in the whole $\mathbb{R}^3$, because a can be taken arbitrarily large.

Since the vorticities $\mathbf{curl}\,\mathbf{U}(\,.\,,0)$ and $\mathbf{curl}\,\mathbf{B}(\,.\,,0)$ also vanish, one can apply the same arguments as in [6], based on the backward uniqueness, to the equations for $\mathbf{curl}\,\mathbf{U}$ and $\mathbf{curl}\,\mathbf{B}$ and deduce that $\mathbf{curl}\,\mathbf{U} = \mathbf{curl}\,\mathbf{B} = \mathbf{0}$ in $\mathbb{R}^3 \times (-S,0)$. This, together with the conditions $\operatorname{div}\mathbf{U} = \operatorname{div}\mathbf{B} = 0$, implies that $\mathbf{U}(\,.\,,s)$ and $\mathbf{B}(\,.\,,s)$ are for all $s \in (-S,0)$ gradients of harmonic functions in $\mathbb{R}^3$. However, since

$$\operatorname*{ess\,sup}_{-S<\tau<s} \big(\|\mathbf{U}(\,.\,,\tau)\|^2_{2;\,\mathrm{unif}} + \|\mathbf{B}(\,.\,,\tau)\|^2_{2;\,\mathrm{unif}}\big) \;<\; \infty$$

(which follows from (3.2) and (3.9)), the only possibility is $\mathbf{U} = \mathbf{B} = \mathbf{0}$ in $\mathbb{R}^3 \times (-S,0)$. This is, however, in contradiction with the conclusions of paragraph 3.7. Consequently, $(\mathbf{x}_0, t_0)$ cannot be a singular point of the solution $(\mathbf{u}, \mathbf{b}, p)$. The proof of Theorem 1, under condition (i), is completed. ☐

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Appendix

In this appendix, we gather the proofs of technical lemmas, Lemma 1, 2, 3 and 4.

Proof of Lemma 1

Taking the divergence to (2.8), we obtain

$$-\Delta\widehat{p}^{(k)} \;=\; \partial_i\partial_j\big(\widehat{u}_i^{(k)}\widehat{u}_j^{(k)}\big) - \partial_i\partial_j\big(\widehat{b}_i^{(k)}\widehat{b}_j^{(k)}\big) - \operatorname{div}\mathbf{f}^{(k)},$$

where $\widehat{u}_i^{(k)}$ and $\widehat{b}_i^{(k)}$ ($i = 1,2,3$) are the components of $\widehat{\mathbf{u}}^{(k)}$ and $\widehat{\mathbf{b}}^{(k)}$, respectively. Then we have

$$\widehat{p}^{(k)}(\mathbf{y},s) = I_1 - I_2 - I_3$$

where

$$I_1 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{z}-\mathbf{y}|}\,\partial_i\partial_j\big(\widehat{u}_i^{(k)}(\mathbf{z},s)\widehat{u}_j^{(k)}(\mathbf{z},s)\big)\;\mathrm{d}\mathbf{z},$$
$$I_2 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{z}-\mathbf{y}|}\,\partial_i\partial_j\big(\widehat{b}_i^{(k)}(\mathbf{z},s)\widehat{b}_j^{(k)}(\mathbf{z},s)\big)\;\mathrm{d}\mathbf{z},$$
$$I_3 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{z}-\mathbf{y}|}\,\operatorname{div}\mathbf{f}^{(k)}(\mathbf{z},s)\;\mathrm{d}\mathbf{z}.$$

We shall show that

$$I_1 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathbb{K}(\mathbf{z}-\mathbf{y}) : \big[\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)\otimes\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)\big]\;\mathrm{d}\mathbf{z} - \frac{1}{3}\,|\mathbf{u}^{(k)}(\mathbf{y},s)|^2. \tag{4.5}$$

Integrating by parts, we rewrite I_1 as

$$\begin{aligned} I_1 &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\Big(\frac{\partial}{\partial z_j}\,\frac{1}{|\mathbf{z}-\mathbf{y}|}\Big)\,\big[\widehat{u}_i^{(k)}(\mathbf{z},s)\,\partial_i\widehat{u}_j^{(k)}(\mathbf{z},s)\big]\;\mathrm{d}\mathbf{z} \\ &= \lim_{\epsilon\to 0+}\,\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{y})}\frac{z_j-y_j}{|\mathbf{z}-\mathbf{y}|^3}\,\big[\widehat{u}_i^{(k)}(\mathbf{z},s)\,\partial_i\widehat{u}_j^{(k)}(\mathbf{z},s)\big]\;\mathrm{d}\mathbf{z} \\ &= \lim_{\epsilon\to 0+}\,\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_\epsilon(\mathbf{y})}k_{ij}(\mathbf{z}-\mathbf{y})\,\big[u_i^{(k)}(\mathbf{z},s)\,u_j^{(k)}(\mathbf{z},s)\big]\;\mathrm{d}\mathbf{z} \\ &\quad+\lim_{\epsilon\to 0+}\,\frac{1}{4\pi}\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{z_j-y_j}{|\mathbf{z}-\mathbf{y}|^3}\,\big[\widehat{u}_i^{(k)}(\mathbf{z},s)\,\widehat{u}_j^{(k)}(\mathbf{z},s)\big]\,\frac{y_i-z_i}{|\mathbf{z}-\mathbf{y}|}\;\mathrm{d}_{\mathbf{z}}S. \end{aligned}$$

The last term becomes

$$\begin{aligned} &\frac{1}{4\pi}\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{z_j-y_j}{|\mathbf{z}-\mathbf{y}|^3}\,\big[\widehat{u}_i^{(k)}(\mathbf{z},s)\,\widehat{u}_j^{(k)}(\mathbf{z},s)\big]\,\frac{y_i-z_i}{|\mathbf{z}-\mathbf{y}|}\;\mathrm{d}_{\mathbf{z}}S \\ &= -\frac{1}{4\pi}\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^4}\,u_i^{(k)}(\mathbf{z},s)\,u_j^{(k)}(\mathbf{z},s)\;\mathrm{d}_{\boldsymbol{z}}S \\ &= -\frac{1}{4\pi}\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^4}\,\big[u_i^{(k)}(\mathbf{z},s)\,u_j^{(k)}(\mathbf{z},s)-u_i^{(k)}(\mathbf{y},s)\,u_j^{(k)}(\mathbf{y},s)\big]\;\mathrm{d}_{\boldsymbol{z}}S \\ &\quad-\frac{1}{4\pi}\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^4}\,u_i^{(k)}(\mathbf{y},s)\,u_j^{(k)}(\mathbf{y},s)\;\mathrm{d}_{\boldsymbol{z}}S \\ &= -\frac{1}{4\pi}\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^4}\,u_i^{(k)}(\mathbf{y},s)\,u_j^{(k)}(\mathbf{y},s)\;\mathrm{d}_{\boldsymbol{z}}S+o(\epsilon) \end{aligned}$$

since

$$\sup_{|\mathbf{z}-\mathbf{y}|=\epsilon}\big|u_i^{(k)}(\mathbf{z},s)\,u_j^{(k)}(\mathbf{z},s)-u_i^{(k)}(\mathbf{y},s)\,u_j^{(k)}(\mathbf{y},s)\big|=o(\epsilon).$$

Using the spatial continuity of $\widehat{u}_i^{(k)}\,\widehat{u}_j^{(k)}$ at the point $(\mathbf{y},s)$, we rewrite the last integral as

$$-\frac{1}{4\pi}|\mathbf{u}^{(k)}(\mathbf{y},s)|^2\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^4}\,\frac{u_i^{(k)}(\mathbf{y},s)}{|\mathbf{u}^{(k)}(\mathbf{y},s)|}\,\frac{u_j^{(k)}(\mathbf{y},s)}{|\mathbf{u}^{(k)}(\mathbf{y},s)|}\;\mathrm{d}_{\boldsymbol{z}}S.$$

To obtain (4.5), it suffices to show that

$$\int_{|\mathbf{z}-\mathbf{y}|=\epsilon}\frac{(z_i-y_i)(z_j-y_j)}{|\mathbf{z}-\mathbf{y}|^4}\,\frac{u_i^{(k)}(\mathbf{y},s)}{|\mathbf{u}^{(k)}(\mathbf{y},s)|}\,\frac{u_j^{(k)}(\mathbf{y},s)}{|\mathbf{u}^{(k)}(\mathbf{y},s)|}\;\mathrm{d}_{\boldsymbol{z}}S=\frac{4\pi}{3}.$$

We may assume that $\mathbf{u}^{(k)}(\mathbf{y},s)/|\mathbf{u}^{(k)}(\mathbf{y},s)|=(1,0,0)$ and then use the transformation $\mathbf{x}=\mathbf{z}-\mathbf{y}$ to compute the last integral as follows:

$$\begin{aligned} \int_{|\mathbf{x}|=\epsilon}\frac{x_1^2}{|\mathbf{x}|^4}\;\mathrm{d}_{\boldsymbol{x}}S &= \frac{1}{\epsilon^4}\int_{|\mathbf{x}|=\epsilon}x_1^2\;\mathrm{d}_{\boldsymbol{x}}S \\ &= \frac{1}{\epsilon^4}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\epsilon^2\,\cos^2\varphi\,\cos^2\vartheta\,\epsilon^2\,\cos\vartheta\;\mathrm{d}\vartheta\,\mathrm{d}\varphi \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{\pi/2} \cos^3\vartheta \, \mathrm{d}\vartheta \int_0^{2\pi} \cos^2\varphi \, \mathrm{d}\varphi \\
&= 2 \times \frac{2}{3} \times \pi = \frac{4\pi}{3}.
\end{aligned}
$$

By the same way, we can calculate I_2. Thus,

$$
\begin{aligned}
\widehat{p}^{(k)}(\mathbf{y}, s) \; = \; & \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathbb{K}(\mathbf{z}-\mathbf{y}) : \widehat{\mathbf{M}}^{(k)}(\mathbf{z}, s) \, \mathrm{d}\mathbf{z} - \frac{1}{3} \left(|\widehat{\mathbf{u}}^{(k)}(\mathbf{y}, s)|^2 - |\widehat{\mathbf{b}}^{(k)}(\mathbf{y}, s)|^2 \right) \\
& - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{z}-\mathbf{y}|} \operatorname{div} \mathbf{f}^{(k)}(\mathbf{z}, s) \, \mathrm{d}\mathbf{z}
\end{aligned}
\tag{4.6}
$$

where

$$
\widehat{\mathbf{M}}^{(k)}(\mathbf{z}, s) = \widehat{\mathbf{u}}^{(k)}(\mathbf{z}, s) \otimes \widehat{\mathbf{u}}^{(k)}(\mathbf{z}, s) - \widehat{\mathbf{b}}^{(k)}(\mathbf{z}, s) \otimes \widehat{\mathbf{b}}^{(k)}(\mathbf{z}, s)
$$

Finally, we get the decomposition of pressure (2.23) by splitting the first integral as follows:

$$
\begin{aligned}
& \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathbb{K}(\mathbf{z}-\mathbf{y}) : \widehat{\mathbf{M}}^{(k)}(\mathbf{z}, s) \, \mathrm{d}\mathbf{z} \\
& = \frac{1}{4\pi} \int_{B_2(\mathbf{x})} \mathbb{K}(\mathbf{z}-\mathbf{y}) : \widehat{\mathbf{M}}^{(k)}(\mathbf{z}, s) \, \mathrm{d}\mathbf{z} \\
& \quad + \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B_2(\mathbf{x})} \big[\mathbb{K}(\mathbf{z}-\mathbf{y}) - \mathbb{K}(\mathbf{z}-\mathbf{x}) \big] : \widehat{\mathbf{M}}^{(k)}(\mathbf{z}, s) \, \mathrm{d}\mathbf{z} \\
& \quad + \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B_2(\mathbf{x})} \mathbb{K}(\mathbf{z}-\mathbf{x}) : \widehat{\mathbf{M}}^{(k)}(\mathbf{z}, s) \, \mathrm{d}\mathbf{z}.
\end{aligned}
$$

This completes the proof of Lemma 1. □

Proof of Lemma 2

Applying the Calderon–Zygmund theorem and the cut-off function procedure (with a cut–off function equal to 1 in $B_{3/2}(\mathbf{x})$ and supported in $B_2(\mathbf{x})$), we obtain

$$
\begin{aligned}
& \left\| \int_{B_2(\mathbf{x})} \mathbb{K}(\mathbf{z}-\mathbf{y}) : \big[\widehat{\mathbf{u}}^{(k)}(\mathbf{z}, s) \otimes \widehat{\mathbf{u}}^{(k)}(\mathbf{z}, s) \big] \, \mathrm{d}\mathbf{z} \right\|_{3/2;\, B_{3/2}(\mathbf{x})} \\
& \lesssim \; \|\widehat{\mathbf{u}}^{(k)}(\,.\,, s)\|_{3;\, B_2(\mathbf{x})}^2 \\
& \lesssim \; \|\widetilde{\mathbf{u}}^{(k)}(\,.\,, s)\|_{3;\, B_2(\mathbf{x})}^2 + \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,, s)\|_{3;\, B_2(\mathbf{x})}^2 \\
& \lesssim_M \; \|\widetilde{\mathbf{u}}^{(k)}(\,.\,, s)\|_{3;\, B_2(\mathbf{x})}^2 + 1.
\end{aligned}
$$

Since the part of the integral, containing $\widehat{\mathbf{b}}^{(k)}(\mathbf{z}, s) \otimes \widehat{\mathbf{b}}^{(k)}(\mathbf{z}, s)$, can be estimated similarly, we get the estimate (2.24)

To get (2.25) we shall show the following estimate

$$
|\mathbb{K}(\mathbf{z}-\mathbf{y}) - \mathbb{K}(\mathbf{z}-\mathbf{x})| \; \lesssim \; \frac{|\mathbf{y}-\mathbf{x}|}{|\mathbf{z}-\mathbf{x}|^4}. \tag{4.7}
$$

Let $|\mathbf{y}-\mathbf{x}|<3/2$ and $|\mathbf{z}-\mathbf{x}|>2$. An elementary computation shows

$$\begin{aligned}
\mathbb{K}(\mathbf{z}-\mathbf{y})-\mathbb{K}(\mathbf{z}-\mathbf{x}) &= \nabla_{\mathbf{z}}^2\Big(\frac{1}{|\mathbf{z}-\mathbf{y}|}-\frac{1}{|\mathbf{z}-\mathbf{x}|}\Big) \\
&= \nabla_{\mathbf{z}}^2\int_0^1 \frac{\partial}{\partial\zeta}\,\frac{1}{|(\mathbf{z}-\mathbf{x})+\zeta(\mathbf{x}-\mathbf{y})|}\;\mathrm{d}\zeta \\
&= \int_0^1 \nabla_{\mathbf{z}}^2\,\frac{(\mathbf{z}-\mathbf{x})+\zeta(\mathbf{x}-\mathbf{y})}{|(\mathbf{z}-\mathbf{x})+\zeta(\mathbf{x}-\mathbf{y})|^3}\cdot(\mathbf{x}-\mathbf{y})\;\mathrm{d}\zeta.
\end{aligned}$$

Since $|\mathbf{y}-\mathbf{x}|<3/2$ and $|\mathbf{z}-\mathbf{x}|>2$, we have for all $0\leq\zeta\leq 1$,

$$|(\mathbf{z}-\mathbf{x})+\zeta(\mathbf{x}-\mathbf{y})|\geq|\mathbf{z}-\mathbf{x}|-|\mathbf{x}-\mathbf{y}|\geq|\mathbf{z}-\mathbf{x}|-\frac{3}{2}\geq\frac{1}{4}|\mathbf{z}-\mathbf{x}|.$$

Therefore,

$$|\mathbb{K}(\mathbf{z}-\mathbf{y})-\mathbb{K}(\mathbf{z}-\mathbf{x})|\;\leq\;\sup_{0\leq\zeta\leq 1}\bigg|\nabla_{\mathbf{z}}^2\,\frac{(\mathbf{z}-\mathbf{x})+\zeta(\mathbf{x}-\mathbf{y})}{|(\mathbf{z}-\mathbf{x})+\zeta(\mathbf{x}-\mathbf{y})|^3}\bigg|\,|\mathbf{x}-\mathbf{y}|\lesssim\;\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{z}-\mathbf{x}|^4}.$$

Now, using (4.7), we obtain (2.25) from the computation

$$\begin{aligned}
|\widehat{p}_{\mathbf{x}}^{2(k)}(\mathbf{y},s)| &\lesssim \int_{\mathbb{R}^3\smallsetminus B_2(\mathbf{x})}\frac{|\mathbf{y}-\mathbf{x}|}{|\mathbf{z}-\mathbf{x}|^4}\,\big(|\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)|^2+|\widehat{\mathbf{b}}^{(k)}(\mathbf{z},s)|^2\big)\;\mathrm{d}\mathbf{z} \\
&= \sum_{i=0}^{\infty}\int_{2^{i+1}<|\mathbf{z}-\mathbf{x}|<2^{i+2}}\frac{|\mathbf{y}-\mathbf{x}|}{|\mathbf{z}-\mathbf{x}|^4}\,\big(|\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)|^2+|\widehat{\mathbf{b}}^{(k)}(\mathbf{z},s)|^2\big)\;\mathrm{d}\mathbf{z} \\
&\lesssim \sum_{i=0}^{\infty}2^{-4i}\int_{2^{i+1}<|\mathbf{z}-\mathbf{x}|<2^{i+2}}\big(|\widehat{\mathbf{u}}^{(k)}(\mathbf{z},s)|^2+|\widehat{\mathbf{b}}^{(k)}(\mathbf{z},s)|^2\big)\;\mathrm{d}\mathbf{z} \\
&\lesssim \sum_{i=0}^{\infty}2^{-4i}\,2^{3i}\big(\|\widehat{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2+\|\widehat{\mathbf{b}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2\big) \\
&\lesssim \|\widehat{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2+\|\widehat{\mathbf{b}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2.
\end{aligned}$$

The estimate (2.26) is obvious.

Finally, (2.27) follows from the computation

$$\begin{aligned}
\big|d^{(k)}(s)\big| &\lesssim \int_{\rho_5<|\mathbf{x}_0+\lambda_k\mathbf{z}|<\rho_6}\frac{1}{|\mathbf{z}-\mathbf{y}|}\,\lambda_k^3\,\big|\mathrm{div}_{\mathbf{z}}\,\mathbf{f}\big(\mathbf{x}_0+\lambda_k\mathbf{z},t_0+\lambda_k^2 s\big)\big|\;\mathrm{d}\mathbf{z} \\
&= \int_{\rho_5<|\mathbf{x}_0+\lambda_k\mathbf{z}|<\rho_6}\frac{1}{|\mathbf{z}-\mathbf{y}|}\,\lambda_k^4\,\big|[\mathrm{div}\,\mathbf{f}]\big(\mathbf{x}_0+\lambda_k\mathbf{z},t_0+\lambda_k^2 s\big)\big|\;\mathrm{d}\mathbf{z} \\
&\leq \lambda_k^4\,\|\mathrm{div}\,\mathbf{f}(\,.\,,t_0+\lambda_k^2 s)\|_{\infty;\,\mathbb{R}^3}\int_{\rho_5<|\mathbf{x}_0+\lambda_k\mathbf{z}|<\rho_6}\frac{\mathrm{d}\mathbf{z}}{|\mathbf{z}-\mathbf{y}|} \\
&\lesssim \lambda_k^2\,\|\mathrm{div}\,\mathbf{f}(\,.\,,t_0+\lambda_k^2 s)\|_{\infty;\,\mathbb{R}^3}\,(\rho_6^2-\rho_5^2)
\end{aligned}$$

since $\int_{\rho_5<|\mathbf{x}_0+\lambda_k\mathbf{z}|<\rho_6}\frac{\mathrm{d}\mathbf{z}}{|\mathbf{z}-\mathbf{y}|}\;\leq\;\frac{2\pi}{\lambda_k^2}\,(\rho_6^2-\rho_5^2)$. The right hand side is integrable with respect to s in $(-S,0)$ with the power at least μ for any $\mu\in(1,2)$. (See paragraph 2.1, item (b), for the explanation where μ comes from.) Hence we have $d^{(k)}\in L^{\mu}(-S,0)$ for any $\mu\in(1,2)$. This completes the proof of Lemma 2. □

Proof of Lemma 3

We note the inequality $\|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathbb{R}^3} \lesssim \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,-S)\|_{2;\,\mathbb{R}^3}$, which follows from inequality (A) in [9, p. 190]. For a.a. $s \in (-S,0)$, we have

$$\begin{aligned}
\|\widetilde{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 &\lesssim \|\widehat{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathbb{R}^3}^2 + \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathbb{R}^3}^2 \\
&\lesssim \|\widehat{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathbb{R}^3}^2 + \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,-S)\|_{2;\,\mathbb{R}^3}^2 \\
&= \|\widehat{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,B_{\rho_6/\lambda_k}(\mathbf{0})}^2 + \|\widehat{\mathbf{u}}^{(k)}(\,.\,,-S)\|_{2;\,B_{\rho_6/\lambda_k}(\mathbf{0})}^2 \\
&\lesssim \lambda_k^{-1}\left(\|\mathbf{u}(\,.\,,t)\|_{2;\,B_{\rho_6}(\mathbf{x}_0)}^2 + \|\widehat{\mathbf{u}}^{(k)}(\,.\,,-S)\|_{3;\,B_{\rho_6/\lambda_k}(\mathbf{0})}^2\right) \\
&\lesssim \lambda_k^{-1}
\end{aligned}$$

where we used the inequality $\|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,s)\|_{2;\,\mathbb{R}^3} \lesssim \|\mathbf{w}_{\mathbf{u}}^{(k)}(\,.\,,-S)\|_{2;\,\mathbb{R}^3}$ in [9, p. 190]. By the same way we also have $\|\widetilde{\mathbf{b}}^{(k)}(\,.\,,s)\|_{2;\,\mathrm{unif}}^2 \lesssim \lambda_k^{-1}$. Hence we obtain (2.28)

For a.a. $s \in (-S,0)$, we have, by the Sobolev and Hölder inequality,

$$\begin{aligned}
&\sup_{\mathbf{x}\in\mathbb{R}^3}\int_{s_0}^{s} \|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{3;\,B_1(\mathbf{x})}^3\,\mathrm{d}\tau \\
&\le \sup_{\mathbf{x}\in\mathbb{R}^3}\int_{s_0}^{s} \|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^{3/2}\,\|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{6;\,B_1(\mathbf{x})}^{3/2}\,\mathrm{d}\tau \\
&\lesssim \sup_{\mathbf{x}\in\mathbb{R}^3}\int_{s_0}^{s} \|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^{3/2}\left(\|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^2 + \|\nabla\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^2\right)^{3/4}\mathrm{d}\tau \\
&\lesssim \int_{s_0}^{s}\alpha_k^{3/2}(\tau)\,\mathrm{d}\tau + \sup_{\mathbf{x}\in\mathbb{R}^3}\int_{s_0}^{s}\alpha_k^{3/4}(\tau)\|\nabla\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,B_1(\mathbf{x})}^{3/2}\,\mathrm{d}\tau \\
&\lesssim \int_{s_0}^{s}\alpha_k^{3/2}(\tau)\,\mathrm{d}\tau + \left(\int_{s_0}^{s}\alpha_k^{3}(\tau)\,\mathrm{d}\tau\right)^{1/4}\beta_k^{3/4}(s_0,s).
\end{aligned}$$

Since we can get the same estimate for $\widetilde{\mathbf{b}}^{(k)}(\,.\,,\tau)$, we obtain (2.29).

Using the estimates in Lemma 2, we obtain (2.30)

$$\begin{aligned}
\delta_k(s_0,s) &\lesssim_M \sup_{\mathbf{x}\in\mathbb{R}^3}\int_{s_0}^{s}\left(\|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{3;\,B_2(\mathbf{x})}^3 + \|\widetilde{\mathbf{b}}^{(k)}(\,.\,,\tau)\|_{3;\,B_2(\mathbf{x})}^3 + 1\right)\mathrm{d}\tau \\
&\quad + \int_{s_0}^{s}\left(\|\widetilde{\mathbf{u}}^{(k)}(\,.\,,\tau)\|_{2;\,\mathrm{unif}}^3 + \|\widetilde{\mathbf{b}}^{(k)}(\,.\,,\tau)\|_{2;\,\mathrm{unif}}^3 + 1\right)\mathrm{d}\tau \\
&\lesssim \gamma_k(s_0,s) + \int_{s_0}^{s}\alpha_k^{3/2}(\tau)\,\mathrm{d}\tau + (s-s_0).
\end{aligned}$$

This completes the proof of Lemma 3. □

Proof of Lemma 4

Let ϕ be an infinitely differentiable function in $\mathbb{R}^3$ with values in $[0,1]$, supported in $B_{3/2}(\mathbf{0})$ and equal to 1 in $B_1(\mathbf{0})$. For $\mathbf{x} \in \mathbb{R}^3$, we denote $\phi_{\mathbf{x}}(\mathbf{y}) := \phi(\mathbf{y}-\mathbf{x})$. Recall that $(\mathbf{u},\mathbf{b},p)$ has been supposed to be a suitable weak solution to the system (1.1)–(1.3), $(\widetilde{\mathbf{u}}^{(k)}, \widetilde{\mathbf{b}}^{(k)}, \widehat{p}^{(k)})$ is a suitable weak solution to the system (2.13), (2.14), completed by the equations $\operatorname{div}\widetilde{\mathbf{u}}^{(k)} = \operatorname{div}\widetilde{\mathbf{b}}^{(k)} = 0$ in $\mathbb{R}^3 \times (-S,0)$, with the initial conditions (2.18). Thus it satisfies a localized energy inequality, analogous to (1.10). Considering the test function in such an inequality in the form of a product of $\phi_{\mathbf{x}}$ with a function of s, and applying a standard limit procedure to the function of s, one can deduce that the functions $\widetilde{\mathbf{u}}^{(k)}$, $\widetilde{\mathbf{b}}^{(k)}$, $\widehat{p}^{(k)}$, $\mathbf{w}_{\mathbf{u}}^{(k)}$ and $\mathbf{w}_{\mathbf{b}}^{(k)}$ satisfy the inequality for a.a. $s_0 \in [-S,0)$ and all $s \in (s_0,0)$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \phi_{\mathbf{x}} \, (|\widetilde{\mathbf{u}}^{(k)}(\mathbf{y},s)|^2 + |\widetilde{\mathbf{b}}^{(k)}(\mathbf{y},s)|^2) \, \mathrm{d}\mathbf{y} + 2\int_{s_0}^{s} \int_{\mathbb{R}^3} \phi_{\mathbf{x}} \left(\nu \, |\nabla \widetilde{\mathbf{u}}^{(k)}|^2 + \xi \, |\nabla \widetilde{\mathbf{b}}^{(k)}|^2\right) \mathrm{d}\mathbf{y} \\
&\leq \int_{\mathbb{R}^3} \phi_{\mathbf{x}} \, (|\widetilde{\mathbf{u}}^{(k)}(\mathbf{y},s_0)|^2 + |\widetilde{\mathbf{b}}^{(k)}(\mathbf{y},s_0)|^2) \, \mathrm{d}\mathbf{y} + \int_{s_0}^{s} \int_{\mathbb{R}^3} I_1^{(k)} + I_2^{(k)} + I_3^{(k)} + I_4^{(k)} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\tau,
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
I_1^{(k)} &= (\nu \, |\widetilde{\mathbf{u}}^{(k)}|^2 + \xi \, |\widetilde{\mathbf{b}}^{(k)}|^2) \Delta\phi_{\mathbf{x}} + \left(|\widetilde{\mathbf{u}}^{(k)}|^2 + |\widetilde{\mathbf{b}}^{(k)}|^2 + 2\widehat{p}^{(k)}\right) \widetilde{\mathbf{u}}^{(k)} \cdot \nabla\phi_{\mathbf{x}}, \\
I_2^{(k)} &= 2\,[(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}}] \, [\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)} + \widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)}] \\
&\quad - 2\,[(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}}] \, [(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \widetilde{\mathbf{u}}^{(k)} + (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \widetilde{\mathbf{b}}^{(k)}] \\
&\quad + (\widetilde{\mathbf{b}}^{(k)} \cdot \nabla\phi_{\mathbf{x}}) \, (\widetilde{\mathbf{u}}^{(k)} \cdot \widetilde{\mathbf{b}}^{(k)}) + (|\widetilde{\mathbf{u}}^{(k)}|^2 + |\widetilde{\mathbf{b}}^{(k)}|^2) \, (\mathbf{w}_{\mathbf{u}}^{(k)} \cdot \nabla\phi_{\mathbf{x}}), \\
I_3^{(k)} &= 2\widetilde{\mathbf{u}}^{(k)} \cdot [\nabla\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)} + \nabla\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)}] \, \phi_{\mathbf{x}} \\
&\quad - 2\,[\widetilde{\mathbf{b}}^{(k)} \cdot \nabla\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)} + \widetilde{\mathbf{b}}^{(k)} \cdot \nabla\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)}] \, \phi_{\mathbf{x}} \\
&\quad - 2\mathbf{w}_{\mathbf{b}}^{(k)} \cdot [\nabla\widetilde{\mathbf{u}}^{(k)} \cdot \widetilde{\mathbf{b}}^{(k)} + \nabla\widetilde{\mathbf{b}}^{(k)} \cdot \widetilde{\mathbf{u}}^{(k)}] \, \phi_{\mathbf{x}}, \\
I_4^{(k)} &= 2\mathbf{w}_{\mathbf{u}}^{(k)} \cdot [\nabla\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)} + \nabla\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)}] \, \phi_{\mathbf{x}} \\
&\quad - 2\mathbf{w}_{\mathbf{b}}^{(k)} \cdot [\nabla\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)} + \nabla\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)}] \, \phi_{\mathbf{x}}, \\
I_5^{(k)} &= 2\phi_{\mathbf{x}} \, \widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{f}^{(k)} + 2\phi_{\mathbf{x}} \, \widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{g}^{(k)}.
\end{aligned}$$

Note that one can also formally derive (4.8) in this way: multiplying equation (2.13) by $2\phi_{\mathbf{x}} \, \widetilde{\mathbf{u}}^{(k)}$, integrating on $\mathbb{R}^3 \times (s_0,s)$ (where $-S \leq s_0 < s < 0$) and then integrating by parts, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^3} \phi_{\mathbf{x}} \, |\widetilde{\mathbf{u}}^{(k)}(\mathbf{y},s)|^2 \, \mathrm{d}\mathbf{y} - \int_{\mathbb{R}^3} \phi_{\mathbf{x}} \, |\widetilde{\mathbf{u}}^{(k)}(\mathbf{y},s_0)|^2 \, \mathrm{d}\mathbf{y} + 2\nu \int_{s_0}^{s} \int_{\mathbb{R}^3} \phi_{\mathbf{x}} \, |\nabla\widetilde{\mathbf{u}}^{(k)}|^2 \, \mathrm{d}\mathbf{y} \, \mathrm{d}\tau \\
&= \int_{s_0}^{s} \int_{\mathbb{R}^3} J_1^{(k)} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\tau.
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
J_1^{(k)} := {} & \nu \, |\widetilde{\mathbf{u}}^{(k)}|^2 \Delta\phi_{\mathbf{x}} + 2\widehat{p}^{(k)} \, \widetilde{\mathbf{u}}^{(k)} \cdot \nabla\phi_{\mathbf{x}} + |\widetilde{\mathbf{u}}^{(k)}|^2 \, (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}} \\
& + 2\,[(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}}] \, (\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)}) + 2(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{w}_{\mathbf{u}}^{(k)} \, \phi_{\mathbf{x}} \\
& - 2(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla\widetilde{\mathbf{u}}^{(k)} \cdot (\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \, \phi_{\mathbf{x}}
\end{aligned}$$

$$- 2[(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}}]\,[(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \widetilde{\mathbf{u}}^{(k)}] + 2\phi_{\mathbf{x}}\widetilde{\mathbf{u}}^{(k)} \cdot \mathbf{f}^{(k)}.$$

Similarly, testing equation (2.14) by $2\phi_{\mathbf{x}}\widetilde{\mathbf{b}}^{(k)}$, one gets

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi_{\mathbf{x}}\,|\widetilde{\mathbf{b}}^{(k)}(\mathbf{y}, s)|^2\,\mathrm{d}\mathbf{y} - \int_{\mathbb{R}^3} \phi_{\mathbf{x}}\,|\widetilde{\mathbf{b}}^{(k)}(\mathbf{y}, s_0)|^2\,\mathrm{d}\mathbf{y} + 2\xi \int_{s_0}^{s}\int_{\mathbb{R}^3} \phi_{\mathbf{x}}\,|\nabla\widetilde{\mathbf{b}}^{(k)}|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \\ &= \int_{s_0}^{s}\int_{\mathbb{R}^3} J_2^{(k)}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau.\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}J_2^{(k)} := {}& \xi\,|\widetilde{\mathbf{b}}^{(k)}|^2\Delta\phi_{\mathbf{x}} + |\widetilde{\mathbf{b}}^{(k)}|^2\,(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}} \\ &+ 2[(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}}]\,(\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)}) + 2(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \nabla\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{w}_{\mathbf{b}}^{(k)}\,\phi_{\mathbf{x}} \\ &- 2(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla\widetilde{\mathbf{b}}^{(k)} \cdot (\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)})\,\phi_{\mathbf{x}} \\ &- 2[(\widetilde{\mathbf{b}}^{(k)} + \mathbf{w}_{\mathbf{b}}^{(k)}) \cdot \nabla\phi_{\mathbf{x}}]\,[(\widetilde{\mathbf{u}}^{(k)} + \mathbf{w}_{\mathbf{u}}^{(k)}) \cdot \widetilde{\mathbf{b}}^{(k)}] + 2\phi_{\mathbf{x}}\widetilde{\mathbf{b}}^{(k)} \cdot \mathbf{g}^{(k)}.\end{aligned}$$

Summing (4.9) and (4.10) and integrating by parts, we obtain (4.8).

We have from the definition

$$\int_{\mathbb{R}^3} \phi_{\mathbf{x}}\,(|\widetilde{\mathbf{u}}^{(k)}(\mathbf{y}, s_0)|^2 + |\widetilde{\mathbf{b}}^{(k)}(\mathbf{y}, s_0)|^2)\,\mathrm{d}\mathbf{y} \;\lesssim\; \alpha(s_0). \tag{4.11}$$

Let $\chi := \chi_{B_{3/2}(\mathbf{x})}$ (the characteristic function of $B_{3/2}(\mathbf{x})$). We have by Young's inequality

$$\begin{aligned}|I_1^{(k)}| &\lesssim\; (|\widetilde{\mathbf{u}}^{(k)}|^2 + |\widetilde{\mathbf{b}}^{(k)}|^2 + |\widetilde{\mathbf{u}}^{(k)}|^3 + |\widetilde{\mathbf{b}}^{(k)}|^2|\widetilde{\mathbf{u}}^{(k)}| + |\widehat{p}^{(k)}||\widetilde{\mathbf{u}}^{(k)}|)\chi \\ &\lesssim\; (|\widetilde{\mathbf{u}}^{(k)}|^2 + |\widetilde{\mathbf{b}}^{(k)}|^2 + |\widetilde{\mathbf{u}}^{(k)}|^3 + |\widetilde{\mathbf{b}}^{(k)}|^3 + |\widehat{p}^{(k)}|^{3/2})\chi\end{aligned}$$

and hence

$$\int_{s_0}^{s}\int_{\mathbb{R}^3} |I_1^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \lesssim\; \int_{s_0}^{s} \alpha_k(\tau)\,\mathrm{d}\tau + \gamma_k(s_0, s) + \delta_k(s_0, s) + \mathcal{F}^{(k)}(s_0, s). \tag{4.12}$$

Using the inequality, $abc \le a^3 + b^3 + c^3$, we have

$$\begin{aligned}|I_2^{(k)}| \lesssim\; &\{(|\widetilde{\mathbf{u}}^{(k)}| + |\mathbf{w}_{\mathbf{u}}^{(k)}|)(|\widetilde{\mathbf{u}}^{(k)}||\mathbf{w}_{\mathbf{u}}^{(k)}| + |\widetilde{\mathbf{b}}^{(k)}||\mathbf{w}_{\mathbf{b}}^{(k)}|) \\ &\quad + (|\widetilde{\mathbf{b}}^{(k)}| + |\mathbf{w}_{\mathbf{b}}^{(k)}|)[(|\widetilde{\mathbf{b}}^{(k)}| + |\mathbf{w}_{\mathbf{b}}^{(k)}|)|\widetilde{\mathbf{u}}^{(k)}| + (|\widetilde{\mathbf{u}}^{(k)}| + |\mathbf{w}_{\mathbf{u}}^{(k)}|)|\widetilde{\mathbf{b}}^{(k)}|] \\ &\quad + |\widetilde{\mathbf{b}}^{(k)}||\widetilde{\mathbf{u}}^{(k)}||\widetilde{\mathbf{b}}^{(k)}| + (|\widetilde{\mathbf{u}}^{(k)}|^2 + |\widetilde{\mathbf{b}}^{(k)}|^2)|\mathbf{w}_{\mathbf{u}}^{(k)}|\}\chi \\ \lesssim\; &(|\widetilde{\mathbf{u}}^{(k)}|^3 + |\widetilde{\mathbf{b}}^{(k)}|^3 + |\mathbf{w}_{\mathbf{u}}^{(k)}|^3 + |\mathbf{w}_{\mathbf{b}}^{(k)}|^3)\chi\end{aligned}$$

and hence by means of (2.21) and (2.22)

$$\int_{s_0}^{s}\int_{\mathbb{R}^3} |I_2^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \lesssim_M\; \gamma_k(s_0, s) + (s - s_0). \tag{4.13}$$

We have

$$|I_3^{(k)}| \lesssim \{|\widetilde{\mathbf{u}}^{(k)}||\nabla\widetilde{\mathbf{u}}^{(k)}||\mathbf{w}_{\mathbf{u}}^{(k)}| + |\widetilde{\mathbf{u}}^{(k)}||\nabla\widetilde{\mathbf{b}}^{(k)}||\mathbf{w}_{\mathbf{b}}^{(k)}| \\ + |\widetilde{\mathbf{b}}^{(k)}||\nabla\widetilde{\mathbf{u}}^{(k)}||\mathbf{w}_{\mathbf{b}}^{(k)}| + |\widetilde{\mathbf{b}}^{(k)}||\nabla\widetilde{\mathbf{b}}^{(k)}||\mathbf{w}_{\mathbf{u}}^{(k)}|\}\chi$$

We use the Hölder inequality and the Sobolev inequality to obtain that

$$\begin{aligned}
&\int_{s_0}^{s}\int_{B_{3/2}(\mathbf{x})} |\widetilde{\mathbf{u}}^{(k)}|\,|\nabla\widetilde{\mathbf{u}}^{(k)}|\,|\mathbf{w}_{\mathbf{u}}^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \\
&\lesssim \int_{s_0}^{s} \|\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}^{2/5}\,\|\widetilde{\mathbf{u}}^{(k)}\|_{6;\,B_{3/2}(\mathbf{x})}^{3/5}\,\|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}\,\mathrm{d}\tau \\
&\lesssim \int_{s_0}^{s} \|\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}^{2/5}\left(\|\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})} + \|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}\right)^{8/5}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}\,\mathrm{d}\tau \\
&\lesssim \int_{s_0}^{s} \alpha_k(\tau)^{1/5}\left(\alpha_k(\tau) + \|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}^{2}\right)^{4/5}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}\,\mathrm{d}\tau \\
&\lesssim \left(\int_{s_0}^{s}\left(\alpha_k(\tau) + \|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}^{2}\right)\mathrm{d}\tau\right)^{4/5}\left(\int_{s_0}^{s}\alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau\right)^{1/5} \\
&\lesssim \left(\int_{s_0}^{s}\alpha_k(\tau)\,\mathrm{d}\tau + \beta_k(s_0,s)\right)^{4/5}\left(\int_{s_0}^{s}\alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau\right)^{1/5}.
\end{aligned}$$

Since the integral of other three terms can be estimated in the same way, we obtain

$$\begin{aligned}
&\int_{s_0}^{s}\int_{\mathbb{R}^3} |I_3^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \\
&\lesssim_M \left(\int_{s_0}^{s}\alpha_k(\tau)\,\mathrm{d}\tau + \beta_k(s_0,s)\right)^{4/5}\left(\int_{s_0}^{s}\alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau\right)^{1/5}.
\end{aligned} \tag{4.14}$$

We have

$$|I_4^{(k)}| \lesssim \{|\mathbf{w}_{\mathbf{u}}^{(k)}||\nabla\widetilde{\mathbf{u}}^{(k)}||\mathbf{w}_{\mathbf{u}}^{(k)}| + |\mathbf{w}_{\mathbf{u}}^{(k)}||\nabla\widetilde{\mathbf{b}}^{(k)}||\mathbf{w}_{\mathbf{b}}^{(k)}| + |\mathbf{w}_{\mathbf{b}}^{(k)}||\nabla\widetilde{\mathbf{u}}^{(k)}||\mathbf{w}_{\mathbf{b}}^{(k)}|\}\chi.$$

We use the Hölder inequality to obtain that

$$\begin{aligned}
&\int_{s_0}^{s}\int_{B_{3/2}(\mathbf{x})} |\mathbf{w}_{\mathbf{u}}^{(k)}||\nabla\widetilde{\mathbf{u}}^{(k)}||\mathbf{w}_{\mathbf{u}}^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \\
&\lesssim \int_{s_0}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{4;\,B_{3/2}(\mathbf{x})}\,\|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{4;\,B_{3/2}(\mathbf{x})}\,\mathrm{d}\tau \\
&\lesssim \left(\int_{s_0}^{s}\|\nabla\widetilde{\mathbf{u}}^{(k)}\|_{2;\,B_{3/2}(\mathbf{x})}^{2}\,\mathrm{d}\tau\right)^{1/2}\left(\int_{s_0}^{s}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{4;\,B_{3/2}(\mathbf{x})}^{4}\,\mathrm{d}\tau\right)^{1/2} \\
&\lesssim_M \beta_k^{1/2}(s_0,s)\,(s-s_0)^{1/4}
\end{aligned}$$

since

$$\int_{s_0}^{s}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{4;\,B_{3/2}(\mathbf{x})}^{4}\,\mathrm{d}\tau \le \int_{s_0}^{s}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{3;\,B_{3/2}(\mathbf{x})}^{3/2}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5/2}\,\mathrm{d}\tau$$

$$
\begin{aligned}
&\lesssim_M \int_{s_0}^{s} \|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5/2}\,\mathrm{d}\tau \le (s-s_0)^{1/2}\bigg(\int_{s_0}^{s}\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau\bigg)^{1/2}\\
&\lesssim_M (s-s_0)^{1/2}.
\end{aligned}
$$

Since the integral of other two terms can be estimated in the same way, we obtain

$$
\int_{s_0}^{s}\int_{\mathbb{R}^3}|I_4^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \lesssim_M \beta_k^{1/2}(s_0,s)\,(s-s_0)^{1/4}. \tag{4.15}
$$

By the Cauchy's inequality we have

$$
\begin{aligned}
|I_5^{(k)}| &\lesssim (|\widetilde{\mathbf{u}}^{(k)}||\mathbf{f}^{(k)}| + |\widetilde{\mathbf{b}}^{(k)}||\mathbf{g}^{(k)}|)\chi\\
&\le \big(|\mathbf{f}^{(k)}|^2 + |\mathbf{g}^{(k)}|^2\big)^{1/2}\big(|\widetilde{\mathbf{u}}^{(k)}|^2 + |\widetilde{\mathbf{b}}^{(k)}|^2\big)^{1/2}\chi.
\end{aligned}
$$

Hence

$$
\int_{s_0}^{s}\int_{\mathbb{R}^3}|I_5^{(k)}|\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau \lesssim \int_{s_0}^{s}\big(\|\mathbf{f}^{(k)}(\,.\,,\tau)\|_\infty + \|\mathbf{g}^{(k)}(\,.\,,\tau)\|_\infty\big)\,\alpha_k^{1/2}(\tau)\,\mathrm{d}\tau. \tag{4.16}
$$

Estimating now the right hand side of (4.8) by means of (4.11)–(4.16), considering the supremum of the left hand side over $\mathbf{x}\in\mathbb{R}^3$, we obtain that for a.a $s_0\in[-S,0)$ and all $s\in(s_0,0)$,

$$
\begin{aligned}
&\alpha_k(s) + \beta_k(s_0,s)\\
&\lesssim \alpha_k(s_0) + \int_{s_0}^{s}\alpha_k(\tau)\,\mathrm{d}\tau + \gamma_k(s_0,s) + \delta_k(s_0,s) + (s-s_0)\\
&\quad + \bigg(\int_{s_0}^{s}\alpha_k(\tau)\,\mathrm{d}\tau + \beta_k(s_0,s)\bigg)^{4/5}\bigg(\int_{s_0}^{s}\alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau\bigg)^{1/5}\\
&\quad + \beta_k^{1/2}(s_0,s)\,(s-s_0)^{1/4} + \mathcal{F}^{(k)}(s_0,s).
\end{aligned}
$$

Using (2.29) and (2.30) we have

$$
\begin{aligned}
&\alpha_k(s) + \beta_k(s_0,s)\\
&\lesssim \alpha_k(s_0) + \int_{s_0}^{s}(\alpha_k(\tau) + \alpha_k^{3/2}(\tau))\,\mathrm{d}\tau + \bigg(\int_{s_0}^{s}\alpha_k^3(\tau)\,\mathrm{d}\tau\bigg)^{1/4}\beta_k^{3/4}(s_0,s) + (s-s_0)\\
&\quad + \bigg(\int_{s_0}^{s}\alpha_k(\tau)\,\mathrm{d}\tau + \beta_k(s_0,s)\bigg)^{4/5}\bigg(\int_{s_0}^{s}\alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau\bigg)^{1/5}\\
&\quad + \beta_k^{1/2}(s_0,s)\,(s-s_0)^{1/4} + \mathcal{F}^{(k)}(s_0,s).
\end{aligned} \tag{4.17}
$$

Finally, we note that for any $\varepsilon>0$ and $0<\theta<1$ there is $C_{\varepsilon,\theta}$ such that

$$
a^\theta b^{1-\theta} \le \varepsilon a + C_{\varepsilon,\theta} b.
$$

We apply this estimate to the terms having $\beta_k(s_0,s)$ so that (4.17) becomes

$$
\begin{aligned}
\alpha_k(s) + \beta_k(s_0,s) \lesssim\ & \alpha_k(s_0) + \varepsilon\beta_k(s_0,s) + \int_{s_0}^{s}(\alpha_k(\tau) + \alpha_k^3(\tau))\,\mathrm{d}\tau + (s-s_0)^{1/2}\\
&+ \int_{s_0}^{s}\alpha_k(\tau)\,\|\mathbf{w}_{\mathbf{u}}^{(k)}\|_{5;\,B_{3/2}(\mathbf{x})}^{5}\,\mathrm{d}\tau + \mathcal{F}^{(k)}(s_0,s).
\end{aligned}
$$

Now, we fix $\varepsilon > 0$ so small that the term $\varepsilon\beta_k(s_0, s)$ is absorbed by the left hand side. The constants A, B, C_M can be determined by means of all constants, used in (4.11)–(4.17). This completes the proof of Lemma 4. □

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J. Neustupa, INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, ŽITNÁ 25, 115 67 PRAGUE 1, CZECH REPUBLIC

E-mail address, J. Neustupa: neustupa@math.cas.cz

M. Yang (Corresponding author), DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, 50 YONSEI-RO SEODAEMUN-GU, SEOUL, REPUBLIC OF KOREA

E-mail address, M. Yang: m.yang@yonsei.ac.kr