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Joint numerical ranges:
Recent advances and applications

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JOINT NUMERICAL RANGES:
RECENT ADVANCES AND APPLICATIONS
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ABSTRACT. We present a survey of some recent results concerning joint numerical ranges of n-tuples of Hilbert space operators, accompanied with several new observations and remarks. Thereafter, numerical ranges techniques will be applied to various problems of operator theory. In particular, we discuss problems concerning orbits of operators, diagonals of operators and their tuples, and pinching problems. Lastly, motivated by known results on the numerical radius of a single operator, we examine whether, given bounded linear operators $T_1, \dots, T_n$ on a Hilbert space H, there exists a unit vector $x \in H$ such that $|\langle T_j x, x \rangle|$ is "large" for all $j = 1, \dots, n$.

1. FOREWORD

The numerical range of a linear operator on a normed linear space is a subset of the scalar field that reflects not only the algebraic structure of the space but also the norm structure. The theory of joint numerical ranges, studying the joint behavior of several operators, is a developing area with a spectacular growth over the last years. The purpose of this article is to collect some recent contributions to this theory, mostly due to the authors and reflecting their tastes. Occasionally, we give proofs, which mainly serve just an illustration of some typical arguments. Of course, the notes below are very far from any complete account.

2. NUMERICAL RANGES AND ESSENTIAL NUMERICAL RANGES

In this section, after introducing the central notions of this survey, we present all instrumental theorems that will be applied in the sequel.

Let H be a (complex) Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $B(H)$ be the space of all bounded linear operators on H.

Definition 2.1. For $T \in B(H)$ we define the numerical range

$$W(T) = \{\langle Tx, x \rangle : x \in H, \ \|x\| = 1\}$$

and the numerical radius

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle| : x \in H, \ \|x\| = 1\}.$$

The geometry of $W(T)$ encodes the structure of T, and as far as $W(T)$ is easily computable, at least in approximate sense, it makes $W(T)$ an important object of operator theory. A good discussion of $W(T)$ from a historical perspective and with emphasis on its fine geometric properties can be found in [44], see also [45] and [54].

Let us summarize some of very basic properties of the numerical range and numerical radius of a single operator. The basic reference for numerical ranges are [13] and [14]. For a more recent treatment one may consult the book [43] and the references therein.

For a bounded linear operator T on H we denote by $\sigma(T)$ its spectrum and by $r(T)$ its spectral radius. All of the statements in the next theorem can be found in [43, Chapter 1].

Theorem 2.2. *Let $T \in B(H)$. Then the following holds.*

    (1) *(The Toeplitz-Hausdorff Theorem) $W(T)$ is a convex subset of $\mathbb{C}$.*
    (2) *One has*

$$(2.1) \qquad\qquad\qquad \mathrm{conv}\, \sigma(T) \subset \overline{W(T)},$$

       where $\mathrm{conv}\, \sigma(T)$ denotes the convex hull of the spectrum of T.
    (3) *$T^* = T$ if and only if $W(T) \subset \mathbb{R}$.*
    (4) *If T is normal then $w(T) = \|T\| = r(T)$ and $\overline{W(T)} = \mathrm{conv}\, \sigma(T)$.*
    (5) *One has $\|T\| \geq w(T) \geq \frac{1}{2}\|T\|$. So $w(\cdot)$ is a norm equivalent to $\|\cdot\|$.*

The convexity of $W(T)$ is a heart matter of the whole of theory of numerical ranges, and underpins many of its directions. The result is due to Toeplitz [82] and Hausdorff [47], and has got a number of proofs since then. Three of them can be found in a recent survey [15]. The property (4) is the first illustration of the interplay between $W(T)$ and the spectrum of T. There is a plethora of statements of that kind. To give a couple of samples, recall that if T is normal, then every extreme point of $W(T)$ is an eigenvalue of T, in a similar vein, for any $T \in B(H)$ a corner point of the boundary $\partial W(T)$ (where a parametrization of $W(T)$ fails to be differentiable) is an eigenvalue of T as well, see e.g. a discussion in [43, p. 21].

For the sake of completeness we mention also some deeper properties of the numerical ranges. All of them but the last one are elaborated e.g. in [43, Chapter 2].

Theorem 2.3.   (6) *For every $n \in \mathbb{N}$ one has $w(T^n) \leq w(T)^n$.*
    (7) *One has $w(T) \leq 1$ if and only if there exists a unitary 2-dilation, i.e., there exists a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$*

such that

$$(2.2) \qquad\qquad T^n = 2P_H U^n|_H, \qquad n \geq 1.$$

(8) *There exists a universal constant K such that*

$$(2.3) \qquad\qquad \|p(T)\| \leq K \, \sup\{|p(z)| : z \in W(T)\}$$

for all $T \in B(H)$ and all polynomials p.

The striking estimate (2.3) was obtained in [21] developing the ideas from [29]. It led to a number of developments and applications in mathematics including the theory of partial differential equations and numerical analysis. Some of them are thoroughly presented in [22]. The estimate implies, for instance, that any $T \in B(H)$ is similar to an operator having a normal $\partial W(T)$-dilation, or that T has a so-called skew-normal dilation on $\partial W(T)$ (where $\partial W(T)$ stands for the boundary of $W(T)$.) For more details on applications of (2.3) to dilation theory we refer to [21], [22] and [75]. The famous open problem posed by Crouzeix is finding the best constant $K = K_{\text{best}}$ in (2.3). Crouzeix conjectured that $K_{\text{best}} = 2$, and, as of now, the best known result is $K = 1 + \sqrt{2}$. See [23] for the proof of this result, and also the recent papers [9], [19] and [77] for further references, alternative proofs and related statements. There is a number of close works on *mapping properties* of numerical ranges, resembling (6) in Theorem 2.3, but concerning a more subtle problem on how numerical ranges behave under appropriate holomorphic maps. For some of the pertinent references one may consult [43], although there are stronger and more recent results. We avoid a discussion of them in this article.

While the spectrum is preserved under the similarity transformation, the behavior of $W(T)$ under similarities can be basically arbitrary modulo a constraint in (2) of Theorem 2.2. The next result clarifying this claim was proved by Williams [84].

Theorem 2.4. (i) *Let $T \in B(H)$ and let $S \subset \mathbb{C}$ be an open convex set such that $\sigma(T) \subset S$. Then there is an invertible $R \in B(H)$ such that $W(RTR^{-1}) \subset S$. As a corollary,*

$$(2.4) \qquad \text{conv} \, (\sigma(T)) = \cap \left\{ \overline{W(RTR^{-1})} : R \in B(H), \, R \, \text{is invertible} \right\}.$$

(ii) *Let $T \in B(H)$ and let S be a compact subset of $\mathbb{C}$. Then there is an invertible $R \in B(H)$ such that $W(RTR^{-1}) \supset S$.*

The relation (2.4) was obtained by Hildebrandt in [51]. It shows, in particular, that the inclusion in (2.1) is in a sense best possible. In another paper by Hildebrandt [50], it was shown that $\text{conv} \, \sigma(T) = \overline{W(T)}$ if and only if the latter set is the spectral set of T. This further clarifies (2.1).

To finish this section, we touch several issues which are not so popular in the literature, but seem to be important. First, remark that, in general, it is not known which subsets of $\mathbb{C}$ can be realized as numerical ranges $W(T)$

of a general $T \in B(H)$ or, for example, compact operators on an infinite-dimensional Hilbert spaces H. The problem seems to be extremely hard and only scattered results are available. For example, the set $\{z \in \mathbb{C} : |z| < 1\} \cup \{e^{i\theta} : \theta \text{ irrational}\}$ is unattainable as $W(T)$, and a half disc can not be $W(T)$ for a compact T on infinite-dimensional H. For more on this, including the examples above, see e. g. [1] and [76]. From a slightly different, geometric perspective, the problem of realization of numerical ranges by operators on $\mathbb{C}^n$ was studied in [48].

It is also of importance to know which subsets of H may substitute the unit sphere to keep the conclusion of the Toeplitz-Hausdorff theorem. Curiously, this question has essentially escaped an attention of experts. For finite-dimensional H, a deep study of that problem was realized in [35]. In particular, it was proved there that the unit sphere in H can be replaced by a closed annulus or by a "slice" of $W(T)$ if T is selfadjoint.

In another interesting paper [41], the authors showed that for $T \in B(\mathbb{C}^n)$ one can introduce a so-called numerical measure μ_T on $W(T)$ being a push-forward of the Haar measure on the unit sphere in $\mathbb{C}^n$ under the map $x \to \langle Tx, x \rangle$. The support of μ_T coincides with $W(T)$, and it seems that the numerical measure μ_T captures a lot of information on $W(T)$. Such an approach has non-trivial applications to the study of partial differential equations. However, it seems, it did not attract an attention it deserves.

The theory of numerical ranges is intimately related to optimization theory and convex analysis. However, the interconnections between these subjects seem to be not exploited enough by the operator-theoretical community. For sample papers discussing a number of them, see e.g. [52] and [31].

There are hundreds of papers on numerical ranges, their properties and various generalizations, including descriptions of numerical ranges for particular classes of operators. However, while for some classes, like Toeplitz operators, a detailed information is available, the other (even quite close) classes, such as Hankel operators, have not received an adequate treatment. We are only aware of an old paper [83] treating the numerical ranges of Hankel operators among other things. Thus, a lot is still to be done.

3. JOINT NUMERICAL RANGES, JOINT ESSENTIAL NUMERICAL RANGES AND THEIR RELATIVES

For an n-tuple $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ we define *the joint numerical range* as

$$W(\mathcal{T}) = W(T_1, \ldots, T_n) = \{(\langle T_1 x, x \rangle, \ldots, \langle T_n x, x \rangle) : x \in H, \|x\| = 1\}.$$

The set $W(\mathcal{T})$ can be identified with a subset of $\mathbb{R}^{2n}$ if one identifies the n-tuple $\mathcal{T}$ with the $(2n)$-tuple $(\operatorname{Re} T_1, \operatorname{Im} T_1, ..., \operatorname{Re} T_n, \operatorname{Im} T_n)$ of selfadjoint operators, where the real (imaginary) part of an operator $T \in B(H)$ is defined by $\operatorname{Re} T = \frac{T+T^*}{2}$ and $\operatorname{Im} T = \frac{T-T^*}{2i}$. Such an identification is often useful, however certain statements require an additional care.

To deal with operator tuples, it would be convenient to introduce the following notation. For $x, y \in H$ and $n \in \mathbb{N}$ we will be writing

$$\langle \mathcal{T}x, y \rangle = (\langle T_1 x, y \rangle, \ldots, \langle T_n x, y \rangle) \in \mathbb{C}^n \quad \text{and} \quad \mathcal{T}x = (T_1 x, \ldots, T_n x) \in H^n.$$

Similarly, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we write $\mathcal{T} - \lambda = (T_1 - \lambda_1, \ldots, T - \lambda_n)$. Moreover, If $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ and $R_1, R_2 \in B(H)$, then we define

$$R_1 \mathcal{T} R_2 := (R_1 T_1 R_2, \ldots, R_1 T_n R_2).$$

For any subspace $M \subset H$ we denote by P_M the orthogonal projection on M.

The noncommutativity of operator entries of $\mathcal{T}$ makes the study of $W(\mathcal{T})$ much more demanding than in the setting of a single operator, and it leads to a number of new geometric phenomena. In particular, the joint numerical range of an n-tuple of operators is, in general, not convex if $n \geq 2$. as the next example from [63, Example 1.1] shows.

Example 3.1. Let $m \geq 2$ and $n \geq 4$. Let $T_i \in B(\mathbb{C}^m), 1 \leq i \leq n$, be defined as $T_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus O_{m-2}, T_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus O_{m-2} \ T_3 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus O_{n-2},$

$T_4 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus O_{m-2}$, and if $n \geq 5$, we let T_j to be the identity on $\mathbb{C}^m$ for all $5 \leq j \leq n$. Then the operators $T_i, 1 \leq i \leq m$, are selfadjoint, but $W(\mathcal{T})$ is not convex, as shown in [61].

Clearly, a similar example can be given for T_j's acting on an infinite-dimensional Hilbert space.

Apparently, the fact that the joint numerical range may fail to be convex was known already to Hausdorff [47]. The example was first given in [8]. See also [34, p. 33-34] for its variation, and for the claim that T_1, T_2, and T_3 as above were first produced by Halmos for similar purposes.

The n-tuples of operators of the form $(T, T^2, \ldots, T^n)$, where T is a fixed operator $T \in B(H)$, are of particular interest since they allow one to link the theory of joint numerical ranges to fine properties of T. Such tuples will be of primary importance in the next section. Note that the geometry of the corresponding numerical ranges $W(T, \ldots, T^n)$ is still far from being understood. In particular, we do not know whether $W(T, \ldots, T^n)$ is always convex if $n \geq 2$.

While $W(\mathcal{T})$ is not convex in general, it still posses some traces of convexity. As shown in [27], with any two points it contains an ellipsoid (perhaps degenerate) joining them. See also Theorem 3.6 below.

There are several instances when $W(T_1, \ldots, T_n)$ is convex. For example, $W(T_1, \ldots, T_n)$ is convex if $n = 3$ and $T_j, 1 \leq j \leq 3$, are selfadjoint and $\dim H \geq 3$, see [45]. This leads, in particular, to the following curious observation made in [62].

Proposition 3.2. *Let* $T \in B(H)$, $\dim H \geq 3$. *Then the set*

$$DW(T) := \left\{ (\langle Tx, x \rangle, \|Tx\|^2) : x \in H, \|x\| = 1 \right\} \subset \mathbb{C} \times \mathbb{R}$$

is convex.

Proof. It suffices to observe that $\mathcal{D}W(T)$ can be identified with the numerical range of the triple $(\operatorname{Re}T, \operatorname{Im}T, T^*T)$ of selfadjoint operators on H. If $\dim H \geq 3$, then this set is convex, as mentioned above. $\qquad\square$

Thus, if $x, y \in H$, $\|x\| = \|y\| = 1$, and $t \in [0,1]$, then there exists a unit vector $u \in H$ such that

$$\langle Tu, u\rangle = t\langle Tx, x\rangle + (1-t)\langle Ty, y\rangle \qquad \text{and} \qquad \|Tu\|^2 = t\|Tx\|^2 + (1-t)\|Ty\|^2.$$

The set $DW(T)$ is called Davis-Wielandt shell in the literature. Clearly, its geometry provides more information on T than the usual numerical range $W(T)$, which is just a projection of $DW(T)$ on the first coordinate. In particular, the normality of $T \in B(\mathbb{C}^n)$ can be described solely in terms of $DW(T)$. For more details on $DW(T)$ one may consult [62] or [65] and the references cited therein.

An interesting illustration of the interplay between $\langle Tx, x\rangle$ and $\|Tx\|^2$ for $x \in H$, and thus the structure of $DW(T)$, is provided by Garske's theorem,[42]. It says that if $T \in B(H)$, then

$$\sup_{\{x : \|x\|=1\}} \left(\|Tx\|^2 - |\langle Tx, x\rangle|^2\right) \geq R^2,$$

where R is the radius of the smallest disk containing the spectrum of T. Moreover, as shown in [12], if T is normal, then above inequality becomes equality. For a tuples version of this result see Section 4.

The notion of the Davis-Wielandt shell is closely related to the notion of maximal numerical range, which has been also studied in the literature. Among very first papers in this direction, we mention [80] and [38].

We note that for a pair of bounded operators $(T_1, T_2) \in B(H)^2$ another joint numerical range $\mathcal{W}(T_1, T_2)$ is introduced in the apparently forgotten paper [7]. In particular, while $W(T_1, T_2)$ is not in general convex, it was proved in [7] that the set $\mathcal{W}(T_1, T_2) := \{(\langle T_1 x, y\rangle, \langle T_2 x, y\rangle) : \|x\| \cdot \|y\| \leq 1\}$ is convex.

Another instance when the convexity of $W(T_1, ..., T_n)$ still survives arise when T has certain special algebraic structure. As a simple illustration one may recall that $W(T_1, ..., T_n)$ is convex if $T_j, 1 \leq j \leq n$, are (bounded) commuting normal operators on H, [24]. Here the assumption of commutativity is essential. Without commutativity it is no longer true.

Example 3.3. There exist normal operators $A, B \in B(\mathbb{C}^2)$ such that $W(A, B)$ is not convex. Indeed, let A and B be given by the next matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Then $A^* = A$, B is normal and

$$\operatorname{Re}B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \operatorname{Im}B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

So $W(A, B)$ is identified with $W(A, \operatorname{Re} B, \operatorname{Im} B)$, which is easy to show to be non-convex, see [14, p. 138].

Having an arbitrary tuple $(T_1, ..., T_n) \in B(H)^n$ one my create a commuting n-tuple by setting $T_1^0 := T_1 \otimes I_2 \otimes \cdots \otimes I_n$, $T_2^0 := I_1 \otimes T_2 \otimes I_3 \otimes \cdots \otimes I_n, \cdots, \ldots$, on the tensor product space $H \otimes \cdots \otimes H$ of n copies of H. Then, as shown in [25], the joint numerical range of $(T_1^0, ..., T_n^0)$ is the Cartesian product of their respective numerical ranges $W(T_j), 1 \le j \le n$, and thus convex.

Finally, we mention yet another situation when the joint numerical range is convex. Let H be a separable infinite-dimensional Hilbert space. Denote by $S_2(H)$ the set of all Hilbert-Schmidt operators on H, and by $S_1(H)$ the set of all trace-class operators on H. Then $S_2(H)$ with the inner product given by $\langle X, Y \rangle_{S_2} = \operatorname{trace}(X^*Y), X, Y \in S_2$, is again a Hilbert space, while S_1 a Banach space with the norm defined as $\|X\|_{S_1} = \operatorname{trace}(X^*X)^{1/2}, X \in S_1(H)$. We will return to $S_1(H), S_2(H)$ and similar classes in Section 5.3. For $T \in B(H)$ denote by $\widehat{T} : S_2(H) \to S_2(H)$ the operator defined by $\widehat{T}(X) = TX, X \in S_2(H)$. As for subsets of $\mathbb{C}$, if $S \subset \mathbb{C}^n$, then the convex hull of S is denoted by $\operatorname{conv} S$. The next statement seems to be new.

Theorem 3.4. *Let* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. *Then*

$$W(\widehat{\mathcal{T}}) = \operatorname{conv} W(\mathcal{T}).$$

Proof. Let $\lambda \in W(\widehat{\mathcal{T}})$. Then there exists $X \in S_2$ with $\|X\|_{S_2} = 1$ and

$$\lambda = \langle TX, X \rangle = \operatorname{trace}(X^*TX) = \operatorname{trace}(TY),$$

where $Y = XX^* \ge 0$ and $\|Y\|_{S_1} = 1$. Then there exists an orthonormal basis $(e_n)_{n=1}^\infty$ in H and a sequence $(\alpha_n)_{n=1}^\infty \subset [0, \infty)$ such that $\sum_{n=1}^\infty \alpha_n = 1$ and $Y = \sum_{n=1}^\infty \alpha_n e_n \otimes e_n$. Hence

$$\lambda = \sum_{n=1}^\infty \alpha_n \langle \mathcal{T} e_n, e_n \rangle.$$

Clearly $\lambda \in \operatorname{conv} W(\mathcal{T})$. Hence $W(\widehat{\mathcal{T}}) \subset \operatorname{conv} W(\mathcal{T})$.

We show now that $\operatorname{conv} W(\mathcal{T}) \subset W(\widehat{\mathcal{T}})$. Let $\mu \in \operatorname{conv} W(\mathcal{T}) \subset \mathbb{C}^n$ and we will identify further $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Then μ is a convex combination of at most $2n + 1$ elements of $W(\mathcal{T})$, i.e.,

$$\mu = \sum_{j=1}^{2n+1} \beta_j \langle Tx_j, x_j \rangle, \qquad \beta_j \ge 0, \quad \sum_{j=1}^{2n+1} \beta_j = 1,$$

for some unit vectors $x_1, \ldots, x_{2n+1} \in H$. Let $M = \bigvee_{j=1}^{2n+1} x_j$. Then $\mu \in \operatorname{conv} W(P_M \mathcal{T} P_M)$. Note that

$$\operatorname{conv} W(P_M \mathcal{T} P_M) = \big\{ f(\mathcal{T}) : f \in B(M)^*, \|f\| = f(I_M) = 1 \big\}$$
$$= \big\{ \operatorname{trace}(P_M \mathcal{T} P_M Y) : Y \in B(M), \|Y\|_{S_1} = \operatorname{tr} Y = 1 \big\}.$$

The assumption $\|Y\|_{S_1} = \text{trace}(Y) = 1$ implies that $Y \geq 0$ and that Y can be represented as

$$Y = \sum_{j=1}^{2n+1} \alpha_j e_j \otimes e_j \qquad \alpha_j \geq 0, \quad \sum_{j=1}^{2n+1} \alpha_j = 1,$$

for some orthonormal system $(e_j)_{j=1}^{2n+1}$ in M. Thus we have $\mu \in W(\widehat{\mathcal{T}})$. $\square$

Before passing to the next main object of our studies, we mention that the following problem seems to be open. Recall that for $S \subset \mathbb{C}^n$ its polynomial (convex) hull $\hat{S}$ is defined as

$$\hat{S} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{z \in S} |p(z)| \text{ for all polynomials } p\}.$$

Accordingly, S is polynomially convex if $S = \hat{S}$.

Problem 3.5. Let $\mathcal{T} \in B(H)^n$. Is $W(\mathcal{T})$ polynomially convex? If not, then what are sufficient conditions on $\mathcal{T}$ to ensure such a property?

Similarly to the spectral theory of linear operators, there is an "essential version" of the notion of the joint numerical range. It will play the central role in our subsequent considerations.

Let $\dim H = \infty$ and $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. We define *the joint essential numerical range* $W_e(\mathcal{T})$ of $\mathcal{T}$ as as the set of all n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there exists an orthonormal sequence $(x_k)_{k=1}^{\infty} \subset H$ with

$$\lim_{k \to \infty} \langle T_j x_k, x_k \rangle = \lambda_j, \qquad j = 1, \ldots, n.$$

The joint essential numerical range $W_e(\mathcal{T})$ can be also defined as

$$W_e(\mathcal{T}) = \bigcap \overline{W(T_1 + K_1, \ldots, T_n + K_n)},$$

where the intersection is taken over all n-tuples $(K_1, \ldots, K_n)$ of compact operators on H. For the proof of the equivalence of these definitions one may consult to [70, Theorem 2]. For $n = 1$, the essential numerical range was introduced and studied in depth in the influential paper [36]. A Banach algebra counterpart of the essential numerical range was defined earlier in [79].

Being an approximate version of $W(\mathcal{T})$, the set $W_e(\mathcal{T})$ appeared to be better adopted to the use of spectral theory for $\mathcal{T}$, and its invariance under compact perturbations illustrates the specifics of $W_e(\mathcal{T})$ very well. Moreover, in contrast to $W(\mathcal{T})$, the joint essential numerical range is always convex. Some of the geometric properties of $W(\mathcal{T})$ are summarized in the next result taken from [63]. Recall that a set $S \subset \mathbb{C}^n$ is star-shaped if there is a point in S, called star center, that can be connected by a line segment with any other point in S. It is known that a star-shaped set is simply connected.

Theorem 3.6. *Let $\mathcal{T} \in B(H)^n$. Then $W_e(\mathcal{T})$ is a compact convex subset of the star-shaped set $\overline{W(\mathcal{T})}$. Moreover, each element in $W_e(\mathcal{T})$ is a star center of $\overline{W(\mathcal{T})}$.*

The analogy to spectral theory mentioned above may serve as a good intuition, however the corresponding relations between $W(\mathcal{T})$ and $W_e(\mathcal{T})$ are more involved than those for their spectral counterparts. Note that for $T \in B(H)$ one may have $W_e(T) \cap W(T) = \emptyset$ and points from $W(T) \cap W_e(T)$ may not be star-centers for $W(T)$. Moreover, for any $n \geq 2$, the set $W(\mathcal{T})$ may be not convex even if $\overline{W(\mathcal{T})}$ is convex, see [63]. The difference between $W(\mathcal{T})$ and $W_e(\mathcal{T})$ is illustrated by the fact that while realizing convex sets by numerical ranges is a hard open problem, for any compact convex set $S \subset \mathbb{C}^n$ there exist n-tuples $\mathcal{T}_1$ and $\mathcal{T}_2$ from $B(H)^n$ such that $S = W_e(\mathcal{T}_1) = \overline{W(\mathcal{T}_2)}$, see [63, Corollary 5.4].

Apparently the convexity of $W_e(T_1, \ldots, T_n)$ was first proved in [11, Lemma 3.1] (where even a more general result can be found), see also [63] for a different proof and further penetrating study of $W_e(\mathcal{T})$, including its geometric properties, stability under perturbations, examples, etc.

Let us present yet another argument yielding the convexity of $W_e(\mathcal{T})$ and based on the next simple but useful observation.

Note that if $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$, $\lambda \in \mathbb{C}^n$ belongs to $W_e(\mathcal{T})$ if and only if for every $\delta > 0$ and every subspace $M \subset H$ of finite codimension there exists a unit vector $x \in M$ such that $||\langle \mathcal{T}x, x \rangle - \lambda||_{\mathbb{C}^n} < \delta$. The observation was used without proof in [72] and the proof of its non-trivial implication was given in [73, Lemma 4.1], see also [73, Proposition 5.5]). We justify the "only if" implication, and omit the other implication whose proof is straightforward. To this aim, note that if $\lambda \in W_e(\mathcal{T})$, then there exists an orthonormal sequence $(x_k)_{k=1}^{\infty}$ in H such that $\langle \mathcal{T}x_k, x_k \rangle \to \lambda$, $k \to \infty$. If $M \subset H$ is a subspace of a finite codimension, then $\|P_{M^{\perp}}x_k\| \to 0$, and so $\|P_M x_k - x_k\| \to 0$ as $k \to \infty$. Setting $u_k = \frac{P_M x_k}{\|P_M x_k\|}$, $k \geq 1$, we infer that $(u_k)_{k=1}^{\infty} \subset M$, $\lim_{k \to \infty} \|u_k - x_k\| = 0$ and $\lim_{k \to \infty} \langle \mathcal{T}u_k, u_k \rangle = \lambda$, hence the claim follows.

Theorem 3.7. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. Then $W_e(\mathcal{T})$ is a closed convex subset of $\mathbb{C}^n$.*

Proof. Either of the two equivalent definitions of $W_e(\mathcal{T})$ implies immediately that $W_e(\mathcal{T})$ is a closed set.

To prove the convexity, let $\lambda, \mu \in W_e(\mathcal{T})$ and $t \in [0, 1]$. Assume that $M \subset H$ is a subspace of finite codimension. By the observation above, there exists an orthonormal sequence $(x_k)_{k=1}^{\infty}$ in M such that $\lim_{k \to \infty} \langle \mathcal{T}x_k, x_k \rangle = \lambda$. Similarly, we can construct inductively an orthonormal sequence $(y_k)_{k=1}^{\infty}$ such that for every $k \in \mathbb{N}$,

$$y_k \in M \cap \left\{ x_m, T_j x_m, T_j^* x_m : 1 \leq j \leq n, \ 1 \leq m \leq k \right\}^{\perp}$$

and $\lim_{k\to\infty}\langle \mathcal{T}y_k, y_k\rangle = \mu$. Let $u_k = \sqrt{t}x_k + \sqrt{1-t}y_k$, $k \in \mathbb{N}$. Clearly $(u_k)_{k=1}^\infty$ is an orthonormal sequence in M and

$$\lim_{k\to\infty}\langle \mathcal{T}u_k, u_k\rangle = t\lim_{k\to\infty}\langle \mathcal{T}x_k, x_k\rangle + (1-t)\lim_{k\to\infty}\langle \mathcal{T}y_k, y_k\rangle = t\lambda + (1-t)\mu.$$

Hence $t\lambda + (1-t)\mu \in W_e(\mathcal{T})$. $\qquad\square$

Thus, the joint essential numerical range $W_e(\mathcal{T})$ has better geometric properties than the joint numerical range $W(\mathcal{T})$. On the other hand, the joint numerical range $W(\mathcal{T})$ provides more information about the n-tuple $\mathcal{T} = (T_1, \ldots, T_n)$, and is more explicit. By mere definitions, $W_e(\mathcal{T}) \subset \overline{W(\mathcal{T})}$, but in general $W_e(\mathcal{T})$ is not contained in $W(\mathcal{T})$. One can show that for $\mathcal{T} \in B(H)^n$,

$$(3.1) \qquad \overline{\mathrm{conv}}\, W(\mathcal{T}) = \mathrm{conv}\,\big(W(\mathcal{T}) \cup W_e(\mathcal{T})\big),$$

see [73, Theorem 5.1] and the discussion preceding it. This is a generalization for operator tuples of the famous theorem due to Lancaster for $n = 1$.

Lancaster's theorem yields quite useful results, e.g. descriptions of the situations, when $W(T)$ is closed or open. Nevertheless, the presence of $W(\mathcal{T})$ on both sides of (3.1) makes the equality somewhat implicit. So, it is a natural question which part of $W_e(\mathcal{T})$ is contained in $W(\mathcal{T})$. The next theorem proved in [73, Corollary 4.2] provides a partial answer. For $S \subset \mathbb{C}^n$ denote by $\mathrm{Int}\,S$ its topological interior.

Theorem 3.8. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. Then*

$$\mathrm{Int}\,W_e(\mathcal{T}) \subset W(\mathcal{T}).$$

Moreover, if $\mu \in \mathrm{Int}\,W_e(\mathcal{T})$ then for every subspace $M \subset H$ of a finite codimension there exists $x \in M$ such that $\|x\| = 1$ and

$$\big(\langle T_1 x, x\rangle, \ldots, \langle T_n x, x\rangle\big) = \mu.$$

As a consequence, we can find a joint *diagonal compression* for $T_1, \ldots, T_n$ to an infinite-dimensional subspace of H, see [72, Corollary 4.3]. Recall that for $T \in B(H)$ the problem of characterizing $\lambda \in \mathbb{C}$ such that $PTP = \lambda P$ for an infinite rank projection P was posed in [36, p. 190]. The special case of $n = 1$ of the following statement was proved in [3, p. 440].

Corollary 3.9. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{Int}\,W_e(\mathcal{T})$. Then there exists an infinite-dimensional subspace $L \subset H$ such that*

$$P_L T_j P_L = \lambda_j P_L, j = 1, \ldots, n,$$

where P_L is the orthogonal projection on L.

Proof. By Theorem 3.8, there exists a unit vector $x_1 \in H$ such that $\langle \mathcal{T}x_1, x_1\rangle = \lambda$. Construct inductively a sequence $(x_k)_{k=1}^\infty \subset H$ of unit vectors such that

$$x_{k+1} \perp \{x_m, T_j x_m, T_j^* x_m : 1 \le j \le n, 1 \le m \le k\} \quad \text{and} \quad \langle \mathcal{T}x_k, x_k\rangle = \lambda$$

for all $k \in \mathbb{N}$, using the fact that the span of $\{x_m, T_j x_m, T_j^* x_m : 1 \le j \le n, 1 \le m \le k\}$ is a subspace of finite dimension. Let L be the closed

linear span of $(x_k)_{k=1}^{\infty}$. Clearly L is an infinite-dimensional subspace with an orthonormal basis $(x_k)_{k=1}^{\infty}$. Let $y \in L$. Then, in view of our construction of $(x_k)_{k=1}^{\infty}$, it is easy to see that $\langle \mathcal{T}y, y \rangle = \lambda ||y||^2$. Since the choice of y is arbitrary, $P_L T_j P_L = \lambda_j P_L$, for all $1 \leq j \leq n$. $\qquad\square$

Closely related to the notion of joint essential numerical range are higher rank numerical ranges. These numerical ranges have been studied intensively, e.g. in connection with the quantum computing, see [64] and the references therein.

Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ and $1 \leq k \leq \infty$. We define the *k-th rank numerical range* $W_k(\mathcal{T})$ of $\mathcal{T}$ as the set of all $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there exists a subspace $L \subset H$, with dim $L = k$ satisfying

$$P_L T_j P_L = \lambda_j P_L, \qquad j = 1, \ldots, n.$$

(Note that $W_k(\mathcal{T})$ are usually denoted by $\Lambda_k(\mathcal{T})$ in the literature, while the notation $W_k(\mathcal{T})$ is used for so-called k-th numerical ranges. However, we preferred the more intuitive notation above.)

Observe that $W_1(\mathcal{T})$ is the usual joint numerical range and

$$(3.2) \qquad\qquad W_1(\mathcal{T}) \supset W_2(\mathcal{T}) \supset \cdots \supset W_\infty(\mathcal{T}).$$

For $k \in \mathbb{N}$, the set $W_k(\mathcal{T})$ is, in general, not convex, but it is always non-empty and star shaped. At the same time, it is easy to see that the *infinite numerical range* $W_\infty(\mathcal{T})$ can be empty even if $n = 1$ (by considering an injective positive definite compact operator T_1), but $W_\infty(\mathcal{T})$ is always convex. In a sense, $W_e(\mathcal{T})$ is an approximate version of $W_\infty(\mathcal{T})$. This is summarized in the theorem below.

Theorem 3.10. *Let* dim $H = \infty$ *and* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$.

(i) *For every $k \in \mathbb{N}$, the set $W_k(\mathcal{T})$ is not empty and star-shaped. The star center is any point in $W_m(\mathcal{T})$ with $m > k(2n+1)$.*

(ii) *The set $W_\infty(\mathcal{T})$ is convex.*

(iii) *One has*

$$W_e(\mathcal{T}) = \bigcap_{k=1}^{\infty} \overline{W_k(\mathcal{T})} \quad and \quad W_\infty(T) = \bigcap_{k=1}^{\infty} W_k(\mathcal{T}).$$

The proofs of this and other related statements were given in [73, Section 5] by means of a unified approach based on essential numerical ranges. A different proof of a statement slightly less general then (i) can be found in [64, Proposition 4.1]. The convexity of $W_\infty(\mathcal{T})$ can be proved in the same way as Theorem 3.7, see [73, Theorem 5.6]). We refer to [64, Theorem 4.2] and [78] for different and earlier proofs of that result. A preceding and different proof of (iii) can be found in [64, Theorem 4.2] and [64, Corollary 4.5].

Using Theorem 3.8 on joint compressions and the definition of $W_e(\mathcal{T})$, we can further clarify the structure of $W_\infty(\mathcal{T})$.

Theorem 3.11. *Let* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. *Then*

$$\text{Int}\, W_e(\mathcal{T}) \subset W_\infty(\mathcal{T}) \subset W_e(\mathcal{T}).$$

The statement above is a good illustration of importance of $\text{Int}\, W_e(\mathcal{T})$ in the theory of numerical ranges. It is natural to ask when $\text{Int}\, W_e(\mathcal{T})$ is nonempty. The following statement proved in [73, Proposition 5.10] expresses this property in algebraic terms.

Theorem 3.12. *Let* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. *The following are equivalent:*

  (i) $\text{Int}\, W_e(\mathcal{T}) \neq \emptyset$
 (ii) $\text{Int}\, W_\infty(\mathcal{T}) \neq \emptyset$
(iii) *the operators* $\text{Re}\, T_1, \text{Im}\, T_1, \ldots, \text{Re}\, T_n, \text{Im}\, T_n$ *are linearly independent in the real vector space of all self-adjoint operators modulo the real vector space generated by selfadjoint compact operators and real multiples of the identity.*
      More precisely, if $c, t_1, \ldots, t_{2n}$ *are real numbers such that*

$$\sum_{j=1}^{n} (t_{2n-1}\text{Re}\, T_j + t_{2n}\text{Im}\, T_j) + cI$$

  is compact, then $t_1 = \cdots = t_{2n} = 0$.

Having an approximate character, essential numerical ranges and sets of similar nature are traditionally expressed as intersections of numerical ranges of tuples over appropriate classes of perturbations, usually compact ones. Somewhat surprisingly, in the following result obtained in [73, Theorem 5.8], $W_e(\mathcal{T})$ is described by means of *unions* of the infinite numerical ranges of compact perturbations $\mathcal{T} + \mathcal{K}$, where $\mathcal{K}$ is an n-tuple of compact operators.

Theorem 3.13. *Let* $\mathcal{T} \in B(H)^n$. *Then the following holds.*

  (i) $W_e(\mathcal{T}) = \bigcup_{\mathcal{K} \in \mathcal{K}(H)^n} W_\infty(\mathcal{T} + \mathcal{K})$, *where* $\mathcal{K}(H)$ *denotes the ideal of compact operators on* H.
 (ii) *There exists an* n-*tuple* $\mathcal{K}$ *of compact operators such that*

$$W_e(\mathcal{T}) = \overline{W_\infty(\mathcal{T} + \mathcal{K})}.$$

4. Joint numerical ranges and spectrum

As it was mentioned in Theorem 2.2 (2), $\text{conv}\, \sigma(T) \subset \overline{W(T)}$ for each single operator $T \in B(H)$. Unfortunately, for non-commuting tuples there is no convenient joint spectrum, although the notion of joint numerical range can be defined properly. On the other hand, for commuting tuples there are many, comparatively useful definitions of spectrum (Taylor, Harte, approximate point spectrum, surjective spectrum, ...), which, in general, may differ from each other. However, all reasonable spectra in this setting have the same convex hull, [69, Chapter III].

For definitiveness, we assume below that for $\mathcal{T} \in B(H)^n$, the notation $\sigma(\mathcal{T})$ stands for the Harte spectrum of $\mathcal{T}$. The next theorem due to V. Wrobel is a version of Theorem 2.2, (2) for tuples of operators, see [85, Theorem 2.2].

Theorem 4.1 (Wrobel, 1988)**.** *Let* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ *be a commuting n-tuple, then*

$$\operatorname{conv} \sigma(\mathcal{T}) \subset \overline{W(\mathcal{T})}.$$

A drawback of Theorem 4.1 is that $\operatorname{conv} \sigma(\mathcal{T})$ only approximates points from $W(\mathcal{T})$ rather than matches them. Generalizing Theorem 4.1 and extending Theorem 3.8, we describe in [73, Theorem 4.2] "big" subsets of $W(\mathcal{T})$ itself in spectral terms. Note that in the next Theorem the operators T_j are not necessarily commuting.

Theorem 4.2. *Let* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. *Then*

$$\operatorname{Int} \operatorname{conv} \big(W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T}) \big) \subset W(\mathcal{T}),$$

where $\sigma_p(\mathcal{T})$ *is the point spectrum of* $\mathcal{T}$, *that is, the set of all n-tuples* $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ *such that* $\bigcap_{j=1}^n \ker(T_j - \lambda_j) \neq \{0\}$.

Using simple properties of polynomial hulls and the coincidence of convex hulls for several joint spectras, one derives the following corollary, [73, Corollary 4.3].

Corollary 4.3. *Let* $T_1, \ldots, T_n \in B(H)$ *be mutually commuting operators. Then for* $\mathcal{T} = (T_1, \ldots, T_n)$ *one has*

$$\operatorname{Int} \operatorname{conv} \sigma(\mathcal{T}) \subset W(\mathcal{T}).$$

Note in passing that there is a partial analogue of Theorem 2.4 for operator tuples, obtained in [37]. For commuting operators $T_1, \ldots, T_n$ and for every $j, 1 \leq j \leq n$, let S_j be an open convex set containing $\sigma(T_j)$. Then there exists an invertible operator R such that $W(RT_jR^{-1}) \subset S_j$ for every j. The commutativity assumption cannot be removed here. The case $n = 1$ was considered in Theorem 2.4, (i). However the description of $\operatorname{conv} \sigma(\mathcal{T})$ in terms of similarities as in Theorem 2.4 is apparently missing in the literature.

It is curious to note that Garske's theorem mentioned in Section 2, generalizes to tuples of operators. As proved in [33], if $T_1, \ldots, T_n \in B(H)$ are mutually commuting, then

$$\sup_{\{x: \|x\|=1\}} \left(\sum_{j=1}^n \|T_j x\|^2 - \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right) \geq R^2,$$

where R is the radius of the smallest ball contaning the (Harte) joint spectrum of $(T_1, \ldots, T_n)$. Moreover, the above inequality becomes equality if $T_j, 1 \leq j \leq n$, are mutually commuting normal operators on H.

The joint numerical ranges $W(T, \ldots, T^n)$ of powers of a single operator have certainly their own specifics. The next consequence of Corollary 4.3

proved in [72, Theorem 4.6] is instrumental in all of our applications of the theory of joint numerical ranges. Note that it allows one to deal with the polynomial hull $\hat{\sigma}(T) \subset \mathbb{C}$ rather than the much less transparent set conv $\sigma(T, ..., T^n) \subset \mathbb{C}^n$. Recall that $\hat{\sigma}(T)$ can be described as the union of $\sigma(T)$ with all bounded components of the complement $\mathbb{C} \setminus \sigma(T)$.

Theorem 4.4. *Let $T \in B(H)$ and $\lambda \in \operatorname{Int} \hat{\sigma}(T)$. Then*

$$(\lambda, \lambda^2, \ldots, \lambda^n) \in \operatorname{Int} W_e(T, T^2, \ldots, T^n),$$

for all $n \in \mathbb{N}$.

5. Several applications of joint numerical ranges

Now we turn to several applications of joint numerical ranges to other problems in operator theory found recently in [72], [73], and [74]. The general ideology developed in [72]-[74] is that to every $T \in B(H)$ one associates an n-tuple $\mathcal{T}_n := (T, ..., T^n)$, and tries to uncover fine properties of T in terms of the structure of the sets

$$\sigma(T), \quad W(\mathcal{T}_n), \quad \text{and} \quad W_e(\mathcal{T}_n), \quad n \in \mathbb{N}.$$

rather than a single set $W(T)$. While the study of an operator T in terms of asymptotic or algebraic properties of its powers is a rather standard approach going back to the birth of operator theory, this idea of invoking the sequences of numerical ranges $W(\mathcal{T}_n)$ and $W_e(\mathcal{T}_n)$ had not been exploited until recent time.

Apart from the papers [72]-[74] developing the approach above, one may mention [27], where very particular results on $W(T, ..., T^n)$ were obtained for $T \in B(\mathbb{C}^n)$.

In this section, we present various generalizations and improvements of notorious operator-theoretical results accomplished by using numerical ranges techniques.

5.1. **Circles in the spectrum.** First, we characterize the circle structure in the spectrum of a bounded linear operator linking in this manner several statements from ergodic theory, harmonic analysis and spectral theory. We start with an old and elegant theorem of W. Arveson proved in his PhD thesis, see [4]. Among motivations for the result is a classical Rokhlin Lemma for measure preserving transformations, one of the building blocks of ergodic theory. The result can be considered as a spatial version of the lemma.

Theorem 5.1 (Arveson, 1966). *Let $U \in B(H)$ be a unitary operator. The following statements are equivalent:*

(i) *$\sigma(U) = \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle in the complex plane;*
(ii) *for every $n \in \mathbb{N}$ there is a unit vector $x \in H$ such that $x, Ux, \ldots, U^n x$ are orthogonal.*

Note that the second condition in the above theorem can be reformulated as

$$(0, \ldots, 0) \in W(U, U^2, \ldots, U^n)$$

for every $n \in \mathbb{N}$. This suggests the use of numerical ranges for tuples and, motivates our studies in Section 4. In view of the results in Section 4 the implication (i)$\Rightarrow$(ii) can be generalized as follows.

Theorem 5.2. *Let $T \in B(H)$, $0 \in \operatorname{Int} \hat{\sigma}(T)$. Then for every $n \in \mathbb{N}$ there exists an infinite-dimensional subspace $L \subset H$ such that*

$$P_L T^j P_L = 0, \qquad j = 1, \ldots, n.$$

Proof. From Theorem 4.4, we have that $(0, \ldots, 0) \in \operatorname{Int} W_e(T, T^2, \ldots, T^n)$ and Corollary 3.9 concludes the proof. $\qquad \square$

Since for unitary operators T one has $\sigma(T) \subset \mathbb{T}$, the above corollary can be further sharpened.

Theorem 5.3. *Let $U \in B(H)$ be a unitary operator. The following statements are equivalent:*

   (i) *$\sigma(U) = \mathbb{T}$;*
   (ii) *for every $n \in \mathbb{N}$ there exists an infinite-dimensional subspace $L \subset H$ such that*

$$P_L U^j P_L = 0, \qquad j = 1, \ldots, n.$$

The ultimate general form of Arveson's Theorem 5.1 seems to be the following statement, [72, Theorem 1.1].

Theorem 5.4. *Let T be a bounded linear operator on H, such that the spectral radius $r(T) \leq 1$. The following statements are equivalent.*

   (i) *$\mathbb{T} \subset \sigma(T)$.*
   (ii) *For all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $x \in H$ such that*

$$|\langle T^m x, T^j x \rangle| < \varepsilon, \qquad 0 \leq m, j \leq n-1, m \neq j,$$

   and

$$\frac{1}{2} \leq \|T^j x\| \leq 2, \qquad 0 \leq j \leq n-1.$$

   (iii) *For all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $x \in H$ such that*

$$(5.1) \qquad\qquad x \perp T^j x, \qquad 1 \leq j \leq n-1,$$

$$(5.2) \qquad\qquad |\langle T^m x, T^j x \rangle| < \varepsilon, \qquad 1 \leq m, j \leq n-1, m \neq j,$$

$$(5.3) \qquad 1 - \varepsilon < \|T^j x\| < 1 + \varepsilon, \qquad 0 \leq j \leq n-1,$$

   and

$$(5.4) \qquad\qquad\qquad \|T^n x - x\| < \varepsilon.$$

Thus presence of $\mathbb{T}$ in the spectrum of T can be characterized by either "almost-orthonormality" relations in (ii), or by the same kind relations as in (ii) strengthened by adding orthogonality relation in (5.1) and "almost-periodicity" property in (5.3). Note that one can drop the assumption $r(T) \leq 1$ assuming in (i) that $\mathbb{T}$ is contained in the so-called essential approximate spectrum $\sigma_{\pi e}(T)$ of T. This version of Theorem 5.4 can be found in [72, Theorem 5.2]. It includes Theorem 5.4, since $\mathbb{T} \subset \partial\sigma(T)$ implies $\mathbb{T} \subset \sigma_{\pi e}(T)$. In fact, the properties of $\sigma_{\pi e}(T)$ are quite crucial in the proof of Theorem 5.4. For a detailed study of essential spectra of bounded linear operators, including the setting of operator tuples and approximate spectrum, one may consult [69, Chapter III]. Moreover, dealing with the unit circle in Theorem 5.4 is just a matter of normalization, and a containment of any circle from $\mathbb{C}$ in $\sigma(T)$ can be treated in the same way.

5.2. Numerical ranges and asymptotics of weak orbits.

Another motivation for the study of the circle structure of the spectrum stems from an interplay of ergodic theory and harmonic analysis. Recall that a positive measure ν on the unit circle $\mathbb{T}$ is called Rajchman if its Fourier coefficients $((\mathcal{F}\nu)(n))_{n\in\mathbb{Z}}$ satisfy $(\mathcal{F}\nu)(n) \to 0$, $|n| \to \infty$. While this class of measures is crucial in many chapters of analysis and appears frequently in the literature, no handy characterization of it is available so far, see e.g. [68] for discussions of results and problems behind it. In his studies of weak mixing properties of dynamical systems, D. Hamdan proved in [46] that if ν is Rajchman, then $\operatorname{supp}\nu = \mathbb{T}$ if and only if for every $\epsilon > 0$ there exists a positive $f \in L^1(\mathbb{T}, \nu)$ such that ν-Fourier coefficients of f given by $(\mathcal{F}_\nu f)(n) = \int_{\mathbb{T}} z^n f(z)\, d\nu(z), n \in \mathbb{Z}$, are *uniformly* small in the sense that $\sup_{n\in\mathbb{Z}} |(\mathcal{F}_\nu f)(n)| < \epsilon$. Note that if $(Uf)(z) = zf(z)$, then U is unitary on $L^2(\mathbb{T}, \nu)$, $U^n \to 0$ in the weak operator topology, and $\sigma(U) = \operatorname{supp}\nu$. This operator-theoretical interpretation of the Hamdan's result on Rajchman measures leads to the following theorem proved in [46] for unitary operators induced by measure preserving transformations.

Theorem 5.5 (Hamdan 2013). *Let $U \in B(H)$ be a unitary operator such that $U^n \to 0$ in the weak operator topology. The following statements are equivalent.*

(i) $\sigma(U) = \mathbb{T}$.
(ii) *For every $\varepsilon > 0$ there exists a unit vector $x \in H$ such that*

$$(5.5) \qquad \sup_{n\geq 1} |\langle U^n x, x\rangle| < \varepsilon.$$

At first sight, the condition (ii) requiring a uniform smallness of the week orbit of U looks enigmatic. However, the very definition of joint numerical ranges suggest interpreting the smallness in terms of $W(U, ..., U^n), n \in \mathbb{N}$. Vaguely, if one thinks of a numerical range $W(U, U^2, ..., U^n, ...) \subset \ell^\infty(\mathbb{C})$ for the sequence $(U^n) \subset B(H)$, then one may reword (ii) as saying that

$W(U, U^2, ..., U^n, ...)$ has the zero limit point. (See [20] for a definition and further discussion of joint numerical ranges for operator sequences.)

Using the properties of joint numerical ranges, Theorem 5.5 has been extended in [72, Corollary 6.3 and Remark 6.4] to the setting of general bounded operators on H. Moreover it was shown in [72] that one is allowed to take elements x in (ii) from a specified infinite-dimensional subspace. Namely, the following result was obtained.

Theorem 5.6. *Let $T \in B(H)$, and let $T^n \to 0$ in the weak operator topology. Suppose that $0 \in \text{Int}\,\hat{\sigma}(T)$. Then for every $\varepsilon > 0$ there exists an infinite-dimensional subspace L of H such that*

$$\sup_{n \geq 1} \|P_L T^n P_L\| \leq \varepsilon \qquad and \qquad \lim_{n \to \infty} \|P_L T^n P_L\| = 0.$$

In particular, this is true if the assumption $0 \in \text{Int}\,\hat{\sigma}(T)$ is replaced by $\mathbb{T} \subset \sigma(T)$.

If T is unitary then the statement above can be improved. The following result generalizes Theorem 5.5 (by using a completely different approach than that of [46]). For its proof see [72, Corollary 6.5].

Corollary 5.7. *Let T be a unitary operator on H such that $T^n \to 0$ in the weak operator topology. Then the following conditions are equivalent.*

  (i) *$\sigma(T) \supset \mathbb{T}$;*
 (ii) *For every $\varepsilon > 0$ there exists $x \in H$, $\|x\| = 1$, with*

$$\sup_{n \geq 1} |\langle T^n x, x \rangle| < \varepsilon.$$

(iii) *For every $\varepsilon > 0$ there exists an infinite-dimensional subspace $L \subset H$ such that*

$$\sup_{n \geq 1} \|P_L T^n P_L\| \leq \varepsilon \qquad and \qquad \lim_{n \to \infty} \|P_L T^n P_L\| = 0.$$

5.3. **Diagonals of operators.** Let $T \in B(H)$. Assume for definitiveness that H is an *infinite-dimensional* separable Hilbert space. The problem we address in this section is how to describe all possible diagonals of T, i.e., all sequences $(d_k)_{k=1}^{\infty}$ such that $d_k = \langle Tu_k, u_k \rangle$ for all $k \in \mathbb{N}$ and some orthonormal basis $(u_k)_{k=1}^{\infty}$ in H. The problem appears naturally in many situations and has been studied intensively. A related, second problem is how to describe all possible diagonals of operators in a given class. These problems are naturally connected with the numerical range and its subsets, since the entries constituting diagonals of T belong to $W(T)$. For a detailed discussion of some motivations for such studies we refer to the introduction in [74]. Here we just quote a claim from [32] speculating that "the diagonal of an operator carries more information about the operator than its relatively small size (compare to the "fat" matrix representation of the operator) may suggest."

Most of research on diagonals concentrated on the second problem. Answering a question by A. Gillespie, C. K. Fong showed that in [39] that for any $(d_k)_{k=1}^\infty \in \ell^\infty$ there exists $T \in B(H)$, $T^4 = 0$ such that $(d_k)_{k=1}^\infty$ is a diagonal of T. (As it was remarked later by Herrero the exponent 4 can be replaced by 2.) His proof was based on a deep theorem due to Fan stating that if for $T \in B(H)$ one has $0 \in \text{Int } W_e(T)$, then T admits a zero diagonal. (There was flaw in Fan's argument which was corrected recently in [67].) While Fong's result led to a further research on diagonals, it remained essentially the only result of this kind for a long while.

In the beginning of this century, being motivated by problems from the theory of C^*-algebras, R. V. Kadison described in [55, 56] the diagonals for a class of selfadjoint projections on H. Kadison's elegant result can be stated as follows.

Theorem 5.8 (Kadison, 2002). *A sequence $(d_k)_{k=1}^\infty$ is a diagonal of some orthogonal projection if and only if $(d_k)_{k=1}^\infty \subset [0,1]$ and if the sums $a := \sum_{d_k < 1/2} d_k$ and $b := \sum_{d_k \geq 1/2}(1 - d_k)$ satisfy either $a + b = \infty$ or $a - b \in \mathbb{Z}$.*

(Note that the situation changes dramatically if one drops the orthogonality assumption. The set of diagonals of idempotents on H then fills the whole of ℓ^∞, see [66].) Almost immediately, W. Arveson extended in [6] the realm of Kadison's considerations to normal operators with finite spectrum, see also [5]. Kadison-Arveson's perspective generated an activity on characterizing the set of diagonals for several classes of operators: selfadjoint, unitary or normal under various spectral assumptions, and gave rise to a number of deep results. As a sample we mentioned the next recent characterization of diagonals for a class of unitary operators, see [53].

Theorem 5.9 (Jasper, Loreaux, Weiss, 2018). *A complex-valued sequence $(d_k)_{k=1}^\infty$ is a diagonal of a unitary operator if and only if $\sup_{k \geq 1} |d_k| \leq 1$ and*

$$2(1 - \inf_{k \geq 1} |d_k|) \leq \sum_{k=1}^\infty (1 - |d_k|).$$

The descriptions of diagonals for operator classes became a part of a long research program realized by Bownik, Jasper, Kaftal, Loreaux, Weiss, and others. For some of their achievements, see [17], [18], [57], [58], [66], [53] and the citations in these papers. There is also a separate and similar direction in the setting of C^*-algebras. We omit a discussion of it and refer e.g. to [59], [60] and the references therein.

An inspiration to our studies was the paper [49] by D. Herrero, motivated in part by [39] and addressing a more demanding problem of description of diagonals for a fixed operator. Let us first introduce some notation. For a separable infinite-dimensional space H and $T \in B(H)$ let $\mathcal{D}(T)$ be the set of all diagonals of T, i.e., all sequences $(\langle Tu_k, u_k \rangle)$ for some orthonormal basis (u_k) in H. Extending this notation for n-tuples $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$,

we set

$$\mathcal{D}(\mathcal{T}) = \{(\langle \mathcal{T} u_k, u_k \rangle) : (u_k)_{k=1}^{\infty} \text{ is an orthonormal basis in } H\},$$

where $\langle \mathcal{T} x, y \rangle = (\langle T_1 x, y \rangle, \ldots, \langle T_n x, y \rangle)$. Uncovering the role of the interior of $W_e(T)$ in the study of diagonals, Herrero proved in [49] the next nice result.

Theorem 5.10 (Herrero, 1991). *Let $T \in B(H)$. If the sequence $(d_k)_{k=1}^{\infty}$ belongs to the interior $\operatorname{Int} W_e(T)$ of $W_e(T)$ and $(d_k)_{k=1}^{\infty}$ has a limit point in $\operatorname{Int} W_e(T)$, then $(d_k)_{k=1}^{\infty} \in \mathcal{D}(T)$.*

Now, assume again that the diagonal belongs to the interior of $W_e(T)$. In an effort to extend Herrero's theorem, we introduced a Blaschke-type condition $\sum_{k=1}^{\infty} \operatorname{dist} \{d_k, \partial W_e(\mathcal{T})\} = \infty$ on the size of diagonal $(d_k)_{k=1}^{\infty}$ near the boundary of $W_e(T)$. In view of Theorems 5.8 and 5.9 that condition looks quite natural. To deal with Blaschke-type assumptions on $(d_k)_{k=1}^{\infty}$, we proposed in [74] a general and new method for constructing a big part of diagonals that works in a variety of different settings, including operator tuples and operator-valued diagonals. See [74, Theorem 1.1] and the comments following it.

Theorem 5.11. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$, $(d_k)_{k=1}^{\infty} \subset \operatorname{Int} W_e(\mathcal{T})$ and*

$$\sum_{k=1}^{\infty} \operatorname{dist} \{d_k, \partial W_e(\mathcal{T})\} = \infty.$$

Then $(d_k)_{k=1}^{\infty} \in \mathcal{D}(\mathcal{T})$.

The theorem allows a slightly more general formulation involving tuples of sefadjoint operators on H. Note that for a selfadjoint operator $T \in B(H)$ the spectrum of T may contain an interval, but the interior of $W(T)$ could be empty in this case. So, in order not to miss several situations of interest, one should deal with the notion of *relative* interior. For more details see [74].

If $n = 1$ and $W_e(T)$ coincides with the closed unit disc $\overline{\mathbb{D}}$, then the assumption $\sum_{k=1}^{\infty} \operatorname{dist} \{d_k, \partial W_e(\mathcal{T})\} = \infty$ reduces to the negation of the classical Blaschke condition $\sum_{k=1}^{\infty} (1 - |d_k|) = \infty$, and this explains our terminology.

The Blaschke-type assumption in Corollary 3.9 is, in some sense, the best possible as the next example from [49] shows.

Example 5.12. Let $T \in B(H)$ be the unilateral shift and $(d_k)_{k=1}^{\infty} \subset \mathbb{D}$. Then $(d_k)_{k=1}^{\infty} \in \mathcal{D}(T)$ if and only if

$$\sum_{k=1}^{\infty} (1 - |d_k|) = \infty.$$

The technique from [74] led also to a version of Theorem 3.9 for n-tuples of the form $(T, T^2, \ldots, T^n)$. The version was based on the auxiliary estimate given below, [74, Lemma 4.9].

Proposition 5.13. *Let* $T \in B(H)$, $n \in \mathbb{N}$ *and let* $\mathcal{T}_n = (T, T^2, \ldots, T^n)$. *If* $\lambda \in \operatorname{Int} \widehat{\sigma}(T)$ *then* $(\lambda, \lambda^2, \ldots, \lambda^n) \in \operatorname{Int} W_e(\mathcal{T})$ *and*

$$\operatorname{dist}\{(\lambda, \lambda^2, \ldots, \lambda^n), \partial W_e(\mathcal{T}_n)\} \geq 2^{-n}\operatorname{dist}^n\{\lambda, \partial\widehat{\sigma}(T)\}.$$

Note that to formulate and to prove such a result for tuples of powers one has to invoke (the polynomial hull of) $\sigma(T)$, rather than $W_e(T)$. Its proof can be found in [74, Corollary 4.11].

Theorem 5.14. *Let* $T \in B(H)$, $n \in \mathbb{N}$ *and let* $(\lambda_k)_{k=1}^{\infty} \subset \operatorname{Int} \widehat{\sigma}(T)$ *satisfy*

$$\sum_{k=1}^{\infty} \operatorname{dist}^n\{\lambda_k, \partial\widehat{\sigma}(T)\} = \infty.$$

Then there exists and orthonormal basis $(u_k)_{k=1}^{\infty}$ *in* H *such that*

$$\langle T^j u_k, u_k \rangle = \lambda_k^j, \qquad k \in \mathbb{N}, j = 1, \ldots, n.$$

Proof. Let $\mathcal{T}_n = (T, T^2, \ldots, T^n)$. Since $W_e(\mathcal{T}_n)$ is convex, by Theorem 4.4,

$$W_e(\mathcal{T}_n) \supset conv\{(\lambda, \ldots, \lambda^n) : \lambda \in \operatorname{Int} \widehat{\sigma}(T)\}.$$

By Proposition 5.13, $(\lambda_k, \ldots, \lambda_k^n) \in \operatorname{Int} W_e(\mathcal{T}_n)$ for all $k \in \mathbb{N}$, and

$$\sum_{k=1}^{\infty} \operatorname{dist}\{(\lambda_k, \lambda_k^2, \ldots, \lambda_k^n), \partial W_e(\mathcal{T}_n)\} = \infty.$$

So the statement follows from Theorem 5.11. $\square$

An interesting interplay between the assumption $0 \in W_e(T)$ and the structure of $\mathcal{D}(T)$ was discovered by Q. Stout in his studies of so-called Schur algebras. These are commutative Banach algebras of infinite matrices defined by Schur multiplication, i.e., the termwise product of the matrix representations of operators of a Hilbert space (given an orthonormal basis). Q. Stout in [81] proved that the condition of zero belonging to the essential numerical range of $T \in B(H)$ is equivalent to several properties revealing the structure of a Schur algebra. The next result, due to Q. Stout ([81, Theorem 2.3]), relates the essential numerical range of T to its diagonals.

Theorem 5.15 (Stout, 1981). *Let* $T \in B(H)$ *and* $0 \in W_e(T)$. *For each sequence of positive numbers* $(\alpha_k)_{k=1}^{\infty} \notin \ell_1$ *there exists an orthonormal basis* $(u_k)_{k=1}^{\infty}$ *in* H *such that*

$$(5.6) \qquad\qquad |\langle Tu_k, u_k \rangle| \leq |\alpha_k|$$

for all $k \in \mathbb{N}$.

This theorem generalizes an older result of J. Anderson that arises in the study of commutators of operators. Recall that $0 \in W_e(T)$ if and only if there exists $(d_k)_{k=1}^{\infty}$ in $\mathcal{D}(T)$ such that $(d_k)_{k=1}^{\infty}$ belongs to $c_0(\mathbb{N})$. Anderson proved that $0 \in W_e(T)$ is in fact equivalent to the existence of a p-summable sequence in $\mathcal{D}(T)$ for every $p > 1$.

Corollary 5.16 (Anderson, 1971)**.** *Let $T \in B(H)$, $0 \in W_e(T)$ and $p > 1$. Then there exists an orthonormal basis $(u_k)_{k=1}^{\infty}$ in H such that*

$$\sum_{k=1}^{\infty} |\langle Tu_k, u_k \rangle|^p < \infty.$$

A similar statement (with a different proof) was used in [40, Theorem 4.1] in the study of operator norms.

In [74], we extended Theorem 5.15 to tuples of operators. Also, we showed that any sequence $(\lambda_k)_{k=1}^{\infty} \subset W_e(\mathcal{T})$ can be approximated by a diagonal in the sense of (5.6), whereas Stout's statement treats just zero sequences. For the proof of the following theorem see [74, Theorem 1.2].

Theorem 5.17. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. For every $(\lambda_k)_{k=1}^{\infty} \subset W_e(\mathcal{T})$ and every $(\alpha_k)_{k=1}^{\infty} \notin \ell_1$ there exists an orthonormal basis $(u_k)_{k=1}^{\infty}$ in H such that*

$$\big\| \langle \mathcal{T} u_k, u_k \rangle - \lambda_k \big\| \leq |\alpha_k|$$

for all $k \in \mathbb{N}$.

Note that the following question of Stout and related to Theorem 5.15 seems to be not yet answered. Given an operator $T \in B(H)$ such that $0 \in W_e(T)$, and a sequence $(a_n)_{n=1}^{\infty}$ of positive numbers which is not in ℓ_2, does there exist a basis $(e_n)_{n=1}^{\infty}$ and a bijection π from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ such that $|\langle Te_n, e_m \rangle| < a_{\pi(n,m)}$ for all n and m ? (Apparently, there is a misprint in the formulation of this question in [81].) It was also asked in [81] whether for an arbitrary $T \in B(H)$ there exist a basis $(e_n)_{n=1}^{\infty}$ and $S \in B(H)$ such that $|\langle Te_n, e_m \rangle| = \langle Se_n, e_m \rangle$ for all n and m. The question is still open. While formally there are no numerical ranges involved here, we feel our technique could be useful here as well.

An importance of Theorem 5.17 can be illustrated by the fact that it yields a description of the set of diagonals $D(\mathcal{T})$ up to p-Schatten class perturbations of $\mathcal{T}$, that is the set $\mathcal{D}(\mathcal{T} + \mathcal{K})$, where $\mathcal{K}$ is an n-tuple of operators from the Schatten class $S_p(H)$. We recall that for $1 \leq p < \infty$, $S_p(H)$ is the Banach space equipped with the norm $||T||_{S_p} = (\sum_{n=1}^{\infty} |s_n|^p)^{1/p}$, for $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ the singular values of T. (Observe that we have already encountered the spaces $S_2(H)$ and $S_1(H)$ in Section 3.) In this more general setting, we are able to construct the diagonals satisfying weakened Blaschke-type conditions (see [74, Corollary 5.1]). Moreover, the diagonals of perturbations may not necessarily belong to $W_e(\mathcal{T})$ but should only approximate $W_e(\mathcal{T})$ good enough, where the rate of approximation is determined by the Schatten class of perturbations. This is a far reaching generalization of [49, Theorem, (ii) and (iii)].

Corollary 5.18. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ and $p > 1$. Let $(\lambda_k)_{k=1}^{\infty} \subset \mathbb{C}^n$ satisfy*

$$\sum_{k=1}^{\infty} \mathrm{dist}^{\,p}\{\lambda_k, W_e(\mathcal{T})\} < \infty.$$

Then there exists an n-tuple of operators $\mathcal{K} = (K_1, \ldots, K_n)$ with K_j from the Schatten class $S_p(H)$, $1 \le j \le n$, such that $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{T} + \mathcal{K})$.

Our results on compact perturbations provide several characterizations of the subset

$$\mathcal{D}_{\mathrm{const}}(\mathcal{T}) := \left\{ \lambda \in \mathbb{C}^n : (\lambda, \lambda, \ldots) \in \mathcal{D}(\mathcal{T}) \right\}$$

consisting of constant diagonals. Understanding the structure of $\mathcal{D}_{\mathrm{const}}(\mathcal{T})$ for a fixed $\mathcal{T}$ and relating it to the structure of $W_e(\mathcal{T})$ was a natural next step. Clearly we have that

$$\mathrm{Int}\, W_e(\mathcal{T}) \subset \mathcal{D}_{\mathrm{const}}(\mathcal{T}) \subset W_e(\mathcal{T}).$$

By Theorem 3.7, the set $W_e(\mathcal{T})$ is convex, and since the interior of a convex set is convex, so is $\mathrm{Int}\, W_e(\mathcal{T})$. However, the question whether $\mathcal{D}_{const}(\mathcal{T})$ is convex is still open, although we have a positive answer if $n = 1$ (unpublished). This problem has been raised by J.-C. Bourin in [16].

5.4. **Block operator diagonals.** A natural generalization of diagonals are block diagonals. Block diagonals arise in a variety of issues from operator theory ranging from the study of quasitriangularity and quasidiagonality to the investigations of unitary and similarity orbits and their spans. Being unable even to touch them, we mention the paper [26] as a sample, where block diagonals and numerical ranges appeared to be crucial in the latter circle of problems. Note that the block diagonals are sometimes called "pinchings" in the literature.

As we will see below, the study of block diagonals is intimately related to essential numerical ranges. However, besides the numerical ranges structure, one has to use new operator-theoretical constructions somewhat similar to dilations. Their description however falls out of the scope of this survey, and we refer to [74] for more explanations and details.

The following result was proved in [16].

Theorem 5.19 (Bourin 2003)**.** *Let $T \in B(H)$ with $W_e(T) \supset \overline{\mathbb{D}}$. Let L_k, $k \in \mathbb{N}$, be separable Hilbert spaces (finite or infinite-dimensional), and let $C_k \in B(L_k)$ be contractions satisfying $\sup_k \|C_k\| < 1$. Then there exist projections P_{K_k}, $k \in \mathbb{N}$, onto mutually orthogonal subspaces $K_k \subset H$ such that*

$$\bigoplus_{k=1}^\infty K_k = H$$

and $P_{K_k} T P_{K_k}$ is unitarily equivalent to C_k, for all $k \in \mathbb{N}$.

In [74], we extended Theorem 5.19 to the setting of tuples and replace the uniform contractivity condition on operator diagonal by a more general assumption of Blaschke's type, see [74, Theorem 1.3]). Such an assumption is, in general, necessary even for scalar diagonals as Example 5.12 shows.

Theorem 5.20. *Let $T \in B(H)$ with $W_e(T) \supset \overline{\mathbb{D}}$. Let L_k, $k \in \mathbb{N}$, be separable Hilbert spaces (finite or infinite-dimensional) and let $C_k \in B(L_k)$ be*

contractions satisfying $\sum_{k=1}^{\infty}(1 - \|C_k\|) = \infty$. Then there exist projections P_{K_k}, $k \in \mathbb{N}$, onto mutually orthogonal subspaces $K_k \subset H$ such that

$$H = \bigoplus_{k=1}^{\infty} K_k,$$

and $P_{K_k}T|_{K_k}$ is unitarily equivalent to C_k, for all $k \in \mathbb{N}$.

Replacing the numerical range condition $W_e(T) \supset \overline{\mathbb{D}}$ in Theorem 5.20 by the spectral assumption $\widehat{\sigma}(T) \supset \overline{\mathbb{D}}$, we can put Theorem 5.20 in a more demanding context of tuples of powers of T. For a sequence of Hilbert space contractions $(C_k)_{k=1}^{\infty}$ with norms not approaching 1 too fast, the following statement, proved in [74, Theorem 6.3], yields pinchings $(C_k, \ldots, C_k^n)$ for a tuple $(T, \ldots, T_n)$, $T \in B(H)$, if the spectrum of T is sufficiently large. It would be instructive to note the analogy to Theorem 5.14.

Theorem 5.21. *Let $T \in B(H)$, $\widehat{\sigma}(T) \supset \overline{\mathbb{D}}$, $n \in \mathbb{N}$. Let L_k, $k \in \mathbb{N}$, be separable Hilbert spaces, and let $C_k \in B(L_k)$, $k \in \mathbb{N}$, be contractions such that $\sum_{k=1}^{\infty}(1 - \|C_k\|)^n = \infty$. Then there are mutually orthogonal subspaces K_k, $k \in \mathbb{N}$, of H such that*

$$H = \bigoplus_{k=1}^{\infty} K_k,$$

and $P_{K_k}(T, \ldots, T^n)P_{K_k}$ is unitarily equivalent to $(C_k, \ldots, C_k^n)$ (in an entry-wise sense) for all $k \in \mathbb{N}$.

6. On joint numerical radius

All results of this section are contained in [71] and [30].

Recall that one of the basic properties of the numerical radius is the inequality

$$w(T) \geq \frac{1}{2}\|T\|$$

for all operators $T \in B(H)$. Equivalently, for every $\varepsilon > 0$ there exists $x \in H$, $\|x\| = 1$ such that

$$|\langle Tx, x\rangle| > \frac{1}{2}\|T\| - \varepsilon.$$

If, moreover, $\dim H < \infty$, then there exists $x \in H$, $\|x\| = 1$, such that $|\langle Tx, x\rangle| \geq \frac{1}{2}\|T\|$.

In this section we will discuss an analogous property for n-tuples of operators. We consider the following problem:

Problem 6.1. *Let $T_1, \ldots, T_n \in B(H)$. Does there exist $x \in H$, $\|x\| = 1$, such that $|\langle T_j x, x\rangle|$ is "large" for all $j = 1, \ldots, n$?*

The problem is closely related to the so-called Tarski's plank problem on covering of a convex body in $\mathbb{R}^n$ by strips, its solution by Bang and further developments by Ball and others. For a nice review of this subject, one may consult the survey [10].

We start with a couple of useful reductions. First, dealing with Problem 6.1, one may assume that $\dim H < \infty$. Indeed, it suffices to note that for $(T_1, ..., T_n) \in B(H)^n$ one has

$$W(T_1, ..., T_n) = \bigcup_P W(PT_1P, ..., PT_nP),$$

where P runs over all finite-rank orthogonal projections (or merely over orthogonal projections of rank not exceeding $n + 1$). Second, replacing the operators T_j by their real and imaginary parts $\operatorname{Re} T_j$ and $\operatorname{Im} T_j$, and noting that

$$|\langle T_j x, x \rangle| \geq \max \left\{ |\langle \operatorname{Re} T_j x, x \rangle|, |\langle \operatorname{Im} T_j x, x \rangle| \right\}, \qquad x \in H,$$

we may consider (without much loss of generality) only tuples of selfadjoint operators.

Thus we may study the following reformulation of Problem 6.1:

Problem 6.2. What is the best constant c_n with the following property: if $\dim H < \infty$, and $T_1, \ldots, T_n \in B(H)$ are selfadjoint operators, then there exists $x \in H$, $\|x\| = 1$, such that

$$|\langle T_j x, x \rangle| \geq c_n \|T_j\|, \qquad j = 1, \ldots, n.$$

If the operators T_j are not only selfadjoint, but also positive semi-definite, then one can obtain the next precise answer.

Proposition 6.3. Let $T_1, \ldots, T_n \in B(H)$ be such that $T_j \geq 0, 1 \leq j \leq n$. Let $\dim H < \infty$. Then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x \rangle| \geq \frac{1}{n} \|T_j\|, \qquad j = 1, \ldots, n.$$

The constant $1/n$ is the best.

For selfadjoint operators $T_j, 1 \leq j \leq n$, the exact estimate is known only for $n = 2$ and $n = 3$.

Theorem 6.4.   (i) *Let $T_1, T_2 \in B(H)$, $T_j^* = T_j, j = 1, 2$, and let $\dim H < \infty$. Then there exists a unit vector $x \in H$ such that*

$$|\langle T_j x, x \rangle| \geq \frac{1}{3} \|T_j\|, \qquad j = 1, 2.$$

(ii) *Let $T_1, T_2, T_3 \in B(H)$, $T_j^* = T_j, j = 1, 2, 3$, and let $\dim H < \infty$. Then there exists a unit vector $x \in H$ such that*

$$|\langle T_j x, x \rangle| \geq \frac{1}{5} \|T_j\|, \qquad j = 1, 2, 3.$$

The estimates above are the best possible.

Problem 6.2 and Theorem 6.4 are closely related to the following open, purely geometric question.

Problem 6.5. Let $x^{(1)}, \ldots, x^{(n)} \in [-1, 1]^n$, $x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$ and $x_k^{(k)} = 1$ $(k = 1, \ldots, n)$. Does there exists $y = (y_1, \ldots, y_n)$ from the convex hull of $x^{(k)}, 1 \leq k \leq n$, such that $|y_k| \geq \frac{1}{2n-1}$ for all $k = 1, \ldots, n$?

A positive answer to this problem would allow us to set $c_n = \frac{1}{2n-1}$ in the estimate in Problem 6.2. It is known that the answer is indeed positive for $n = 2$ and $n = 3$, and this is used in the proof of Theorem 6.4. As remarked in [71, Example 8], the estimate $\frac{1}{2n-1}$ in Problem 6.5 cannot be improved. Indeed, let $n \in \mathbb{N}$ and let $u_j = (u_{j1}, \ldots, u_{jn}) \in \mathbb{R}^n$ be defined by $u_{jj} = 1, j = 1, \ldots, n$, and $u_{jk} = -\frac{1}{2n-1}$, for $1 \leq j, k \leq n, j \neq k$. If $v = (v_1, \ldots, v_n)$ is an arbitrary vector from the convex hull of $\{u_1, \ldots u_n\}$, then $\min_{1 \leq k \leq n} |v_k| \leq \frac{1}{2n-1}$. In general, for $n \geq 4$, it is only known that there exits $y \in Q$ from the convex hull of $x^{(k)}, 1 \leq k \leq n$, with $|y_k| \geq \frac{1}{2n\sqrt{n}}$ for all k.

The last property can be applied in all situations where an appropriate numerical range of $(T_1, \ldots, T_n)$ is convex. Apart from the situations discussed in Section 3, we mention the following set-up. Let $\mathcal{A}$ be a unital Banach algebra, and let $a_1, \ldots, a_n \in \mathcal{A}$. Define the *algebraic numerical range*

$$V(a_1, \ldots, a_n, \mathcal{A}) = \{(f(a_1), \ldots, f(a_n)) : f \in \mathcal{A}^*, \|f\| = 1 = f(1_{\mathcal{A}})\}.$$

Then $V(a_1, \ldots, a_n, \mathcal{A})$ is a closed convex subset of $\mathbb{C}^n$, see [13, p. 23].

Thus the partial answer to Problem 6.2 leads, for instance, to the following results.

Theorem 6.6. (i) Let $T_1, \ldots, T_n \in B(H)$ be commuting selfadjoint operators, and let $c \in (0, 1/2)$. Then there exists $x \in H, \|x\| = 1$, such that

$$|\langle T_j x, x \rangle| \geq \frac{c}{n\sqrt{n}} \|T_j\|, \qquad j = 1, \ldots, n.$$

(ii) Let $\mathcal{A}$ be a unital Banach algebra, $a_1, \ldots, a_n \in \mathcal{A}$. Then there exists $f \in \mathcal{A}^*, \|f\| = f(1_{\mathcal{A}}) = 1$, such that

$$|f(a_j)| \geq \frac{\|a_j\|}{2en\sqrt{n}}, \qquad j = 1, \ldots, n,$$

where e is the Euler constant.

One can also prove an asymptotic version of the estimate treated in this section.

Theorem 6.7. Let $T_1, \ldots, T_n \in B(H)$. Then there exists an orthonormal sequence $(x_k)_{k=1}^{\infty} \subset H$ such that

$$\lim_{k \to \infty} |\langle T_j x_k, x_k \rangle| \geq \frac{\|T_j\|_e}{4n\sqrt{n}}, \qquad j = 1, \ldots, n,$$

where $\|T\|_e := \inf\{\|T - K\| : K \in K(H)\}$.

The best known estimate for the joint numerical range of general operators is the following result.

Theorem 6.8. Let $\dim H < \infty$, $n \in \mathbb{N}$, let $T_1, \ldots, T_n \in B(H)$. Then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x \rangle| \geq \frac{\|T_j\|}{4n^2}, \qquad j = 1, \ldots, n.$$

If the operators $T_1, \ldots, T_n \in B(H)$ are selfadjoint, then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x \rangle| \geq \frac{\|T_j\|}{2n^2}, \qquad j = 1, \ldots, n.$$

So for the constant c_n in Problem 6.2, there is still a large gap between the plausible upper estimate $\frac{1}{2n-1}$ (verified in several particular cases) and the lower estimate $\frac{1}{4n^2}$. We conjecture that c_n should be proportional to $\frac{1}{n}$.

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