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Abstract If T is a Kreiss bounded operator on a Banach space, then $\|T^n\| = O(n)$. Forty years ago Shields conjectured that in Hilbert spaces, $\|T^n\| = O(\sqrt{n})$. A negative answer to this conjecture was given by Spijker, Tracogna and Welfert in 2003. We improve their result and show that this conjecture is not true even for uniformly Kreiss bounded operators. More precisely, for every $\varepsilon > 0$ there exists a uniformly Kreiss bounded operator T on a Hilbert space such that $\|T^n\| \sim (n+1)^{1-\varepsilon}$ for all $n \in \mathbb{N}$. On the other hand, any Kreiss bounded operator on Hilbert spaces satisfies $\|T^n\| = O(\frac{n}{\sqrt{\log n}})$.

We also prove that the residual spectrum of a Kreiss bounded operator on a reflexive Banach space is contained in the open unit disc, extending known results for power bounded operators. As a consequence we obtain examples of mean ergodic Hilbert space operators which are not Kreiss bounded.

Keywords Kreiss boundedness · Cesàro mean · mean ergodic

Mathematics Subject Classification (2010) MSC 47A10 · MSC 47A35

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1 Introduction

Throughout this paper X stands for a complex Banach space, the symbol $B(X)$ denotes the space of all bounded linear operators acting on X, and X^* is the dual of X.

Definition 1 For an operator $T \in B(X)$ we have three notions of Kreiss boundedness, ordered by strength:

1. T is *strong Kreiss bounded* if there exists $C > 0$ such that
$$\|(\lambda I - T)^{-k}\| \leq \frac{C}{(|\lambda| - 1)^k} \quad \text{for all } k \in \mathbb{N} \text{ and } |\lambda| > 1;$$
2. T is *uniformly Kreiss bounded* if there exists $C > 0$ such that
$$\left\| \sum_{k=0}^{n} \lambda^{-k-1} T^k \right\| \leq \frac{C}{|\lambda| - 1} \quad \text{for all } n \in \mathbb{N} \text{ and } |\lambda| > 1;$$
3. T is *Kreiss bounded* if there exists $C > 0$ such that
$$\|(\lambda I - T)^{-1}\| \leq \frac{C}{|\lambda| - 1} \quad \text{for all } |\lambda| > 1.$$

Given $T \in B(X)$ and $n \geq 0$, we denote the *Cesàro mean* by
$$M_n(T) := \frac{1}{n+1} \sum_{k=0}^{n} T^k.$$

We recall some definitions concerning the behavior of the sequence of Cesàro means $(M_n(T))$.

Definition 2 A linear operator T on a Banach space X is called

1. *mean ergodic* if $M_n(T)$ converges in the strong operator topology of X;
2. *Cesàro bounded* if the sequence $(M_n(T))_{n\in\mathbb{N}}$ is bounded;
3. *absolutely Cesàro bounded* if there exists a constant $C > 0$ such that
$$\frac{1}{N+1} \sum_{j=0}^{N} \|T^j x\| \leq C\|x\| \ ,$$
for all $x \in X$ and $N \in \mathbb{N}$.

An operator T is said to be *power bounded* if there is a $C > 0$ such that $\|T^n\| < C$ for all n.

The first example of a mean ergodic operator which is not power-bounded was given by Hille ([8], where $\|T^n\| \sim n^{1/4}$). An example of a mean ergodic operator T on $L^1(\mathbb{Z})$ with $\limsup_n \|T^n\|/n > 0$ was obtained in [10] (certainly, $\|T^n x\|/n \to 0$ for every $x \in L^1(\mathbb{Z})$).

Remark 1 1. In [15, Corollary 3.2], it is proved that an operator T is uniformly Kreiss bounded if and only if there is a constant C such that

$$\|M_n(\lambda T)\| \leq C \quad \text{for all } n \in \mathbb{N} \text{ and } |\lambda| = 1. \tag{1}$$

2. In [6], it was shown that every strong Kreiss bounded operator is uniformly Kreiss bounded. The converse is not true, see [15, Section 5]. Moreover, McCarthy (see [14], [18]) proved that if T is strong Kreiss bounded then $\|T^n\| \leq Cn^{1/2}$ (see also [12, Theorem 2.1]) and gave also an example of a strong Kreiss bounded operator which is not power bounded.
3. Denote by

$$M_n^{(2)}(T) := \frac{2}{(n+1)(n+2)} \sum_{j=0}^{n} (j+1) M_j(T)$$

the second Cesàro mean. It is easy to see that

$$M_n^{(2)}(T) = \frac{2}{(n+1)(n+2)} \sum_{j=0}^{n} (n+1-j) T^j.$$

In [20], it was proved that T is Kreiss bounded if and only if there is a constant C such that

$$\|M_n^{(2)}(\lambda T)\| \leq C \quad \text{for all } n \in \mathbb{N} \text{ and } |\lambda| = 1. \tag{2}$$

There exist Kreiss bounded operators which are not Cesàro bounded, and conversely [22].
4. An operator T is called Möbius bounded if its spectrum is contained in the closed unit disc and $\varphi(T)$ is uniformly bounded on the set of the automorphism of the unit disc. By [18], T is a Möbius bounded operator if and only if it is Kreiss bounded.
5. On finite-dimensional spaces, the classes of Kreiss bounded operators and power bounded operators coincide.
6. By (1), any absolutely Cesàro bounded operator is uniformly Kreiss bounded.

Let X be the space of all bounded analytic functions f on the unit disc in the complex plane such that the derivative f' belongs to the Hardy space H^1, endowed with the norm

$$\|f\| := \|f\|_\infty + \|f'\|_{H^1} .$$

Then the multiplication operator M_z acting on X is Kreiss bounded but it fails to be power bounded. Moreover, this operator is not uniformly Kreiss bounded (see [20]).

Let V be the Volterra operator acting on $L^p[0,1]$, $1 \leq p \leq \infty$ defined by

$$(Vf)(t) = \int_0^t f(s)ds \qquad (f \in L^p(0,1)).$$

Then $I - V$ is uniformly Kreiss bounded. For $p = 2$ it is even power bounded (see [15]).

It is immediate that any power bounded operator is absolutely Cesàro bounded. In general, the converse is not true.

Let $1 \le p < \infty$ and let $e_n, n \in \mathbb{N}$ be the standard basis in $\ell^p(\mathbb{N})$. The following theorem yields examples of absolutely Cesàro bounded operators with different behavior on $\ell^p(\mathbb{N})$.

Theorem 1 *[3, Theorem 2.1] Let T be the weighted backward shift on $\ell^p(\mathbb{N})$ with $1 \le p < \infty$ defined by $Te_1 := 0$ and $Te_k := w_k e_{k-1}$ for $k > 1$. If $w_k := \left(\frac{k}{k-1}\right)^\alpha$ with $0 < \alpha < \frac{1}{p}$, then T is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$ and is not power bounded.*

For $p = 2$, the adjoint of the operator in Theorem 1 is uniformly Kreiss bounded, mean ergodic but not absolutely Cesàro bounded.

In [9], Kornfeld and Kosek constructed for every $\delta \in (0, 1)$ a positive mean ergodic operator T on L^1 with $\|T^n\| \sim n^{1-\delta}$. By positivity, T is absolutely Cesàro bounded.

Since

$$T^n = (n+1)M_n(T) - nM_{n-1}(T), \tag{3}$$

any Cesàro bounded operator satisfies that $\|T^n\| = O(n)$.

In the following picture we summarize the implications among the above definitions.

$$\begin{tikzcd}
& \text{Power bounded} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Strong Kreiss bounded} \arrow[dr, Rightarrow] & & \text{Absolutely Ces\`aro bounded} \arrow[dl, Rightarrow] \\
& \text{Uniformly Kreiss bounded} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Kreiss bounded} \arrow[dr, Rightarrow] & & \text{Ces\`aro bounded} \arrow[dl, Rightarrow] \\
& \|T^n\| = O(n) &
\end{tikzcd}$$

Fig. 1 Implications among different definitions related with Kreiss bounded and Cesàro bounded operators on Banach spaces.

2 About the Shields conjecture on Hilbert spaces

If T is a Kreiss bounded operator in a Banach space, then $\|T^n\| \le Cn$ [12, (2.4)]. By Nevanlinna [16, Theorem 6], there are Kreiss bounded operators T on Banach spaces with $\|T^n\| \ge C'n$ for some $C' > 0$.

In [18], Shields conjectured that any Kreiss bounded Hilbert space operator T satisfies $\|T^n\| = O(\sqrt{n})$. A negative answer to this conjecture was given in [19] where it was shown that for every $\varepsilon > 0$ there exists a Kreiss bounded Hilbert space operator T such that the norms of its powers $\|T^n\|$ grow as fast as $n^{1-\varepsilon}$.

In this section we improve this result and construct an operator with similar properties, which is even uniformly Kreiss bounded.

In the same paper [18] Shields mentioned without proof that if T is a Kreiss bounded Hilbert space operator such that the sequence of norms $(\|T^n\|)$ is increasing and there is a unit vector x such that $\|T^n x\| \geq \|T^n\|/2$ for all n then $\|T^n\| = O(\sqrt{n})$ (such properties would satisfy the first natural attempt to disprove the Shields conjecture). For the sake of completeness we give a proof of this result. We need to prove the following lemma.

Lemma 1 *Let $(a_k)_{k=0}^{\infty}$ be an increasing sequence of non-negative numbers, $B > 0$, and let $\sum_{n=0}^{\infty} a_k^2 r^{2k} \leq B/(1-r)^2$ for $0 \leq r < 1$. Then $a_n = O(\sqrt{n})$.*

Proof Since $\sum_{n=0}^{\infty} a_k^2 r^{2k} \leq \dfrac{B}{(1-r)^2}$, multiplying both sides by $1 - r^2$, we see that

$$a_0^2 + \sum_{k=1}^{\infty} (a_k^2 - a_{k-1}^2) r^{2k} \leq \frac{2B}{1-r}.$$

Now since $\{a_k\}_{k \in \mathbb{N}}$ is increasing, we have

$$r^{2n}\Big(a_0^2 + \sum_{k=1}^{n} (a_k^2 - a_{k-1}^2)\Big) \leq a_0^2 + \sum_{k=1}^{n} (a_k^2 - a_{k-1}^2) r^{2k} \leq \frac{2B}{1-r}.$$

Set $r = e^{-1/n}$. We conclude that

$$a_n^2 = a_0^2 + \sum_{k=1}^{n} (a_k^2 - a_{k-1}^2) \leq B' n$$

for some constant B'. Thus

$$a_n = O(\sqrt{n}).$$

Theorem 2 *Let T be a Kreiss bounded operator on a Hilbert space such that $\{\|T^n\|\}_{n=0}^{\infty}$ is increasing and suppose that there exist a unit vector x and a constant A such that $\|T^n\| \leq A\|T^n x\|$ for all n. Then $\|T^n\| = O(\sqrt{n})$.*

Proof Let $f(z) = \sum_{k=0}^{\infty} T^k z^k$. Since T is Kreiss bounded we have $\|f(z)\| \leq \frac{C}{1-|z|}$ for all $|z| < 1$. If $y \in H$ with $\|y\| = 1$ then

$$\sum_{k=0}^{\infty} r^{2n} \|T^n y\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta}) y\|^2 d\theta \leq \frac{C^2}{(1-r)^2}.$$

Since there exists a unit vector x and a constant A such that $\|T^n\| \le A\|T^n x\|$ for all n, we have

$$\sum_{k=0}^{\infty} r^{2n}\|T^n\|^2 \le A^2 \sum_{k=0}^{\infty} r^{2n}\|T^n x\|^2 \le \frac{A^2C^2}{(1-r)^2}.$$

Now by Lemma 1, we obtain $\|T^n\| = O(\sqrt{n})$.

The Shields conjecture $\|T^n\| = O(n^{1/2})$ is true for some subclasses of Kreiss bounded operators:

1. If T is a strong Kreiss bounded operator on a Banach space, then $\|T^n\| = O(n^{1/2})$, see [14].
2. If T is an absolutely Cesàro bounded operator on a Hilbert space, then $\|T^n\| = o(n^{1/2})$ and moreover for all ε there exist absolutely Cesàro bounded operators on $\ell^2(\mathbb{N})$ such that $\|T^n\| = O(n^{1/2-\varepsilon})$ [3].
3. See [5] for other classes of Kreiss bounded operators where the Shields conjecture is true.

Now we construct a uniformly Kreiss bounded operator which disproves the Shields conjecture.

Theorem 3 *Let $0 < \eta < 1/2$. Then there exists a constant $c > 0$ with the following property: for each $N \in \mathbb{N}$ there exists an operator T_N acting on a $2N$-dimensional Hilbert space H_N such that*

$$\|T_N^{2N-1}\| = N^{2\eta},$$

$$T_N \text{ is a weighted shift satisfying } \|T_N\| = 2^\eta,$$

$$\|M_n(T_N)\| \le c \text{ for every } n \in \mathbb{N}.$$

Proof Let H_N be the Hilbert space with an orthonormal basis $e_1, \dots, e_{2N}$. Let

$$w_j = j^\eta \qquad (j = 1, \dots, N)$$

and

$$w_j = \frac{N^{2\eta}}{(2N-j+1)^\eta} \qquad (j = N+1, \dots, 2N).$$

Consider the weighted shift T_N on H_N defined by

$$T_N e_j = \frac{w_{j+1}}{w_j} e_{j+1} \qquad (j = 1, \dots, 2N-1)$$

and $T_N e_{2N} = 0$.

Note that $w_1 = 1$, $w_N = w_{N+1} = N^\eta$ and $w_{2N} = N^{2\eta}$. Then $\|T^{2N-1}\| = \|T^{2N-1}e_1\| = w_{2N} = N^{2\eta}$.

Clearly $\|T_N\| = \max\left\{\frac{w_{j+1}}{w_j} : 1 \le j \le 2N-1\right\} = 2^\eta$.

Let $n \in \mathbb{N}$. We have

$$\|M_n(T_N)\| = \sup\left\{\left|\langle M_n(T_N)x, y\rangle\right| : x, y \in H_N, \|x\| = \|y\| = 1\right\}.$$

Let $x = \sum_{j=1}^{2N} \alpha_j e_j$, $y = \sum_{j=1}^{2N} \beta_j e_j$, $\|x\|^2 = \sum_{j=1}^{2N} |\alpha_j|^2 = 1$ and $\|y\|^2 = \sum_{j=1}^{2N} |\beta_j|^2 = 1$. Let $x = x_1 + x_2$ where $x_1 = \sum_{n=1}^{N} \alpha_j e_j$ and $x_2 = \sum_{j=N+1}^{2N} \alpha_j e_j$. Similarly, $y = y_1 + y_2$, where $y_1 = \sum_{j=1}^{N} \beta_j e_j$ and $y_2 = \sum_{j=N+1}^{2N} \beta_j e_j$.

We have

$$|\langle M_n(T_N)x, y\rangle| \leq A + B + C \tag{1}$$

where

$$A = |\langle M_n(T_N)x_1, y_1\rangle|,$$

$$B = |\langle M_n(T_N)x_2, y_2\rangle|$$

and

$$C = |\langle M_n(T_N)x_1, y_2\rangle|.$$

To estimate A, B and C we need two simple lemmas.

Claim 1. There exists a constant c_1 such that

$$\frac{1}{n+1} \sum_{j=1}^{\infty} \sum_{j \leq j' \leq j+n} \gamma_j \delta_{j'} \frac{j'^\eta}{j^\eta} \leq c_1$$

for all n, $\gamma_j, \delta_{j'} \geq 0 \quad (j, j' \in \mathbb{N})$ with $\sum_j \gamma_j^2 = \sum_{j'} \delta_{j'}^2 = 1$.

Proof. Let H be the Hilbert space with an orthonormal basis $f_j \quad (j \in \mathbb{N})$. Consider the weighted shift $V \in B(H)$ defined by $V f_j = \left(\frac{j+1}{j}\right)^\eta f_{j+1}$. Let $u = \sum_{j=1}^{\infty} \gamma_j f_j$ and $v = \sum_{j=1}^{\infty} \delta_{j'} f_{j'}$ with $\|u\| = \|v\| = 1$. By [3, Corollary 2.4], V is uniformly Kreiss bounded. So there exists a constant c_1 such that

$$c_1 \geq \|M_n(V)\| \geq \left|\langle M_n(V)u, v\rangle\right| = \frac{1}{n+1} \sum_{j=1}^{\infty} \sum_{j \leq j' \leq j+n} \gamma_j \delta_{j'} \frac{j'^\eta}{j^\eta}.$$

Claim 2. There exists a constant $c_2 > 0$ such that

$$\sum_{j=1}^{M} j^{-2\eta} \leq c_2 M^{1-2\eta}$$

for all M.

Proof. We have

$$\sum_{j=1}^{M} j^{-2\eta} \leq 1 + \int_1^M t^{-2\eta} dt = 1 + \left[\frac{t^{1-2\eta}}{1-2\eta}\right]_1^M \leq c_2 M^{1-2\eta}$$

for some constant $c_2 > 0$ independent of M.

Continuation of the proof of Theorem 3:

We have

$$A=\frac{1}{n+1}\sum_{\substack{j\le j'\le j+n\\ j'\le N}}\alpha_j\bar\beta_{j'}\frac{w_{j'}}{w_j}\le\frac{1}{n+1}\sum_{\substack{j\le j'\le j+n\\ j'\le N}}|\alpha_j|\cdot|\beta_{j'}|\frac{j'^\eta}{j^\eta}\le c_1 \tag{2}$$

by Claim 1.

Similarly,

$$B=\frac{1}{n+1}\sum_{\substack{N<j\le j'\le j+n,\\ j'\le 2N}}\alpha_j\bar\beta_{j'}\frac{w_{j'}}{w_j}\le\frac{1}{n+1}\sum_{\substack{N<j\le j'\le j+n,\\ j'\le 2N}}|\alpha_j|\cdot|\beta_{j'}|\Bigl(\frac{2N-j+1}{2N-j'+1}\Bigr)^\eta$$

$$\le\frac{1}{n+1}\sum_{1\le s'\le s\le\min\{s'+n,N\}}|\alpha_{2N-s+1}|\cdot|\beta_{2N-s'+1}|\Bigl(\frac{s}{s'}\Bigr)^\eta.$$

Setting $\gamma_s=|\alpha_{2N-s+1}|$, $\delta_{s'}=|\beta_{2N-s'+1}|$ in Claim 1, we get

$$B\le c_1\|x_2\|\cdot\|y_2\|\le c_1. \tag{3}$$

To estimate C, we distinguish two cases:

Let $n+1\ge\frac{N}{2}$. Then

$$\begin{aligned}C&=\frac{1}{n+1}\sum_{\substack{j\le j'\le j+n\\ j\le N<j'}}\alpha_j\bar\beta_{j'}\frac{w_{j'}}{w_j}\le\frac{2}{N}\Bigl(\sum_{j=1}^N\frac{|\alpha_j|}{w_j}\Bigr)\cdot\Bigl(\sum_{j'>N}|\beta_{j'}|w_{j'}\Bigr)\\
&\le\frac{2}{N}\Bigl(\sum_{j=1}^N|\alpha_j|^2\Bigr)^{1/2}\cdot\Bigl(\sum_{j=1}^N w_j^{-2}\Bigr)^{1/2}\cdot\Bigl(\sum_{j'>N}|\beta_{j'}|^2\Bigr)^{1/2}\cdot\Bigl(\sum_{j'>N}w_{j'}^2\Bigr)^{1/2}\\
&\le\frac{2}{N}\|x_1\|\cdot\Bigl(\sum_{j=1}^N j^{-2\eta}\Bigr)^{1/2}\cdot\|y_2\|\cdot N^{2\eta}\cdot\Bigl(\sum_{s=1}^N s^{-2\eta}\Bigr)^{1/2}\\
&\le\frac{2c_2}{N}N^{2\eta}N^{1-2\eta}=2c_2.\end{aligned} \tag{4}$$

Let $n+1<\frac{N}{2}$. Then

$$C=\frac{1}{n+1}\sum_{\substack{j\le j'\le j+n\\ j\le N<j'}}\alpha_j\bar\beta_{j'}\frac{w_{j'}}{w_j}\le\frac{2}{n+1}\Bigl(\sum_{j=N-n+1}^{N}|\alpha_j|\Bigr)\cdot\Bigl(\sum_{j'=N+1}^{N+n}|\beta_{j'}|\Bigr)$$

since

$$\begin{aligned}&\max\Bigl\{\frac{w_{j'}}{w_j}:j\le j'\le j+n, j\le N<j'\Bigr\}\\
=&\frac{w_{N+n}}{w_{N-n+1}}=\frac{N^{2\eta}}{(N-n+1)^\eta(N-n+1)^\eta}\le\frac{N^{2\eta}}{(N/2)^{2\eta}}=2^{2\eta}\le 2.\end{aligned}$$

So

$$C \leq \frac{2}{n+1}\|x_1\| \cdot \sqrt{n} \cdot \|y_2\| \cdot \sqrt{n} \leq 2. \tag{5}$$

Hence by (1), (2), (3),(4) and (5) we have $\|M_n(T_N)\| \leq c$ for all n, where $c = 2c_1 + \max\{2, 2c_2\}$.

Theorem 4 *Let $\varepsilon > 0$. Then there exists a uniformly Kreiss bounded operator T on a Hilbert space such that $\|T^k\| \geq \frac{1}{3}(k+1)^{1-\varepsilon}$ for all $k \in \mathbb{N}$.*

Proof Choose $\eta \in \left(\frac{1-\varepsilon}{2}, \frac{1}{2}\right)$. Let

$$T = \bigoplus_{N=1}^{\infty} T_N,$$

where $T_N \quad (N \in \mathbb{N})$ are the operators constructed in Theorem 3.

Clearly T is a bounded linear operator, $\|T\| = 2^\eta < \sqrt{2}$.

For each $n \in \mathbb{N}$ we have

$$\|M_n(T)\| = \sup_N \|M_n(T_N)\| \leq c.$$

Since T is a weighted shift, λT is unitarily equivalent to T for each $\lambda \in \mathbb{C}$, $|\lambda| = 1$ [17, Corollary 2]. Hence T is uniformly Kreiss bounded.

For each $N \in \mathbb{N}$ we have

$$\|T^{2N-1}\| \geq \|T_N^{2N-1}\| = N^{2\eta} > N^{1-\varepsilon} \geq \frac{1}{3}(2N)^{1-\varepsilon}$$

and

$$\|T^{2N}\| \geq \|T_{N+1}^{2N}\| \geq \frac{\|T_{N+1}^{2N+1}\|}{\|T_{N+1}\|} \geq \frac{(N+1)^{2\eta}}{2^\eta} > \frac{(N+1)^{1-\varepsilon}}{\sqrt{2}} > \frac{1}{3}(2N+1)^{1-\varepsilon}.$$

Hence $\|T^k\| \geq \frac{1}{3}(k+1)^{1-\varepsilon}$ for all $k \in \mathbb{N}$.

On the other hand, we prove a small improvement of the general estimate of norms of powers of Kreiss bounded operators on Hilbert spaces. The proof follows the argument of [3, Theorem 2.3] for uniformly Kreiss bounded operators with some necessary modifications.

Theorem 5 *Let T be a Kreiss bounded operator in a Hilbert space. Then $\|T^n\| = O\Big(\frac{n}{\sqrt{\log n}}\Big)$.*

Proof In [20] it was proved that T is Kreiss bounded if and only if there is a constant $C' > 0$ such that

$$\|M_n^{(2)}(\lambda T)\| \leq C' \quad \text{for } n = 0, 1, 2, \cdots \text{ and } |\lambda| = 1.$$

Thus there exists a constant $C > 0$ such that

$$\Big\| \sum_{j=0}^{N-1} (N-j)(\lambda T)^j \Big\| \le CN^2$$

for all $\lambda, |\lambda| = 1$ and $N \ge 1$.

We need several claims.

Claim Let $x \in H$, $\|x\| = 1$. Then

$$\sum_{j=0}^{N-1} \|T^j x\|^2 \le 16C^2 N^2$$

for all $N \ge 1$.

Proof Consider the normalized Lebesgue measure on the unit circle. We have

$$N^2 \sum_{j=0}^{N-1} \|T^j x\|^2 \le \sum_{j=0}^{2N-1} (2N-j)^2 \|T^j x\|^2$$

$$= \int_{|\lambda|=1} \Big\| \sum_{j=0}^{2N-1} (2N-j)(\lambda T)^j x \Big\|^2 d\lambda \le 16C^2 N^4.$$

So $\sum_{j=0}^{N-1} \|T^j x\|^2 \le 16C^2 N^2$.

Claim Let $0 < M < N$ and $x \in H$, $\|x\| = 1$, $T^N x \ne 0$. Then

$$\sum_{j=0}^{M-1} \frac{\|T^N x\|^2}{\|T^{N-j} x\|^2} \le 16C^2 M^2.$$

Proof Set $y = T^N x$. Since T^* is also Kreiss bounded with the same constant, we have

$$16C^2 M^2 \|y\|^2 \ge \sum_{j=0}^{M-1} \|T^{*j} y\|^2 \ge \sum_{j=0}^{M-1} \Big| \Big\langle T^{*j} y, \frac{T^{N-j} x}{\|T^{N-j} x\|} \Big\rangle \Big|^2$$

$$= \sum_{j=0}^{M-1} \Big| \Big\langle y, \frac{T^N x}{\|T^{N-j} x\|} \Big\rangle \Big|^2 = \|y\|^2 \sum_{j=0}^{M-1} \frac{\|T^N x\|^2}{\|T^{N-j} x\|^2}.$$

Hence

$$\sum_{j=0}^{M-1} \frac{\|T^N x\|^2}{\|T^{N-j} x\|^2} \le 16C^2 M^2.$$

Claim Let $N \in \mathbb{N}$, $x \in H$, $\|x\| = 1$ and $T^N x \ne 0$. Then

$$\sum_{j=0}^{N-1} \frac{1}{\|T^j x\|} \ge \frac{\sqrt{N}}{4C}.$$

Proof We have

$$\sum_{j=1}^{N-1}\|T^jx\| \le \Big(\sum_{j=0}^{N-1}\|T^jx\|^2\Big)^{1/2}\cdot\sqrt{N} \le 4CN^{3/2}.$$

Thus

$$N=\sum_{j=0}^{N-1}\frac{\sqrt{\|T^jx\|}}{\sqrt{\|T^jx\|}} \le \Big(\sum_{j=0}^{N-1}\|T^jx\|\Big)^{1/2}\cdot\Big(\sum_{j=0}^{N-1}\frac{1}{\|T^jx\|}\Big)^{1/2}$$

and

$$\sum_{j=0}^{N-1}\frac{1}{\|T^jx\|} \ge \frac{N^2}{\sum_{j=0}^{N-1}\|T^jx\|} \ge \frac{N^2}{4CN^{3/2}}=\frac{\sqrt{N}}{4C}.$$

Claim Let $0<M_1<M_2<N$, $\|x\|=1$ and $T^Nx\neq 0$. Then

$$\sum_{j=M_1}^{M_2-1}\frac{\|T^{N-j}x\|^2}{\|T^Nx\|^2} \ge \frac{(M_2-M_1)^2}{16C^2M_2^2}.$$

Proof Let $a_j=\frac{\|T^{N-j}x\|^2}{\|T^Nx\|^2}$. By Claim 2,

$$\sum_{j=M_1}^{M_2-1}\frac{1}{a_j} \le \sum_{j=0}^{M_2-1}\frac{1}{a_j} \le 16C^2M_2^2.$$

We have

$$M_2-M_1=\sum_{j=M_1}^{M_2-1}\frac{\sqrt{a_j}}{\sqrt{a_j}} \le \Big(\sum_{j=0}^{N-1}a_j\Big)^{1/2}\cdot\Big(\sum_{j=0}^{N-1}\frac{1}{a_j}\Big)^{1/2}$$

and

$$\sum_{j=M_1}^{M_2-1}a_j \ge (M_2-M_1)^2\cdot\Big(\sum_{j=0}^{N-1}\frac{1}{a_j}\Big)^{-1} \ge \frac{(M_2-M_1)^2}{16C^2M_2^2}.$$

Continuation of the proof of Theorem 5.

Let $K\in\mathbb{N}$ and $2^{K+1}<N\le 2^{K+2}$. Let $x\in H$, $\|x\|=1$ and $T^Nx\neq 0$.

For $|\lambda|=1$ let $y_\lambda=\sum_{j=0}^{2N-1}\frac{(\lambda T)^jx}{\|T^jx\|}$. Then

$$\int_{|\lambda|=1}\|y_\lambda\|^2\mathrm{d}\lambda=2N$$

and

$$\int_{|\lambda|=1}\Big\|\sum_{j=0}^{2N-1}(2N-j)(\lambda T)^jy_\lambda\Big\|^2\mathrm{d}\lambda \le 16C^2N^4\int_{|\lambda|=1}\|y_\lambda\|^2\mathrm{d}\lambda \le 32C^2N^5.$$

On the other hand,

$$\int_{|\lambda|=1}\Big\|\sum_{j=0}^{2N-1}(2N-j)(\lambda T)^j y_\lambda\Big\|^2 \mathrm{d}\lambda=\int_{|\lambda|=1}\Big\|\sum_{j=0}^{2N-1}(2N-j)(\lambda T)^j\sum_{r=0}^{2N-1}\frac{(\lambda T)^r x}{\|T^r x\|}\Big\|^2\mathrm{d}\lambda$$

$$=\int_{|\lambda|=1}\Big\|\sum_{j=0}^{4N-2}(\lambda T)^j x\sum_{r=0}^{\min\{j,2N-1\}}\frac{2N-j+r}{\|T^r x\|}\Big\|^2\mathrm{d}\lambda$$

$$=\sum_{j=0}^{4N-2}\|T^j x\|^2\Big(\sum_{r=0}^{\min\{j,2N-1\}}\frac{2N-j+r}{\|T^r x\|}\Big)^2\geq\sum_{j=N-2^K}^{N-1}\|T^j x\|^2\Big(\sum_{r=0}^{j}\frac{N}{\|T^r x\|}\Big)^2$$

$$\geq N^2\sum_{j=N-2^K}^{N-1}\|T^j x\|^2\Big(\frac{\sqrt{N-2^K}}{4C}\Big)^2\geq\frac{N^3}{32C^2}\sum_{j=N-2^K}^{N-1}\|T^j x\|^2$$

$$\geq\frac{N^3}{32C^2}\|T^N x\|^2\sum_{k=0}^{K-1}\sum_{j=N-2^{k+1}}^{N-2^k-1}\frac{\|T^j x\|^2}{\|T^N x\|^2}=\frac{N^3}{32C^2}\|T^N x\|^2\sum_{k=0}^{K-1}\sum_{j=2^k+1}^{2^{k+1}}\frac{\|T^{N-j}x\|^2}{\|T^N x\|^2}$$

$$\geq\frac{N^3}{32C^2}\|T^N x\|^2\sum_{k=0}^{K-1}\frac{2^{2k}}{16C^2\cdot 2^{2k+2}}=\frac{N^3}{2^{11}C^4}\|T^N x\|^2 K.$$

Thus we have

$$\|T^N x\|^2\leq\frac{2^{16}C^6N^2}{K}\leq\frac{2^{16}C^6N^2}{\log_2 N-2}.$$

Hence $\|T^N\|=O(\frac{N}{\sqrt{\log N}})$.

In the next diagram we show graphically the implications among various definitions related with Kreiss boundedness on Hilbert spaces and corresponding known estimates for the growth of $\|T^n\|$.

$$\begin{tikzcd}
 & {} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Strong Kreiss bounded } (\|T^n\|=O(n^{1/2})) \arrow[dr, Rightarrow] & & \text{Abs. Cesàro bounded } (\|T^n\|=o(n^{1/2})) \arrow[dl, Rightarrow] \\
 & \text{Uniformly Kreiss bounded} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Kreiss bounded} \arrow[d, Rightarrow] & & \text{Mean ergodic} \arrow[d, Rightarrow] \\
\|T^n\|=o(n) & & \|T^n x\|=o(n), \forall x
\end{tikzcd}$$

Fig. 2 Implications among different definitions related with Kreiss bounded on Hilbert spaces.

Example 1 Derriennic [4] gave an example of a mean ergodic operator T on a real Hilbert space for which $n^{-1}\|T^n\|$ does not converge to zero, and such that T^* is not mean ergodic, only weakly mean ergodic (i.e., the Cesàro means converge weakly). In [23, Example 3.1] it was shown that the operator

$$T := \begin{pmatrix} B & B-I \\ 0 & B \end{pmatrix}$$

acting on the Hilbert spaces $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, where B is the backward shift in $l^2(\mathbb{N})$, is mean ergodic and $n^{-1}\|T^n\| \geq 2$. As a consequence of Theorem 5, T is an example of a mean ergodic operator acting on a Hilbert spaces, which is not Kreiss bounded.

By [1, Remark 3.1], in Banach spaces there exists a Kreiss bounded operator such that

$$\lim_{n\to\infty} \|M_{n+1}(T) - M_n(T)\| \neq 0.$$

However, in Hilbert spaces this is not possible.

Theorem 6 *If T is a Kreiss bounded operator on a Hilbert space then*

$$\lim_{n\to\infty} \|M_{n+1}(T) - M_n(T)\| = 0.$$

Proof We have

$$\frac{n+2}{n+1} M_{n+1}(T) - M_n(T) = \frac{1}{n+1} T^{n+1}.$$

So

$$M_{n+1}(T) - M_n(T) = \frac{1}{n+1} T^{n+1} - \frac{1}{n+1} M_{n+1}(T).$$

Now if T is a Kreiss bounded operator on a Hilbert space, then we have $\|T^n\| = o(n)$ by Theorem 5 and $\|M_n(T)\| = O(\log(n+2))$ (see, [20, Theorem 6.1, 6.2]). Thus

$$\lim_{n\to\infty} \|M_{n+1}(T) - M_n(T)\| = 0.$$

3 On residual spectrum of Kreiss bounded operators

The following characterization of ergodic operators was proved in [11].

Theorem 7 *[11, Theorem 2.1.3, page 73] An operator T in a Banach space X is mean ergodic if and only if it is Cesàro bounded, $\|T^n x\| = o(n)$ for all $x \in X$ and*

$$X = N(I-T) \oplus \overline{R(I-T)}$$

Recall that the residual spectrum $\sigma_R(T)$ of an operator $T \in B(X)$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective and $\overline{R(T-\lambda)} \neq X$.

Corollary 1 *If T is mean ergodic in a Banach space, then* $1 \notin \sigma_R(T)$

Corollary 2 *[3, Corollary 2.5], [3, Corollary 2.7] Let T be a bounded operator on a Banach space X. If*

1. *either T is a uniformly Kreiss bounded operator on a Hilbert space,*
2. *or T is an absolutely Cesàro bounded operator on a reflexive Banach space,*

then λT is mean ergodic for all $\lambda, |\lambda| = 1$. Consequently, $\sigma_R(T) \subset \mathbb{D}$.

The next result generalizes the above observation as well as the results of [13], [6] for power bounded operators.

Theorem 8 *If T is a Kreiss bounded operator on a reflexive Banach space then $\sigma_R(T) \subset D$.*

Proof If T is a Kreiss bounded operator on a Banach space X, then $\|M_n^{(2)}(\lambda T)\|$ are uniformly bounded and $\|M_n(\lambda T)\| = O(log(n+2))$ for all $\lambda \in \mathbb{T}$ (see, [20, Theorem 6.1, 6.2]).

Now, since X is a reflexive Banach space, $M_n^{(2)}(\lambda T)$ converge strogly in $X = N(I - \lambda T) \oplus \overline{R(I - \lambda T)}$ (see, [21, Theorem 2.1]).

So for all $\lambda \in \mathbb{T}$, we have $X = N(\overline{\lambda} I - T) \oplus \overline{R(\overline{\lambda} I - T)}$. Hence if $\overline{\lambda} \notin \sigma_p(T)$, then $X = \overline{R(\overline{\lambda} I - T)}$ and $\overline{\lambda} \notin \sigma_R(T)$.

Thus $\sigma_R(T) \subset D$.

The condition on the residual spectrum is optimal. The forward shift in $l^2(\mathbb{N})$ is a power bounded operator with residual spectrum equal to the open unit disc.

Example 2 There exists a power bounded operator T on $c_0(\mathbb{N})$ such that $1 \in \sigma_R(T)$.

Proof The operator $T : c_0(\mathbb{N}) \to c_0(\mathbb{N})$ defined by

$$T(a_1, a_2, a_3, \cdots) = (a_1, a_1, a_2, a_3, \cdots)$$

is power-bounded and $1 \in \sigma_R(T)$.

Example 3 There exists a Kreiss bounded operator T on a non-reflexive Banach space, which is not power bounded and $1 \in \sigma_R(T)$.

Proof Let X denote the Banach space of analytic functions f in the open unit disc and continuous on the boundary, such that f' belongs to the Hardy space H^1, equipped with the norm

$$\|f\| := \|f\|_\infty + \|f'\|_1$$

If M_z denotes the multiplication operator and $T = \frac{1}{2}(I + M_z)$, then T is a Kreiss bounded operator, see [16, Example 4].

Moreover, $N(I-T) = \{0\}$ and $R(I-T)$ is not dense because every function in this closure necessarily verifies that $f(1) = 0$. Thus $1 \in \sigma_R(T)$.

Proposition 1 *There exists a Cesàro bounded operator T on a Hilbert space which is mean ergodic, $N(T+I)\neq\{0\}$ and $N(T^*+I)=\{0\}$.*

Proof Let H be the Hilbert space with an orthonormal basis $e_j \quad (j=0,1,\dots)$. Let $\varepsilon_j=2^{-j}, c_j=1-\varepsilon_j^2 \quad (j\geq 1)$. Let

$$T=-\begin{pmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \cdots \\ 0 & c_1 & 0 & 0 & \cdots \\ 0 & 0 & c_2 & 0 & \cdots \\ 0 & 0 & 0 & c_3 & \cdots \\ & & \cdots & & \end{pmatrix}$$

Clearly $Te_0=-e_0$, so $N(T+I)\neq\{0\}$. We have $(T+I)e_j=(1-c_j)e_j-\varepsilon_je_0=\varepsilon_j^2e_j-\varepsilon_je_0$. So $(T+I)(-\varepsilon_j^{-1}e_j)=e_0-\varepsilon_je_j\to e_0$. Thus $e_0\in\overline{R(T+I)}$ and it is easy to see that $\overline{R(T+I)}=H$. Hence $N(I+T^*)=\{0\}$.

For $n\geq 1$ we have

$$((T)^n)_{j,j}=(-c_j)^n \qquad (j\geq 0),$$

$$((T)^n)_{0,j}=(-1)^n\varepsilon_j(1+c_j+c_j^2+c_j^{n-1})=(-1)^n\varepsilon_j\frac{1-c_j^n}{1-c_j} \qquad (j\geq 1)$$

and $((T)^n)_{i,j}=0$ otherwise.

So

$$T^n=(-1)^n\begin{pmatrix} 1 & \varepsilon_1(1+c_1+\cdots+c_1^{n-1}) & \varepsilon_2(1+\cdots+c_2^{n-1}) & \cdots \\ 0 & c_1^n & 0 & \cdots \\ 0 & 0 & c_2^n & \cdots \\ & \cdots & & \cdots \end{pmatrix}$$

We show that $n^{-1}\|T^n\|\to 0$. Let $\delta>0$. Find j_0 such that $\varepsilon_{j_0}<\delta$ and n_0 satisfying $c_{j_0}^{n_0\delta}<\delta$ and $2n_0^{-1}<\delta$.

Let $n\geq n_0$. Clearly $|(T^n)_{j,j}|\leq 1$ for all $j\geq 0$.

If $j\geq j_0$ then

$$|(T^n)_{0,j}|=\varepsilon_j(1+c_j+\cdots+c_j^{n-1})\leq\varepsilon_jn=2^{-j}n<\frac{\delta n}{2^{j-j_0}}.$$

If $1\leq j<j_0$ then

$$|(T^n)_{0,j}|=\varepsilon_j(1+c_j+\cdots+c_j^{n-1})=\varepsilon_j\sum_{0\leq i<n\delta}c_j^i+\varepsilon_j\sum_{n\delta\leq i\leq n-1}c_j^i\leq\varepsilon_j(n\delta+1)+\varepsilon_jn\delta.$$

Hence

$$\begin{aligned}\|T^n\|&\leq\sup_j|(T^n)_{j,j}|+\sum_{j=1}^{\infty}|(T^n)_{0,j}|\\ &\leq 1+\delta n\sum_{j=j_0}^{\infty}2^{-j+j_0}+\sum_{j=1}^{j_0-1}\big(\varepsilon_j(n\delta+1)+\varepsilon_jn\delta\big)\\ &\leq 1+2\delta n+\sum_{j=1}^{j_0-1}2^{-j}(2n\delta+1)\leq 5\delta n.\end{aligned}$$

Hence $n^{-1}\|T^n\| \to 0$.

We show that T is Cesàro bounded.

Let $M_{2k}(T) = (2k+1)^{-1}(I + T + T^2 + \cdots + T^{2k})$. It is easy to see that

$$|(M_{2k}(T))_{j,j}| \le 1.$$

For each $k \ge 1$ we have

$$\left|(M_{2k}(T))_{0,j}\right| = \frac{\varepsilon_j}{(2k+1)(1-c_j)}\left|-(1-c_j) + (1-c_j^2) - \cdots + (1-c_j^{2k})\right|$$

$$= \frac{\varepsilon_j}{(2k+1)(1-c_j)}\left|c_j - c_j^2 + \cdots - c_j^{2k}\right| = \frac{\varepsilon_j}{2k+1}\left(c_j + c_j^3 + \cdots + c_j^{2k-1}\right) \le \frac{\varepsilon_j}{2}.$$

So

$$\|M_{2k}(T)\| \le 1 + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{2} = \frac{3}{2}.$$

Since

$$M_{2k+1}(T) = \frac{2k+1}{2k+2} M_{2k}(T) + \frac{T^{2k+1}}{2k+2},$$

T is Cesàro bounded. Moreover, as $n^{-1}\|T^n\| \to 0$, T is mean ergodic.

Corollary 3 *There exists a mean ergodic operator T on a Hilbert space such that $\sigma_R(T) \cap \partial\mathbb{D} \ne \emptyset$.*

Example 4 By Theorem 8, the operator of Corollary 3 is another example of a mean ergodic operator on a Hilbert space, which is not Kreiss bounded.

4 Questions

As consequence of results of this paper we gave two examples in Hilbert spaces of mean ergodic operators which are not Kreiss bounded. By [3, Corollary 2.5], all uniformly Kreiss bounded operators on Hilbert spaces are mean ergodic. However, the following problem is open:

Question 1 Does there exist a Kreiss bounded operator on a Hilbert space which is not mean ergodic?

Observe that by Theorem 5, the above problem is equivalent to the question whether there exists a Kreiss bounded Hilbert space operator which is not Cesàro bounded.

If the answer of the above question is positive then such an operator is not uniformly Kreiss bounded. If the answer is negative then it is natural to ask

Question 2 Does there exist a Kreiss bounded Hilbert space operator which is not uniformly Kreiss bounded?

Another open question is whether it is possible to generalize the Jacobs-de Leeuw-Glicksberg theorem for uniformly Kreiss bounded operators.

Question 3 Let X be a reflexive Banach space and $T \in B(X)$ a uniformly Kreiss bounded operator such that $n^{-1}T^n x \to 0$ for all $x \in X$. Is it true that X can be decomposed as

$$X = \bigvee_{|\lambda|=1} N(T-\lambda) \oplus \bigcap_{|\lambda|=1} \overline{R(T-\lambda)} \quad ?$$

Clearly for Hilbert space operators the condition $n^{-1}T^n x \to 0$ is satisfied automatically.

It is an open question due to Aleman and Suciu [2], p.279 whether each uniformly Kreiss bounded operator T on a Banach space X satisfies the condition $\|n^{-1}T^n\| \to 0$.

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