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σ-lacunary actions of Polish groups

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σ-LACUNARY ACTIONS OF POLISH GROUPS

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ABSTRACT. We show that every essentially countable orbit equivalence relation induced by a continuous action of a Polish group on a Polish space is σ-lacunary. In combination with [4] we obtain a straightforward proof of the result from [3] that every essentially countable equivalence relation that is induced by an action of abelian non-archimedean Polish group is Borel reducible to $\mathbb{E}_0$, i.e., it is essentially hyperfinite.

We say that an equivalence relation E on a Polish space X is *Borel reducible* to an equivalence relation F on a Polish space Y, and write $E \leq_B F$, if there is a Borel map $\psi : X \to Y$ such that

$$(x, y) \in E \iff (\psi(x), \psi(y)) \in F$$

for every $x, y \in X$. An equivalence relation F on Y is *countable* if $|[y]_F| \leq \aleph_0$ for every $y \in Y$. We follow [10] and say that an equivalence relation E on a Polish space X

(A) is *essentially countable* if there is a countable Borel equivalence relation F on some Polish space Y such that $E \sim_B F$, i.e., $E \leq_B F$ and $F \leq_B E$,

(B) admits a Borel *countable complete section* if there is a Borel set $B \subseteq X$ such that $[B]_E = X$ and $|B \cap [x]_E| \leq \aleph_0$ for every $x \in X$.

If we assume that E is a Borel equivalence relation, then (B) $\Rightarrow$ (A) by the Lusin–Novikov Theorem, see [7, Theorem 18.10].

Let $G \curvearrowright X$ be a continuous action of a Polish group G on a Polish space X. We denote as E^X_G the orbit equivalence relation defined as $(x, y) \in E^X_G \iff (\exists g \in G)\ g \cdot x = y$. If we have such an action, then we say that X is a Polish G-space. It follows from [10, Theorem 3.6] that if E^X_G satisfies (A), then E^X_G satisfies (B). It is natural to ask if we can find a Borel countable complete section with additional properties. Following [12] we say that E^X_G is

(C) *σ-lacunary* if there are sequences of Borel sets $\{B_n\}_{n<\omega}$ and $\{V_n\}_{n<\omega}$ such that $\bigcup_{n<\omega} B_n$ is a countable complete section of E^X_G, $V_n \subseteq G$ is an open neighbourhood of 1_G and B_n is V_n-lacunary for every $n \in \mathbb{N}$, i.e., if $g \cdot x = y$ for some $g \in V_n$ and $x, y \in B_n$, then $x = y$.

It follows from [9] that in the case when G is a locally compact Polish group, then (A) and (C) are equivalent. Main result of this paper is the following statement.

Theorem 0.1. *Let G be a Polish group, X be a Polish G-space and suppose that E^X_G is essentially countable. Then E^X_G is σ-lacunary.*

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There are some other similar concepts in the literature. Following [10], we say that

(D) E_G^X is *reducible to countable* if there is a countable Borel equivalence relation F on some Polish space Y such that $E_G^X \leq_B F$,

(E) E_G^X admits *countable invariants* if there is a Polish space Y and a Borel map $\varphi : X \to Y$ such that $|\varphi([x]_{E_G^X})| \leq \aleph_0$ for every $x \in X$ and $\varphi([x]_{E_G^X}) \cap \varphi([y]_{E_G^X}) = \emptyset$ whenever $(x, y) \notin E_G^X$ (see [6, Section 7.6.]).

Next we summarize what is known about these concepts. It is easy to see that (A) $\Rightarrow$ (D) $\Rightarrow$ (E) and (C) $\Rightarrow$ (B). Moreover, (A) and (D) implies that E_G^X is Borel. If we suppose that E_G^X is a Borel equivalence relation, then (E) $\Rightarrow$ (D) by [6, Lemma 7.6.1.]. Altogether, combination of Theorem 0.1 and the discussion above yields that, if E_G^X is Borel, then all the concepts are equivalent.

In fact, we show that if E_G^X satisfies (E), then E_G^X is Borel and satisfies (C). As a corollary we get

$$\text{(A)} \Leftrightarrow \text{(D)} \Leftrightarrow \text{(E)} \Rightarrow \text{(C)} \Rightarrow \text{(B)}$$

without assuming that E_G^X is Borel.

1. Application

Let G be a Polish group and X be a Polish G-space. Suppose that E_G^X satisfies (A). It is natural to ask if there is a connection between properties of G and the position of E_G^X in the Borel reducibility among countable Borel equivalence relations. For example, we say that E_G^X is *essentially hyperfinite* if $E_G^X \sim_B F$ where F is a countable Borel equivalence relation induced by a Borel action of $\mathbb{Z}$. Variations of the following question appeared in [3, Conjecture 8.4] or [6, Question 5.7.5].

Question 1.1. *Let E_G^X be an essentially countable orbit equivalence relation induced by a continuous action of an* **abelian** *Polish group G on a Polish space X. Is it true that E_G^X is essentially hyperfinite?*

The answer is affirmative in the case when the abelian Polish group G is discrete, see [4], non-archimedean, see [3], and locally compact, see [2].

Next result is derived directly from Theorem 0.1, we note that it is a variation on [5, Theorem 7.3].

Theorem 1.2. *Let G be a non-archimedean Polish group that admits two-sided invariant metric and X be a Polish G-space. Suppose that E_G^X is essentially countable. Then there is a sequence of open normal subgroups $\{N_n\}_{n\in\mathbb{N}}$ and a continuous actions $H_n = G/N_n \curvearrowright X_n$ where X_n is a Polish space such that*

$$E_G^X \leq_B \bigoplus_{n\in\mathbb{N}} E_{H_n}^{X_n}.$$

First we need a variation of an unpublished result of Conley and Dufloux, see [10, Theorem 3.11]. They considered locally compact groups but not necessarily with two-sided invariant metric.

Lemma 1.3. *Let G be a Polish group that admits a two-sided invariant metric and X be a Polish G-space such that E_G^X is a Borel equivalence relation. Let $V \subseteq G$ be an open symmetric conjugacy-invariant neighbourhood of 1_G and $B \subseteq X$ be a Borel V-lacunary complete section of E_G^X, i.e., $g \cdot x = y$ for $g \in V$ and $x, y \in B$ implies that $x = y$. Then there is $C \supseteq B$ a Borel V-lacunary complete section of E_G^X such that $V^2 \cdot C = X$.*

Proof. Fix some countable dense subset $\{g_i\}_{i\in\mathbb{N}} \subseteq G$. Let $C_0 = B$ and define inductively $C_{i+1} = C_i \cup (g_i \cdot C_i \setminus V \cdot C_i)$. We claim that $C = \bigcup_{i\in\mathbb{N}} C_i$ works as required.

First note that C_i is a countable section for every $i \in \mathbb{N}$. We show by induction that C_i is Borel for every $i \in \mathbb{N}$. If $i = 0$, then it follows from the assumption on B. Suppose that C_i is Borel.

Claim. *The set $T_i = \{(g \cdot x, x) \in X \times X : x \in C_i \ \&\ g \in V\}$ is Borel .*

Proof. The assumption that E_G^X is Borel together with [1, Theorem 7.1.2] gives that the assignment $y \mapsto \mathrm{stab}(y) = \{g \in G : g \cdot y = y\}$ is Borel. Note that $\mathrm{stab}(y)$ is non-empty closed subset of G and by [7, Theorem 12.13] there is a Borel map $y \mapsto (g_{j,y})_{j\in\mathbb{N}}$ such that $(g_{j,y})_{j\in\mathbb{N}}$ is dense subset of $\mathrm{stab}(y)$ for every $y \in X$. We claim that the relation

$$R_V = \{(y, x) \in X \times X : \exists g \in V \ g \cdot x = y\} = \{(g \cdot x, x) \in X \times X : x \in X \ \&\ g \in V\}$$

is Borel. It is clearly analytic by the definition. We show that the complement is analytic as well, we have

$$(y, x) \notin R_V \ \Leftrightarrow\ (y, x) \notin E_G^X \ \vee (\exists h \in G \ h \cdot x = y \ \wedge\ \forall j \in \mathbb{N} \ g_{j,y} \cdot h \notin V).$$

Finally, note that $T_i = X \times C_i \cap R_V$. □

It is easy to see that T_i has countable vertical sections. This is because $(T_i)_y = \{x \in X : (y, x) \in T_i\} \subseteq C_i \cap [y]_{E_G^X}$ for every $y \in X$ and we know that C_i is a countable section. By Lusin–Novikov Theorem [7, Theorem 18.10] we have that $V \cdot C_i$, which is equal to the projection of T_i to the first coordinate, is a Borel set and so is the set C_{i+1}. This gives immediately that C is Borel.

Suppose that $x \in X$. Then there is $y \in C_0$, $h \in G$ and $i \in \mathbb{N}$ such that $h \cdot y = x$ and $h^{-1} \cdot g_i \in V$. Let $z = g_i \cdot y$. Then either $z \in C_{i+1}$ and therefore $x \in V \cdot z$, or there is $z_0 \in C_i$ such that $z \in V \cdot z_0$ and then we have $x \in V^2 \cdot z_0$. This shows that $X = V^2 \cdot C$.

It remains to show that C is V-lacunary. We show by induction that C_i is V-lacunary for every $i \in \mathbb{N}$. It clearly holds for $i = 0$. Let $x, y \in C_{i+1}$ and suppose that $y \in V \cdot x$. If $x, y \notin C_i$, then there is $x_0, y_0 \in C_i$ such that $g_i \cdot x_0 = x$ and $g_i \cdot y_0 = y$. Then we have $y_0 \in g_i^{-1} \cdot V \cdot g_i \cdot x_0 = V \cdot x_0$ because V is conjugacy invariant and therefore $x = y$ by the inductive assumption. If $x \in C_i$, then $y \in C_i$ by the definition of C_{i+1}. Again, the inductive assumption gives $x = y$ and that finishes the proof. □

Proof of Theorem 1.2. Using Theorem 0.1 we get sequences $\{B_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ where B_n is V_n-lacunary Borel section and V_n is an open neighbourhood of identity. By the assumption on G we find $N_n \subseteq V_n$ an open normal subgroup and by Lemma 1.3 $X_n \supseteq B_n$ an N_n-lacunary Borel section such that $N_n^2 \cdot X_n = N_n \cdot X_n = [B_n]_{E_G^X}$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $(gN_n) \in H_n = G/N_n$ and $x \in X_n$ define $(gN_n) \star x$ to be the unique element of X_n in $g \cdot N_n \cdot x$. It follows from the maximality and N_n-lacunarity of X_n that this is a well-defined map and it can be easily verified that it is an action $H_n \curvearrowright X_n$. Moreover, it follows from Lusin–Novikov Theorem [7, Theorem 18.10] that the action is Borel for every $n \in \mathbb{N}$. Another use of Lusin–Novikov Theorem [7, Theorem 18.10] gives the desired reduction. □

Corollary 1.4. [3] *Let G be an abelian non-archimedean Polish group and X be a Polish G-space such that E^X_G is essentially countable. Then E^X_G is essentially hyperfinite.*

Proof. This is a combination of Theorem 1.2 and [4, Corollary 8.2]. □

2. Proof of Theorem 0.1

Let X be a Polish space and F an equivalence relation on X. We denote as $[x]_F$ the F-equivalence class of $x \in X$. Let G be a Polish group that acts continuously on a Polish space X and E^X_G be the corresponding orbit equivalence relation on X. We denote the σ-ideal of meager subsets of G as $\mathcal{M}_G$. For $A \subseteq X$ we define $G(x, A) = \{g \in G : g \cdot x \in A\}$ and $E^A_G = E^X_G \upharpoonright A \times A$.

We say that $C \subseteq X$ is a *G-lg (G-locally globally) comeager* set if $G \setminus G(x, C) \in \mathcal{M}_G$ for every $x \in X$. Using the category quantifier $\forall^*$, for comeager many, this can be equivalently stated as

$$\forall x \in X \ \forall^* g \in G \ (g \cdot x \in C).$$

Note that the collection of G-lg comeager sets is closed under supersets and countable intersections. If G is countable, then the only G-lg comeager set is X. Even though we do not need it here, we remark that it follows from [7, Theorem 8.41] that if $C \subseteq X$ is a Borel G-lg comeager set, then C is comeager in X. This might serve as an explanation for the word "globally" in the definition. More generally, a Borel set $C \subseteq X$ is G-lg comeager if and only if it is comeager in every finer Polish topology on X such that the action of G is continuous.

Next we collect the technical statements that we need in the proof.

Proposition 2.1. *Let $C \subseteq X$ be a Borel G-lg comeager set. Then $E^C_G \sim_B E^X_G$.*

Proposition 2.2. *Let $F \subseteq E^X_G$ be a Borel equivalence relation on X such that each E^X_G-class contains at most countably many F-classes. Then there is a Borel G-lg comeager set $C \subseteq X$ such that $G(x, C \cap [x]_F)$ is relatively open in $G(x, C)$ for every $x \in C$, i.e., for every $x \in C$ there is $V \subseteq G$ open neighbourhood of 1_G such that $V \cdot x \cap C \subseteq [x]_F \cap C$.*

Proposition 2.3. *Let $F \subseteq E^X_G$ be a Borel equivalence relation on X such that each E^X_G-class contains at most countably many F-classes. Then E^X_G is Borel.*

Proposition 2.4. [7, Theorem 18.6][8, Theorem 18.6*][11, Proof of Lemma 3.7] *Let Y, X be standard Borel spaces and $P \subseteq Y \times X$ be Borel with $A = \operatorname{proj}_Y(P)$. Let $y \in A \mapsto I_y$ be a map assigning to each $y \in A$ a σ-ideal of subsets of P_y such that:*

(i) *For each Borel $R \subseteq P$, there is a $\mathbf{\Sigma}^1_1$ set $S \subseteq Y$ and a $\mathbf{\Pi}^1_1$ set $T \subseteq Y$ such that*

$$y \in A \ \Rightarrow \ [R_y \in I_y \ \Leftrightarrow \ y \in S \ \Leftrightarrow \ y \in T],$$

(ii) $y \in A \Rightarrow \ P_y \notin I_y$.

Then there is a Borel uniformization of P and, in particular, A is Borel.

Proof of Theorem 0.1. Suppose that E^X_G satisfies (E). We show that E^X_G is Borel and satisfies (C).

Let $\varphi : X \to Y$ be as in (E). Define $F = (\varphi^{-1} \times \varphi^{-1})(=_Y)$, i.e., $(x, y) \in F$ if and only if $\varphi(x) = \varphi(y)$. Then it follows from (E) that F is a Borel equivalence relation and every E^X_G-class contains at most countably many F-classes. By Proposition 2.3 we have that E^X_G is Borel and by Proposition 2.2 we find a Borel G-lg comeager set $C \subseteq X$ such that $G(x, C \cap [x]_F)$ is relatively open in $G(x, C)$ for every $x \in C$.

Next we want to apply Proposition 2.4. Define $P \subseteq Y \times X$ as $P = \{(\varphi(x), x) : x \in C\}$, $A = \mathrm{proj}_Y(P)$ and the assignment $\varphi(x) \in A \mapsto I_{\varphi(x)}$ as

$$B \in I_{\varphi(x)} \ \Leftrightarrow \ G(x, B \cap C \cap [x]_F) \in \mathcal{M}_G$$

where $x \in C$ and $B \subseteq C \cap [x]_F$.

We verify the assumptions of Proposition 2.4. It is easy to see that P is a Borel set because it is just the reversed graph of the Borel function $\varphi \restriction C : C \to Y$. Let $x, y \in C$ such that $(x, y) \in F$, i.e., $\varphi(x) = \varphi(y)$. Especially, there is $g \in G$ such that $g \cdot x = y$. Let $B \subseteq C \cap [x]_F$. Note that $G(y, B \cap C \cap [x]_F) \cdot g = G(x, B \cap C \cap [x]_F)$. This implies that the assignment $\varphi(x) \in A \mapsto I_{\varphi(x)}$ is well-defined and it is easy to see that $I_{\varphi(x)}$ is an σ-ideal of subsets of $P_{\varphi(x)}$ for every $x \in C$. Moreover, since $V \cdot x \cap C \subseteq C \cap [x]_F = P_{\varphi(x)}$ for some open set $1_G \in V \subseteq G$ and C is G-lg comeager we have that $P_{\varphi(x)} \notin I_{\varphi(x)}$ for every $x \in C$. It remains to show that (ii) in Proposition 2.4 holds as well. To this end pick a Borel set $R \subseteq P$. Define the set R' as

$$R' = \{(r, s) \in X \times X : r, s \in C \ \& \ (\varphi(r), s) \in R\}.$$

Note that R' is Borel and we have $R'_{r_0} = R'_{r_1}$ whenever $r_0, r_1 \in C$ such that $(r_0, r_1) \in F$. Then for $r \in C$ we have

$$(*) \qquad R_{\varphi(r)} \in I_{\varphi(r)} \ \Leftrightarrow \ G(r, R_{\varphi(r)} \cap C \cap [r]_F) \in \mathcal{M}_G \ \Leftrightarrow \ G(r, R'_r) \in \mathcal{M}_G$$

because $R_{\varphi(r)} = R'_r \subseteq C \cap [r]_F$. It follows from [7, Theorem 16.1] together with (*) that the sets

$$\mathcal{Z}_0 = \{r \in C : G(r, R'_r) \in \mathcal{M}_G\} \ \& \ \mathcal{Z}_1 = \{r \in C : G(r, R'_r) \notin \mathcal{M}_G\}$$

are Borel and $F \restriction C \times C$-invariant. Set $S = \varphi(\mathcal{Z}_0)$ and $T = Y \setminus \varphi(\mathcal{Z}_1)$. Then $S \subseteq Y$ is $\mathbf{\Sigma}^1_1$ and $T \subseteq Y$ is $\mathbf{\Pi}^1_1$ because φ is a Borel map and the rest follows again from (*).

Having verified the assumptions of Proposition 2.4, we get that the set A is Borel and there is a Borel map $f : A \to C$ such that $(y, f(y)) \in P$ for every $y \in A$. It is easy to see that $(f(y), f(z)) \notin F$ for every $y \neq z \in A$ because $\varphi(f(y)) = y$ for every $y \in A$ by the definition of P. Especially, f is injective and $\varphi \circ f : A \to A$ is the identity on A. It follows that $D=f(A)\subseteq C$ is a Borel countable complete section of E^X_G and a transversal of the equivalence relation $F\restriction C\times C$ on C.

Pick any decreasing sequence $\{V_n\}_{n\in\mathbb{N}}$ of open neighbourhoods of 1_G such that $\{1_G\}=\bigcap_{n\in\mathbb{N}}V_n$. Define

$$B_n=\{x\in D: V_n\cdot x\cap C\subseteq [x]_F\}.$$

We claim that $\{B_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ is the sequence from (C). It follows from the fact that $G(x,C\cap[x]_F)$ is relatively open in $G(x,C)$ for every $x\in C$ together with the fact that $D\subseteq C$ that $D=\bigcup_{n\in\mathbb{N}}B_n$. The definition of B_n together with the fact that D is a transversal of $F\restriction C\times C$ implies that if $g\cdot x=y$ for some $g\in V_n$ and $x,y\in B_n$, then $x=y$. It remains to show that B_n is Borel for every $n\in\mathbb{N}$. To see this first note that the set

$$C_n=\{(x,g)\in D\times V_n:(x,g\cdot x)\in F\}$$

is Borel because F and D are Borel sets. Then we have

$$B_n=\{x\in D:(\forall^* g\in V_n)(x,g)\in C_n\}$$

and the set on the right-hand side is Borel by [7, Theorem 16.1]. This finishes the proof. □

3. Technical Proofs

Proof of Proposition 2.1. Define $D=\{(x,g)\in X\times G: g\cdot x\in C\}$. Then D is a Borel set, $\operatorname{proj}_X(D)=X$ and $D_x\notin\mathcal{M}_G$ for every $x\in X$. By [7, Theorem 18.6] or Proposition 2.4 there is a Borel function $f:X\to G$ such that $(x,f(x))\in D$ for every $x\in X$. The function

$$F(x)=f(x)\cdot x$$

is the desired Borel reduction from E^X_G to E^C_G. □

Proof of Proposition 2.2. Let $\{V_n\}_{n\in\mathbb{N}}$ be an open basis at 1_G made of symmetric sets such that $V_{n+1}\cdot V_{n+1}\subseteq V_n$. Define

$$C=\{x\in X:(\exists n\in\mathbb{N})(\forall^* g\in V_n)\ (x,g\cdot x)\in F\}.$$

It follows from [7, Theorem 16.1] that C is a Borel set. Let $x\in C$ and $n\in\mathbb{N}$ such that $(x,g\cdot x)\in F$ for comeager many $g\in V_n$. Take $g\in G(x,C)\cap V_{n+1}$. Then we have that $V_{n+1}\cdot g\subseteq V_n$ and therefore $(x,h\cdot g\cdot x)\in F$ for comeager many $h\in V_{n+1}$. By the choice of g we have that $g\cdot x\in C$ and by the definition of C we find $n'\in\mathbb{N}$ such that $(g\cdot x,h'\cdot g\cdot x)\in F$ for comeager many $h'\in V_{n'}$. This shows that $(x,g\cdot x)\in F$ and as a consequence that $G(x,C\cap[x]_F)$ is relatively open in $G(x,C)$.

It remains to show that C is G-lg comeager in G. Suppose that there is $x\in X$ such that $G(x,C)$ is not comeager. By [7, Theorem 8.26] there is an open set $U\subseteq G$ such that $G(x,C)$ is meager in U. Let $\{\mathfrak{f}_i\}_{i\in\mathbb{N}}$ be an enumeration of the F-classes that are subset of $[x]_{E^X_G}$. Define $D_i=G(x,\mathfrak{f}_i)$. It follows that D_i has the Baire property for every $i\in\mathbb{N}$ and that $U\subseteq\bigcup_{i\in\mathbb{N}}D_i$. Another use of [7, Theorem 8.26] gives an open set $V\subseteq U$ and $i\in\mathbb{N}$ such that D_i is comeager in V. In another words $h\cdot x\in\mathfrak{f}_i$ for comeager many $h\in V$. Pick $g\in(V\cap D_i)\setminus G(x,C)$ and $n\in\mathbb{N}$ such that $V_n\cdot g\subseteq V$. Then we have that D_i is comeager in $V_n\cdot g$ and $g\cdot x\in\mathfrak{f}_i$. This shows that there are comeager many $h\in V_n$ such that $(g \cdot x, h \cdot g \cdot x) \in F$. That is a contradiction with $g \notin G(x, C)$ and that finishes the proof. $\square$

Proof of Proposition 2.3. Let $C \subseteq X$ be as in Proposition 2.2. We claim that

$$(x,y) \in E_G^C \ \Leftrightarrow\ (\exists^*(a,b) \in G \times G)\ (a \cdot x, b \cdot y) \in F \tag{1}$$

for every $x, y \in C$.

Let $x, y \in C$. If x, y satisfies the right-hand side of (1), then $(x, y) \in E_G^C$ because $F \subseteq E_G^X$. Suppose, on the other hand, that $(x, y) \in E_G^C$. By the definition of E_G^C and C we find an open set $1_G \in V \subseteq G$ and $g \in G$ such that $g \cdot x = y$ and $V \cdot y \cap C \subseteq [y]_F \cap C$. Note that $W = G(y, V \cdot y \cap C) = V \cap G(y, C)$ is nonmeager and $a \cdot y \in [y]_F$ for every $a \in W$. The set $W \cdot g \times W$ is nonmeager in $G \times G$. Let $(a \cdot g, b) \in W \cdot g \times W$. Then we have $a \cdot g \cdot x = a \cdot y \in [y]_F \cap C$ and $b \cdot y \in [y]_F \cap C$ by the definition of W. This shows that x, y satisfies the right-hand side of (1).

It remains to show that the right-hand side of (1) defines a Borel set. The set

$$R = \{(x, y, g, h) \in C \times C \times G \times G : (g \cdot x, h \cdot y) \in F\}$$

is Borel because F is a Borel equivalence relation and C is a Borel set. This implies by [7, Theorem 16.1] that E_G^C is a Borel equivalence relation and Proposition 2.1 finishes the proof. $\square$

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