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Invariant half-spaces for rank-one perturbations

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INVARIANT HALF-SPACES FOR RANK-ONE PERTURBATIONS

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Abstract. If T is a bounded linear operator acting on an infinite-dimensional Banach space and $\varepsilon > 0$, then there exists and operator F of rank at most one with $\|F\| < \varepsilon$ such that $T - F$ has an invariant subspace of infinite dimension and codimension. This improves results of Tcaciuc and other authors.

1. Introduction

The invariant subspace problem is the most important problem in operator theory. It is the question whether each bounded linear operator on a complex Banach space has a nontrivial closed invariant subspace. The problem is still open for operators on Hilbert spaces, or more generally, on reflexive Banach spaces. In the class of non-reflexive Banach spaces negative examples were given by Enflo [4] and Read [10].

It is easy to see that each operator on a non-separable Banach space has a nontrivial invariant subspace. Similarly, all operators on a finite-dimensional Banach space of dimension at least two have eigenvalues, and so nontrivial invariant subspaces. So the question makes sense only in separable infinite-dimensional Banach spaces.

Inspired by the invariant subspace problem, the following question was studied intensely: given a Banach space operator T, does there exists a "small" perturbation F such that $T - F$ has an invariant subspace?

It is easy to see that for each bounded linear operator T on a Banach space X there exists a rank-one operator F such that $T - F$ has a one-dimensional invariant subspace. Indeed, take any non-zero vector $x \in X$ and a rank-one operator F on X such that $Fx = Tx$. Then $(T - F)x = 0$ and so $T - F$ has the one-dimensional invariant subspace generated by x.

So the proper question is: does every operator T have a "small" perturbation F such that $T - F$ has an invariant subspace of infinite dimension and codimension?

For short, closed subspaces of infinite dimension and codimension are called half-spaces.

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The first result in this direction was proved by Brown and Pearcy [3]:

Theorem 1.1. *Let T be an operator on a separable infinite-dimensional Hilbert space H and let $\varepsilon > 0$. Then there exists a compact operator K on H such that $\|K\| < \varepsilon$ and $T - K$ has an invariant half-space.*

The question has been then studied by a number of authors, see e.g. [1], [8], [9], [11]. The research culminated by [11], where the following results were proved.

Theorem 1.2. *Let T be an operator on an infinite-dimensional Banach space X. Then:*

(i) ([11], Theorem 1.1) *there exists an operator F of rank at most one such that $T - F$ has an invariant half-space;*

(ii) ([11], Theorem 4.3) *if $\varepsilon > 0$, then there exists an operator F of finite rank with $\|F\| < \varepsilon$ such that $T - F$ has an invariant half-space.*

In this note we complete and unify the above results and show that for any operator T acting on an infinite-dimensional Banach space X and $\varepsilon > 0$ there exists an operator F of rank at most one with $\|F\| < \varepsilon$ such that $T - F$ has an invariant half-space. The proof uses modified techniques from [9] and [11].

2. Preliminaries

For a (complex) Banach space X, we denote by X^* its dual.

If $M \subset X$ is a subset, then the annihilator $M^\perp$ is defined by

$$M^\perp = \{x^* \in X^* : \langle m, x^* \rangle = 0 \text{ for all } m \in M\}.$$

Clearly $M^\perp$ is a w^*-closed subspace of X^*.

Similarly, for a subset $M' \subset X^*$ define the preannihilator ${}^\perp M'$ by

$${}^\perp M' = \{x \in X : \langle x, m^* \rangle = 0 \text{ for all } m^* \in M'\}.$$

Clearly ${}^\perp M'$ is a weakly closed (and so closed) subspace of X.

A sequence $(x_n)_{n=1}^\infty$ in X is called basic if any vector $x \in \bigvee_{n=1}^\infty x_n$ can be written uniquely as $x = \sum_{n=1}^\infty \alpha_n x_n$ for some complex coefficients α_n. Then there exist functionals $x_n^* \in X^*$ $(n \in \mathbb{N})$ such that $\langle x_n, x_j^* \rangle = \delta_{n,j}$ (the Kronecker symbol) for all $n, j \in \mathbb{N}$ and $\sup\{\|x_j^*\| : j \in \mathbb{N}\} < \infty$.

Recall that any infinite-dimensional Banach space contains a basic sequence. If $(x_n)_{n=1}^\infty$ is a basic sequence in a Banach space X and $A \subset \mathbb{N}$ an infinite subset such that $\mathbb{N} \setminus A$ is infinite then it is easy to see that $\bigvee_{n \in A} x_n$ is a subspace of infinite dimension and codimension. In particular, in any infinite-dimensional Banach space there is a half-space.

The basic result about the existence of basic sequences is the following criterion, see [7] or [2], Theorem 1.5.6. Recall that a sequence of vectors in a Banach space is called semi-normalized if it is bounded and bounded away from zero.

Theorem 2.1. (Kadets-Pełczyński) *Let $(x_n)_{n=1}^\infty$ be a semi-normalized sequence in a Banach space X. Then the following conditions are equivalent:*

(i) *$(x_n)_{n=1}^\infty$ fails to contain a basic subsequence;*
(ii) *the weak closure $\{x_n : n \in \mathbb{N}\}^{-w}$ is weakly compact and fails to contain 0.*

A dual version of this criterion can be found in [5] or [6], Theorem III.1 and Remark III.1.

Theorem 2.2. *If $(x_n^*)_{n=1}^\infty$ is a semi-normalized sequence in a dual Banach space X^* and 0 is a weak*-cluster point of $\{x_n^*\}_{n=1}^\infty$ then there exist a basic subsequence $\{y_k^*\}_{k=1}^\infty$ of $(x_n^*)_{n=1}^\infty$ and a bounded sequence $(y_k)_{k=1}^\infty$ in X such that $\langle y_k, y_j^*\rangle = \delta_{k,j}$ for all $k, n \in \mathbb{N}$.*

Denote by $B(X)$ the algebra of all (bounded linear) operators on a Banach space X.

Let $T \in B(X)$. Denote by $N(T)$ the kernel of T, $N(T) = \{x \in X : Tx = 0\}$, and by $R(T) = TX$ the range of T. Write $R^\infty(T) = \bigcap_{k=1}^\infty R(T^k)$. Clearly $R^\infty(T)$ is a (not necessarily closed) linear manifold.

Clearly $N(T) \subset N(T^2) \subset N(T^3) \subset \cdots$. Denote by $\operatorname{asc} T$ the ascent of T, $\operatorname{asc} T = \min\{k : N(T^{k+1}) = N(T^k)\}$ (if no such k exists then we set $\operatorname{asc} T = \infty$). It is easy to see that if $\operatorname{asc} T = k < \infty$ then $N(T^j) = N(T^k)$ for all $j \geq k$.

Denote by $\mathcal{K}(X)$ the closed two-sided ideal of all compact operators on X. For $T \in B(X)$ let $\|T\|_e$ be the essential norm of T, $\|T\|_e = \inf\{\|T+K\| : K \in \mathcal{K}(X)\}$. Let $\sigma_e(T)$ be the essential spectrum of an operator $T \in B(X)$, $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$. It is well known that $\sigma_e(T)$ is the spectrum of the class $T + \mathcal{K}(X)$ in the Calkin algebra $B(X)/\mathcal{K}(X)$.

3. Main result

For short, we use the following definition.

Definition 3.1. Let T be an operator acting on an infinite-dimensional separable Banach space X. We say that T has property (P) if for every $\varepsilon > 0$ there exists an operator $F \in B(X)$ of rank at most one such that $\|F\| < \varepsilon$ and $T - F$ has an invariant half-space.

Proposition 3.2. *Let X be a separable infinite-dimensional Banach space, let $T \in B(X)$, $0 \in \sigma_e(T) \cap \partial\sigma(T)$ and $\operatorname{asc} T < \infty$. Then T has property (P).*

Proof. If $\dim N(T) = \infty$ then any half-subspace of $N(T)$ is invariant for T, and so T has property (P).

So we may assume that $\dim N(T) < \infty$. Let $\varepsilon > 0$. Let $k = \operatorname{asc} T < \infty$ and $E = N(T^k)$. Then E is a finite-dimensional subspace of X, $\dim E \leq k \cdot \dim N(T)$. Let $M \subset X$ be a complement of E, $X = E \oplus M$. Let P_M be the projection onto M along E. Then $P_E := I - P_M$ is the projection onto E along M.

Find a sequence $(\lambda_n)\subset\mathbb{C}\setminus\sigma(T)$ such that $\lambda_n\to 0$. Since $0\in\sigma_e(T)$, we have $\lim_{n\to\infty}\|(T-\lambda_n)^{-1}\|_e=\infty$. Consequently,

$$\lim_{n\to\infty}\|P_M(T-\lambda_n)^{-1}\|=\infty.$$

By the Banach-Steinhaus uniform boundedness theorem, there exists a vector $u\in X$, $\|u\|=1$ such that

$$\sup\bigl\{\|P_M(T-\lambda_n)^{-1}u\|:n\in\mathbb{N}\bigr\}=\infty.$$

Without loss of generality we may assume that $\lim_{n\to\infty}\bigl\|P_M(T-\lambda_n)^{-1}u\bigr\|=\infty$. For $n\in\mathbb{N}$ set

$$x_n=\frac{P_M(T-\lambda_n)^{-1}u}{\|P_M(T-\lambda_n)^{-1}u\|}.$$

We have $\|x_n\|=1$ and

$$\begin{aligned}Tx_n=\quad&\lambda_nx_n+(T-\lambda_n)x_n=\lambda_nx_n+\frac{(T-\lambda_n)(I-P_E)(T-\lambda_n)^{-1}u}{\|P_M(T-\lambda_n)^{-1}u\|}\\ =\quad&\lambda_nx_n+\frac{u}{\|P_M(T-\lambda_n)^{-1}u\|}-\frac{(T-\lambda_n)P_E(T-\lambda_n)^{-1}u}{\|P_M(T-\lambda_n)^{-1}u\|},\end{aligned}\tag{3.1}$$

where $\lambda_nx_n+\frac{u}{\|P_M(T-\lambda_n)^{-1}u\|}\to 0$ as $n\to\infty$ and

$$\frac{(T-\lambda_n)P_E(T-\lambda_n)^{-1}u}{\|P_M(T-\lambda_n)^{-1}u\|}\in E$$

since $TE\subset E$.

We show that the sequence $(x_n)_{n=1}^\infty$ has a basic subsequence. Suppose the contrary. By Theorem 2.1, $\{x_n:n\in\mathbb{N}\}^{-w}$ is weakly compact and does not contain 0. By the Eberlein-Smulian theorem, there exists a weakly convergent subsequence (x_k) of (x_n), $x_k\xrightarrow{w}x$ and $x\neq 0$. Then $Tx_k\xrightarrow{w}Tx$ and $Tx\in E=N(T^k)$ by (3.1). So $x\in N(T^{k+1})=N(T^k)=E$. By definition, $x_k\in M$ for all k, and so $x\in M$. Hence $x=0$, a contradiction.

So the set $\{x_n:n\in\mathbb{N}\}$ contains a basic sequence. By passing to a subsesquence if necessary we may assume that $(x_n)_{n=1}^\infty$ is a basic sequence in M. Let $(x_n^*)_{n=1}^\infty\subset M^*$ be the corresponding biorthogonal sequence, $\langle x_n,x_j^*\rangle=\delta_{n,j}$ for all $n,j\in\mathbb{N}$.

Set $y_n^*=x_n^*P_M\in X^*\quad(n\in\mathbb{N})$. Then $y_n^*\in E^\perp$, $\langle x_n,y_j^*\rangle=\delta_{n,j}$ for all $n,j\in\mathbb{N}$ and $c:=\sup\{\|y_j^*\|:j\in\mathbb{N}\}<\infty$. Without loss of generality we may assume that $\sum_{n=1}^\infty\|P_M(T-\lambda_{2n})^{-1}u\|^{-1}<\varepsilon/c$.

Set $F=u\otimes\Bigl(\sum_{n=1}^\infty\frac{y_{2n}^*}{\|P_M(T-\lambda_{2n})u\|}\Bigr)$. Then F is an operator of rank one and

$$\|F\|=\|u\|\cdot\Bigl\|\sum_{n=1}^\infty\frac{y_{2n}^*}{\|P_M(T-\lambda_{2n})u\|}\Bigr\|\le\sum_{n=1}^\infty\frac{c}{\|P_M(T-\lambda_{2n})^{-1}u\|}<\varepsilon.$$

Let $L = \left(\bigvee_{n=1}^{\infty} x_{2n}\right) \vee E$. Clearly $\dim L \geq \dim \bigvee_{n=1}^{\infty} x_{2n} = \infty$. Furthermore, $\operatorname{codim} \bigvee_{n=1}^{\infty} x_{2n} = \infty$ and since $\dim E < \infty$, we have $\operatorname{codim} L = \infty$. Hence L is a half-space and $(T-F)E = TE \subset E \subset L$. Furthermore, for $n \in \mathbb{N}$ we have

$$\begin{aligned}(T-F)x_{2n} &= \lambda_{2n}x_{2n} + \frac{u}{\|P_M(T-\lambda_{2n})^{-1}u\|} - \frac{(T-\lambda_{2n})P_E(T-\lambda_{2n})^{-1}u}{\|P_M(T-\lambda_{2n})^{-1}u\|} \\ &\quad - \frac{u}{\|P_M(T-\lambda_{2n})^{-1}u\|} \\ &= \lambda_{2n}x_{2n} - \frac{(T-\lambda_{2n})P_E(T-\lambda_{2n})^{-1}u}{\|P_M(T-\lambda_{2n})^{-1}x_{2n}\|} \in L.\end{aligned}$$

□

The dual result is also true.

Proposition 3.3. *Let X be a separable infinite-dimensional Banach space, let $T \in B(X)$, $0 \in \sigma_e(T) \cap \partial\sigma(T)$ and $\operatorname{asc} T^* < \infty$. Then T has property (P).*

Proof. If $\dim N(T^*) = \infty$ then $\operatorname{codim} \overline{R(T)} = \infty$. If $\dim R(T) = \infty$ then $\overline{R(T)}$ is a half-space invariant for T. If $\dim R(T) < \infty$ then $\dim N(T) = \infty$ and any half-subspace of $N(T)$ is invariant for T. So T has property (P).

So we may assume that $\dim N(T^*) < \infty$. Let $\varepsilon > 0$. Let $k = \operatorname{asc} T^* < \infty$. Let $E' = N(T^{*k})$. Then E' is a finite-dimensional subspace of X^*.

We have $\overline{R(T^k)} = {}^{\perp}E'$. So $\operatorname{codim} \overline{R(T^k)} < \infty$. Let $E \subset X$ be a complement of $\overline{R(T^k)}$, $X = E \oplus \overline{R(T^k)}$. Let $M' = E^{\perp}$. Then M' is a w^*-closed subspace of X^* and $X^* = M' \oplus E'$. Let $P_{M'}$ be the projection onto M' along E'. Then $P_{E'} := I - P_{M'}$ is the projection onto E' along M'.

Find a sequence $(\lambda_n) \subset \mathbb{C} \setminus \sigma(T) = \mathbb{C} \setminus \sigma(T^*)$ such that $\lambda_n \to 0$. Since $0 \in \sigma_e(T) = \sigma_e(T^*)$, we have $\lim_{n\to\infty} \|(T^* - \lambda_n)^{-1}\|_e = \infty$. Consequently,

$$\lim_{n\to\infty} \|P_{M'}(T^* - \lambda_n)^{-1}\| = \infty.$$

By the Banach-Steinhaus uniform boundedness theorem, there exists a vector $u^* \in X^*$, $\|u^*\| = 1$ such that

$$\sup\bigl\{\|P_{M'}(T^* - \lambda_n)^{-1}u^*\| : n \in \mathbb{N}\bigr\} = \infty.$$

Without loss of generality we may assume that $\lim_{n\to\infty}\bigl\|P_{M'}(T^* - \lambda_n)^{-1}u^*\bigr\| = \infty$. For $n \in \mathbb{N}$ set

$$y_n^* = \frac{P_{M'}(T^* - \lambda_n)^{-1}u^*}{\|P_{M'}(T^* - \lambda_n)^{-1}u^*\|}.$$

We have $\|y_n^*\| = 1$ and

$$\begin{aligned}T^*y_n^* &= \lambda_n y_n^* + (T^* - \lambda_n)y_n^* = \lambda_n y_n^* + \frac{(T^* - \lambda_n)(I - P_{E'})(T^* - \lambda_n)^{-1}u^*}{\|P_{M'}(T^* - \lambda_n)^{-1}u^*\|} \\ &= \lambda_n y_n^* + \frac{u^*}{\|P_{M'}(T^* - \lambda_n)^{-1}u^*\|} - \frac{(T^* - \lambda_n)P_{E'}(T^* - \lambda_n)^{-1}u^*}{\|P_{M'}(T^* - \lambda_n)^{-1}u^*\|},\end{aligned}$$

where $\lambda_n y_n^* + \frac{u^*}{\|P_{M'}(T^*-\lambda_n)^{-1}u^*\|} \to 0$ as $n \to \infty$ and

$$\frac{(T^*-\lambda_n)P_{E'}(T^*-\lambda_n)^{-1}u^*}{\|P_{M'}(T^*-\lambda_n)^{-1}u^*\|} \in E'$$

since $T^*E' \subset E'$.

Since X is a separable Banach space, the closed unit ball in X^* with the w^*-topology is metrizable and compact. So (y_n^*) has a w^*-convergent subsequence. Without loss of generality we may assume that $y_n^* \xrightarrow{w^*} y^*$. So $T^*y_n^* \xrightarrow{w^*} T^*y^* \in E' = N(T^{*k})$. Hence $y^* \in N(T^{*k+1}) = N(T^{*k}) = E'$. However, clearly $y^* \in M'$, and so $y^* = 0$. By Theorem 2.2, $(y_n^*)_{n=1}^\infty$ contains a basic subsequence. Without loss of generality we may assume that $(y_n^*)_{n=1}^\infty$ is basic. Let $(y_n)_{n=1}^\infty \subset X$ be a bounded sequence satisfying $\langle y_n, y_j^*\rangle = \delta_{n,j} \quad (n,j \in \mathbb{N})$.

Let $Q : X = E \oplus \overline{R(T^k)} \to \overline{R(T^k)}$ be the canonical projection onto $\overline{R(T^k)}$ along E. Let $x_n = Qy_n \quad (n \in \mathbb{N})$. Then $x_n \in \overline{R(T^k)}$ for all n, $\langle x_n, y_j^*\rangle = \delta_{n,j} \quad (n,j \in \mathbb{N})$ and $c := \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$.

By passing to a subsequence if necessary we can assume that

$$\sum_{n=1}^\infty \frac{1}{\|P_{M'}(T^*-\lambda_n)^{-1}u^*\|} < \frac{\varepsilon}{c}.$$

Define operator $F = \Bigl(\sum_{n=1}^\infty \frac{x_{2n}}{\|P_{M'}(T^*-\lambda_{2n})u^*\|}\Bigr) \otimes u^*$. Then F is an operator of rank one and

$$\|F\| = \|u^*\| \cdot \Bigl\|\sum_{n=1}^\infty \frac{x_{2n}}{\|P_{M'}(T^*-\lambda_{2n})u^*\|}\Bigr\| \le \sum_{n=1}^\infty \frac{c}{\|P_{M'}(T^*-\lambda_{2n})^{-1}u^*\|} < \varepsilon.$$

Let

$$L' = \Bigl(\bigvee_{n=1}^\infty y_{2n}^*\Bigr) \vee E' \subset X^*$$

and

$$L = {}^\perp L' = {}^\perp\Bigl(\bigvee_{n=1}^\infty y_{2n}^*\Bigr) \cap {}^\perp E'.$$

Clearly ${}^\perp\bigl(\bigvee_{n=1}^\infty y_{2n}^*\bigr) \supset \{x_{2j+1} : j \in \mathbb{N}\}$. So $\dim {}^\perp\bigl(\bigvee_{n=1}^\infty y_{2n}^*\bigr) = \infty$. Moreover, $\operatorname{codim} {}^\perp E' < \infty$, and so $\dim L = \infty$.

Similarly, $x_{2j} \notin {}^\perp\bigl(\bigvee_{n=1}^\infty y_{2n}^*\bigr) \supset L$. So $\operatorname{codim} L = \infty$ and L is a half-space.

We show that $(T-F)L \subset L$. Let $z \in L = {}^\perp\bigl(\bigvee_{n=1}^\infty y_{2n}^*\bigr) \cap {}^\perp E'$. To show that $(T-F)z \in L$ we must show that

$$\langle (T-F)z, y_{2j}^*\rangle = 0$$

for all $j \in \mathbb{N}$ and

$$\langle (T-F)z, y^*\rangle = 0$$

for all $y^* \in E'$.

We have

$$\langle (T-F)z, y_{2n}^*\rangle = \langle z, T^*y_{2n}^*\rangle - \langle Fz, y_{2n}^*\rangle$$

$$= \Big\langle z, \lambda_{2n} y_{2n}^* + \frac{u^*}{\|P_{M'}(T^*-\lambda_{2n})^{-1}u^*\|} - \frac{(T^*-\lambda_{2n})P_{E'}(T^*-\lambda_{2n})^{-1}u^*}{\|P_{M'}(T^*-\lambda_{2n})^{-1}u^*\|}\Big\rangle$$

$$-\frac{\langle z, u^*\rangle}{\|P_{M'}(T^*-\lambda_{2n})^{-1}u^*\|} = 0.$$

Furthermore, for $y^* \in E'$ we have

$$\langle (T-F)z, y^*\rangle = \langle z, T^*y^*\rangle - \langle z, u^*\rangle \cdot \Big\langle \sum_{n=1}^{\infty} \frac{x_{2n}}{\|P_{M'}(T^*-\lambda_{2n})^{-1}u^*\|}, y^*\Big\rangle = 0$$

since $T^*y^* \in E'$, $z \in {}^{\perp}E'$ and $x_{2n} \in {}^{\perp}E'$ for all $n \in \mathbb{N}$. Hence $(T-F)L \subset L$, and so T has property (P). □

Lemma 3.4. *Let $T \in B(X)$, $\dim N(T) < \infty$. Then $TR^\infty(T) = R^\infty(T)$.*

Proof. Clearly $TR^\infty(T) \subset R^\infty(T)$.

Let $x \in R^\infty(T)$. Suppose on the contrary that $x \notin TR^\infty(T)$.

Let $n = \dim N(T)$. Set $k_0 = 0$. Since $x \in R^\infty(T) \subset R(T)$, there exists $y_0 \in X$ such that $Ty_0 = x$. Since $x \notin TR^\infty(T)$, there exists $k_1 \in \mathbb{N}$ such that $y_0 \notin R(T^{k_1})$.

Since $x \in R^\infty(T) \subset R(T^{k_1+1})$, there exists $y_1 \in R(T^{k_1})$ with $Ty_1 = x$. Since $x \notin TR^\infty(T)$, there exists $k_2 > k_1$ such that $y_1 \notin R(T^{k_2})$.

Inductively we can find vectors $y_1, \dots, y_n, y_{n+1}$ and numbers $k_1 < k_2 < \dots k_{n+1}, k_{n+2}$ such that $Ty_j = x$ and $y_j \in R(T^{k_j}) \setminus R(T^{k_{j+1}})$ for $j = 1, \dots, n+1$.

Set $u_j = y_j - y_0 \quad (j = 1, \dots, n+1)$. Clearly $Tu_j = 0$ for all $j = 1, \dots, n+1$. We show that the vectors $u_1, \dots, u_{n+1}$ are linearly independent. Suppose that $\sum_{j=1}^{n+1} \alpha_j u_j = 0$ for some coefficients α_j. We have

$$0 = \sum_{j=1}^{n+1} \alpha_j y_j - y_0 \sum_{j=1}^{n+1} \alpha_j,$$

where $\sum_{j=1}^{n+1} \alpha_j y_j \in R(T^{k_1})$ and $y_0 \notin R(T^{k_1})$. So $\sum_{j=1}^{n+1} \alpha_j = 0$.

Let j_0 be the smallest index such that $\alpha_{j_0} \neq 0$. Then

$$0 = \sum_{j=j_0}^{n+1} \alpha_j y_j \in \alpha_{j_0} y_{j_0} + R(T^{k_{j_0+1}}).$$

Since $y_{j_0} \notin R(T^{k_{j_0+1}})$, we have $\alpha_{j_0} = 0$. So $\alpha_j = 0$ for all j and the vectors $u_1, \dots, u_{n+1}$ are linearly independent elements in $N(T)$, a contradiction with the assumption that $\dim N(T) = n$.

Hence $x \in TR^\infty(T)$. □

Theorem 3.5. *Let X be an infinite-dimensional Banach space, let $T \in B(X)$ and $\varepsilon > 0$. Then there exists an operator $F \in B(X)$ of rank at most one such that $\|F\| < \varepsilon$ and $T - F$ has an invariant half-space.*

Proof. Without loss of generality we may assume that X is separable.

Let $\lambda \in \sigma_e(T)$ satisfy $|\lambda| = \max\{|\mu| : \mu \in \sigma_e(T)\}$. Then there are only countably many elements $\mu \in \sigma(T)$ satisfying $|\mu| > |\lambda|$. So there exists a sequence $(\lambda_n) \subset \mathbb{C} \setminus \sigma(T)$ such that $\lambda_n \to \lambda$.

Replacing T by $T - \lambda$ we may assume without loss of generality that $\lambda = 0$.

So we may assume that $0 \in \sigma_e(T) \cap \partial\sigma(T)$.

By Proposition 3.2, we may assume that $\operatorname{asc} T = \infty$. Clearly we may assume that $\dim N(T) < \infty$; otherwise any half-subspace of $N(T)$ is invariant for T. We have

$$N(T) \supset R(T) \cap N(T) \supset R(T^2) \cap N(T) \supset \cdots ,$$

where $R(T^j) \cap N(T) \neq \{0\}$ for all $j \in \mathbb{N}$. So there exists $j_0 \in \mathbb{N}$ such that

$$R(T^j) \cap N(T) = R(T^{j_0}) \cap N(T)$$

for all $j \geq j_0$. Hence $R(T^{j_0}) \cap N(T) = R^\infty(T) \cap N(T)$ and there exists a nonzero vector $y_0 \in R^\infty(T) \cap N(T)$.

By Lemma 3.4, we have $TR^\infty(T) = R^\infty(T)$. So we can find inductively vectors $y_j \in R^\infty(T) \quad (j \in \mathbb{N})$ such that $Ty_j = y_{j-1} \quad (j \geq 1)$ and $Ty_0 = 0$.

By Proposition 3.3, we can assume that $\operatorname{asc} T^* = \infty$. Similarly we can find vectors $y_0^*, y_1^*, y_2^*, \dots \in X^*$ such that $y_0^* \neq 0$, $T^* y_j^* = y_{j-1}^* \quad (j \in \mathbb{N})$ and $T^* y_0^* = 0$.

Let $L = \bigvee_{j=0}^\infty y_j$. Clearly $TL \subset L$. Vectors $y_j \quad (j \geq 0)$ are linearly independent. Indeed, suppose that $\sum_{j=0}^\infty \alpha_j y_j = 0$ for some finite sum. Suppose that $\alpha_{j_0} \neq 0$ and $\alpha_j = 0$ for all $j > j_0$. Then

$$0 = T^{j_0} \sum_{j=0}^\infty \alpha_j y_j = \alpha_{j_0} y_0.$$

So $\alpha_{j_0} = 0$, a contradiction. So $\dim L = \infty$.

For $j, k = 0, 1, \dots$ we have

$$\langle y_j, y_k^* \rangle = 0$$

since $y_j \in R^\infty(T) \subset R(T^{k+1})$ and $y_k^* \in N(T^{*k+1})$. So $L^\perp \supset \bigvee\{y_k^* : k = 0, 1, \dots\}$ where the vectors y_k^* are linearly independent as above. Hence $\dim L^\perp = \infty$ and so $\operatorname{codim} L = \infty$.

Hence T has an invariant half-space. □

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