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Distinguished metrizable spaces E and injective tensor products $C_p(X) \otimes_\epsilon E$

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Distinguished metrizable spaces E and injective tensor products $C_p(X) \otimes_\varepsilon E$

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Abstract. The paper continues the study on the distinguished property of the space $C_p(X)$ (of real-valued continuous functions over a Tychonoff space X in the pointwise topology) under the formation of tensor products towards the following research directions: (i) the injective tensor product $C_p(X) \otimes_\varepsilon E$ of $C_p(X)$ and a real locally convex space E (and its completion $C_p(X) \widehat{\otimes}_\varepsilon E$), and (ii) the space $C_p(X, E)$ of all E-valued continuous functions (with a normed space E) endowed with the pointwise topology. This work leads also to a new characterization of distinguished Fréchet locally convex spaces E. Indeed, we show (among the others) that if $C_p(X)$ is metrizable, then E is distinguished if and only if metrizable $C_p(X) \otimes_\varepsilon E$ is distinguished.

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1. Introduction

Following Dieudonné and Schwartz [**7**] a locally convex space (lcs) E is called *distinguished* if E is a large subspace of its weak* bidual E''. In other words, E is distinguished if and only if each bounded set in $(E'', \sigma(E'', E))$ is contained in the $\sigma(E'', E)$-closure of a bounded set in E. Equivalently, E is distinguished if and only if the strong dual of E (i. e., the topological dual of E endowed with the strong topology) is barrelled, see [**17**, 8.7.1]. We refer the reader to survey articles [**2**] and [**3**] about distinguished metrizable and Fréchet spaces (i. e., metrizable and complete lcs).

In [**8, 9, 10, 12, 15**] we have studied the distinguished property of the space $C_p(X)$. It turns out (see below) that $C_p(X)$ is distinguished if and only if $C_p(X)$ is a large subspace of $\mathbb{R}^X$. In [**15**] we showed that $C_p(X)$ is distinguished if and only if X is a Δ-space in the sense of Knight [**16**], and we provided several applications of that fact. Equivalently, $C_p(X)$ is distinguished if for each countable partition $\{X_k : k \in \mathbb{N}\}$ of X into nonempty pairwise disjoint sets there are open sets $\{U_k : k \in \mathbb{N}\}$ with $X_k \subseteq U_k$ for each $k \in \mathbb{N}$ such that each point $x \in X$ belongs to U_n for only finitely many $n \in \mathbb{N}$, [**12**, Definition 4].

Some part of the research around distinguished Fréchet lcs was related to the stability of this property under the formation of tensor products. For example, Grothendieck proved already [**13**] that if E is a nuclear Fréchet space and and F is a Fréchet space which is either nuclear, Schwartz, Montel, reflexive, quasinormable, or distinguished, then the complete projective tensor product $E \widehat{\otimes}_\pi F$ satisfies the same property.

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This line of research about projective and injective tensor products of distinguished Fréchet spaces has been continued by several specialists, see for example [**4, 5**] (and references therein). Note that in general, if E and F are distinguished Fréchet spaces, neither $E\,\widehat{\otimes}_{\epsilon}\,F$ nor $E\,\widehat{\otimes}_{\pi}\,F$ need to be distinguished [**4, 6**].

In the present paper we extend our study on the distinguished property of spaces $C_p(X)$ under the formation of tensor products. Among other results, we prove that $C_p(X)\otimes_{\varepsilon} E$ is distinguished if both $C_p(X)$ is distinguished and $\mathbb{R}^{(X)}\otimes_{\varepsilon} E'_{\beta}$ is barrelled (Corollary 6).

This leads to a characterization of distinguished Fréchet spaces E: If X is countable, then metrizable $C_p(X)\otimes_{\varepsilon} E$ (as well as its completion $C_p(X)\ \widehat{\otimes}_{\varepsilon}\,E$) is distinguished if and only if E is distinguished (Theorem 14). Recall that $C_p(X)$ is metrizable if and only if X is countable. On the other hand, Díaz and Domański [**4**] constructed a distinguished Fréchet space E such that $E\,\widehat{\otimes}_{\varepsilon}\,\ell_1$ is not distinguished.

For any Tychonoff space X and any normed space E, the space $C_p(X,E)$ is distinguished if and only if $C_p(X)\otimes_{\varepsilon} E$ is distinguished (Corollary 21). So, if X is countable and E normed, $C_p(X,E)$ is distinguished (Corollary 22). A corresponding result (due to Díaz and Domański [**4**, Corolary 2.5]) states that the space $C_k(K,E)$ of all continuous functions defined on a compact Hausdorff space K with values in a reflexive Fréchet space (and endowed with the compact-open topology) is distinguished.

In what follows X is an infinite Tychonoff space and all locally convex spaces are supposed to be real and Hausdorff. $\mathbb{R}^{(X)}$ designates the locally convex direct sum of $|X|$ real lines. The symbol '$\simeq$' indicates some canonical linear isomorphism or homeomorphism.

2. Preliminaries

First we recall the concept of the distinguished property for $C_p(X)$ spaces and collect together some results on distinguished and nondistinguished $C_p(X)$ spaces which have been obtained in [**8, 10, 9**].

Let X be an infinite Tychonoff space. If A is a pointwise bounded subset of $C(X)$ we set $\phi_A := \sup\{|f| : f\in A\}$, where the supremum is with respect to the ordering of $\mathbb{R}^X$, i.e., $f\le g$ if $f(x)\le g(x)$ for every $x\in X$. In other words$\phi_A(x)=\sup\{|f(x)| : f\in A\}$ for all $x\in X$. If $0\le g\in\mathbb{R}^X$, we designate by P_g the closed and bounded subset of $\mathbb{R}^X$ defined by $P_g=\left\{h\in\mathbb{R}^X : |h|\le g\right\}$. Let us say that a set $\mathcal{M}$ in $\mathbb{R}^X$ is *cofinal* if for each $g\in\mathbb{R}^X$ there is $f\in\mathcal{M}$ with $|g|\le|f|$, and denote by $\mathrm{Cof}\left(\mathbb{R}^X\right)$ the least cardinality of a cofinal set $\mathcal{M}$ in $\mathbb{R}^X$.

We represent by $bn(X)$ the so-called *borne number* of X [**9**, Section 4], which is defined as the least cardinality of a fundamental family of bounded sets in $C_p(X)$.

Lemma 1. *If A is a bounded set in $C_p(X)$, there is another bounded set Q_A in $C_p(X)$ such that $P_{\phi_A}\subseteq\overline{Q_A}$, where the closure is in $\mathbb{R}^X$.*

Proof. For each finite set Λ in A, set $\varphi_\Lambda(x) := \max_{f\in\Lambda}|f(x)|$ for every $x\in X$. Then $B=\{\varphi_\Lambda : \Lambda\subseteq A,\ |\Lambda|<\aleph_0\}$ is a bounded set in $C_p(X)$ and its closure $\overline{B}$ in $\mathbb{R}^X$ clearly contains ϕ_A. Now P_{ϕ_A} is contained in the $\mathbb{R}^X$ closure of $Q_A := \{f\in C(X) : |f|\le\varphi$ for some $\varphi\in B\}$ which is a bounded set in $C_p(X)$ (see [**9**, Lemma 13]). ☐

Theorem 2. *The following conditions are equivalent.*

1. *The space $C_p(X)$ is distinguished.*

2. *There is a family of bounded sets $\mathcal{F}$ in $C_p(X)$ with $\{\phi_A : A \in \mathcal{F}\}$ cofinal in $\mathbb{R}^X$.*
3. *$C_p(X)$ is a large subspace of $\mathbb{R}^X$.*
4. *The strong dual $L_\beta(X)$ of $C_p(X)$ carries the strongest locally convex topology.*

Proof. The bidual of $C_p(X)$ contains the subspace E_0 of all functions of finite support, and if $g \in \mathbb{R}^X$ there is a bounded set P in E_0 with $g \in \overline{P}$, closure in $\mathbb{R}^X$. If (1) holds, there is a bounded set Q in $C_p(X)$ with $P \subseteq \overline{Q}$. So, there is a family $\mathcal{F}$ of bounded sets in $C_p(X)$ such that if $h \in \mathbb{R}^X$ there is $A \in \mathcal{F}$ with $h \in \overline{A}$. As $\sup\{|f(x)| : f \in \overline{A}\} = \phi_A$, clearly $|h| \leq \phi_A$. Thus $\{\phi_A : A \in \mathcal{F}\}$ is cofinal in $\mathbb{R}^X$. If (2) holds, there is a family $\mathcal{F}$ of bounded sets in $C_p(X)$ with $\{\phi_A : A \in \mathcal{F}\}$ cofinal in $\mathbb{R}^X$. So, if P is a bounded set in $\mathbb{R}^X$ and we set $g(x) := \sup\{|g(x)| : g \in P\}$ for all $x \in X$, there is $A \in \mathcal{F}$ with $g \leq \phi_A$. Hence, Lemma 1 provides a bounded set Q_A in $C_p(X)$ such that $P \subseteq P_{\phi_A} \subseteq \overline{Q_A}$. Now, if (3) holds, the strong topology $\beta(L(X), C(X))$ of $L(X)$ coincides with $\beta(L(X), \mathbb{R}^X)$, the strongest locally convex topology. Finally, if statement (4) holds then $L_\beta(X)$ is barrelled, thus $C_p(X)$ is distinguished. □

Corollary 3. *The following statements hold*

1. *If $C_p(X)$ is distinguished, then $bn(X) \geq$ Cof $(\mathbb{R}^X)$.*
2. *If $bn(X) \leq |X|$ then $C_p(X)$ is not distinguished,* [**9**, Theorem 28].
3. *If Y is a subspace of X and $C_p(X)$ is distinguished, then $C_p(Y)$ is distinguished.*
4. *If $|X| = \aleph_0$ then $C_p(X)$ is distinguished,* [**8**, Theorem 3.3].
5. *If X is a continuous injective image of Y and $C_p(X)$ is distinguished, then $C_p(Y)$ is distinguished.*

Proof. (1) Let $\mathcal{M}$ be a fundamental family of bounded sets in $C_p(X)$ with $|\mathcal{M}| = bn(X)$. If $C_p(X)$ is distinguished, by the statement (2) of Theorem 2 there is a family of bounded sets $\mathcal{F}$ in $C_p(X)$ such that $\{\phi_A : A \in \mathcal{F}\}$ is cofinal in $\mathbb{R}^X$. If $A \in \mathcal{F}$ there is $Q_A \in \mathcal{M}$ such that $A \subseteq Q_A$, which implies that $\phi_A \leq \phi_{Q_A}$. This means that $\{\phi_Q : Q \in \mathcal{M}\}$ is cofinal in $\mathbb{R}^X$. Thus Cof $(\mathbb{R}^X) \leq |\{\phi_Q : Q \in \mathcal{M}\}| \leq bn(X)$, as stated.

(2) Assume that $bn(X) = |X|$ and let κ be the first ordinal of cardinality $|X|$. Denote by $\{x : 0 \leq \alpha < \kappa\}$ the elements of X. If $\mathcal{F}$ is a fundamental family of bounded sets with $|\mathcal{F}| = bn(X) = |X|$, let $\mathcal{F} = \{A_\alpha : 0 \leq \alpha < \kappa\}$ and set $\phi_\alpha := \sup\{|f| : f \in A_\alpha\}$. The function $f \in \mathbb{R}^X$ defined by $f(x_\alpha) := \phi_\alpha(x_\alpha) + 1$, verifies that $|f| \nleq \phi_\alpha$ for all $0 \leq \alpha < \kappa$, so $\{\phi_\alpha : 1 \leq \alpha < \kappa\}$ is not cofinal, which contradicts statement (2) of Theorem 2.

(3) If $C_p(X)$ is distinguished, statement (4) of Theorem 2 asserts that $L_\beta(X)$ carries the strongest locally convex topology. Since the map $T : C_p(X) \to C_p(Y)$ given by $Tf = f|_Y$ is continuous and onto, its adjoint $T^* : L_\beta(Y) \to L_\beta(X)$ is a linear homeomorphism. So, $L_\beta(Y)$ carries the strongest locally topology and statement (4) of Theorem 2 applies.

(4) If $|X| = \aleph_0$, $C_p(X)$ is a dense subspace of the metric space $\mathbb{R}^X$. For $g \in \mathbb{R}^X$ choose a sequence $Q_g := \{g_n : n \in \mathbb{N}\}$ in $C(X)$ such that $g_n \to g$ in $\mathbb{R}^X$. Clearly Q_g is bounded in $C_p(X)$ and $|g| \leq \phi_{Q_g}$, which shows that $\mathcal{F} := \{Q_g : g \in \mathbb{R}^X\}$ is a family of bounded sets in $C_p(X)$ such that $\{\phi_Q : Q \in \mathcal{F}\}$ is cofinal in $\mathbb{R}^X$. So, the second statement of Theorem 2 applies.

(5) Let ξ be the original topology of X. By assumption there exists on X a stronger topology γ such that (X, γ) is homeomorphic with Y. Since $C_p(X, \gamma) \subseteq C_p(X, \xi) \subseteq \mathbb{R}^X$, it is enough to apply Theorem 2 to get that $C_p(X, \xi)$ is distinguished. □

For concrete examples of distinguished and nondistinguished $C_p(X)$ spaces, if X is a cosmic space with $|X| = 2^{\aleph_0}$ then $C_p(X)$ is nondistinguished, so $C_p(\mathbb{R})$, $C_p([0,1])$, $C_p(\text{Cantor set})$ are nondistinguished. Also $C_p(\beta\mathbb{N})$, $C_p(\beta\mathbb{Q})$, $C_p(\mathbb{M})$ and $C_p(\mathbb{S})$, where $\mathbb{M}$ and $\mathbb{S}$ are the Michael and the Sorgenfrey lines, respectively, are nondistinguished. Either if $|X| \leq \aleph_0$ or if X is discrete then $C_p(X)$ is countable, so $C_p(\mathbb{Q})$ and $C_p(\mathcal{D}(\mathfrak{m}))$, where $\mathcal{D}(\mathfrak{m})$ denotes the discrete space of cardinality $\mathfrak{m}$, are distinguished. In addition, if X is a scattered Eberlein compact space, then $C_p(X)$ is distinguished. So, if $\mathcal{D}(\mathfrak{m})^\natural$ denotes the one-point compactification of $\mathcal{D}(\mathfrak{m})$ then $C_p(\mathcal{D}(\mathfrak{m})^\natural)$ is distinguished. Also $C_p(X)$ is distinguished whenever X is strongly splittable (see [**9**]).

3. Distinguished $C_p(X) \otimes_\varepsilon E$ spaces

If X is a Tychonoff space, $L(X)$ denotes the algebraic dual of $C_p(X)$. We designate by $L_p(X)$ or $L_\beta(X)$ the space $L(X)$ equipped with the weak* topology $\sigma(L(X), C(X))$ or with the strong topology $\beta(L(X), C(X))$, respectively. Likewise, if E is a lcs, we denote by E'_β the strong dual of E, i.e., the dual E' of E equipped with the strong topology $\beta(E', E)$. If E and F are lcs $E \otimes_\varepsilon F$ represents the *injective* tensor product of E and F and $E \otimes_\pi F$ the *projective* tensor product. A typical neighborhood of the origin in $E'_\beta \otimes_\pi F'_\beta$ is $\pi(A, B) := \operatorname{acx}(A^0 \otimes B^0)$, where A is a bounded set in E and B a bounded set in F. A typical neighborhood of the origin in $E'_\beta \otimes_\varepsilon F'_\beta$ is $\varepsilon(A, B) := (A^{00} \otimes B^{00})^0$, where A is a bounded set in E and B a bounded set in F. Here $A^0 \subseteq E'$, $A^{00} \subseteq E''$, $B^0 \subseteq F'$, $B^{00} \subseteq F''$ and $(A^{00} \otimes B^{00})^0 \subseteq E' \otimes F'$. If E carries the weak topology, by [**18**, 41.3 (9) and 45.1 (2)] we have $(E \otimes_\varepsilon F)' \simeq (E \otimes_\pi F)' \simeq E' \otimes F'$.

Lemma 4. *Let E and F be lcs. If E carries the weak topology, the strong topology $\beta(E' \otimes F', E \otimes F)$ on $E' \otimes F'$ is stronger than the injective topology of $E'_\beta \otimes F'_\beta$.*

Proof. Since E carries the weak topology, $(E \otimes_\varepsilon F)' \simeq E' \otimes F'$. Let us consider a typical neighborhood of the origin in $E'_\beta \otimes_\varepsilon F'_\beta$ of the form $\varepsilon(A, B)$, where A is a bounded absolutely convex set in E and B a bounded absolutely convex set in F. By the bipolar theorem $A^{00} \otimes B^{00} = \overline{A} \otimes \overline{B}$ where the closure $\overline{A}$ is in the weak* topology $\sigma(E'', E')$ of E'' and the closure $\overline{B}$ is in the weak* topology $\sigma(F'', F')$ of F''. Since the weak topology $\sigma(E'' \otimes F'', E' \otimes F')$ on $E'' \otimes F''$ of the dual pair $\langle E'' \otimes F'', E' \otimes F' \rangle$ is coarser than the injective topology $E''(\text{weak}^*) \otimes_\varepsilon F''(\text{weak}^*)$, due to the latter is also a topology of the dual pair $\langle E'' \otimes F'', E' \otimes F' \rangle$, the continuity of the map

$$\otimes : E''(\text{weak}^*) \times F''(\text{weak}^*) \to (E'' \otimes F'', \sigma(E'' \otimes F'', E' \otimes F'))$$

ensures that

$$\overline{A} \otimes \overline{B} = \otimes(\overline{A}, \overline{B}) \subseteq \overline{\otimes(A, B)} = \overline{A \otimes B},$$

the latter closure in $\sigma(E'' \otimes F'', E' \otimes F')$. So $\overline{A \otimes B} \supseteq A^{00} \otimes B^{00}$, which yields

$$(A \otimes B)^0 = \left(\overline{A \otimes B}\right)^0 \subseteq \left(A^{00} \otimes B^{00}\right)^0$$

where the outer polar is in $E' \otimes F'$. Since $A \otimes B$ is a bounded set in $E \otimes_\varepsilon F$, the inclusion $(A \otimes B)^0 \subseteq \varepsilon(A, B)$ shows that $\beta(E' \otimes F', E \otimes F)$ is stronger than the injective topology of $E'_\beta \otimes F'_\beta$, as stated. $\square$

A lcs F has the *bounded approximation property* (b.a.p. for short) if there is an equicontinuous net $\{T_d : d \in D\}$ of finite-rank continuous linear maps from F into F that converges pointwise on F to the identity $id_F : F \to F$.

Theorem 5. *Let E and F be lcs, where E carries the weak topology. If τ_ε and τ_π denote the injective and projective topologies of $E'_\beta \otimes F'_\beta$, the following hold*

1. *If $E'_\beta \otimes_\varepsilon F'_\beta$ is barrelled, then $\tau_\varepsilon = \beta(E' \otimes F', E \otimes F)$ and $E \otimes_\varepsilon F$ is distinguished.*
2. *If $E'_\beta \otimes_\pi F'_\beta$ is barrelled then $\tau_\varepsilon \leq \beta(E' \otimes F', E \otimes F) \leq \tau_\pi$.*
3. *If $E \otimes_\varepsilon F$ is distinguished and F has the b.a.p., then $\tau_\pi \leq \beta(E' \otimes F', E \otimes F)$.*
4. *If $E'_\beta \otimes_\varepsilon F'_\beta$ is barrelled and F has the b.a.p. then $\tau_\varepsilon = \beta(E' \otimes F', E \otimes F) = \tau_\pi$.*

Proof. (1) Since E carries the weak topology, Lemma 4 asserts that the strong topology $\beta(E' \otimes F', E \otimes F)$ on $E' \otimes F' \simeq (E \otimes_\varepsilon F)'$ is stronger than the injective topology τ_ε of $E'_\beta \otimes F'_\beta$. Now, the polars M^0 in $E' \otimes F'$ of the bounded sets M in $E \otimes_\varepsilon F$ compose a base of neighborhoods of the origin of the strong topology $\beta(E' \otimes F', E \otimes F)$. We claim that each M^0 is τ_ε-closed in $E'_\beta \otimes_\varepsilon F'_\beta$. In fact, the dual H of $E'_\beta \otimes_\varepsilon F'_\beta$ verifies that $E \otimes F \subseteq E'' \otimes F'' \subseteq H$. Since M^0 is obviously $\sigma(E' \otimes F', E \otimes F)$-closed, it is $\sigma(E' \otimes F', H)$-closed. Hence M^0 is τ_ε-closed, as stated. As by assumption $E'_\beta \otimes_\varepsilon F'_\beta$ is barrelled and M^0 is a barrel in $E'_\beta \otimes_\varepsilon F'_\beta$, it follows that M^0 is a neighborhood of the origin in $E'_\beta \otimes_\varepsilon F'_\beta$. This shows that $\beta(E' \otimes F', E \otimes F) = \tau_\varepsilon$. Thus $(E \otimes_\varepsilon F)'_\beta$ is barrelled and $E \otimes_\varepsilon F$ is distinguished.

(2) According to Lemma 4, relation $\tau_\varepsilon \leq \beta(E' \otimes F', E \otimes F)$ always holds whenever E carries the weak topology. On the other hand, if G denotes the dual of $E'_\beta \otimes_\pi F'_\beta$, the fact that $E \otimes F \subseteq E'' \otimes F'' \subseteq G$ yields $\beta(E' \otimes F', G) \geq \beta(E' \otimes F', E \otimes F)$. As we are assuming that $E'_\beta \otimes_\pi F'_\beta$ is barrelled, then $\beta(E' \otimes F', G) = \tau_\pi$, so $\beta(E' \otimes F', E \otimes F) \leq \tau_\pi$.

(3) Let $\varphi \in \mathcal{B}(E' \times F')$ and denote by $\dot{\varphi}$ its linearization. Let $\{T_d : d \in D\}$ be an equicontinuous net in $\mathcal{L}(F'_\beta, F'_\beta)$ consisting of continuous linear maps of finite ranks that converges pointwise to the identity map $id_{F'}$. Since $T_d(F')$ is finite-dimensional, $E'_\beta \otimes_\varepsilon T_d(F) = E'_\beta \otimes_\pi T_d(F)$. Hence the linear forms $\dot{\varphi} \circ (id_E \otimes T_d)$ are continuous on $E'_\beta \otimes_\varepsilon F'_\beta$. So, $\{T_d : d \in D\}$ equicontinuity entails that $M := \{\dot{\varphi} \circ (id_E \otimes T_d) : d \in D\}$ is a net in $(E'_\beta \otimes_\varepsilon F'_\beta)' \subseteq L := (E' \otimes F', \beta(E' \otimes F', E \otimes F))'$ pointwise bounded on $E' \otimes F'$. Thus, the barrelledness of $(E' \otimes F', \beta(E' \otimes F', E \otimes F)) = (E \otimes_\varepsilon F)'_\beta$ ensures that the weak* closure $\overline{M}$ of M in L is weak* compact. But $\dot{\varphi} \circ (id_E \otimes T_d) \to \dot{\varphi}$ pointwise on $E' \otimes F'$, so $\varphi \in \overline{M} \subseteq L$. This shows that $(E'_\beta \otimes_\varepsilon F'_\beta)' \simeq \mathcal{B}(E' \times F') \subseteq L$. Since we have by assumption that $\beta(E' \otimes F', E \otimes F) = \beta(E' \otimes F', L)$, one has $\beta(E' \otimes F', E \otimes F) \geq \beta(E' \otimes F', \mathcal{B}(E' \times F')) \geq \tau_\pi$.

(4) This is consequence of the previous statements. □

Corollary 6. *Assume that $C_p(X)$ is distinguished. If $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$ is barrelled, the space $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. If $C_p(X)$ is distinguished then $L_\beta(X) = \mathbb{R}^{(X)}$, so that our assumption ensures that $L_\beta(X) \otimes_\varepsilon E'_\beta$ is barrelled. Thus $C_p(X) \otimes_\varepsilon E$ is distinguished by the first statement of Theorem 5. □

Example 7. *If X and Y are countable then $C_p(X) \otimes_\varepsilon C_p(Y)$ is distinguished.* Indeed, if X and Y are countable, both $C_p(X)$ and $C_p(Y)$ are distinguished by the fourth statement

Corollary 3. Hence $C_p(Y)'_\beta \simeq \mathbb{R}^{(\mathbb{N})}$ by part four of Theorem 2. Hence

$$\mathbb{R}^{(\mathbb{N})} \otimes_\varepsilon C_p(Y)'_\beta = \mathbb{R}^{(\mathbb{N})} \otimes_\varepsilon \mathbb{R}^{(\mathbb{N})} = \mathbb{R}^{(\mathbb{N})} \otimes_\pi \mathbb{R}^{(\mathbb{N})},$$

which is barrelled since $\mathbb{R}^{(\mathbb{N})}$ is a barrelled , [**18**, 41.4.(8) a)], and Corollary 6 applies.

Remark 8. *A necessary condition for $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$ to be barrelled is that E contains no complemented copy of $\mathbb{R}^{(\mathbb{N})}$.* If E contains a complemented copy of $\mathbb{R}^{(\mathbb{N})}$ then E'_β contains a complemented copy of $\mathbb{R}^{\mathbb{N}}$. Since $\mathbb{R}^{(\mathbb{N})}$ is a complemented subspace of $\mathbb{R}^{(X)}$, it follows that the non barrelled space $\mathbb{R}^{(\mathbb{N})} \otimes_\varepsilon \mathbb{R}^{\mathbb{N}}$, [**14**, 15.5.2 Remark], is linearly homeomorphic to a complemented subspace of $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$. Consequently $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$ cannot be barrelled.

Corollary 9. *If X is countable and E is a distinguished metrizable lcs, or if $C_p(X)$ is distinguished and E finite-dimensional, then $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. In the first case $\mathbb{R}^{(X)}$ is nuclear, so that $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta = \mathbb{R}^{(X)} \otimes_\pi E'_\beta$. Now $\mathbb{R}^{(X)}$ and E'_β, as strong duals of distinguished metrizable spaces, are barrelled (DF)-spaces. So, the projective tensor product $\mathbb{R}^{(X)} \otimes_\pi E'_\beta$ is barrelled (see [**18**, 41.4.(8) a)]). In the second case E is nuclear, so $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta = \mathbb{R}^{(X)} \otimes_\pi E'_\beta \simeq \left(\mathbb{R}^{(X)}\right)^{\dim E}$is also barrelled. ☐

According to [**4**] the Fréchet space $C_k(K, E)$ is distinguished if E is reflexive Fréchet and K compact. Hence $\ell_\infty \otimes_\varepsilon E \simeq C_k(\beta\mathbb{N}) \otimes_\varepsilon E$ is distinguished as being a (dense) large subspace of $C_k(\beta\mathbb{N}, E)$. If $(c_0)_p$ denotes the space of all sequences in $\mathbb{R}$ converging to 0 endowed with the topology of $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{N}^{\#} = \mathbb{N} \cup \{\xi\}$ for $\xi \notin \mathbb{N}$ is the one-point compactification of $\mathbb{N}$, then $C_p(\mathbb{N}^{\#})$ is linearly isomorphic to $(c_0)_p$, so we have

Corollary 10. *For a metrizable lcs E the space $(c_0)_p \otimes_\varepsilon E$ is distinguished if and only if E is distinguished.*

Corollary 11. *Let X be a Tychonoff space and E be a normed space. If $C_p(X)$ is distinguished, then $\tau_\varepsilon \le \beta(L(X) \otimes E', C(X) \otimes E) \le \tau_\pi$.*

Proof. If $C_p(X)$ is distinguished, then $L_\beta(X) = \mathbb{R}^{(X)}$ is barrelled. Since E'_β is a Banach space, $L_\beta(X) \otimes_\pi E'_\beta$ is barrelled by [**11**, Theorem 1.6.6]. So the corollary is a straightforward consequence of the second statement of Theorem 5. ☐

We need the following two lemmas for the next Theorem 14.

Lemma 12. *Let X be a Tychonoff space and E be a Fréchet space. If X is countable, then $C_p(X) \otimes_\pi E$ is a large subspace of $\mathbb{R}^X \widehat{\otimes}_\pi E$.*

Proof. Since $\mathbb{R}^X$ is a nuclear Fréchet space and E a Fréchet space, if M is a bounded set in $\mathbb{R}^X \widehat{\otimes}_\pi E$ then, according to [**14**, 21.5.8 Theorem], there are bounded sets $A \subseteq \mathbb{R}^X$ and $B \subseteq E$ such that $M \subseteq \overline{\operatorname{acx}(A \otimes B)}$, where the closure is in $\mathbb{R}^X \widehat{\otimes}_\pi E$. On the other hand, as $C_p(X)$ is distinguished by part (4) of Corollary 3, statement (3) of Theorem 2 provides a bounded set $Q \subseteq C_p(X)$ such that $A \subseteq \overline{Q}$, where the closure is in $\mathbb{R}^X$. Hence

$$M \subseteq \overline{\operatorname{acx}(A \otimes B)}^{\mathbb{R}^X \widehat{\otimes}_\pi E} \subseteq \overline{\operatorname{acx}(\overline{Q}^{\mathbb{R}^X} \otimes B)}^{\mathbb{R}^X \widehat{\otimes}_\pi E} = \overline{\operatorname{acx}(Q \otimes B)}^{\mathbb{R}^X \widehat{\otimes}_\pi E} \tag{3.1}$$

because of the continuity of the map $\otimes : \mathbb{R}^X \times E \to \mathbb{R}^X \otimes_\pi E$ guarantees that

$$\overline{Q}^{\mathbb{R}^X} \otimes B \subseteq \overline{Q}^{\mathbb{R}^X} \otimes \overline{B}^E \subseteq \overline{Q \otimes B}^{\mathbb{R}^X \otimes_\pi E} \subseteq \overline{\operatorname{acx}(Q \otimes B)}^{\mathbb{R}^X \widehat{\otimes}_\pi E}.$$

Since $\operatorname{acx}(Q \otimes B)$ is a bounded set in $C_p(X) \otimes_\pi E$, relation (3.1) shows that $C_p(X) \otimes_\pi E$ is a large subspace of $\mathbb{R}^X \widehat{\otimes}_\pi E$. $\square$

If F is a lcs and λ a perfect sequence space, let $\lambda\{F\}$ be the space of all sequences $y = (y_n)_{n\in\mathbb{N}}$ in F such that $\sum_{n=1}^{\infty} |u_n|\, p(y_n) < \infty$ for each $\mathfrak{u} = (u_n)_{n\in\mathbb{N}} \in \lambda^\times$ and each continuous seminorm p on F. The *normal topology* of $\lambda\{F\}$ is defined by the seminorms $\pi_{\mathfrak{u},p}(y) = \sum_{n=1}^{\infty} |u_n|\, p(y_n)$, where $\mathfrak{u} \in \lambda^\times$ and p is a continuous seminorm on F. Recall that $\lambda\{\widehat{F}\} \simeq \lambda \widehat{\otimes}_\pi F$ when λ is equipped with the normal topology (see [**18**, 41.7]). In particular, if λ coincides with $\mathbb{R}^{(\mathbb{N})}$ and F is complete then $\mathbb{R}^{(\mathbb{N})}\{F\} \simeq \mathbb{R}^{(\mathbb{N})} \widehat{\otimes}_\pi F$.

Lemma 13. *If X is a countable Tychonoff space and E a Fréchet space, then*

$$(C_p(X) \otimes_\varepsilon E)'_\beta \simeq (C_p(X) \,\widehat{\otimes}_\varepsilon E)'_\beta \simeq \mathbb{R}^{(\mathbb{N})}\left\{E'_\beta\right\} \simeq L_\beta(X) \otimes_\pi E'_\beta. \tag{3.2}$$

Proof. Since $C_p(X)$, as a subspace of the nuclear space $\mathbb{R}^X$, is nuclear, Lemma 12 applies to get that $C_p(X) \otimes_\varepsilon E = C_p(X) \otimes_\pi E$ is a large subspace of $\mathbb{R}^X \widehat{\otimes}_\pi E = \mathbb{R}^X \widehat{\otimes}_\varepsilon E$. This implies that

$$(C_p(X) \otimes_\varepsilon E)'_\beta = (\mathbb{R}^X \widehat{\otimes}_\varepsilon E)'_\beta \simeq (C_p(X) \,\widehat{\otimes}_\varepsilon E)'_\beta,$$

where the last isomorphism holds because $C_p(X) \,\widehat{\otimes}_\varepsilon E$ is linearly homeomorphic to $\mathbb{R}^X \widehat{\otimes}_\varepsilon E$, [**18**, 44.5.(1)]. Bearing in mind that $\mathbb{R}^X$ is a nuclear Fréchet space, [**14**, 21.5.9 Theorem] guarantees that

$$(\mathbb{R}^X \widehat{\otimes}_\varepsilon E)'_\beta = (\mathbb{R}^X \widehat{\otimes}_\pi E)'_\beta \simeq L_\beta(X) \,\widehat{\otimes}_\pi E'_\beta \simeq \mathbb{R}^{(\mathbb{N})}\left\{E'_\beta\right\},$$

As $(C_p(X) \otimes_\varepsilon E)' \simeq L(X) \otimes E'$, necessarily $L_\beta(X) \otimes_\pi E'_\beta = L_\beta(X) \,\widehat{\otimes}_\pi E'_\beta$. So, relations (3.2) hold. $\square$

The following theorem characterizes distinguished Fréchet spaces. Recall that $C_p(X)$ is metrizable if and only if X is countable.

Theorem 14. *If X is a countable Tychonoff space and E a Fréchet space, are equivalent*

1. *E is distinguished.*
2. *Metrizable space $C_p(X) \otimes_\varepsilon E$ is distinguished.*
3. *Fréchet space $C_p(X) \,\widehat{\otimes}_\varepsilon E$ is distinguished.*

Proof. Implication (1) $\Rightarrow$ (2) is included in Corollary 9. On the other hand, if condition (1) holds, $L_\beta(X) \otimes_\pi E'_\beta$ is barrelled by [**18**, 41.4.(8) a)] and Lemma 13 ensures that $C_p(X) \,\widehat{\otimes}_\varepsilon E$ is distinguished. Consequently, (1) implies both (2) and (3). Since E is linearly homeomorphic to a complemented subspace of $C_p(X) \otimes_\pi E$ and $C_p(X) \,\widehat{\otimes}_\pi E$, either (2) or (3) imply (1). $\square$

If E and F are locally convex space, following [**14**, 15.3] we set

$$\mathcal{L}(E, (F'_\beta)'_{\text{cont}}) = \bigcup \left\{ \mathcal{L}(E, (F'_\beta)'_{A^{00}}) : \forall \text{ absolutely convex bounded set } A \text{ in } F \right\},$$

where $(F'_\beta)'_{A^{00}}$ indicates the Banach space consisting of the linear span in the bidual F'' of F of the compact set $\overline{A} = A^{00}$ in F'' (weak*), equipped with the Minkowski functional of A^{00} as a norm.

Proposition 15. *Let E be a lcs. If $C_p(X)$ is distinguished then*

$$\left(L_\beta(X) \otimes_\pi E'_\beta\right)' \simeq L(L_\beta(X), \left(E'_\beta\right)'_{\text{cont}}).$$

Proof. Since $C_p(X)$ is distinguished, $L_\beta(X)$ carries the strongest locally convex topology by part (4) of Theorem 2. Consequently, by [**14**, 15.3.3 Proposition] one has

$$\left(L_\beta(X) \otimes_\pi E'_\beta\right)' \simeq \mathcal{B}\left(L_\beta(X) \times E'_\beta\right) \simeq \mathcal{L}(L_\beta(X), \left(E'_\beta\right)'_{\text{cont}}).$$

But $\mathcal{L}(L_\beta(X), \left(E'_\beta\right)'_{\text{cont}}) = L(L_\beta(X), \left(E'_\beta\right)'_{\text{cont}})$, since $L_\beta(X)$ is a locally convex space with the strongest locally convex topology. □

Corollary 16. *If X is a countable Tychonoff space and E is a Fréchet space, the bidual $(C_p(X) \otimes_\varepsilon E)''$ of $C_p(X) \otimes_\varepsilon E$ is linearly isomorphic to $L(L_\beta(X), \left(E'_\beta\right)'_{\text{cont}})$, i. e., $(C_p(X) \otimes_\varepsilon E)'' \simeq L(L_\beta(X), \left(E'_\beta\right)'_{\text{cont}})$.*

Proof. According to Lemma 13, we have $(C_p(X) \otimes_\varepsilon E)'_\beta \simeq L_\beta(X) \otimes_\pi E'_\beta$. Since $C_p(X)$ is distinguished, Proposition 15 yields the conclusion. □

4. Distinguished $C_p(X, E)$ spaces

If E is a real lcs, we denote by $C_p(X, E)$ the linear space of all E-valued continuous functions $C(X, E)$ equipped with the pointwise topology, and by $C_k(X, E)$ we designate $C(X, E)$ endowed with the compact-open topology. As usual, if $E = \mathbb{R}$ we write $C_p(X)$ and $C_k(X)$, instead of $C_p(X, \mathbb{R})$ and $C_k(X, \mathbb{R})$. Observe that $F \in C(X, E)$ if $x_d \to x$ in X implies that $F(x_d) \to F(x)$ in E, and if Δ is a finite (compact) set in X and V a neighborhood of the origin in E, a neighborhood of the origin $W[\Delta, V]$ in the space $C_p(X, E)$ (resp. in $C_k(X, E)$) consists of those $F \in C(X, E)$ such that $F(x) \in V$ for each $x \in \Delta$. A set Q is bounded in $C_p(X, E)$ if for each $x \in X$ and each continuous seminorm p in E one has that $\sup_{F \in Q} p(F(x)) < \infty$.

Proposition 17. *If X is a Tychonoff space and Y a discrete space, are equivalent*

1. *The space $C_p(X)$ is distinguished*
2. *The space $C_p(X, \mathbb{R}^Y)$ is distinguished*
3. *The space $C_p(X \times Y)$ is distinguished.*

Proof. $1 \Rightarrow 2$. Note that $C_p\left(X, \mathbb{R}^Y\right)$ is linearly homeomorphic to $C_p(X, \mathbb{R})^Y = C_p(X)^Y$ via the canonical map S given by $((SF)(y))(x) = (F(x))(y)$, [**1**, 0.3.3 Proposition]. Since the strong dual of the product space $C_p(X)^Y$ is linearly homeomorphic to the locally convex direct sum $\oplus_{y \in Y} L_\beta(X_y)$ with $X_y = X$ for every $y \in Y$, and the locally convex direct sum of barrelled spaces is barrelled, if $C_p(X)$ is distinguished, it follows that $(C_p(X))^Y$ is distinguished.

$2 \Rightarrow 3$. The canonical map $T : C_p(X \times Y) \to C_p(X, \mathbb{R}^Y)$ given by $(Tf)(x) = f_x$, where $f_x(y) = f(x, y)$ for each $y \in Y$, is linear, well-defined and one-to-one. If $F \in C(X, \mathbb{R}^Y)$, define as usual $f(x, y) = (F(x))(y)$ for each $(x, y) \in X \times Y$. Since if $(x_d, y_d) \to (x, y)$ in $X \times Y$ there is some k such that $y_d = y$ for $d \geq k$, it can be easily seen that $f(x_d, y_d) \to f(x, y)$. Hence $f \in C(X \times Y)$, which proves that T is onto. Clearly, T is a linear homeomorphism from $C_p(X \times Y)$ onto $C_p(X, \mathbb{R}^Y)$.

$3 \Rightarrow 1$. Since X is homeomorphic to a (closed) subspace of $X \times Y$, if $C_p(X \times Y)$ is distinguished, then $C_p(X)$ is distinguished by Corollary 3. □

Remark 18. If we regard $X \times Y$ as the topological sum of $|Y|$ copies of X, the implication $1 \Rightarrow 3$ is also consequence of [**9**, Proposition 19].

The following proposition identifies up to a linear isomorphism the dual of the space $C_p(X, E)$ for an arbitrary Tychonoff space X and any lcs E.

Proposition 19. *If X is a Tychonoff space and E a lcs, the dual of $C_p(X, E)$ is algebraically isomorphic to $L(X) \otimes E'$, i. e., $C_p(X, E)' \simeq L(X) \otimes E'$.*

Proof. The canonical map $T : C_p(X) \otimes_\varepsilon E \to C_p(X, E)$ given by $(T\varphi)(x) = \sum_{i=1}^n f_i(x) u_i$, where $\sum_{i=1}^n f_i \otimes u_i$ is a representation of $\varphi \in C(X) \otimes E$, is linear and one-to-one. If $\Delta \subseteq X$ is finite, V a neighborhood of zero in E and q_V its Minkowski functional, the equality

$$\sup_{x \in \Delta} q_V(T\varphi(x)) = \sup\left\{\left|\sum_{i=1}^n f_i(x)\langle v, u_i\rangle\right| : x \in U^0, v \in V^0\right\} = \varepsilon_{U,V}(\varphi)$$

where $\varepsilon_{U,V}$ is the ε-seminorm defined by $U = \Delta^0$ and V, shows that T is a linear homeomorphism from $C_p(X) \otimes_\varepsilon E$ into the subspace $\operatorname{Im} T$ of $C_p(X, E)$. Now $\operatorname{Im} T$ is a dense linear subspace of $C_p(X, E)$. For if $F \in C(X, E)$ and $\Delta = \{x_1, \ldots, x_n\}$ is a nonempty finite set in X, choosing $g_i \in C(X)$ with $0 \leq g_i \leq 1$ such that $g_i(x_i) = 1$ and $g_i(x_j) = 0$ if $i \neq j$, and setting

$$\varphi_{F,\Delta} := \sum_{i=1}^n g_i \otimes F(x_i) \in C(X) \otimes E.$$

clearly $T\varphi_{F,\Delta}(x) = F(x)$ for every $x \in \Delta$, which shows that $C_p(X) \otimes_\varepsilon E$ is linearly homeomorphic to a dense linear subspace $\operatorname{Im} T$ of $C_p(X, E)$. Now, since $(C_p(X) \otimes_\varepsilon E)' \simeq L(X) \otimes E'$, we get $C_p(X, E)' \simeq L(X) \otimes E'$. □

Clearly $\langle w, F\rangle = \sum_{i=1}^p \langle v_i, F(x_i)\rangle$ for $w = \sum_{i=1}^p \delta_{x_i} \otimes v_i \in L(X) \otimes E'$ and $F \in C(X, E)$.

Theorem 20. *Let X be a Tychonoff space. If E is a normed space, then $C_p(X) \otimes_\varepsilon E$ is linearly homeomorphic to a large subspace of $C_p(X, E)$.*

Proof. If V is a neighborhood of the origin in E, the canonical linear homeomorphism $T : C_p(X) \otimes_\varepsilon E \to C_p(X, E)$ mentioned above tell us that $\sup_{x \in \Delta} q_V(T\varphi(x)) = \varepsilon_{U,V}(\varphi)$ for every $x \in \Delta$, where q_V is the Minkowski functional of V and $U = \Delta^0$. Let us show that $\operatorname{Im} T$ is a large subspace of $C_p(X, E)$.

Let Q be a bounded set in $C_p(X, E)$. Let $F \in Q$, $\Delta = \{x_1, \ldots, x_n\}$ and $\epsilon > 0$ be given. Choose n pairwise disjoint open sets U_i with $x_i \in U_i$, and for each $1 \leq i \leq n$ select $h_{F,i} \in C(X)$ such that $0 \leq h_{F,i} \leq 1$ with $h_{F,i}(x_i) = 1$ and $h_{F,i}(x) = 0$ if $x \notin U_i$. If $F(x_i) \neq \mathbf{0}$ define $g_{F,i}(x) = \inf\left\{h_{F,i}(x), \|F(x)\| \|F(x_i)\|^{-1}\right\}$, then put

$$\varphi_{F,\Delta} = \sum_{i=1}^n g_{F,i} \otimes F(x_i) \in C(X) \otimes E.$$

Since $T\varphi_{F,\Delta}(x) = F(x)$ for every $x \in \Delta$, we have $\left\|T\varphi_{F,\Delta}(x) - F(x)\right\| = 0 < \epsilon$ for every $x \in \Delta$. Clearly, the set

$$P := \{T\varphi_{F,\Delta} : F \in Q,\ \Delta \subseteq X,\ \Delta \text{ finite}\}$$

is contained in $\operatorname{Im} T$ and verifies that $Q \subseteq \overline{P}$, closure in $C_p(X, E)$. In addition, one has

$$\left\|T\varphi_{F,\Delta}(x)\right\| \leq \sum_{i=1}^{n} g_{F,i}(x)\left\|F(x_i)\right\| \leq \|F(x)\|$$

for every $x \in X$, which guarantees that P is pointwise bounded. □

Corollary 21. *Let X be a Tychonoff space and let E be a normed space. Then $C_p(X, E)$ is distinguished if and only if $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. By Theorem 20 the strong topology $\beta(L(X) \otimes E', C(X, E))$ of the dual $L(X) \otimes E'$ of $C_p(X, E)$ coincides with the topology $\beta(L(X) \otimes E', C(X) \otimes E)$. Thus the strong dual of $C_p(X, E)$ is barrelled if and only if the strong dual of $C_p(X) \otimes_\varepsilon E$ is barrelled, i. e., if and only if $C_p(X) \otimes_\varepsilon E$ is distinguished. □

Corollary 22. *Let $C_p(X)$ be distinguished and E be a normed space. If $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$ is barrelled then $C_p(X, E)$ is distinguished. So, if X is countable, $C_p(X, E)$ is distinguished.*

Proof. If $C_p(X)$ be distinguished, the space $C_p(X) \otimes_\varepsilon E$ is distinguished by Corollary 6. Since E is normed, $C_p(X, E)$ is distinguished by Corollary 21. If X is countable, then $\mathbb{R}^{(X)}$ is both barrelled and nuclear. So $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta = \mathbb{R}^{(X)} \otimes_\pi E'_\beta$ is barrelled by [**11**, Theorem 1.6.6]. □

Data availability

Not applicable

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Code availability

Not applicable

Authors' contributions

Both authors have contributed in the same proportion (results and writing) to the final manuscript.

References

1 Arkhangel'skiĭ, A. V., *Topological Function Spaces*, Kluwer, Dordrecht, 1992.

2 Bierstedt, D., Bonet, J., *Density conditions in Fréchet and (DF)-spaces*, Rev. Mat. Univ. Complut. Madrid **2** (1989), 59–75.

3 Bierstedt, D., Bonet, J., *Some aspects of the modern theory of Fréchet spaces*, RACSAM **97** (2003), 159–188.

4 Díaz, J. C., Domański, P., *On the Injective Tensor Product of Distinguished Fréchet spaces*, Math. Nachr. **198** (1999), 41–50.

5 Díaz, J. C., Minarro, M. A. *Distinguished Fréchet spaces and projective tensor products*, Doga Mat. **14**, (1990) 191–208.

6 Dierolf, S., *On spaces of continuous mappings between locally convex spaces*, Habilitationsschrift, Universität München, 1984.

7 Dieudonne, J., Schwartz, L., *La dualite dans les espaces (F) et (LF)* Ann. Inst. Fourier (Grenoble) **1** (1949), 61–101.

8 Ferrando, J. C., Kąkol, J., *Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces*, J. Convex. Anal. **26** (2019), 1337–1346.

9 Ferrando, J. C., Kąkol, J., Leiderman, A., Saxon, S. A., *Distinguished $C_p(X)$ spaces*, RACSAM (2021) **115**:27.

10 Ferrando, J. C., Kąkol, J., Saxon, S. A., *Examples of Nondistinguished Function Spaces $C_p(X)$*, J. Convex. Anal. **26** (2019), 1347–1348.

11 Ferrando, J. C., López-Pellicer, M., Sánchez Ruiz, L. M., *Metrizable Barrelled Spaces*, Longman, Harlow, 1995.

12 Ferrando, J. C., Saxon, S. A., *If not distinguished, is $C_p(X)$ even close*?, Proc. Amer. Math. Soc, to appear.

13 Grothendieck, A., *Produits tensoriels topologiques et espaces nucleaires*, Mem. Amer. Math. Soc. **16** (1955).

14 Jarchow, H., *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

15 Kąkol, J., Leiderman, A., *A characterization of X for which spaces $C_p(X)$ are distinguished and its applications*, Proc. Amer. Math. Soc, to appear.

16 Knight, R. W., *Δ-Sets*, Trans. Amer. Math. Soc. **339** (1993), 45–60.

17 Köthe, G., *Topological Vector Spaces I*, Springer-Verlag, Berlin, 1983.

18 Köthe, G., *Topological Vector Spaces II*, Springer-Verlag, Berlin, 1979.

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