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# Abstract evolution systems

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**Abstract**

We introduce the concept of an abstract evolution system, which provides a convenient framework for studying generic mathematical structures and their properties. Roughly speaking, an *evolution system* is a category endowed with a selected class of morphisms called *transitions*, satisfying certain natural conditions. We illustrate our ideas by a series of examples from several areas of mathematics.

Evolution systems can also be viewed as a generalization of abstract rewriting systems, where the partially ordered set is replaced by a category. In our setting, the *process* of rewriting plays a nontrivial role, whereas in rewriting systems only the result of a reduction/rewriting is relevant. An analogue of Newman's Lemma holds in our setting, although the proof is a bit more delicate, nevertheless, still based on Huet's idea using well founded induction.



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# Contents




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# 1 Introduction

Let us imagine that a certain evolution or transition system is given, starting from some initial state $E_0$ and having the *amalgamation property*, namely, for every two transitions $f$, $g$ from the same state $z$ it is possible to make several further transitions $f_1, \dots, f_n$, $g_1, \dots, g_m$ so that the compositions

$$f_n \circ \ldots \circ f_1 \circ f \qquad \text{and} \qquad g_m \circ \ldots \circ g_1 \circ g$$

are the same, in particular, leading to the same state. It is then natural to expect that there exists a special infinite process (evolution) accumulating all possible states and in some sense recording all possible transitions. Specifically, denoting a fixed transition $f$ from $A$ to $B$ by $A \xrightarrow{f} B$, an infinite evolution process can be represented as an infinite diagram of the form

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots$$

where $A_0$ is the initial state $E_0$. Saying that such a process *absorbs* all possible transitions means that for every $n$, given a transition $A_n \xrightarrow{f} Y$, there exist $m > n$ and a sequence of transitions

$$Y \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_k} Y_{k+1} = A_m$$

such that the composition $f_{m-1} \ldots f_n$ is the same as $g_k \circ \ldots \circ g_0 \circ f$ or at least approximates $g_k \circ \ldots \circ g_0 \circ f$ with a given in advance error. It turns out that if a process with the absorption property exists, it is essentially unique. This means that every two processes with the absorption property are isomorphic, which in turn means that there is a way of "jumping" between the first and the second one infinitely many times (technical details are explained below).

It is rather evident that the proper framework for describing and studying evolution systems comes from category theory. Namely, the objects are the system states while the arrows are transitions or, more generally, compositions of transitions. Category theory offers elegant and quite strong, yet at the same time manageable, tools. Perhaps one of the basics is the concept of a functor. In order to describe an infinite evolution process, it suffices to use functors from the set of non-negative integers $\omega$ (more traditionally denoted by $\mathbb{N}$) viewed as a category in which the objects are natural numbers and arrows are pairs of the form $\langle n, m \rangle$ with $n \le m$.

Actually, a significant power of category theory lies within the notion of *colimit* of a functor, unifying concepts like *supremum* in partially ordered sets, Cartesian products, unions of families of structures, and so on. In particular, every functor from the natural numbers has its colimit in a suitable, possibly bigger, category.

While it is possible to investigate infinite evolutions in their "pure" form, it is more convenient and more natural to look at their colimits, identifying them with the isomorphism classes of certain objects in a bigger category. As mentioned above, actually we get an isomorphism class of a single object, which typically has many symmetries. This is due to *homogeneity*, saying that every transition between two states of evolution processes with the absorption property can be extended to (or at least approximated by) an isomorphism between these processes.

Our goal is to introduce and study abstract evolution systems, focusing on evolutions with the absorption property. In particular, we show obvious connections with the theory of universal homogeneous structures. We also present several illustrative examples.

The note is organized as follows. We start with motivating examples (Section 2) and we introduce formal definitions in Section 3. Next we present a variant of the theory of universal homogeneous structures (Section 4), with a result involving a natural infinite game. The last Section 6 contains more examples and a discussion of possible further research.

**Historical remarks.** While the concept of an evolution system is formally new (although the ideas are at least as old as the theory of categories), the main results are just adaptation of abstract *Fraïssé theory* of universal homogeneous structures. It was created by Roland Fraïssé [2] in the fifties of the last century, in the language of model theory. Namely, Fraïssé observed that Cantor's theorem characterizing the linearly ordered set of the rational numbers can actually be stated and proved in a much more general setup, using any first-order language. Actually, Fraïssé considered relational structures only, however adding algebraic operations does not change much. The main result of Fraïssé was that, given a suitable class of finite structures satisfying certain natural conditions (including the amalgamation property), there exists a unique countable ultra-homogeneous structure from which one can reconstruct the original class. Ultra-homogeneity (often called just *homogeneity*) means that every isomorphism between finite substructures extends to an automorphism. Fraïssé theory was extended by Jónsson [5] to uncountable structures, with extra cardinal arithmetic assumptions. As it happens, Fraïssé theory actually has a purely category-theoretic nature, although it was formally stated this way (almost forty years after Fraïssé's work) by Droste and Göbel [1] and few years ago explored by the first author [11, 9], also in metric-enriched categories. Let us mention that Fraïssé theory received a lot of attention after the seminal work of Kechrs, Pestov, and Todorcevic [6] that discovers a correspondence between dynamic properties of the automorphism group of the Fraïssé limit and combinatorial properties of the Fraïssé class. Currently there are several lines of research exploring various faces of the theory of universal homogeneous structures, mainly in model theory (including its continuous variant dealing with metric structures) and in pure category theory. Summarizing, mathematical objects resembling Fraïssé limits appear in several areas of mathematics and one of the goals of this note is to present some of them, through the "looking glass" of evolution systems.

# 2 Some examples

Before going into technical details, we now present several motivating examples that could fit into our framework.

## 2.1 Cracking the glass

Let us look at a very natural evolution process starting from a nice "glass" rectangle $S$ (or any other polygon made of glass), whose transitions are formed by breaking the glass into smaller and smaller pieces. A state in this system is a finite family of pairwise disjoint polygons that can be glued together, recovering $S$. By *cracking* a polygon $P$ we mean replacing it by polygons $Q_1, \dots, Q_k$ that play the role of the "pieces" of $P$, namely, all of them can be translated in such a way that the union is $P$ and their interiors are pairwise disjoint. A *transition* from a state $\{P_1, \dots, P_n\}$ to a state $\{Q_1, \dots, Q_m\}$ is cracking each $P_i$ (or just a selected one) and just collecting the pieces together. Actually, one should keep track of the possible gluing back, as it is not unique in general. This can be easily achieved by keeping in mind a continuous surjective mapping $f \colon \bigcup_{i=1}^m Q_i \to \bigcup_{j=1}^n P_j$ performing the gluing, namely, it is an isometry on each $Q_i$ and each $P_j$ is the union of images of some $Q_i$s with pairwise disjoint interiors. So after all, a transition from a state $\{P_1, \dots, P_n\}$ to a state $\{Q_1, \dots, Q_m\}$ is a suitable continuous function from $\bigcup_{i=1}^m Q_i$ onto $\bigcup_{j=1}^n P_j$ that 'memorizes' the cracking in the sense that the function knows how to glue the pieces back. It is rather clear that this transition system has the amalgamation property: Given two crackings of a polygon $P$, there is a very concrete cracking of $P$ refining both, simply by intersecting all the pieces.

The infinite process of cracking the initial glass polygon $S$ leads to a compact planar set $C$, the inverse limit of the sequence of "cracking" mappings. If the process has the absorption property, $C$ is homeomorphic to the well known Cantor set. What is the conclusion? Well, one can say that no matter how nice the initial glass polygon is, after cracking it thoroughly infinitely many times, we always receive the same "dust", namely, the Cantor set whose geometry is rather mediocre, as it lacks nontrivial connected subsets.

Summarizing, there exist natural evolution processes, like the one described above, whose limits might be of different nature and outside of the real world. Nevertheless, investigating such processes and their limits may lead to a better understanding of the original evolution system.

## 2.2 Ribbons

Given a flexible ribbon, it is easy to imagine many possibilities of folding it so that after squeezing (and possibly gluing) the material we obtain another ribbon. Such a transition can be represented by a continuous surjection $f\colon I \to J$, where $I$, $J$ are the two ribbons, namely compact intervals of the real numbers. Since each two such intervals have exactly the same structure, we may assume $I = J = [0, 1]$. The process of folding and squeezing the ribbon without reverting it can be recorded as a continuous surjection $f\colon [0, 1] \to [0, 1]$. Thus, we are dealing with an evolution system where all the states are identical, however the transitions could be quite complicated. This is indeed the case, as the natural limit of the process with the absorption property is the *pseudo-arc*, a rather intriguing planar geometric object. Mathematically, this is the unique, up to homeomorphism, compact connected subset $\mathbb{P}$ of the plane which from a far distance looks like the interval (formally it is called *chainable* or *arc-like*), while at the same time it cannot be written as the union of two connected proper closed subsets. Furthermore, every nontrivial closed connected subset of $\mathbb{P}$ is homeomorphic to $\mathbb{P}$.

The amalgamation property of this system is known under the name *Mountain Climbing Theorem*, saying that for each two reasonable (say, piecewise monotone) continuous surjections $f, g$ from the unit interval onto itself there exist continuous surjections $f'$, $g'$ on the unit interval satisfying $f \circ f' = g \circ g'$. Assuming $f(0) = 0 = g(0)$, $f(1) = 1 = g(1)$, drawing the graphs of $f$ and $g$, we can imagine a mountain and the statement above says that two climbers can go from the bottom to the top on the two different mountain slopes in such a way that at each moment of time their altitudes are the same.

The evolution process described above exhibits the fact that sometimes the states play an inferior role to the transitions that carry all the relevant concrete information about the process, leading to rather surprising structures, again very different from the states of the system. On the other hand, the pseudo-arc contains all the relevant information about possible continuous surjections between closed intervals.

## 2.3 Simplices

There is a clear definition of a finite-dimensional simplex: The convex hull of an affinely independent finite set. So, the 0-dimensional simplex is a point, the 1-dimensional simplex is an interval, the 2-dimensional simplex is a triangle, and so on. All the finite-dimensional simplices could be thought of states of some evolution system. The question is how to describe transitions. The obvious possibility is to consider embeddings onto faces, namely, the $k$-dimensional simplex $\Delta_k$ can be isometrically embedded into any $\Delta_m$ with $m > k$ so that its extreme points are within the extreme points of $\Delta_m$. Let us assume that $m = k + 1$. Then there are exactly $(k+1)! \cdot (k+2) = (k+2)!$ possibilities for such embeddings. This definitely makes sense, nevertheless every infinite process in this system is actually the same: it is a strictly increasing chain of finite-dimensional simplices in which the successor of each simplex is built by adding one more vertex in a new dimension. It turns out that another natural transition from $\Delta_k$ to $\Delta_{k+1}$ can be a pair consisting of the embedding as above together with a fixed affine projection $p\colon \Delta_{k+1} \to \Delta_k$. Note that $p$ is actually determined by choosing a point $x_p \in \Delta_k$. In any case, now our evolution system becomes much more complicated, once we insist on recording the projections. One can explain this approach by assuming that each simplex is actually a very flexible geometric figure, so that choosing a point inside of it one can pull it out, obtaining a more complicated simplex-like figure, recording where the point initially was. The reverse of the procedure of pulling out is affine, of course.

This evolution system clearly has the amalgamation property. An evolution process with the absorption property leads to the *Poulsen simplex*, the unique (up to affine homeomorphism) metrizable simplex (contained in the Hilbert space) whose set of extreme points is dense.

Contrary to the previous examples, the Poulsen simplex contains all finite-dimensional simplices, in fact every inverse limit of finite-dimensional simplices with affine projections is affinely homeomorphic to a face of the Poulsen simplex. The fact that the Poulsen simplex carries more information about the evolution system than the other objects, like the Cantor set or the pseudo-arc is just illusion. An explanation is very simple: The transitions in the system producing the Poulsen simplex are capable of recording the history, namely, each simplex $\Delta_k$ appears as a concrete face of $\Delta_m$ for every $m > k$. This is not the case in the previous example of glass polygons, where the transitions actually change the particles, by breaking them into smaller ones.

## 2.4 Evolving populations

A population can be modeled by a set, possibly with some extra structure. For instance, this extra structure could consist of an ordering representing some hierarchy. Another possibility is adding graph relations representing various connections between the individuals (elements of the population). Summarizing, a *population* is a finite set $X$ together with several binary relations. Each relation could be either a partial ordering imposing some hierarchy on the elements of $X$, or a symmetric relation representing some connections between the elements of $X$. The set of all types of the relations is called the *language* of the structure.

We can start with the empty population $\emptyset$. A *transition* from a population $X$

to $X'$ could be any mapping $e\colon X \to X'$ preserving all the relations in the sense that $(x_0, x_1) \in E$ if and only if $(e(x_0), e(x_1)) \in E$ for every binary relation $E$ that is in the common language of $X$ and $X'$. A more restrictive (and perhaps more natural) option is requiring that $X'$ differs from $X$ by at most one element. The amalgamation property may sometimes fail. It will hold once we take *all* possible finite structures (populations) in a fixed language. Once we add some axioms and restricting the class, the amalgamation property needs to be verified, of course.

In any case, once our class of populations has the amalgamation property, there exists an evolution sequence of populations whose limit is a *random population*, namely, a countably infinite population realizing all possible relations between its finite subsets.

Perhaps a more natural evolution system can be described by specifying the ancestors, namely, a population $A$ consists of two disjoint subsets $♀(A)$ and $♂(A)$ and a transition to another population $B$ is specified by two mappings $m\colon B \to ♀(A)$ and $f\colon B \to ♂(A)$ saying that $m(x)$ is the *mother* and $f(x)$ is the *father* of $x \in B$. Additionally, some relations on the populations can be imposed, for example, not allowing to have ancestors from the same grandparents. The problem here is that populations may deteriorate, namely, given a population $A$, it is always possible to make a transition $f\colon A \to B$ such that $B$ is much smaller than $A$. A reasonable assumption on a population $X$ is that both sets $♀(X)$, $♂(X)$ are nonempty. By this way, there is a minimal population $E_{\min}$ such that $♀(E_{\min}) = \{F\}$ and $♂(E_{\min}) = \{M\}$. Clearly, every population admits a transition to $E_{\min}$.

# 3 Preliminaries

We adopt the convention which many category-theorists support, namely, that arrows are more important than objects. Thus, a category $\mathfrak{C}$ will be identified with its class of arrows and $\mathfrak{C}(A, B)$ will denote the set of all $\mathfrak{C}$-arrows with domain $A$ and codomain $B$. For our purposes it will be sufficient to assume that all categories are locally small, therefore $\mathfrak{C}(A, B)$ is indeed a set, not a proper class. The class of $\mathfrak{C}$-objects will be denoted by $\operatorname{Obj}(\mathfrak{C})$. The composition of $\mathfrak{C}$-arrows $f\colon A \to B$ and $g\colon B \to C$ will be denoted by $g \circ f$.

By a *sequence* in a category $\mathfrak{C}$ we mean a covariant functor from $\mathbb{N}$ (treated as a linearly ordered category) into $\mathfrak{C}$. Sequences will be denoted by $\vec{x}$, $\vec{a}$, etc. Given a sequence $\vec{x}\colon \mathbb{N} \to \mathfrak{C}$, we denote $X_n = \vec{x}(n)$ and $x_n^m$ the bonding arrow from $X_n$ to $X_m$ ($n \le m$). When $X \in \operatorname{Obj}(\mathfrak{C})$ is the colimit of $\vec{x}$, we write $X = \lim \vec{x}$ and we denote by $x_n^\infty$ the colimiting arrows from $X_n$ to $X$. In particular, $x_n^\infty = x_m^\infty \circ x_n^m$ whenever $n \le m$.

We say that $\mathfrak{C}$ has the *amalgamation property* if for every $\mathfrak{C}$-arrows $f$, $g$ with $\operatorname{dom}(f) = \operatorname{dom}(g)$ there exist $\mathfrak{C}$-arrows $f'$, $g'$ such that $f' \circ f = g' \circ g$. In Section 4 we shall consider a specialized variant of the amalgamation property involving transitions.

For undefined notions concerning category theory we refer to Mac Lane's monograph [12].

## 3.1 Evolution systems

An *evolution system* is a structure of the form $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$, where $\mathfrak{V}$ is a category, $E_0$ is a fixed $\mathfrak{V}$-object (called the *origin*) and $\mathscr{T}$ is a class of $\mathfrak{V}$-arrows (its elements are called *transitions*). We are interested in *evolutions* (or *evolution processes*), namely, sequences of the form

$$E_0 \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition. The category $\mathfrak{V}$ serves as the universe of discourse and the minimal assumption here is that every evolution has a colimit in $\mathfrak{V}$. Given a $\mathfrak{V}$-object $X$, we denote

$$\mathscr{T}(X) = \{f \in \mathscr{T} \colon \operatorname{dom}(f) = X\},$$

that is, the set of all transitions with domain $X$. Two transitions $f, g \in \mathscr{T}(X)$ are *isomorphic* if there is an isomorphism $h$ in $\mathfrak{V}$ such that $g = h \circ f$. The relation of being isomorphic will be denoted by $\sim$ (formally, it should be $\sim_X$, as it depends on the object $X$).

Denote by $\langle \mathscr{T} \rangle$ the category generated by $\mathscr{T}$. The $\langle \mathscr{T} \rangle$-arrows will be called *paths*. Specifically, a non-identity path is any arrow of the form $f_0 \circ \ldots \circ f_{n-1}$ where $f_0, \ldots, f_{n-1}$ are transitions. We do not require that identities are transitions.

An object $X$ of $\langle \mathscr{T} \rangle$ will be called *finite* if there exist transitions $f_0, \ldots, f_{n-1}$ such that $f_i \colon X_i \to X_{i+1}$ for $i < n$, $X_0 = E_0$ and $X_n = X$. Such a sequence of transitions can also be called a *path* of length $n$ from $E_0$ to $X$. There may be several paths with the same composition, the minimal length will be called the *size* of $X$, denoted by $\operatorname{size}(X)$. In particular, $\operatorname{size}(E_0) = 0$. Formally, there might be a confusion with the notion of a path, as in graph theory this should be a *sequence* of arrows or transitions, while we have decided to use the name *path* for a composition of transitions and we believe this little inaccuracy will not lead to any misunderstanding.

Finally, we denote by $\mathscr{E}^{\text{fin}}$ the category of all finite objects with paths, namely, compositions of transitions (and identities). The *length* of $f \in \mathscr{E}^{\text{fin}}$ is the minimal $n$ such that $f$ is the composition of $n$ transitions. We agree that isomorphisms have length zero and hence proper transitions (i.e. non-isomorphisms) have length one.

Note that $\mathscr{E}^{\text{fin}}$ can also be regarded as an evolution system (a subsystem of $\mathscr{E}$), suitably restricting the class of transitions, although it would fail the minimal assumption, as typically the colimit of an evolution is not a finite object.

## 3.2 More examples

Below we present some natural examples of evolution systems.

**Example 3.1.** Let $\mathscr{F}$ be a class of finite structures in a fixed first-order language consisting of relations only. It is convenient to assume $\mathscr{F}$ is closed under isomorphisms. Let $\sigma\mathscr{F}$ denote the class of all structures of the form $\bigcup_{n\in\omega} X_n$, where $\{X_n\}_{n\in\omega}$ is a chain in $\mathscr{F}$. Let $\mathfrak{V}$ be the category of all embeddings between structures in $\sigma\mathscr{F}$. Let $\mathscr{T}$ consist of all embeddings of the form $f\colon X \to Y$, where $Y \setminus f[X]$ is a singleton or the empty set. In other words, transitions are one-point extensions and isomorphisms. Finally, $E_0$ might be the empty structure. Clearly, $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$ is an evolution system.

Note that we can also define $\mathfrak{V}$ to be the category of all homomorphisms between $\sigma\mathscr{F}$-objects. Yet another option is to consider embeddings or homomorphisms between arbitrarily large structures that can be built as unions of directed families consisting of structures from $\mathscr{F}$. One can also replace the empty structure by any (possibly large) structure, declaring it to be the origin $E_0$.

Finally, one can generalize this by allowing functions and constants in the language. Now a transition would be an embedding $f\colon X \to Y$ such that $Y$ is generated by $f[X] \cup \{a\}$ for some $a \in Y$. A very concrete example here could be the class of all finite fields, where it is natural to define the origin $E_0$ as the $p$-element field, where $p$ is a fixed prime. By this way, the category $\mathscr{E}^{\mathrm{fin}}$ consists of all finite fields of characteristic $p$.

**Example 3.2.** Let $\mathscr{F}$ be a fixed class of finite nonempty relational structures and consider it as a category where the arrows are epimorphisms. A concrete example could be just finite sets with no extra structure. Define transitions to be epimorphisms $f\colon X \to Y$ such that either $f$ is an isomorphism (a bijection) or else there is a unique $y \in Y$ with a nontrivial $f$-fiber and moreover $f^{-1}(y)$ consists of precisely two points. Define $\mathfrak{V}$ to be the opposite category, so that $f \in \mathfrak{V}$ is an arrow from $Y$ to $X$ if it is an epimorphism from $X$ onto $Y$. Then $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$ is an evolution system, where $E_0$ is a prescribed finite structure in $\mathscr{F}$.

**Example 3.3** (Monoids)**.** A monoid $\mathbb{M} = \langle M, \cdot, 1 \rangle$ is just a category with a single object $M$, whose arrows are the elements of $M$ and $\cdot$ is the composition. It can be turned into an evolution system by selecting any subset of $M$ as the class of transitions. Evolutions may still lead to something new. A concrete example is the multiplicative monoid $\langle \mathbb{Z} \setminus \{0\}, \cdot, 1 \rangle$, where perhaps the most natural choice for the transitions are all prime numbers (plus possibly the identity, that is 1). An evolution may be eventually constant one, which corresponds to a concrete natural number. Otherwise, it corresponds to a so-called *super-natural* number, namely, a formal infinite product of nonnegative powers of primes

$$\prod_{p\in\mathbb{P}} p^{\alpha(p)},$$

where $\mathbb{P}$ denotes the set of all primes and $\alpha(p) \in \mathbb{N} \cup \{\infty\}$. Note that $p^\infty$ means that the prime $p$ occurs infinitely many times in the evolution. Of course, the most complicated evolution corresponds to $\prod_{p \in \mathbb{P}} p^\infty$.

Non-zero integers with multiplication actually encode all embeddings of the group $\langle \mathbb{Z}, + \rangle$ into itself (an embedding is determined by the image of 1). Thus, super-natural numbers correspond to sequences of self-embeddings of $\langle \mathbb{Z}, + \rangle$. Their colimits are torsion-free abelian groups whose all finitely generated subgroups are cyclic. The most complicated one is $\langle \mathbb{Q}, + \rangle$, corresponding to $\prod_{p \in \mathbb{P}} p^\infty$.

**Example 3.4** (Posets)**.** Let $\mathbb{P} = \langle P, \leq \rangle$ be a partially ordered set with a fixed element $\perp$. For simplicity, we may assume $\perp$ is minimal and $\mathbb{P}$ is well founded. In that case it is natural to say that a pair $\langle x, y \rangle$ is a transition if $x < y$ and there is no $z$ with $x < z < y$. By this way, $\mathbb{P}$ becomes an evolution system with origin $\perp$. Recall that every poset (in fact, a quasi-ordered set) is a category in which the arrows are pairs $\langle x, y \rangle$ with $x \leq y$ and identities are pairs $\langle x, x \rangle$. In our case, an evolution is a sequence

$$\perp = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots$$

such that no $z \in P$ is strictly between two consecutive elements. Finite objects are those that can be reached from $\perp$ by finitely many transitions. For instance, if $\mathbb{P}$ is a tree and $\perp$ is its root, then finite objects are those living on the finite levels.

**Example 3.5.** Fix an arbitrary category $\mathfrak{V}$ and fix a $\mathfrak{V}$-object $E_0$. Define $\mathscr{T} = \mathfrak{V}$. Then $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$ is obviously an evolution system. It is perhaps a bit more interesting when $E_0$ is weakly initial in $\mathfrak{V}$ (that is, $\mathfrak{V}(E_0, X) \neq \emptyset$ for every $X \in \operatorname{Obj}(\mathfrak{V})$). In any case, this shows that every category can be easily converted to an evolution system.

The last example indicates that evolution systems are so general that perhaps they lead to nothing interesting. One of our goals is to convince the readers that this is not the case.

# 4 Generic evolutions

In this section we show that, under some natural assumptions, there exists an evolution with the absorption property and it is unique up to isomorphism of the colimits. Next we show that such an evolution is "the most complicated one" and it can be described in terms of a natural infinite game.

Most of the results are adaptations of the classical theory of universal homogeneous structures, due to Fraïssé [2], developed in the fifties of the last century in the context of model theory. Game-theoretic approach is due to Krawczyk and the first author [10, 7]. Throughout this section we assume that $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$ is a fixed evolution system.

**Definition 4.1** (Absorption property)**.** Let $\vec{u}$ be an evolution. We say that $\vec{u}$ has the *absorption property* if for every $n \in \omega$, for every transition $t \colon U_n \to Y$ there are $m > n$ and a path $g \colon Y \to U_m$ such that $g \circ t = u_n^m$.

In other words, any transition going out of the evolution is "absorbed" at some point. The key tool for proving the existence of such an evolution is the amalgamation property, which has been considered many times in pure and applied category theory. Below we state its variant involving transitions, often more convenient to check in concrete examples.

**Definition 4.2** (Amalgamation property)**.** We say that $\mathscr{E}$ has the *finite amalgamation property* (FAP) if for every two transitions $f$, $g$ with $\operatorname{dom}(f) = \operatorname{dom}(g) \in \operatorname{Obj}(\mathscr{E}^{\mathrm{fin}})$ there exist transitions $f'$, $g'$ such that $f' \circ f = g' \circ g$, that is, the following square

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ C & \xrightarrow[\;g'\;]{} & D \end{array}$$

is commutative.

It might seem more natural to define the amalgamation property for all transitions, not restricting to finite objects. On the other hand, the FAP is relevant for the existence of a generic evolution and it does generally not imply the amalgamation property at infinite objects (even though in most of the natural examples the amalgamation property holds everywhere).

The next simple lemma is actually quite crucial.

**Lemma 4.3.** *The FAP implies that $\mathscr{E}^{\mathrm{fin}}$ has the amalgamation property. More precisely, if $f, g$ are $\mathscr{E}^{\mathrm{fin}}$-arrows with $\operatorname{dom}(f) = \operatorname{dom}(g)$ and $f$ is a transition, then there exist $\mathscr{E}^{\mathrm{fin}}$-arrows $f', g'$ such that $f' \circ f = g' \circ g$ and $g'$ is a transition.*

*Proof.* Easy induction on the length of $\mathscr{E}^{\mathrm{fin}}$-arrows. □

The last ingredient needed for the existence of an evolution with the absorption property is some kind of smallness, formally defined below.

**Definition 4.4.** We say that an evolution system $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$ is *essentially countable* if for every finite object $X$ there is a countable set of transitions $\mathscr{F}(X) \subseteq \mathscr{T}(X)$ such that for every transition $f \in \mathscr{T}(X)$ there is an isomorphism $h$ such that $h \circ f \in \mathscr{F}(X)$.

We are now ready to state the main "existential" result.

**Theorem 4.5.** *Assume $\mathscr{E}$ is an essentially countable evolution system that has the finite amalgamation property. Then there exists a unique, up to isomorphism, evolution with the absorption property.*

*Proof.* The existence can be proved by easy induction with a suitable bookkeeping. Uniqueness is a standard back and forth argument. Below we provide some details.

**The existence**. We use the powerful fundamental property of the infinity: The set $\omega$ of non-negative integers can be decomposed into infinitely many infinite sets, say, $\omega = \bigcup_{n\in\omega} B_n$, where each $B_n$ is infinite and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. The sets $B_n$ will be used for bookkeeping.

Namely, we start at the origin $E_0$ by enumerating all transitions from $E_0$ (up to isomorphism) using the numbers from $B_0$. We use the first one to obtain the first step of our evolution $e_0 \colon E_0 \to E_1$. We enumerate all the isomorphic types of transitions from $E_1$, using the numbers from $B_1$.

At step $n$, we take the first transition $t$ from some $E_i$ with $i \leq n$ that was not considered yet and we define $e_n \colon E_n \to E_{n+1}$ as the result of the amalgamation

$$\begin{array}{ccc} E_i & \longrightarrow & E_n \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle e_n} \\ X & \longrightarrow & E_{n+1} \end{array}$$

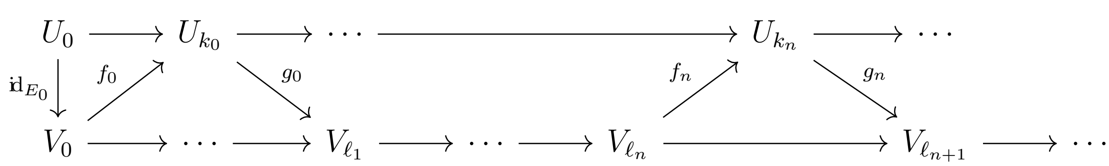
which is possible due to Lemma 4.3. Note that the horizontal arrows in the square above are compositions of transitions, namely, arrows of $\mathscr{E}^{\mathrm{fin}}$. We enumerate all transitions from $E_{n+1}$ (up to isomorphism), using the set $B_{n+1}$.

After infinitely many steps, we obtain an evolution

$$E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots$$

that has the absorption property, because each transition (up to isomorphism) has been taken into account.

**Uniqueness**. Suppose $\vec{u}$, $\vec{v}$ are two evolutions with the absorption property. We start with the absorption of $\vec{u}$, obtaining a suitable $\mathscr{E}^{\mathrm{fin}}$-arrow $f_0 \colon V_0 \to U_{k_0}$. Then we use the absorption property of $\vec{v}$ to obtain a suitable arrow $g_0 \colon U_{k_0} \to V_{\ell_1}$. And so on. Instead of writing the technical details, we present the relevant (infinite) diagram



in which $U_0 = E_0 = V_0$. Note that the sequences $\{f_n\}_{n\in\omega}$ and $\{g_n\}_{n\in\omega}$ converge to pairwise invertible arrows between the colimits $U_\infty = \lim \vec{u}$ and $V_\infty = \lim \vec{v}$, showing that $U_\infty$ and $V_\infty$ are isomorphic. □

**Corollary 4.6.** *Theorem 4.5 is valid if FAP is replaced by the amalgamation property of* $\mathscr{E}^{\mathrm{fin}}$.

*Proof.* Replace $\mathscr{E}$ by a new evolution system in which transitions are arbitrary $\mathscr{E}^{\mathrm{fin}}$-arrows. □

## 4.1 An abstract Banach-Mazur game

Fix a locally countable evolution system $\mathscr{E}$ with the finite amalgamation property. We define the following game $\mathrm{BM}(\mathscr{E}, U)$ for two players, say, *Eve* and *Odd*. Here, $U$ is a fixed $\mathfrak{V}$-object. The rules are as follows. Eve starts the game by choosing a transition $e_0 \colon E_0 \to A_0$. Odd responds with a transition $e_1 \colon A_0 \to A_1$. Eve responds with an transition $e_2 \colon A_1 \to A_2$; Odd responds with a transition $e_3 \colon A_2 \to A_3$. And so on.

Note that the rules for both players are identical. After infinitely many steps, the players build an evolution $\vec{a}$. We say *Odd wins* if the colimit of $\vec{a}$ is isomorphic to $U$. Otherwise, *Eve wins*.

**Definition 4.7.** An object $U$ is *generic* if Odd has a winning strategy in the game $\mathrm{BM}(\mathscr{E}, U)$.

Note that a generic object is unique up to isomorphism (as long as it exists, of course). The reason is simple: Assuming $U$, $V$ are generic, Odd can use a winning strategy aiming at $U$ while at the same time Eve can use strategy Odd's strategy aiming at $V$. Playing such a game, the colimit is isomorphic to both $U$ and $V$.

**Theorem 4.8.** *The following conditions are equivalent.*

(a) *Odd has a winning strategy in* $\mathrm{BM}(\mathscr{E}, U)$.

(b) *$U$ is the colimit of an evolution with the absorption property.*

*Proof.* (b) $\Longrightarrow$ (a) Suppose we are given a finite step of the game

$$E_0 \xrightarrow{e_0} A_0 \xrightarrow{e_1} \cdots \xrightarrow{e_n} A_n$$

and it is Odd's turn (that is, $n$ is even). He chooses $i < n$ and a transition $t \colon A_i \to B$. He responds with a transition $e_{n+1} \colon A_n \to A_{n+1}$ realizing an amalgamation of $t$ and the given path from $A_i$ to $A_n$ (namely, the composition $e_n \circ \ldots \circ e_{i+1}$). This strategy is winning as long as Odd makes a suitable bookkeeping, so that all transitions from $A_i$s are taken into account. This is possible, since $\mathscr{E}$ is locally countable.

(a) $\Longrightarrow$ (b) Suppose Odd has a winning strategy in $\mathrm{BM}(\mathscr{E}, U)$. Eve can use the strategy described above, so that the resulting evolution has the absorption property. This shows that $U$ is the colimit of an evolution with the absorption property. □

By the result above, a *generic evolution* will be the one whose colimit is a generic object in the sense described above. Thus, once our evolution system has the finite amalgamation property and is locally countable, a generic evolution is precisely the one with the absorption property. Without amalgamations, a generic evolution may still exist. In pure category theory this has been treated in [8]. The original Banach-Mazur game was invented by Mazur in the thirties of the last century, it was played with open intervals of the real line. It was later generalized to arbitrary topological spaces by Choquet, therefore it is also known under the name *Choquet game*. We refer to the survey article [14] for detailed information on infinite topological games and to [10, 7] for a recent study of the model-theoretic variant of the Banach-Mazur game. We also refer to the monogrpahs [3, 15] for more general infinite games in model theory.

## 4.2 Cofinality

We now turn to universality, or rather *cofinality*, as the term "universal object" has different meanings in category theory and model theory. So, given a category $\mathfrak{C}$, we say that a $\mathfrak{C}$-object $U$ is *cofinal* if for every $\mathfrak{C}$-object $X$ there is a $\mathfrak{C}$-arrow from $X$ to $U$. This becomes interesting when the $\mathfrak{C}$-arrows are some kinds of monics (in model theory: embeddings). Recall that we are working with a fixed evolution system $\mathscr{E}$ with the finite amalgamation property.

**Theorem 4.9** (Cofinality)**.** *Let $\vec{u}$ be an evolution with the absorption property. Given another evolution $\vec{x}$, there exists a $\mathfrak{V}$-arrow from $\lim \vec{x}$ to $\lim \vec{u}$.*

*Proof.* Our goal is to obtain an infinite sequence of $\mathscr{E}^{\mathrm{fin}}$-arrows $f_0, f_1, \dots$, so that the following infinite diagram

$$\begin{array}{lcccccc} \vec{u}: & U_0 \longrightarrow & U_{k_1} \longrightarrow & \cdots \longrightarrow & U_{k_{n-1}} \longrightarrow & U_{k_n} \longrightarrow & \cdots \\ & {\scriptstyle f_0}\uparrow & {\scriptstyle f_1}\uparrow & & {\scriptstyle f_{n-1}}\uparrow & {\scriptstyle f_n}\uparrow & \\ \vec{x}: & X_0 \longrightarrow & X_1 \longrightarrow & \cdots \longrightarrow & X_{n-1} \longrightarrow & X_n \longrightarrow & \cdots \end{array}$$

is commutative. Given $f_{n-1}$, in order to find $f_n$, we first amalgamate $f_{n-1}$ and $x^n_{n-1} \colon X_{n-1} \to X_n$, obtaining a commutative square

$$\begin{array}{ccc} U_{k_{n-1}} & \xrightarrow{t} & Y \\ {\scriptstyle f_{n-1}}\uparrow & & \uparrow{\scriptstyle g} \\ X_{n-1} & \xrightarrow[x^n_{n-1}]{} & X_n \end{array}$$

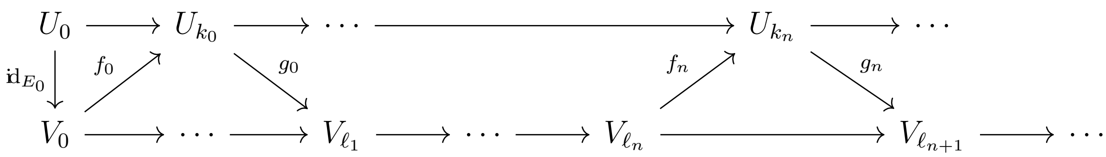
in which $t$ is a transition. Next we use the absorption property of $\vec{u}$ so that $f_n$ is the composition of $g$ and a suitable $\mathscr{E}^{\mathrm{fin}}$-arrow $h$ satisfying $h \circ t = u^{k_n}_{k_{n-1}}$. Finally, the colimiting arrow $f_\infty \colon \lim \vec{x} \to \lim \vec{u}$ witnesses that $\mathfrak{V}(\lim \vec{x}, \lim \vec{u}) \neq \emptyset$. □

## 4.3 Homogeneity

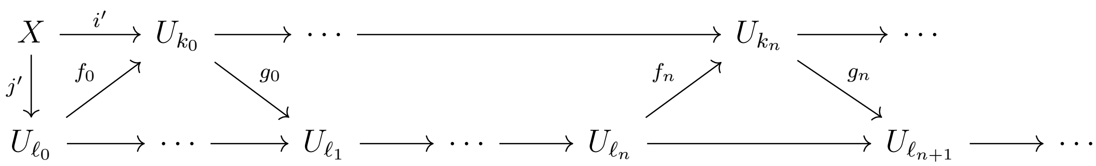
We fix an evolution $\vec{u}$ with the absorption property. Let $U = \lim \vec{u}$. Recall that $u^\infty_n$ denotes the colimiting arrow from $U_n$ to $U$. A *trail* is an arrow of the form $u^\infty_n \circ f$, where $n \in \omega$ and $f \in \mathscr{E}^{\mathrm{fin}}$.

**Theorem 4.10** (Homogeneity)**.** *Assume $X$ is a finite object and $i, j \colon X \to U$ are trails. Then there exists an automorphism $h \colon U \to U$ such that $j = h \circ i$.*

*Proof.* Let us recall the infinite diagram from the proof of uniqueness (Theorem 4.5):




Note that the same inductive arguments can be used when the sequence $\vec{v}$ is replaced by $\vec{u}$ and $\mathrm{id}_{E_0}$ is replaced by any $\mathscr{E}^{\mathrm{fin}}$-arrow. Next, we replace $U_0$ by $X$, knowing that $i = u_{k_0}^{\infty} \circ i'$ and $j = u_{\ell_0^{\infty}} \circ j'$ for some $k_0, \ell_0$. Finally, we have the following infinite diagram




built inductively, using the absorption property. The colimiting arrow is the required automorphism. $\square$

In model theory (see Example 3.1), homogeneity is often phrased as follows: Every isomorphism between finitely generated substructures extends to an automorphism. This holds when the class of structures under consideration is *hereditary*, namely, closed under finitely generated substructures and isomorphisms.

# 5 Confluence and termination

The reader may have already noticed that evolution systems resemble abstract rewriting systems (see e.g. [4]), namely, structures of the form $\mathbb{X} = \langle X, \to \rangle$, where $\to$ is a binary relation, called *rewriting* or *reduction*. The so-called reflexive-transitive closure of $\to$ gives a quasi-ordering of $X$, therefore $\mathbb{X}$ becomes a category. The only missing ingredient is the origin, which could be any element of $X$. Rewriting systems are meant to model processes like reducing certain expressions (term rewriting) or processes (graph rewriting), and so on. A typical feature is *termination* which, in our language, says that there are no evolutions or every evolution "stops" in the sense that from some point on all possible transitions are isomorphisms. Another useful feature is *confluence*, which corresponds precisely to the amalgamation property of the category of finite objects with paths.

We shall now formalize these concepts in the language of evolution systems and we prove a generalization of the celebrated Newman's Lemma (also called the *diamond lemma*, see [13] and [4]) saying that in a terminating system local confluence implies the global one.

**Definition 5.1.** An evolution system $\mathscr{E}$ is *confluent* if the category $\mathscr{E}^{\mathrm{fin}}$ of finite objects with paths has the amalgamation property.

The system $\mathscr{E}$ is called *locally confluent* if for every finite object $X$, for every two transitions $f, g \in \mathscr{T}(X)$ there exist paths $f', g'$ such that $f' \circ f = g' \circ g$.

Note that local confluence is formally weaker than the FAP, while confluence is sufficient for the theory of generic evolutions (by Corollary 4.6). Actually, there is a subtle problem here: Absorption property needs to be refined.

**Definition 5.2.** An evolution $\vec{u}$ has the *path absorption property* if for every $n \in \omega$, for every path $f \colon U_n \to Y$ there exists a path $g \colon Y \to U_m$ with $m > n$, such that $g \circ f = u_n^m$.

One of the motivations of introducing the finite amalgamation property is that it gives equivalence of the absorption property and the path absorption property. The results of Section 4 can be rephrased as follows:

**Theorem 5.3.** *Assume $\mathscr{E}$ is a confluent, locally countable evolution system. Then there exists an evolution $\vec{u}$ with the absorption property. Its colimit $\lim \vec{u}$ is generic in the sense of the Banach-Mazur game introduced in Section 4.1. Furthermore, it is homogeneous with respect to $\mathscr{E}^{\mathrm{fin}}$ and cofinal in $\mathscr{E}^{\sigma}$.*

The proof is actually repeating the arguments from Section 4, replacing transitions by paths.

**Definition 5.4.** A transition is called *trivial* if it is an isomorphism. An evolution system is *terminating* if there is no evolution consisting of nontrivial transitions. An object $X$ is *normalized* if $\mathscr{T}(X)$ consists of isomorphisms (i.e. all transitions from $X$ are trivial).

In other words, an evolution system is terminating if every path starting from the origin ends at a normalized object. The term "normalized" is inspired by the "normal form" in the theory of rewriting systems.

**Definition 5.5** (Isomorphism Amalgamation Property)**.** An evolution system $\mathscr{E}$ has the *isomorphism amalgamation property* (briefly: *IAP*) if for every transition $f \colon X \to Y$, for every isomorphism $h \colon X \to \tilde{X}$ there exist a transition $h' \colon \tilde{X} \to \tilde{Y}$ and an isomorphism $f' \colon Y \to \tilde{Y}$ satisfying $f' \circ f = h' \circ h$.

This property can possibly be called "transferring transitions" as it really says that every transition can be "moved" by an arbitrary isomorphism.

We are now ready to state our variant of Newman's Lemma.

**Theorem 5.6.** *A locally confluent terminating evolution system with the isomorphism amalgamation property, in which all isomorphisms are transitions, is confluent.*

*Proof.* We follow the scheme of Huet's proof of Newman's Lemma [4]. Namely, fix an evolution system $\mathscr{E}$ as in the theorem and define a strict ordering on $\mathscr{E}^{\mathrm{fin}}$ by declaring $X \prec Y$ if there is a path from $X$ to $Y$ and at least one of the transitions on this path is nontrivial. Since $\mathscr{E}$ is terminating, the ordering $\succ$ is well founded on the class of all finite objects. This is indeed a strict ordering, as it is impossible to have both $X \prec Y$ and $Y \prec X$, which would lead to an evolution consisting of nontrivial transitions. Note that $\succ$-minimal elements are precisely the normalized finite objects and they trivially admit amalgamations in $\mathscr{E}^{\mathrm{fin}}$.

Now fix an arbitrary finite object $Z$ and two paths $f\colon Z \to X$, $g\colon Z \to Y$. Let us assume first that $g$ is a transition.

If all the transitions composing $f$ are trivial then $f$ is an isomorphism, therefore it is a transition, so we amalgamate $f$, $g$ easily. Otherwise, $f = \tilde{f} \circ f_0 \circ h$, where $h$ is an isomorphism, $f_0$ is a nontrivial transition and $\tilde{f}$ is a path (possibly an identity). Using the IAP, we amalgamate $g$ and $h$ by a transition and an isomorphism. Thus, we may assume $h = \mathrm{id}_Z$ and $f = \tilde{f} \circ f_0$. Let $\tilde{Z} = \mathrm{dom}(\tilde{f})$. Then $Z \prec \tilde{Z}$.

By local confluence, there are paths $k, \ell$ such that $k \circ f_0 = \ell \circ g$. By inductive hypothesis (over the well founded ordering $\succ$), there exist paths $f', k'$ such that $f' \circ \tilde{f} = k' \circ k$. Finally, $f'$ and $k' \circ \ell$ provide an amalgamation of $f$ and $g$.

This actually completes the proof, as the general case, where $g$ is a path, is settled by an obvious induction on the length. □

**Corollary 5.7.** *Under the assumptions of Theorem 5.6, every evolution is generic.*

*Proof.* Fix an evolution $\vec{e}$ and let $N$ be a natural number such that $E_n$ is normalized. Fix $n < N$ and fix a path $f\colon E_n \to Y$. By confluence, there are paths $f'$, $g'$ such that $f' \circ f = g' \circ g$, where $g = e_n^N\colon E_n \to E_N$. Since $E_N$ is normalized, $g'$ is an isomorphism. Thus $(g')^{-1} \circ f'$ shows that $f$ is absorbed by $\vec{e}$. Finally, any path from $E_k$ with $k \geq N$ is an isomorphism, therefore it is absorbed by an isomorphism. □

**Example 5.8.** Let $\mathfrak{V}$ be the category of linearly ordered sets and let $E_0 = \langle \mathbb{Q}, < \rangle$. Define $\mathscr{T}(E_0)$ to be all automorphism of $\langle \mathbb{Q}, < \rangle$ with one exception: An embedding $e\colon \mathbb{Q} \to \mathbb{Q}$ such that $e[\mathbb{Q}] = \mathbb{Q} \setminus \{0\}$. Define $\mathscr{T}(X) = \mathrm{id}_X$ for every other $\mathfrak{V}$-object $X$. So, there is precisely one nontrivial transition in $\mathscr{E} = \langle \mathfrak{V}, \mathscr{T}, E_0 \rangle$. On the other hand, paths of the form $e \circ h$, where $h$ is an automorphism of $\langle \mathbb{Q}, < \rangle$ are isomorphic to self-embeddings of $\mathbb{Q}$ corresponding to filling an arbitrary single gap. Obviously, there are continuum many such paths, up to isomorphism. Consequently, there is no generic evolution. Note that $\mathscr{E}$ totally fails the IAP.

# 6 Conclusions and future research

It should now be clear that all the examples described in Section 2 can be formally phrased in the language of evolution systems. One of them (ribbons) actually has only the *approximate* amalgamation property, namely, two transitions $f, g$ (continuous surjections of the unit interval) can be completed by transitions $f', g'$ to a square

$$\begin{array}{ccc} \mathbb{I} & \xleftarrow{\ f\ } & \mathbb{I} \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ \mathbb{I} & \xleftarrow[\ g'\ ]{} & \mathbb{I} \end{array}$$

that commutes with a prescribed small error $\varepsilon > 0$. That is, $|f(f'(t)) - g(g'(t))| < \varepsilon$ for every $t \in \mathbb{I}$. Formal treatment of such situations needs a category enriched over complete metric spaces (see [9] for details). On the other hand, in this particular case, one can restrict to piecewise linear surjections, where the amalgamation property (the Mountain Climbing Theorem) holds with no errors.

Another example from Section 2, dealing with simplices, does not have this problem (amalgamation property just holds true), however, there are too many transitions and the absorption property holds with errors. Again, in order to treat it formally, one needs to use metric-enriched categories. On the other hand, in this particular case it is possible to restrict the class of transitions so that the problem disappears. Specifically, given a simplex $\Delta_m$, a transition could be an affine surjection $f\colon \Delta_{m+1} \to \Delta_m$ that is identity on $\Delta_m$ and the "new" vertex is mapped to a point of $\Delta_m$ that has rational barycentric coordinates. By this way our evolution system is locally countable and the theory of Section 4 applies.

Concerning further research, we believe that the concept of evolution systems will open the gate for investigating new aspects of the theory of generic objects and related things, like the complexity of the system, the existence of random evolutions, and so on. In fact, evolution systems without any confluence properties (e.g. trees) are of some interest too. One of the natural lines of research here is searching for tools that would allow comparing and classifying evolution systems, perhaps emphasizing on those admitting generic evolutions.

Another interesting line of research is studying probabilistic approach, especially when the system is locally finite, where it is natural to impose the uniform probability. In this setting, evolution systems can model abstract stochastic processes.

Finally, abstract evolution systems could possibly play some role in automata theory, where an automaton is viewed as a category and transitions are the only way of moving from one state to another. One can then specify a family of objects that would be "accepting" the input, that is, the origin of the system in our terminology.

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