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Bingkang Huang
Šárka Nečasová
Lan Zhang

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ON THE COMPRESSIBLE MICROPOLAR FLUIDS IN A TIME DEPENDENT DOMAIN

BINGKANG HUANG, ŠÁRKA NEČASOVÁ, AND LAN ZHANG

ABSTRACT. We investigate compressible micropolar fluids on a time-dependent domain with slip boundary conditions. Our contribution in this paper is threefold. Firstly, we establish the local existence of the strong solution. Secondly, the global existence of weak solutions is given. The third one is the weak-strong uniqueness principle for slip boundary conditions. There are several new ideas developed by us to overcome the difficulties caused by the coupled terms and slip boundary conditions.

MSC: 35Q35; 76N10

Key words: compressible micropolar fluids, asymmetric stress tensor, slip boundary conditions, strong solution, weak solutions.

1. Introduction and main results

In this paper, we consider a class of compressible fluids with asymmetric stress tensor, introduced in [6] by C.A. Eringen which describes fluids with microstructure. It is a significant generalization of the Navier-Stokes equations and has been extensively studied and applied for modeling rheologically complex liquids such as blood and suspensions by many engineers and physicists. The precise system is described by

$$\begin{cases} \rho_t + \mathrm{div}_x(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}_x(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p = (\mu + \xi)\Delta_x \boldsymbol{u} + (\mu + \lambda - \xi)\nabla_x \mathrm{div}_x \boldsymbol{u} + 2\xi \mathrm{curl}_x \boldsymbol{w}, \\ (\rho \boldsymbol{w})_t + \mathrm{div}_x(\rho \boldsymbol{u} \otimes \boldsymbol{w}) + 4\xi \boldsymbol{w} = (c_a + c_d)\Delta_x \boldsymbol{w} + (c_0 + c_d - c_a)\nabla_x \mathrm{div}_x \boldsymbol{w} + 2\xi \mathrm{curl}_x \boldsymbol{u}, \end{cases} \tag{1.1}$$

where $\rho(t,x)$ is the density, $\boldsymbol{u}(t,x) = (u_1,u_2,u_3)(t,x)$ is the fluid velocity, $\boldsymbol{w}(t,x) = (w_1,w_2,w_3)(t,x)$ is the micro-rotation velocity. We assume that the flow is in the barotropic regime, and we focus on the isentropic case where the relation between p and ρ is given by the constitutive law:

$$p = a\rho^{\gamma}, \tag{1.2}$$

with a a positive constant and the adiabatic constant $\gamma > \frac{3}{2}$, which is a necessary assumption for the existence of a weak solution of compressible fluids (see for example [8]). Also suppose that $a = 1$ for simplicity without losing generality.

We shall specify the stress tensor $\mathbb{S}$ and the couple stress tensor $\mathbb{M}$ as follows

$$\mathbb{S}_{ij} = \lambda \mathrm{div}_x \boldsymbol{u} \delta_{ij} + \mu(u_{i,x_j} + u_{j,x_i}) + \xi(u_{j,x_i} - u_{i,x_j}) - 2\xi \varepsilon_{mij} w_m, \tag{1.3}$$

$$\mathbb{M}_{ij} = c_0 \mathrm{div}_x \boldsymbol{w} \delta_{ij} + c_d(w_{i,x_j} + w_{j,x_i}) + c_a(w_{j,x_i} - w_{i,x_j}), \tag{1.4}$$

where $\delta_{i,j}$ is the Kronecker symbol and ε_{mij} is the Levi-Civita symbol with $i,j,m = 1,2,3$.

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The symmetric part of the stress tensor $\mathbb{S}$ in (1.3) is

$$\mathbb{S}^s_{ij} = \lambda \mathrm{div}_x \boldsymbol{u} \delta_{ij} + \mu(u_{i,x_j} + u_{j,x_i}),$$

which is the stress tensor of compressible Navier–Stokes equations, where λ and μ are the usual viscosity coefficients with $\mu > 0$ and $3\lambda + 2\mu \geq 0$. The constants ξ, c_0, c_a and c_d represent the dynamic micro-rotation viscosity. They satisfy that $\xi > 0$, $c_d > 0$, $c_a > 0$, and $3c_0 + 2c_d \geq 0$.

The fluids we study in this paper are considered to occupy a bounded domain whose boundary is moving in time in a prescribed way. A position of the domain in a time moment t is denoted by Ω_t and we suppose that it is described by a function $\boldsymbol{X} : (0, T) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ in the following way

$$\Omega_t = \big\{x \in \mathbb{R}^3; \;\; \text{there exists } x_0 \in \Omega_0 \;\text{ such that } x = \boldsymbol{X}(t, x_0)\big\} \tag{1.5}$$

where $\Omega_0 \subset \mathbb{R}^3$ is the given initial position of the domain and $\boldsymbol{X}$ is a solution to the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t, x_0) = \boldsymbol{V}(t, \boldsymbol{X}(t, x_0)), \quad \boldsymbol{X}(0, x_0) = x_0, \tag{1.6}$$

for a given vector field $\boldsymbol{V} : (0, T) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$.

Meanwhile, we set $\Gamma_\tau = \partial\Omega_\tau$ and

$$Q_\tau = \cup_{t\in(0,\tau)}\{t\} \times \Omega_t = (0, \tau) \times \Omega_t.$$

Furthermore, we supply the system (1.1) with the Navier's slip boundary conditions. As we know, for compressible Navier–Stokes equations, Navier [28] proposed the boundary condition

$$[\mathbb{S}^s\boldsymbol{n}]_{\tan} + \kappa[\boldsymbol{u} - \boldsymbol{V}]_{\tan}|_{\Gamma_\tau} = 0,$$

where $\boldsymbol{n}(\tau, x)$ denotes the unit outer normal vector to the boundary Γ_τ and $\kappa \geq 0$ is friction coefficient. The opposite limit cases, $\kappa \to 0$ and $\kappa \to \infty$ yield the complete slip boundary condition and the no-slip boundary condition, respectively.

For the system (1.1), the boundary conditions in the Navier-type form were made in [1]. It is natural to use such boundary conditions to describe an interaction between fluids and solid parts. Here we consider a physically reasonable boundary condition with the form of

$$\mathbb{M} \cdot \boldsymbol{n} = \kappa(\boldsymbol{w} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{v}_b),$$

on $\partial\Omega_\tau$, where $\kappa = (\alpha_{ij})$ is a matrix with numeric components and $\boldsymbol{v}_b$ is the velocity of the solid boundary. The opposite limit cases, $\alpha_{ij} \to \infty$, and $\alpha_{ij} \to 0$, yield

$$\boldsymbol{w} = \frac{1}{2}\mathrm{curl}_x \boldsymbol{v}_b \quad and \quad \mathbb{M} \cdot \boldsymbol{n} = 0,$$

on $\partial\Omega_\tau$, respectively. A variety of different boundary conditions can be found in [27].

As the foregoing considerations, we describe Navier's boundary conditions for (1.1) as follows

$$\begin{aligned} &[\mathbb{S}\boldsymbol{n}]_{\tan} + \kappa_1[\boldsymbol{u} - \boldsymbol{V}]_{\tan}|_{\Gamma_\tau} = 0, \\ &[\mathbb{M}\boldsymbol{n}]_{\tan} + \kappa_2[\boldsymbol{w} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{V}]_{\tan}|_{\Gamma_\tau} = 0, \end{aligned} \tag{1.7}$$

where $\kappa_1, \kappa_2 \geq 0$. Meanwhile, we equip (1.7) with the impermeability relations

$$\begin{aligned} &(\boldsymbol{u} - \boldsymbol{V}) \cdot \boldsymbol{n}|_{\Gamma_\tau} = 0, \\ &(\boldsymbol{w} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{V}) \cdot \boldsymbol{n}|_{\Gamma_\tau} = 0. \end{aligned} \tag{1.8}$$

Finally, the initial conditions for strong solutions are described by

$$\rho(0, \cdot) = \rho_0, \quad \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0, \quad \boldsymbol{w}(0, \cdot) = \boldsymbol{w}_0, \tag{1.9}$$

while, for weak solutions, the initial conditions are given by

$$\begin{cases} \rho_0 \in L^\gamma(\mathbb{R}^3), \quad \rho_0 \geq 0, \quad \rho_0|_{\mathbb{R}^3 \setminus \Omega_0} = 0, \\ (\rho \boldsymbol{u})_0 = 0, \quad (\rho \boldsymbol{w})_0 = 0 \quad a.e. \quad on\{\rho_0 = 0\}, \\ \displaystyle\int_{\Omega_0} \frac{1}{\rho_0} (|(\rho \boldsymbol{u})_0|^2 + |(\rho \boldsymbol{w})_0|^2) \mathrm{d}x \leq \infty. \end{cases} \tag{1.10}$$

In fact, if we neglect the effect of micro-rotational velocity, the system (1.1) reduces to the compressible isentropic Navier-Stokes equations. The global existence of weak solutions to the compressible Navier-Stokes equations on the fixed domain was established in the seminal work of P.L. Lions, see [26]. The existence of weak solutions in the moving domain was proved by Feireisl et al. see [11, 12]. Recently, the local existence of the strong solutions on moving domain was shown by Kreml et al. in [25] which generalized the results in [31]. The reader interested in weak solutions can find more results in [7, 8, 9, 13, 21]. What's more, dissipative measure-valued solutions to (1.1) were established by Huang in [19]. See also interesting results by I. Dražić in [3, 4].

Such boundary conditions (1.7)-(1.8) are closely related to the fluid-structure interaction, so let us review the results of this model. One can find the local in-time existence results for the incompressible Navier-Stokes flow past several moving or rotating obstacles by Dintelmann et al. [5]. Moreover, the L^p approach is used for the incompressible and compressible cases in [16, 20]. We also notice that the 3–D magnetohydrodynamic with a slip boundary condition was investigated in [30]. Nowadays, there is quite a large amount of literature on the fluid-rigid body coupled system, we refer to the papers by [15], and the references therein, for some results of the existence of solutions.

To fill the gap in the theory of weak-strong uniqueness, Feireisl et al. [14] defined a suitable weak solution to the compressible Navier-Stokes equations by using a general relative entropy inequality. Then, in [10], the authors obtained the weak-strong uniqueness property with no-slip boundary conditions. Nečasová et al. in [25] generalized the former results on a time depend domain with Navier's slip boundary conditions. We can see [29, 17] for more recent results in this direction. On the other hand, the global well-posedness of compressible Navier-Stokes equations on a moving domain in the L^p-L^q framework could be found in [24]. Another interesting result concerned with a rigid body in a compressible Navier-Stokes-Fourier system was introduced in [18].

Based on the related results on the compressible Navier-Stokes equations, the problem we want to consider is: *is it possible to obtain the local existence of the strong solutions, the global weak solutions, and weak-strong uniqueness principle to the compressible fluids with asymmetric stress tensor on a time-dependent domain?* It can be seen that for such compressible fluids with asymmetric stress tensor in time-dependent domains, there is little literature on the existence of strong solutions or weak solutions for slip boundary conditions. There are several difficulties we need to overcome:

- Concerning the local existence of strong solutions to (1.1) with slip boundary and initial conditions (1.7)-(1.9), we find that the fluid velocity $\boldsymbol{u}(t,x)$ is coupled with micro-rotational velocity $\boldsymbol{w}(t,x)$ in the stress tensor $\mathbb{S}$, and the occurrence of non-linearity of interactions of velocity and micro-rotational velocity.
- When we consider the global existence of weak solutions to (1.1) with slip boundary and initial conditions (1.7)-(1.9), we shall give the precise weak formulation of (1.1) and make sure that the impermeability conditions (1.8) are satisfied in the sense of traces.
- In view of the weak-strong uniqueness principle, it is necessary to construct a relative entropy inequality to measure the distance between weak solutions and strong ones. Moreover, due to slip boundary conditions, some additional difficulties arise.

Considering the aforementioned difficulties, we have corresponding methods to fix them.

- In order to establish the local existence of strong solutions, we construct an iterative scheme that consists of linear continuity, momentum, and micro-rotational velocity equations. After solving the continuity equation on the moving domain, we shall treat micro-rotational velocity equations and momentum equations, respectively. To transfer the moving domain to a fixed one, the Lagrangian coordinates transformation defined by the velocity of the domain is used. Meanwhile, non-homogeneous boundary conditions are arising. Here the treatment for the integration by parts on the boundary should be careful, elaborate treatment on boundary conditions and the smallness of time are used in closing energy estimates. Eventually, the sequence defined by the iterative scheme is a Cauchy sequence and the local existence of strong solutions is proved.

- To utilize the penalization method on formulating weak solutions to system (1.1) on a moving domain with slip boundary conditions, we introduce new singular forcing terms. Some new cancellations play a crucial role in the process of deriving modified energy inequality, and we emphasize that (1.7)-(1.8) are not only reasonable in physical meaning but also rational in mathematical analysis. With uniform bounds coming from elementary energy in hand, we can perform the singular limits to derive weak solutions and similar processes are presented in [2] as long as we artificially choose magnetization to be zero, so we omit these details.
- Inspired by specific boundary conditions (1.7)-(1.8), we choose some suitable test functions to construct the relative entropy inequality. To the best of our knowledge, this is the first result that we consider the compressible fluid with asymmetric tensor on a time-dependent domain with Navier's slip boundary conditions. Though the weak-strong uniqueness for Navier-Stokes equations with slip boundary conditions was established in [25], it is still a non-trivial task to generalize it to the compressible fluids with asymmetric tensor.

The main contents of this paper are organized into ten sections. After this somewhat long introduction, main results, and the definition of function spaces, which constitute Section 1, we will give the iterative scheme in Section 2. Section 3, Section 4, and Section 5 deal with the existence of the strong solution to the linear continuity equation, micro-rotational equations, and momentum equations, respectively, in appropriate function spaces. Section 6 presents the convergence of the iterative scheme and completes the proof of Theorem 1.1. Section 7 and Section 8 are devoted to the global existence of weak solutions to the original problem. The weak-strong uniqueness principle is formulated in Section 9. Lastly, the details of the construction on boundary values and a recalling the fundamental lemma from [11] are presented in Appendix.

We are now in a position to present the main results in this paper.

Firstly, we state the local existence of strong solutions to the system (1.1).

Theorem 1.1. *Let $\Omega_0 \subset \mathbb{R}^3$ be a bounded domain of class of C^3 and $\boldsymbol{V} \in C^4((0,T)\times\mathbb{R}^3)$. Suppose that the initial data satisfies $(\boldsymbol{u},\boldsymbol{w}) \in H^3(\Omega_0)$, $\rho_0 \in H^2(\Omega_0)$ and $0 < \rho_* \leq \rho \leq \rho*$ for any positive constants ρ_*, $\rho*$. Then there exists a sufficiently small time $T > 0$ such that the system* (1.1) *with initial-boundary conditions* (1.7)-(1.9) *admits a unique solution $(\rho,\boldsymbol{u},\boldsymbol{w})$ belonging to the following function spaces*

$$\begin{cases} (\boldsymbol{u},\boldsymbol{w}) \in L^\infty(0,T;H^2(\Omega_t)) \cap L^2(0,T;H^3(\Omega_t)), \\ (\boldsymbol{u}_t,\boldsymbol{w}_t) \in L^\infty(0,T;H^1(\Omega_t)) \cap L^2(0,T;H^2(\Omega_t)), \\ (\boldsymbol{u}_{tt},\boldsymbol{w}_{tt}) \in L^2(0,T;L^2(\Omega_t)), \\ \rho \in L^\infty(0,T;H^2(\Omega_t)), \quad \rho_t \in L^2(0,T;H^1(\Omega_t)). \end{cases} \tag{1.11}$$

Our second result is concerned with the existence of weak solutions to the system (1.1).

Theorem 1.2. *Let $\Omega_0 \in \mathbb{R}^3$ is a bounded domain of class of C^3 and $\boldsymbol{V} \in C^1(0,T;C^4_c(\mathbb{R}^3)$. Let initial data satisfies* (1.10). *Then there exists a weak solution with finite energy to the system* (1.1)-(1.9) *in the sense of* (7.2), (7.4), *and* (7.5).

Finally, we give the weak-strong uniqueness principle for slip boundary conditions.

Theorem 1.3. *Let $\boldsymbol{V} \in C^1([0,T];C^4_c(\mathbb{R}^3))$. Let $(\rho,\boldsymbol{u},\boldsymbol{w})$ be a weak solution to the system* $(1.1)-(1.9)$ *constructed in Theorem* 1.2, *and let $(\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})$ be a strong solution to the problem* $(1.1)-(1.9)$

which emanates from the same initial data satisfying

$$\begin{cases} 0 < \inf_{Q_T} \widehat{\rho} \le \sup_{Q_T} \widehat{\rho} < \infty, \\ \nabla_x \widehat{\rho} \in L^2(0,T;L^q(\Omega_t)), \\ (\nabla_x^2 \widehat{\boldsymbol{u}}, \nabla_x^2 \widehat{\boldsymbol{w}}, \nabla_x \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{w}}) \in L^2(0,T;L^q(\Omega_t)), \end{cases} \tag{1.12}$$

with $q > \max\{3; 6\gamma/(5\gamma-6)\}$. *Then it holds that*

$$\rho = \widehat{\rho}, \quad \boldsymbol{u} = \widehat{\boldsymbol{u}}, \quad \boldsymbol{w} = \widehat{\boldsymbol{w}} \quad a.e. \quad in\ Q_t. \tag{1.13}$$

1.1. **Function spaces.** For any fixed $T > 0$, Ω_t is a bounded domain, for all $t \in [0,T]$. It also means that there exists $R > 0$ such that for all $t \in [0,T]$, one has $\Omega_t \subset B_R(0)$ where $B_R(0)$ is a ball with radius R centered at the origin. According to the assumption, we define the function spaces

$$L^p(0,T;L^q(\Omega_t)) := \{u \in L^p(0,T;L^q(\Omega_t)), u(t,\cdot) = 0 \ in\ B_R(0)\backslash\Omega_t \ for\ a.e.\ t \in (0,T)\},$$

with the norm

$$\|u\|_{L^p(0,T;L^q(\Omega_t))} := \left(\int_0^T \|u(t)\|^p_{L^q(\Omega_t)} \mathrm{d}t \right)^{1/p},$$

for $1 < p < \infty$ and $\|u\|_{L^\infty(0,T;L^q(\Omega_t))} := \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_{L^q(\Omega_t)}$. In the same way, we can define spaces $L^p(0,T;W^{l,q}(\Omega_t))$ and $C([0,T];W^{l,q}(\Omega_t))$.

We also introduce the function spaces which will be used later. For a function f defined on a moving domain $(0,T)\times\Omega_t$, we denote

$$\|f\|_{\mathcal{X}(T)} = \|f\|_{L^\infty(0,T;H^2(\Omega_t))\cap L^2(0,T;H^3(\Omega_t))} + \|f_t\|_{L^\infty(0,T;H^1(\Omega_t))\cap L^2(0,T;H^2(\Omega_t))} + \|f_{tt}\|_{L^2(0,T;L^2(\Omega_t))}. \tag{1.14}$$

For a function $\tilde{f}$ belongs to fixed domain $(0,T)\times\Omega$ where Ω is a fixed domain, we define

$$\|\tilde{f}\|_{\mathcal{Y}(T)} = \|\tilde{f}\|_{L^\infty(0,T;H^2(\Omega))\cap L^2(0,T;H^3(\Omega))} + \|\tilde{f}_t\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \|\tilde{f}_{tt}\|_{L^2(0,T;L^2(\Omega))}. \tag{1.15}$$

2. Iterative scheme

In this subsection, we define an iterative scheme where the continuity equation, momentum equations, and micro-rotation velocity equations are coupled. For this linearized system, we derive suitable estimates to guarantee the convergence of the iterative scheme. We also apply the method of successive approximations adopted in [25] where the authors use Lagrangian transformation determined by the velocity $\boldsymbol{V}$. Moreover, we can give the proof of Theorem 1.1.[1]

We set $\rho_1(t,\boldsymbol{X}(t,x)) := \rho_0(x)$, $\boldsymbol{u}_1(t,\boldsymbol{X}(t,x)) := \boldsymbol{u}_0(x)$ and $\boldsymbol{w}_1(t,\boldsymbol{X}(t,x)) := \boldsymbol{w}_0(x)$ for all $t \in (0,T)$. With $(\rho_n, \boldsymbol{u}_n, \boldsymbol{w}_n)$ in hand, then we use the following iterative scheme to define $(\rho_{n+1}, \boldsymbol{u}_{n+1}, \boldsymbol{w}_{n+1})$.

- We use the linearized continuity equation to obtain ρ_{n+1}

$$\partial_t \rho_{n+1} + \boldsymbol{u}_n \cdot \nabla_x \rho_{n+1} + \rho_{n+1} \mathrm{div}_x \boldsymbol{u}_n = 0, \tag{2.1}$$

with the initial data $\rho_{n+1}(0,x) = \rho_0(x)$ in Ω_0.

- Then, we use the linearized micro-rotation velocity equations to deduce $\boldsymbol{w}_{n+1}$

$$\begin{aligned} &\rho_{n+1}\partial_t \boldsymbol{w}_{n+1} - (c_0 + c_d - c_a)\nabla_x \mathrm{div}_x \boldsymbol{w}_{n+1} - (c_a + c_d)\Delta_x \boldsymbol{w}_{n+1} - 2\xi(\mathrm{curl}_x \boldsymbol{u}_n - 2\boldsymbol{w}_{n+1}) \\ &= -\rho_{n+1}\boldsymbol{w}_n \cdot \nabla_x \boldsymbol{w}_n := \boldsymbol{F}_2(\rho_{n+1}, \boldsymbol{w}_n), \end{aligned} \tag{2.2}$$

with boundary conditions

$$\begin{aligned} (\boldsymbol{w}_{n+1} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{V}) \cdot \boldsymbol{n}|_{\Gamma_t} &= 0, \\ [\mathbb{M}(\nabla_x \boldsymbol{w}_{n+1}) \cdot \boldsymbol{n}]_{\tan} + \kappa_2[\boldsymbol{w}_{n+1} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{V}]_{\tan}|_{\Gamma_t} &= 0, \end{aligned} \tag{2.3}$$

[1]Originally, the method was introduced for fixed domain, see [31].

and initial condition $\boldsymbol{w}_{n+1}(0,x)=\boldsymbol{w}_0(x)$ in Ω_0.

• Lastly, with ρ_{n+1} and $\boldsymbol{w}_{n+1}$ in hand, we use the linearized momentum equations to derive $\boldsymbol{u}_{n+1}$

$$\begin{aligned}&\rho_{n+1}\partial_t\boldsymbol{u}_{n+1}-(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\boldsymbol{u}_{n+1}-(\mu+\xi)\Delta_x\boldsymbol{u}_{n+1}-2\xi\mathrm{curl}_x\boldsymbol{w}_{n+1}\\ =&-\rho_{n+1}\boldsymbol{u}_n\cdot\nabla_x\boldsymbol{u}_n-\nabla_x p(\rho_n):=\boldsymbol{F}_1(\rho_{n+1},\boldsymbol{u}_n),\end{aligned}\tag{2.4}$$

with boundary conditions

$$\begin{aligned}(\boldsymbol{u}_{n+1}-\boldsymbol{V})\cdot\boldsymbol{n}|_{\Gamma_t}&=0,\\ [\mathbb{S}(\nabla_x\boldsymbol{u}_{n+1},\boldsymbol{w}_{n+1})\cdot\boldsymbol{n}]_{\tan}+\kappa_1[\boldsymbol{u}_{n+1}-\boldsymbol{V}]_{\tan}|_{\Gamma_t}&=0,\end{aligned}\tag{2.5}$$

and initial condition $\boldsymbol{u}_{n+1}(0,x)=\boldsymbol{u}_0(x)$ in Ω_0.

3. Linearized continuity equation

In this section, we consider the local existence of strong solutions to the linearized continuity equation on a moving domain. It should be mentioned that we can not go with the continuity equation to the Lagrangian coordinates since we can not close the system (missing regularity for density). Hence, such a problem can be treated in the same way as Proposition 3.1 in [25], we only list some results for the linearized continuity equation without detailed proof.

$$\rho_t+\boldsymbol{u}\cdot\nabla_x\rho+\rho\mathrm{div}_x\boldsymbol{u}=0,\tag{3.1}$$

with $(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}|_{\Gamma_t}=0$.

Introducing the characteristics as

$$X(t,z)=z+\int_0^t\boldsymbol{u}(s,X(s,z))ds,\tag{3.2}$$

then it derives from (3.1) that

$$\rho(t,X(t,z))=\rho_0\exp\left(-\int_0^t\mathrm{div}_x\boldsymbol{u}(s,X(s,z))ds\right).\tag{3.3}$$

We notice that the mapping $X(t,z)$ depends on the velocity $\boldsymbol{u}(t,x)$ while the mapping $\boldsymbol{X}(t,z)$ is associated with $\boldsymbol{V}(t,x)$. When $\boldsymbol{u}(t,x)$ gains enough regularity, we can derive some properties of the mapping $X(t,z)$.

Lemma 3.1. *Assume that $\boldsymbol{u}\in L^2(0,T;H^3(\Omega_t))$. Let $X(t,z)$ be defined by (3.2), i.e. for fixed $t\in(0,T)$ we have $X(t,\cdot):\Omega_0\to\Omega_t$. Then there exists a sufficiently small time $T>0$ such that for all $t\in(0,T)$ there exists an inverse mapping $z(t,\cdot):\Omega_t\to\Omega_0$, i.e. $z(t,X(t,y))=y$ for all $y\in\Omega_0$ and $X(t,z(t,y))=y$ for all $y\in\Omega_t$.*

Furthermore, it holds that

$$\|\nabla_x z(t,x)-\mathbb{I}\|_{L^\infty(Q_\tau)}\le E(\tau),\tag{3.4}$$

and

$$\|\nabla_z^2X(t,z)\|_{L^\infty(0,T;L^4(\Omega_0))}\le\phi(\|\boldsymbol{u}\|_{L^2(0,T;H^3(\Omega_t))}),\tag{3.5}$$

where $\mathbb{I}$ denotes the identity matrix, $E(t)$ is a non-negative function satisfying $E(t)\to0$ as $t\to0^+$ and ϕ is an increasing positive function.

With the aid of Lemma 3.1, we present the solvability of (3.1).

Proposition 3.1. *Suppose that $\rho_0\in H^2(\Omega_0)$, $\boldsymbol{u}\in L^\infty(0,T;H^2(\Omega_t))\cap L^2(0,T;H^3(\Omega_t))$. Then, for sufficiently small time $T>0$, there exists a unique solution $\rho(t,x)$ to the linearized transport equation (3.1) such that*

$$\rho\in C(0,T;H^2(\Omega_t)),\quad\rho_t\in L^2(0,T;H^1(\Omega_t)).\tag{3.6}$$

Moreover, the following estimates hold

$$\|\rho\|_{L^\infty(0,T;H^2(\Omega_t))}\le C\|\rho_0\|_{H^2(\Omega_0)}\phi(\sqrt{T}\|\boldsymbol{u}\|_{L^2(0,T;H^3(\Omega_t))}),\tag{3.7}$$

and

$$\|\rho_t\|_{L^2(0,T;H^1(\Omega_t))} \le C\sqrt{T}\|\rho_0\|_{H^2(\Omega_0)}\phi(\sqrt{T}\|\boldsymbol{u}\|_{L^2(0,T;H^3(\Omega_t))})\|\boldsymbol{u}\|_{L^2(0,T:H^3(\Omega_t))}, \tag{3.8}$$

where $\phi(\cdot)$ is an increasing positive function.

In addition, if $\rho_0 \ge \underline{\rho} > 0$ then it holds that $\rho \ge C(\underline{\rho}, T, \boldsymbol{u}) > 0$.

4. Linearized micro-angular equations

In this section, we derive the local existence of strong solutions to the linearized micro-rotation velocity equations where the fluid velocity $\boldsymbol{u}(t,x)$ is given. More precisely, we shall treat the following linearized micro-angular equations with Navier's slip boundary conditions.

$$\begin{cases} \rho\boldsymbol{w}_t - (c_0 + c_d - c_a)\nabla_x \mathrm{div}_x \boldsymbol{w} - (c_a + c_d)\Delta_x \boldsymbol{w} - 2\xi(\mathrm{curl}_x \boldsymbol{u} - 2\boldsymbol{w}) = \boldsymbol{F}_2, \\ (\boldsymbol{w} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{V}) \cdot \boldsymbol{n}|_{\Gamma_t} = 0, \\ n \cdot \mathbb{M} \cdot \tau_k + \kappa_2(\boldsymbol{w} - \frac{1}{2}\mathrm{curl}_x \boldsymbol{V}) \cdot \tau_k|_{\Gamma_t} = 0, \quad k = 1, 2. \end{cases} \tag{4.1}$$

We establish the local existence of strong solutions to the linearized micro-angular equations (4.1) on a moving domain and leave the proof in the following several subsections.

Proposition 4.1. *Let $T > 0$ be sufficiently small and $\boldsymbol{V}$ satisfies the assumption of Theorem 1.1. Assume that*

$$\begin{cases} \rho \in L^\infty(0,T;H^2(\Omega_t)), \quad \rho_t \in L^2(0,T;H^1(\Omega_t)), \\ \boldsymbol{F}_2 \in L^2(0,T;H^1(\Omega_t)), \quad \boldsymbol{F}_{2,t} \in L^2(0,T;L^2(\Omega_t)), \\ \boldsymbol{F}_2(0) \in H^1(\Omega_0), \quad \boldsymbol{w}_0 \in H^3(\Omega_0), \quad \boldsymbol{u}_0 \in H^3(\Omega_0). \end{cases} \tag{4.2}$$

Then there exists a unique solution $\boldsymbol{w}(t,x)$ to the system (4.1) such that $\boldsymbol{w} \in \mathcal{X}(T)$ and the following estimate holds

$$\begin{aligned} \|\boldsymbol{w}\|_{\mathcal{X}(T)} \le \phi(\|\rho\|_{L^\infty(0,T;H^2(\Omega_t))}, \|\rho_t\|_{L^2(0,T;H^1(\Omega_t))}) \times \Big(\|\boldsymbol{u}\|_{L^2(0,T;H^1(\Omega_t))} + \|\boldsymbol{u}_t\|_{L^2(0,T;H^1(\Omega_t))} \\ + \|\boldsymbol{F}_2\|_{L^2(0,T;H^1(\Omega_t))} + \|\boldsymbol{F}_{2,t}\|_{L^2(0,T;L^2(\Omega_t))} + \|\boldsymbol{F}_2(0)\|_{H^1(\Omega_0)} + \|\boldsymbol{w}_0\|_{H^3(\Omega_0)} \\ + \|\boldsymbol{V}\|_{L^\infty(0,T;H^3(\Omega_t)) \cap L^2(0,T;H^3(\Omega_t))} + \|\boldsymbol{V}_t\|_{L^\infty(0,T;H^2(\Omega_t)) \cap L^2(0,T;H^3(\Omega_t))} + \|\boldsymbol{V}_{tt}\|_{L^2(0,T;H^2(\Omega_t))}\Big), \end{aligned} \tag{4.3}$$

where ϕ is a positive increasing function.

4.1. **Lagrangian coordinates.** We transform the boundary problem (4.1) in the moving domain into a problem defined on a fixed spatial domain $(0,T) \times \Omega_0$ by using the Lagrangian coordinates determined by $\boldsymbol{V}$.

We set

$$\begin{aligned} \tilde{\rho}(t,y) := \rho(t, \boldsymbol{X}(t,y)), \quad \tilde{\boldsymbol{u}}(t,y) := \boldsymbol{u}(t, \boldsymbol{X}(t,y)), \\ \tilde{\boldsymbol{w}}(t,y) := \boldsymbol{w}(t, \boldsymbol{X}(t,y)), \quad \tilde{\boldsymbol{V}}(t,y) := \boldsymbol{V}(t, \boldsymbol{X}(t,y)), \end{aligned} \tag{4.4}$$

with $\boldsymbol{X} = (X_1, X_2, X_3)$. Then, we denote $\boldsymbol{Y}(t,x) = (Y_1, Y_2, Y_3)$ to be the inverse mapping to $\boldsymbol{X}(t,y)$. It holds that for all $t \ge 0$ and $x \in \Omega_t$, $\boldsymbol{X}(t, \boldsymbol{Y}(t,x)) = x$, which gives the following identities

$$\frac{\partial \boldsymbol{X}}{\partial t} + \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{Y}}\frac{\partial \boldsymbol{Y}}{\partial t} = 0, \quad \frac{\partial \boldsymbol{Y}}{\partial t} = -\boldsymbol{V} \cdot \nabla_x \boldsymbol{Y}. \tag{4.5}$$

According to (4.5), we have

$$\frac{\partial w_i}{\partial t} = \frac{\partial \tilde{w}_i}{\partial t} + \nabla_y \tilde{w}_i \cdot \frac{\partial \boldsymbol{Y}}{\partial t} = \frac{\partial \tilde{w}_i}{\partial t} - \nabla_y \tilde{w}_i \cdot (\boldsymbol{V} \cdot \nabla_x \boldsymbol{Y}).$$

Then we can rewrite the i-th component of micro-angular equations $(4.1)_{1,i}$ as follows

$$\tilde\rho\left(\frac{\partial\tilde w_i}{\partial t}-\frac{\partial\tilde w_i}{\partial y_j}V_k\frac{\partial Y_j}{\partial x_k}\right)-(c_0+c_d-c_a)\frac{\partial^2\tilde w_p}{\partial y_k\partial y_l}\frac{\partial Y_l}{\partial x_p}\frac{\partial Y_k}{\partial x_i}-(c_0+c_d-c_a)\frac{\partial\tilde w_p}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_i\partial x_p}$$
$$-(c_a+c_d)\frac{\partial\tilde w_i}{\partial y_k}\Delta_x Y_k-(c_a+c_d)\frac{\partial^2\tilde w_i}{\partial y_k\partial y_l}\frac{\partial Y_k}{\partial x_p}\frac{\partial Y_l}{\partial x_p}-2\xi\left(\varepsilon_{ijk}\frac{\partial\tilde u_k}{\partial y_l}\frac{\partial Y_l}{\partial x_j}-2\tilde w_i\right)=F_{2,i},$$

where we used the Einstein summation convention and the Levi-Civita symbol. Finally, we reformulate the original equations into

$$\begin{aligned}&\tilde\rho\frac{\partial\tilde{\boldsymbol w}}{\partial t}-(c_0+c_d-c_a)\nabla_y\mathrm{div}_y\tilde{\boldsymbol u}-(c_a+c_d)\Delta_y\tilde{\boldsymbol u}-2\xi(\mathrm{curl}_y\tilde{\boldsymbol u}-2\tilde{\boldsymbol w})\\&=\boldsymbol F_2+\tilde\rho\boldsymbol V\cdot\nabla_y\tilde{\boldsymbol w}+\boldsymbol R_2(\tilde\rho,\tilde{\boldsymbol u},\tilde{\boldsymbol w}):=\tilde{\boldsymbol F}_2(\tilde\rho,\tilde{\boldsymbol u},\tilde{\boldsymbol w}),\end{aligned}\tag{4.6}$$

with

$$\begin{aligned}\boldsymbol R_{2,i}(\tilde\rho,\tilde{\boldsymbol u},\tilde{\boldsymbol w})=&\tilde\rho\frac{\partial\tilde w_i}{\partial y_j}V_k\left(\frac{\partial Y_j}{\partial x_k}-\delta_{jk}\right)+(c_0+c_d-c_a)\frac{\partial^2\tilde w_p}{\partial y_k\partial y_l}\left(\frac{\partial Y_l}{\partial x_p}\frac{\partial Y_k}{\partial x_i}-\delta_{lp}\boldsymbol e_k\right)\\&+(c_0+c_d-c_a)\frac{\partial\tilde w_p}{\partial y_k}\frac{\partial^2Y_k}{\partial x_i\partial x_p}+(c_a+c_d)\frac{\partial^2\tilde w_i}{\partial y_k\partial y_l}\left(\frac{\partial Y_k}{\partial x_p}\frac{\partial Y_l}{\partial x_p}-\delta_{lp}\delta_{kp}\right)\\&+(c_a+c_d)\frac{\partial\tilde w_i}{\partial y_k}\Delta_xY_k+2\xi\varepsilon_{ijk}\frac{\partial\tilde u_k}{\partial y_l}\left(\frac{\partial Y_l}{\partial x_j}-\delta_{lj}\right),\end{aligned}\tag{4.7}$$

where $\boldsymbol e_j$ is the j-th unit vector.

The boundary conditions $(4.1)_2$ and $(4.1)_3$ can be rewritten through these new variables (4.4). Here we use (4.4) and $(4.1)_2$ to derive that

$$\begin{aligned}(\tilde{\boldsymbol w}-\frac12\mathrm{curl}_y\tilde{\boldsymbol V})(t,y)\cdot\boldsymbol n(y)=&(\tilde{\boldsymbol w}(t,y)-\frac12\mathrm{curl}_y\tilde{\boldsymbol V})\cdot(\boldsymbol n(y)-\boldsymbol n(\boldsymbol X(t,y)))\\&+\frac12[((\nabla_x\boldsymbol Y-\mathbb I)\nabla_y)\wedge\tilde{\boldsymbol V}]\cdot\boldsymbol n(\boldsymbol X(t,y)):=\boldsymbol d_2(\tilde{\boldsymbol w},\tilde{\boldsymbol V})(t,y),\end{aligned}\tag{4.8}$$

where we use the cross-product symbol $\wedge$ to represent the operator curl. Similarly, we use (4.4) and $(4.1)_3$ to get

$$\begin{aligned}&\left[(c_a+c_d)\nabla_y\tilde{\boldsymbol w}(t,y)+(c_d-c_a)\nabla_y^{\mathsf T}\tilde{\boldsymbol w}\right]\boldsymbol n(y)\cdot\tau_k(y)+\kappa_2(\tilde{\boldsymbol w}(t,y)-\frac12\mathrm{curl}_y\tilde{\boldsymbol V})\cdot\tau_k(y)\\=&\left[(c_a+c_d)\nabla_y\tilde{\boldsymbol w}(t,y)(\mathbb I-\nabla_x\boldsymbol Y)+(c_d-c_a)(\mathbb I-\nabla_x^{\mathsf T}\boldsymbol Y)\nabla_y^{\mathsf T}\tilde{\boldsymbol w}\right]\boldsymbol n(\boldsymbol X(t,y))\cdot\tau_k(\boldsymbol X(t,y))\\&+\left[(c_a+c_d)\nabla_y\tilde{\boldsymbol w}(t,y)+(c_d-c_a)\nabla_y^{\mathsf T}\tilde{\boldsymbol w}\right](t,y)\left[(\boldsymbol n(y)-\boldsymbol n(\boldsymbol X(t,y)))\cdot\tau_k(\boldsymbol X(t,y))\right.\\&\left.+\boldsymbol n(y)\cdot(\tau_k(y)-\tau_k(\boldsymbol X(t,y)))\right]+\kappa_2(\tilde{\boldsymbol w}(t,y)-\frac12\mathrm{curl}_y\tilde{\boldsymbol V})\cdot(\tau_k(y)-\tau_k(\boldsymbol X(t,y)))\\&+\frac{\kappa_2}{2}[((\nabla_x\boldsymbol Y-\mathbb I)\nabla_y)\wedge\tilde{\boldsymbol V}]\cdot\tau_k(\boldsymbol X(t,y)):=\boldsymbol B_2(\tilde{\boldsymbol w},\tilde{\boldsymbol V})(t,y).\end{aligned}\tag{4.9}$$

Finally, we translate the linearized micro-rotation velocity equations (4.1) into a fixed domain $(0,T)\times\Omega_0$,

$$\begin{cases}\tilde\rho\tilde{\boldsymbol w}_t-(c_0+c_d-c_a)\nabla_y\mathrm{div}_y\tilde{\boldsymbol w}-(c_a+c_d)\Delta_y\tilde{\boldsymbol w}-2\xi(\mathrm{curl}_y\tilde{\boldsymbol u}-2\tilde{\boldsymbol w})=\tilde{\boldsymbol F}_2,\\(\tilde{\boldsymbol w}-\frac12\mathrm{curl}_y\boldsymbol V)\cdot\boldsymbol n|_{\Gamma_0}=\boldsymbol d_2,\\\boldsymbol n\cdot\mathbb M\cdot\tau_k+\kappa_2(\tilde{\boldsymbol w}-\frac12\mathrm{curl}_y\boldsymbol V)\cdot\tau_k|_{\Gamma_0}=\boldsymbol B_2,\quad k=1,2.\end{cases}\tag{4.10}$$

4.2. **Solvability of system** (4.10)**.** In this subsection, we give the local existence of strong solutions to (4.10) on a fixed domain $(0,T)\times\Omega_0$.

Lemma 4.1. *Assume that $\boldsymbol{B}_2$ and $\boldsymbol{d}_2$ admit an extension to Ω_0 given by*

$$\begin{aligned} \boldsymbol{w}^b\cdot\boldsymbol{n}|_{\Gamma_0}&=\boldsymbol{d}_2,\\ ([\mathbb{M}(\nabla_y\boldsymbol{w}^b)\boldsymbol{n}]_{\tan}+\kappa_2[\boldsymbol{w}^b]_{\tan})|_{\Gamma_0}&=\boldsymbol{B}_2, \end{aligned}\tag{4.11}$$

such that $\boldsymbol{w}^b\in\mathcal{Y}(t)$. Let $\boldsymbol{V}$ satisfy the assumptions in Theorem 1.1. Suppose that

$$\begin{cases} \tilde\rho\in L^\infty(0,T;H^2(\Omega_0)),\tilde\rho_t\in L^2(0,T;H^1(\Omega_0)),\tilde\rho\geq\underline{\rho}>0,\\ \tilde{\boldsymbol{F}}_2\in L^2(0,T;H^1(\Omega_0)),\tilde{\boldsymbol{F}}_{2,t}\in L^2(0,T;L^2(\Omega_0)),\\ \tilde{\boldsymbol{w}}_0\in H^3(\Omega_0)\quad \tilde{\boldsymbol{u}}_0\in H^2(\Omega_0). \end{cases}\tag{4.12}$$

Then there exists a unique solution $\tilde{\boldsymbol{w}}(t,x)$ to the system (4.10) such that

$$\begin{aligned} \|\tilde{\boldsymbol{w}}\|_{\mathcal{Y}(T)}\leq&\phi(\underline{\rho},\|\tilde\rho\|_{L^\infty(0,T;H^2(\Omega_0))},\|\tilde\rho_t\|_{L^2(0,T;H^1(\Omega_0))})(\left\|\tilde{\boldsymbol{u}}\right\|_{L^2(0,T;H^1(\Omega_0))}+\left\|\tilde{\boldsymbol{u}}_t\right\|_{L^2(0,T;H^1(\Omega_0))}\\ &+\|\tilde{\boldsymbol{F}}_2\|_{L^2(0,T;H^1(\Omega_0))\cap L^\infty(0,T;L^2(\Omega_0))}+\|\tilde{\boldsymbol{F}}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))}+\|\boldsymbol{w}^b\|_{\mathcal{Y}(T)}\\ &+\|\tilde{\boldsymbol{V}}\|_{L^\infty(0,T;H^1(\Omega_0))\cap L^2(0,T;H^2(\Omega_0))}+\|\tilde{\boldsymbol{V}}_t\|_{L^\infty(0,T;H^2(\Omega_0))\cap L^2(0,T;H^3(\Omega_0))}\\ &+\|\tilde{\boldsymbol{V}}_{tt}\|_{L^2(0,T;H^2(\Omega_0))}+\|\tilde{\boldsymbol{w}}_0\|_{H^3(\Omega_0)}+\left\|\tilde{\boldsymbol{u}}_0\right\|_{H^2(\Omega_0)}), \end{aligned}\tag{4.13}$$

where ϕ denotes a positive increasing function.

Proof. In order to remove the influence of inhomogenity of the boundary data $(4.10)_2$ and $(4.10)_3$, we set $\boldsymbol{w}^\dagger=\tilde{\boldsymbol{w}}-\boldsymbol{w}^b$ to obtain

$$\begin{cases} \tilde\rho\partial_t\boldsymbol{w}^\dagger-(c_0+c_d-c_a)\nabla_y\mathrm{div}_y\boldsymbol{w}^\dagger-(c_a+c_d)\Delta_y\boldsymbol{w}^\dagger-2\xi(\mathrm{curl}_y\tilde{\boldsymbol{u}}-2\boldsymbol{w}^\dagger)=\hat{\boldsymbol{F}}_2-\tilde\rho\boldsymbol{w}^b_t,\\ (\boldsymbol{w}^\dagger-\dfrac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}|_{\Gamma_0}=0,\\ \boldsymbol{n}\cdot\mathbb{M}\cdot\tau_k+\kappa_2(\boldsymbol{w}^\dagger-\dfrac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\cdot\tau_k|_{\Gamma_0}=0,\quad k=1,2, \end{cases}\tag{4.14}$$

where

$$\hat{\boldsymbol{F}}_2=\tilde{\boldsymbol{F}}_2+(c_0+c_d-c_a)\nabla_y\mathrm{div}_y\boldsymbol{w}^b+(c_a+c_d)\Delta_y\boldsymbol{w}^b-4\xi\boldsymbol{w}^b.\tag{4.15}$$

The positive frictions contribute to lower-order terms which are easy to estimate, so we omit the effect of frictions and assume that $\kappa_2=0$.

The proof (4.13) is presented by several parts in the below section.

• The estimate of $\|\boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}$.

Multiplying (4.14) by $\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}}$ and integrating over Ω_0, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\tilde\rho|\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}}|^2\mathrm{d}y+\int_{\Omega_0}(T_2\boldsymbol{w}^\dagger-2\xi\mathrm{curl}_y\tilde{\boldsymbol{u}}+4\xi\boldsymbol{w}^\dagger)\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y\\ =&\frac{1}{2}\int_{\Omega_0}\tilde\rho_t|\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}}|^2\mathrm{d}y-\frac{1}{2}\int_{\Omega_0}\tilde\rho\mathrm{curl}_y\tilde{\boldsymbol{V}}_t(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y\\ &+\int_{\Omega_0}\hat{\boldsymbol{F}}_2\cdot(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y-\int_{\Omega_0}\tilde\rho\boldsymbol{w}^b_t\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y\\ :=&\sum_{i=1}^4\mathcal{I}_{1,i}, \end{aligned}\tag{4.16}$$

where we denote $T_2\boldsymbol{w}^\dagger=-(c_0+c_d-c_a)\nabla_y\mathrm{div}_y\boldsymbol{w}^\dagger-(c_a+c_d)\Delta_y\boldsymbol{w}^\dagger$.

For the left-hand side of (4.16), we use (4.14) and integration by parts to obtain

$$\begin{aligned}\int_{\Omega_0}(T_2\boldsymbol{w}^\dagger)\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y=&\int_{\Omega_0}\mathbb{M}:\nabla_y\boldsymbol{w}^\dagger\mathrm{d}y-\frac{1}{2}\int_{\Omega_0}\mathbb{M}:\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}\mathrm{d}y\\ =&(c_a+c_d)\|\nabla_y\boldsymbol{w}^\dagger\|_{L^2(\Omega_0)}^2+(c_0+c_d-c_a)\|\mathrm{div}_y\boldsymbol{w}^\dagger\|_{L^2(\Omega_0)}^2\\ &-\frac{1}{2}\int_{\Omega_0}\mathbb{M}:\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}\mathrm{d}y,\end{aligned}\tag{4.17}$$

then Hölder's inequality leads to

$$\begin{aligned}&\left|-\frac{1}{2}\int_{\Omega_0}\mathbb{M}:\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}\mathrm{d}y\right|\\ &\le\epsilon\|\nabla_y\boldsymbol{w}^\dagger\|_{L^2(\Omega_0)}^2+C\|\nabla_y^2\tilde{\boldsymbol{V}}\|_{L^2(\Omega_0)}^2.\end{aligned}\tag{4.18}$$

Direct calculation shows that

$$\begin{aligned}&\int_{\Omega_0}(-2\xi\mathrm{curl}_y\tilde{\boldsymbol{u}}+4\xi\boldsymbol{w}^\dagger)\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y\\ &=4\xi\int_{\Omega_0}|\boldsymbol{w}^\dagger|^2\mathrm{d}y-2\xi\int_{\Omega_0}\boldsymbol{w}^\dagger\mathrm{curl}_y\tilde{\boldsymbol{V}}\mathrm{d}y-\int_{\Omega_0}2\xi\mathrm{curl}_y\tilde{\boldsymbol{u}}\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y.\end{aligned}\tag{4.19}$$

Applying Hölder's inequality also yields

$$\begin{aligned}&\left|-2\xi\int_{\Omega_0}\boldsymbol{w}^\dagger\mathrm{curl}_y\tilde{\boldsymbol{V}}\mathrm{d}y-\int_{\Omega_0}2\xi\mathrm{curl}_y\tilde{\boldsymbol{u}}\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\mathrm{d}y\right|\\ &\le\epsilon\|\boldsymbol{w}^\dagger\|_{L^2(\Omega_0)}^2+C(\big\|\nabla_y\tilde{\boldsymbol{u}}\big\|_{L^2(\Omega_0)}^2+\|\nabla_y\tilde{\boldsymbol{V}}\|_{L^2(\Omega_0)}^2).\end{aligned}\tag{4.20}$$

For the first term of the right-hand side of (4.16), the following estimate holds

$$|\mathcal{I}_{1,1}|\le C(\underline{\rho})(\|\tilde{\rho}_t\|_{L^3(\Omega_0)}^2\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\|_{L^2(\Omega_0)}^2).\tag{4.21}$$

Then, it follows from Hölder's inequality and the Sobolev inequality that

$$\begin{aligned}|\mathcal{I}_{1,2}|\le&C\big\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\big\|_{L^2(\Omega_0)}\big\|\tilde{\rho}\big\|_{L^\infty(\Omega_0)}^{\frac{1}{2}}\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\big\|_{L^2(\Omega_0)}\\ \le&C\big\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\big\|_{L^2(\Omega_0)}^2\big\|\tilde{\rho}\big\|_{H^2(\Omega_0)}+C\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\big\|_{L^2(\Omega_0)}^2,\end{aligned}\tag{4.22}$$

$$\begin{aligned}|\mathcal{I}_{1,3}|\le&C\big\|\hat{\boldsymbol{F}}_2\big\|_{L^2(\Omega_0)}\big\|\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}}\big\|_{L^2(\Omega_0)}\\ \le&C\big\|\hat{\boldsymbol{F}}_2\big\|_{L^2(\Omega_0)}\big\|\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}}\big\|_{L^6(\Omega_0)}\\ \le&\epsilon\big\|\nabla_y\boldsymbol{w}^\dagger\big\|_{L^2(\Omega_0)}^2+C\big\|\hat{\boldsymbol{F}}_2\big\|_{L^2(\Omega_0)}^2+C\big\|\nabla_y^2\tilde{\boldsymbol{V}}\big\|_{L^2(\Omega_0)}^2,\end{aligned}\tag{4.23}$$

and

$$\begin{aligned}|\mathcal{I}_{1,4}|\le&\big\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\big\|_{L^2(\Omega_0)}\big\|\tilde{\rho}\big\|_{L^\infty(\Omega_0)}^{\frac{1}{2}}\big\|\boldsymbol{w}_t^b\big\|_{L^2(\Omega_0)}\\ \le&\big\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\big\|_{L^2(\Omega_0)}^2\big\|\tilde{\rho}\big\|_{H^2(\Omega_0)}+C\big\|\boldsymbol{w}_t^b\big\|_{L^2(\Omega_0)}^2.\end{aligned}\tag{4.24}$$

Putting above estimates into (4.16), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\|_{L^2(\Omega_0)}^2+\|\boldsymbol{w}^\dagger\|_{L^2(\Omega_0)}^2+\|\nabla_y\boldsymbol{w}^\dagger\|_{L^2(\Omega_0)}^2\\ &\le C(\underline{\rho})(\big\|\tilde{\rho}\big\|_{H^2(\Omega_0)}+\|\tilde{\rho}_t\|_{L^3(\Omega_0)}^2)\|\sqrt{\tilde{\rho}}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\|_{L^2(\Omega_0)}^2\\ &\quad+C(\big\|\nabla_y\tilde{\boldsymbol{u}}\big\|_{L^2(\Omega_0)}^2+\|\hat{\boldsymbol{F}}_2\|_{L^2(\Omega_0)}^2+\|\boldsymbol{w}_t^b\|_{L^2(\Omega_0)}^2+\|\tilde{\boldsymbol{V}}\|_{H^2(\Omega_0)}^2+\|\tilde{\boldsymbol{V}}_t\|_{H^1(\Omega_0)}^2).\end{aligned}\tag{4.25}$$

Applying Grönwall's inequality, we get

$$\begin{aligned}
&\|\boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\boldsymbol{w}^\dagger\|_{L^2(0,T;L^2(\Omega_0))} + \|\nabla_y \boldsymbol{w}^\dagger\|_{L^2(0,T;L^2(\Omega_0))} \\
&\le \phi(\underline{\rho}, \|\tilde{\rho}\|_{L^\infty(0,T;H^2(\Omega_0))}, \|\tilde{\rho}_t\|_{L^2(0,T;H^1(\Omega_0))})(\|\tilde{\boldsymbol{u}}\|_{L^2(0,T;H^1(\Omega_0))} + \|\hat{\boldsymbol{F}}_2\|_{L^2(0,T;L^2(\Omega_0))} \\
&\quad + \|\boldsymbol{w}^b_t\|_{L^2(0,T;L^2(\Omega_0))} + \|\tilde{\boldsymbol{V}}\|_{L^2(0,T;H^2(\Omega_0))} + \|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^1(\Omega_0))} + \|\tilde{\boldsymbol{V}}\|_{L^\infty(0,T;H^1(\Omega_0))}).
\end{aligned} \tag{4.26}$$

• The estimate of $\|\nabla_y \boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}$.

We multiply (4.14) by $(\boldsymbol{w}^\dagger - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})_t + \epsilon T_2 \boldsymbol{w}^\dagger$, where ϵ is sufficiently small, then integrate the resulting equation over Ω_0 to derive

$$\begin{aligned}
&\int_{\Omega_0} \tilde{\rho}|\partial_t \boldsymbol{w}^\dagger|^2 \mathrm{d}y + \int_{\Omega_0} (\boldsymbol{w}^\dagger - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})_t \cdot (T_2 \boldsymbol{w}^\dagger + 4\xi \boldsymbol{w}^\dagger - 2\xi \mathrm{curl}_y \tilde{\boldsymbol{u}})\mathrm{d}y + \epsilon \int_{\Omega_0} |T_2 \boldsymbol{w}^\dagger|^2 \mathrm{d}y \\
&= -\epsilon \int_{\Omega_0} \tilde{\rho} {\boldsymbol{w}^\dagger}_t T_2 \boldsymbol{w}^\dagger \mathrm{d}y + \frac{1}{2}\int_{\Omega_0} \tilde{\rho} {\boldsymbol{w}^\dagger}_t \mathrm{curl}_y \tilde{\boldsymbol{V}}_t \mathrm{d}y + 2\xi\epsilon \int_{\Omega_0} T_2 \boldsymbol{w}^\dagger \cdot (\mathrm{curl}_y \tilde{\boldsymbol{u}} - 2\boldsymbol{w}^\dagger)\mathrm{d}y \\
&\quad + \int_{\Omega_0} (\hat{\boldsymbol{F}}_2 - \tilde{\rho}\partial_t \boldsymbol{w}^b) \cdot \left((\boldsymbol{w}^\dagger - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})_t + \epsilon T_2 \boldsymbol{w}^\dagger \right) \mathrm{d}y \\
&:= \sum_{i=1}^4 \mathcal{I}_{2,i}.
\end{aligned} \tag{4.27}$$

Using the fact that $(\boldsymbol{w}^\dagger - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})_t \cdot \boldsymbol{n} = 0$ and integration by parts, we can obtain

$$\begin{aligned}
&\int_{\Omega_0} (\boldsymbol{w}^\dagger - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})_t \cdot T_2 \boldsymbol{w}^\dagger \mathrm{d}y \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(c_a + c_d)\|\nabla_y \boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)} + (c_0 + c_d - c_a)\|\mathrm{div}_y \boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)} \right] \\
&\quad - \frac{1}{2}(c_0 + c_d - c_a)\int_{\Omega_0} \mathrm{div}_y \mathrm{curl}_y \tilde{\boldsymbol{V}}_t \mathrm{div}_y \boldsymbol{w}^\dagger \mathrm{d}y - \frac{1}{2}(c_a + c_d)\int_{\Omega_0} \nabla_y \mathrm{curl}_y \tilde{\boldsymbol{V}}_t : \nabla_y \boldsymbol{w}^\dagger \mathrm{d}y.
\end{aligned} \tag{4.28}$$

We apply Hölder's inequality to get

$$\begin{aligned}
&\left| -\frac{1}{2}(c_0 + c_d - c_a)\int_{\Omega_0} \mathrm{div}_y \mathrm{curl}_y \tilde{\boldsymbol{V}}_t \mathrm{div}_y \boldsymbol{w}^\dagger \mathrm{d}y - \frac{1}{2}(c_a + c_d)\int_{\Omega_0} \nabla_y \mathrm{curl}_y \tilde{\boldsymbol{V}}_t : \nabla_y \boldsymbol{w}^\dagger \mathrm{d}y \right| \\
&\le C\|\nabla_y \boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)} + C\|\nabla_y^2 \tilde{\boldsymbol{V}}_t\|^2_{L^2(\Omega_0)}.
\end{aligned} \tag{4.29}$$

Direct calculation gives that

$$\begin{aligned}
&\int_{\Omega_0} (\boldsymbol{w}^\dagger - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})_t \cdot (4\xi \boldsymbol{w}^\dagger - 2\xi \mathrm{curl}_y \tilde{\boldsymbol{u}})\mathrm{d}y \\
&= 2\xi \frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)} - 2\xi \int_{\Omega_0} \mathrm{curl}_y \tilde{\boldsymbol{V}}_t \cdot \boldsymbol{w}^\dagger \mathrm{d}y - 2\xi \int_{\Omega_0} \left({\boldsymbol{w}^\dagger}_t - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}} \right) \cdot \mathrm{curl}_y \tilde{\boldsymbol{u}} \mathrm{d}y,
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
&\left| 2\xi \int_{\Omega_0} \mathrm{curl}_y \tilde{\boldsymbol{V}}_t \cdot \boldsymbol{w}^\dagger \mathrm{d}y - 2\xi \int_{\Omega_0} \left({\boldsymbol{w}^\dagger}_t - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}} \right) \cdot \mathrm{curl}_y \tilde{\boldsymbol{u}} \mathrm{d}y \right| \\
&\le C\|\boldsymbol{w}^\dagger\|^2_{L^2(\omega_0)} + C\|\nabla_y \tilde{\boldsymbol{V}}_t\|^2_{L^2(\Omega_0)} + \frac{1}{5}\|\sqrt{\tilde{\rho}} {\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)} + C\|\nabla_y \tilde{\boldsymbol{u}}\|^2_{L^2(\Omega_0)}.
\end{aligned} \tag{4.31}$$

For the right-hand terms of (4.27), we use the smallness of ϵ and Hölder's inequality to get

$$\begin{aligned}
|\mathcal{I}_{2,1}| &\le \epsilon \|\sqrt{\tilde{\rho}} {\boldsymbol{w}^\dagger}_t\|_{L^2(\Omega_0)} \|\tilde{\rho}\|_{L^\infty(\Omega_0)} \|T_2 \boldsymbol{w}^\dagger\|_{L^2(\Omega_0)} \\
&\le \frac{1}{5}\|\sqrt{\tilde{\rho}} {\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)} + 5\epsilon^2 \|\tilde{\rho}\|^2_{H^2(\Omega_0)} \|T_2 \boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)} \\
&\le \frac{1}{5}\|\sqrt{\tilde{\rho}} {\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)} + \frac{1}{4}\epsilon \|T_2 \boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}.
\end{aligned} \tag{4.32}$$

Similarly, it holds that

$$
\begin{aligned}
|\mathcal{I}_{2,2}| &\le \big\|\sqrt{\tilde\rho}\boldsymbol{w}^\dagger{}_t\big\|_{L^2(\Omega_0)}\big\|\tilde\rho\big\|^{\frac12}_{L^\infty(\Omega_0)}\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\big\|_{L^2(\Omega_0)}\\
&\le \frac15\big\|\sqrt{\tilde\rho}\boldsymbol{w}^\dagger{}_t\big\|^2_{L^2(\Omega_0)}+C\big\|\tilde\rho\big\|_{H^2(\Omega_0)}\big\|\nabla_y\tilde{\boldsymbol{V}}_t\big\|^2_{L^2(\Omega_0)},
\end{aligned}
\tag{4.33}
$$

$$
|\mathcal{I}_{2,3}| \le \frac14\epsilon\big\|T_2\boldsymbol{w}^\dagger\big\|^2_{L^2(\Omega_0)}+C\big\|\nabla_y\tilde{\boldsymbol{u}}\big\|^2_{L^2(\Omega_0)}+C\big\|\boldsymbol{w}^\dagger\big\|^2_{L^2(\Omega_0)},
\tag{4.34}
$$

and

$$
\begin{aligned}
&|\mathcal{I}_{2,4}| \le \big\|\hat{\boldsymbol{F}}_2\big\|_{L^2(\Omega_0)}(\epsilon\big\|T_2\boldsymbol{w}^\dagger\big\|_{L^2(\Omega_0)}+\big\|\sqrt{\tilde\rho}\boldsymbol{w}^\dagger{}_t\big\|_{L^2\Omega_0}\big\|\tilde\rho^{-1}\big\|^{\frac12}_{L^\infty(\Omega_0)}+\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\big\|_{L^2(\Omega_0)})\\
&+\big\|\boldsymbol{w}^b_t\big\|_{L^2(\Omega_0)}(\big\|\tilde\rho\big\|^{\frac12}_{L^\infty(\Omega_0)}\big\|\sqrt{\tilde\rho}\boldsymbol{w}^\dagger{}_t\big\|_{L^2(\Omega_0)}+\big\|\tilde\rho\big\|_{L^\infty(\Omega_0)}(\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\big\|_{L^2(\Omega_0)}+\epsilon\big\|T_2\boldsymbol{w}^\dagger\big\|_{L^2(\Omega_0)}))\\
&\le \frac14\epsilon\big\|T_2\boldsymbol{w}^\dagger\big\|^2_{L^2(\Omega_0)}+\frac15\big\|\sqrt{\tilde\rho}\boldsymbol{w}^\dagger{}_t\big\|^2_{L^2(\Omega_0)}+C(\underline{\rho})\big\|\hat{\boldsymbol{F}}_2\big\|^2_{L^2(\Omega_0)}\\
&\quad+C\big\|\tilde\rho\big\|^2_{H^2(\Omega_0)}(\big\|\boldsymbol{w}^b_t\big\|^2_{L^2(\Omega_0)}+\big\|\nabla_y\boldsymbol{V}_t\big\|^2_{L^2(\Omega_0)}).
\end{aligned}
\tag{4.35}
$$

Combining the above estimates, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}+\|\nabla_y\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}\right)+\|\sqrt{\tilde\rho}\boldsymbol{w}^\dagger{}_t\|^2_{L^2(\Omega_0)}+\epsilon\|\nabla_y^2\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}\\
\le& C\left(\big\|\nabla_y\tilde{\boldsymbol{u}}\big\|^2_{L^2(\Omega_0)}+\|\tilde{\boldsymbol{V}}_t\|^2_{H^2(\Omega_0)}+\|\hat{\boldsymbol{F}}_2\|^2_{L^2(\Omega_0)}+\big\|\tilde\rho\big\|^2_{H^2(\Omega_0)}(\|\boldsymbol{w}^b_t\|^2_{L^2(\Omega_0)}+\big\|\nabla_y\tilde{\boldsymbol{V}}_t\big\|^2_{L^2(\Omega_0)})\right)\\
&+C\left(\|\nabla_y\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}+\|\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}\right),
\end{aligned}
\tag{4.36}
$$

then we apply Grönwall's inequality to derive

$$
\begin{aligned}
&\|\boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}+\|\nabla_y\boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}+\|\boldsymbol{w}^\dagger{}_t\|_{L^2(0,T;L^2(\Omega_0))}+\|\nabla_y^2\boldsymbol{w}^\dagger\|_{L^2(0,T;L^2(\Omega_0))}\\
\le&\phi(\underline{\rho},\|\rho\|_{L^\infty(0,T;H^2(\Omega_0))})\Big(\big\|\tilde{\boldsymbol{u}}\big\|_{L^2(0,T;H^1(\Omega_0))}+\|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^2(\Omega_0))}\\
&+\|\hat{\boldsymbol{F}}_2\|_{L^2(0,T;L^2(\Omega_0))}+\|\boldsymbol{w}^b_t\|_{L^2(0,T;L^2(\Omega_0))}\Big).
\end{aligned}
\tag{4.37}
$$

- The estimate of $\big\|\boldsymbol{w}^\dagger{}_t\big\|_{L^\infty(0,T;L^2(\Omega_0))}$.

Next we apply operator ∂_t to (4.14) and multiply by $(\boldsymbol{w}^\dagger-\frac12\mathrm{curl}_y\tilde{\boldsymbol{V}})_t$ to get

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\tilde\rho|\boldsymbol{w}^\dagger{}_t|^2\mathrm{d}y+\int_{\Omega_0}(T_2\boldsymbol{w}^\dagger{}_t-2\xi\mathrm{curl}_y\tilde{\boldsymbol{u}}_t+4\xi\boldsymbol{w}^\dagger{}_t)\cdot(\boldsymbol{w}^\dagger-\frac12\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\mathrm{d}y\\
=&-\frac12\int_{\Omega_0}\rho_t|\boldsymbol{w}^\dagger{}_t|^2\mathrm{d}y+\frac12\int_{\Omega_0}\tilde\rho\boldsymbol{w}^\dagger{}_{tt}\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\mathrm{d}y+\frac12\int_{\Omega_0}\tilde\rho_t\boldsymbol{w}^\dagger{}_t\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\mathrm{d}y\\
&+\int_{\Omega_0}(\hat{\boldsymbol{F}}_{2,t}-\tilde\rho_t\boldsymbol{w}^b_t)\cdot(\boldsymbol{w}^\dagger-\frac12\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\mathrm{d}y-\int_{\Omega_0}\tilde\rho{\boldsymbol{w}^\dagger}^b_{tt}\cdot(\boldsymbol{w}^\dagger-\frac12\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\mathrm{d}y\\
:=&\sum_{i=1}^5\mathcal{I}_{3,i}.
\end{aligned}
\tag{4.38}
$$

We use (4.14) and integration by parts to derive that

$$
\int_{\Omega_0}(\boldsymbol{w}^\dagger-\frac12\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\cdot T_2\boldsymbol{w}^\dagger{}_t\mathrm{d}y=\int_{\Omega_0}\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t):\nabla_y\boldsymbol{w}^\dagger{}_t\mathrm{d}y-\frac12\int_{\Omega_0}\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t):\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\mathrm{d}y,
\tag{4.39}
$$

and Hölder's inequality gives

$$
\begin{aligned}
&\left|-\frac12\int_{\Omega_0}\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t):\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\mathrm{d}y\right|\\
&\le\epsilon\big\|\nabla_y\boldsymbol{w}^\dagger{}_t\big\|^2_{L^2(\Omega_0)}+C\big\|\nabla_y^2\tilde{\boldsymbol{V}}_t\big\|^2_{L^2(\Omega_0)}.
\end{aligned}
\tag{4.40}
$$

Direct calculation shows that

$$\begin{aligned}&\int_{\Omega_0}(-2\xi\mathrm{curl}_y\tilde{\boldsymbol{u}}_t+4\xi{\boldsymbol{w}^\dagger}_t)\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\mathrm{d}y\\&=2\xi\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}^\dagger\|^2_{L^2(\Omega_0)}-2\xi\int_{\Omega_0}\mathrm{curl}_y\tilde{\boldsymbol{u}}_t\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\mathrm{d}y-2\xi\int_{\Omega_0}{\boldsymbol{w}^\dagger}_t\cdot\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\mathrm{d}y,\end{aligned}\tag{4.41}$$

and

$$\begin{aligned}&\left|-2\xi\int_{\Omega_0}\mathrm{curl}_y\tilde{\boldsymbol{u}}_t\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\mathrm{d}y-2\xi\int_{\Omega_0}{\boldsymbol{w}^\dagger}_t\cdot\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\mathrm{d}y\right|\\&\leq\|\nabla_y\tilde{\boldsymbol{u}}_t\|_{L^2(\Omega_0)}(\|\sqrt{\tilde{\rho}}{\boldsymbol{w}^\dagger}_t\|_{L^2(\Omega_0)}\|\rho^{-1}\|^{\frac{1}{2}}_{L^\infty(\Omega_0)}+\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\|_{L^2(\Omega_0)})\\&\quad+\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\|_{L^2(\Omega_0)}\|\sqrt{\tilde{\rho}}{\boldsymbol{w}^\dagger}_t\|_{L^2(\Omega_0)}\|\rho^{-1}\|^{\frac{1}{2}}_{L^2(\Omega_0)}\\&\leq C(\underline{\rho})(\|\nabla_y\tilde{\boldsymbol{u}}_t\|^2_{L^2(\Omega_0)}+\|\sqrt{\tilde{\rho}}{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)}+\|\nabla_y\tilde{\boldsymbol{V}}_t\|^2_{L^2(\Omega_0)}).\end{aligned}\tag{4.42}$$

Similarly, we have

$$\begin{aligned}|\mathcal{I}_{3,1}|\leq&C\|\tilde{\rho}_t\|_{L^3(\Omega_0)}\|\sqrt{\tilde{\rho}}{\boldsymbol{w}^\dagger}_t\|_{L^2(\Omega_0)}\|\tilde{\rho}^{-1}\|^{\frac{1}{2}}_{L^\infty(\Omega_0)}\|{\boldsymbol{w}^\dagger}_t\|_{L^6(\Omega_0)}\\\leq&\epsilon\|\nabla_y{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)}+C(\underline{\rho})\|\tilde{\rho}_t\|^2_{H^1(\Omega_0)}\|\sqrt{\tilde{\rho}}{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)},\end{aligned}\tag{4.43}$$

$$|\mathcal{I}_{3,2}|\leq\delta\|{\boldsymbol{w}^\dagger}_{tt}\|^2_{L^2(\Omega_0)}+C\|\tilde{\rho}\|^2_{H^2(\Omega_0)}\|\nabla_y\tilde{\boldsymbol{V}}_t\|^2_{L^2(\Omega_0)},\tag{4.44}$$

$$|\mathcal{I}_{3,3}|\leq\epsilon\|\nabla_y{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)}+C\|\tilde{\rho}_t\|^2_{L^3(\Omega_0)}\|\nabla_y\tilde{\boldsymbol{V}}_t\|^2_{L^2(\Omega_0)},\tag{4.45}$$

$$\begin{aligned}|\mathcal{I}_{3,4}|\leq&\|\hat{\boldsymbol{F}}_{2,t}\|_{L^2(\Omega_0)}(\|{\boldsymbol{w}^\dagger}_t\|_{L^6(\Omega_0)}+\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\|_{L^2(\Omega_0)})\\&+\|\tilde{\rho}_t\|_{L^3(\Omega_0)}\|\boldsymbol{w}^b_t\|_{L^2(\Omega_0)}(\|{\boldsymbol{w}^\dagger}_t\|_{L^6(\Omega_0)}+\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\|_{L^6(\Omega_0)})\\&\leq\epsilon\|\nabla_y{\boldsymbol{w}^\dagger}_t\|^2_{L^2}+C(\|\hat{\boldsymbol{F}}_{2,t}\|^2_{L^2(\Omega_0)}+\|\tilde{\rho}_t\|^2_{L^3(\Omega_0)}\|\boldsymbol{w}^b_t\|^2_{L^2(\Omega_0)}+\|\tilde{\boldsymbol{V}}_t\|_{H^2(\Omega_0)}),\end{aligned}\tag{4.46}$$

and

$$\begin{aligned}|\mathcal{I}_{3,5}|&\leq\|\boldsymbol{w}^b_{tt}\|_{L^2(\Omega_0)}(\|\sqrt{\tilde{\rho}}\boldsymbol{w}_t\|_{L^2(\Omega_0)}\|\tilde{\rho}\|^{\frac{1}{2}}_{L^\infty(\Omega_0)}+\|\tilde{\rho}\|_{L^\infty(\Omega_0)}\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\|_{L^2(\Omega_0)})\\&\leq\|\sqrt{\tilde{\rho}}\boldsymbol{w}_t\|^2_{L^2(\Omega_0)}+C(\|\boldsymbol{w}^b_{tt}\|^2_{L^2(\Omega_0)}\|\tilde{\rho}\|_{L^\infty(\Omega_0)}+\|\tilde{\rho}\|^2_{L^\infty(\Omega_0)}\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_t\|^2_{L^2(\Omega_0)}).\end{aligned}\tag{4.47}$$

Putting above estimates together, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\tilde{\rho}}\boldsymbol{w}_t\|^2_{L^2(\Omega_0)}+\|{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\omega_0)}+\|\nabla_y{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)}\\&\leq C(\underline{\rho})(1+\|\tilde{\rho}_t\|^2_{H^1(\Omega_0)})\|\sqrt{\tilde{\rho}}{\boldsymbol{w}^\dagger}_t\|^2_{L^2(\Omega_0)}+\delta\|{\boldsymbol{w}^\dagger}_{tt}\|^2_{L^2(\Omega_0)}\\&\quad+C(\|\nabla_y\tilde{\boldsymbol{u}}_t\|^2_{L^2(\Omega_0)}+\|\hat{\boldsymbol{F}}_{2,t}\|^2_{L^2(\Omega_0)}+\|\tilde{\rho}_t\|^2_{H^1(\Omega_0)}\|\boldsymbol{w}^b_t\|^2_{L^2(\Omega_0)}+\|\tilde{\rho}\|_{H^2(\Omega_0)}\|\boldsymbol{w}^b_{tt}\|^2_{L^2(\Omega_0)})\\&\quad+C(1+\|\tilde{\rho}_t\|^2_{H^1(\Omega_0)}+\|\tilde{\rho}\|^2_{H^2(\Omega_0)})\|\tilde{\boldsymbol{V}}_t\|^2_{H^2(\Omega_0)}).\end{aligned}\tag{4.48}$$

By Grönwall's inequality, we obtain

$$\begin{aligned}&\|{\boldsymbol{w}^\dagger}_t\|_{L^\infty(0,T;L^2(\Omega_0))}+\|{\boldsymbol{w}^\dagger}_t\|_{L^2(0,T;L^2(\Omega_0))}+\|\nabla_y{\boldsymbol{w}^\dagger}_t\|_{L^2(0,T;L^2(\Omega_0))}\\&\leq\phi(\|\rho\|_{L^\infty(0,T;H^2(\Omega_0))},\|\rho_t\|_{L^2(0,T;H^2(\Omega_0))})(\|\tilde{\boldsymbol{u}}_t\|^2_{L^2(0,T;H^1(\Omega_0))}+\|\hat{\boldsymbol{F}}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))}\\&\quad+\|\boldsymbol{w}^b_t\|_{L^2(0,T;L^2(\Omega_0))}+\|\boldsymbol{w}^b_{tt}\|_{L^2(0,T;L^2(\Omega_0))}+\|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^2(\Omega_0))})+\delta\|{\boldsymbol{w}^\dagger}_{tt}\|_{L^2(0,T;L^2(\Omega_0))}\triangleq\Phi_1.\end{aligned}\tag{4.49}$$

According to the classical elliptic theory, we have

$$\begin{aligned}\|\nabla^2_y\boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}\leq&\|\boldsymbol{w}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}+\|\nabla_y\tilde{\boldsymbol{u}}\|_{L^\infty(0,T;L^2(\Omega_0))}\\&+\|\hat{\boldsymbol{F}}_2\|_{L^\infty(0,T;L^2(\Omega_0))}+\|\tilde{\rho}\boldsymbol{w}^b_t\|_{L^\infty(0,T;L^2(\Omega_0))}+\|\tilde{\rho}\|_{L^\infty(0,T;L^\infty(\Omega_0))}\Phi_1.\end{aligned}\tag{4.50}$$

Next we show the bound of $\|\boldsymbol{w}^\dagger\|_{L^2(0,T;H^3(\Omega_0))}$.

By applying gradient operator ∇_y on (4.14), it holds that

$$\begin{aligned}\|\nabla_y^3\boldsymbol{w}^\dagger\|_{L^2(0,T;L^2(\Omega_0))} \leq&\|\nabla_y\hat{\boldsymbol{F}}_2\|_{L^2(0,T;L^2(\Omega_0))}+\phi(\|\tilde{\rho}\|_{L^\infty(0,T;H^2(\Omega_0))})\|\boldsymbol{w}^\dagger{}_t+\boldsymbol{w}^b_t\|_{L^2(0,T;H^1(\Omega_0))}\\ &+\|\nabla_y^2\tilde{\boldsymbol{u}}\|_{L^2(0,T;L^2(\Omega_0))}+\|\nabla_y\boldsymbol{w}^\dagger\|_{L^2(0,T;L^2(\Omega_0))}.\end{aligned} \tag{4.51}$$

• The estimate of $\|\boldsymbol{w}^\dagger{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))}$.

We apply ∂_t to (4.14), multiply the resulting equations by $(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}$, and integrate over Ω_0 to get

$$\begin{aligned}&\int_{\Omega_0}\tilde{\rho}|\boldsymbol{w}^\dagger{}_{tt}|^2\mathrm{d}y+\int_{\Omega_0}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\cdot T_2\boldsymbol{w}^\dagger{}_t\mathrm{d}y\\ &+2\xi\int_{\Omega_0}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\cdot(2\boldsymbol{w}^\dagger{}_t-\mathrm{curl}_y\tilde{\boldsymbol{u}}_t)\mathrm{d}y\\ =&\int_{\Omega_0}(\hat{\boldsymbol{F}}_{2,t}-\tilde{\rho}\boldsymbol{w}^b_{tt})\cdot(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\mathrm{d}y+\frac{1}{2}\int_{\Omega_0}\tilde{\rho}\boldsymbol{w}^\dagger{}_{tt}\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\mathrm{d}y\\ &-\int_{\Omega_0}\tilde{\rho}_t(\boldsymbol{w}^\dagger{}_t+\boldsymbol{w}^b_t)(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\mathrm{d}y:=\sum_{i=1}^3\mathcal{I}_{4,i}.\end{aligned} \tag{4.52}$$

Applying integration by parts and the boundary condition $(4.14)_2$, we have

$$\begin{aligned}\int_{\Omega_0}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\cdot T_2\boldsymbol{w}^\dagger{}_t\mathrm{d}y=&\int_{\Omega_0}\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t):\nabla_y\boldsymbol{w}^\dagger{}_{tt}\mathrm{d}y-\frac{1}{2}\int_{\Omega}\mathbb{M}(\nabla_y\boldsymbol{w}_t):\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\mathrm{d}y\\ =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}((c_0+c_d-c_a)|\mathrm{div}_y\boldsymbol{w}^\dagger{}_t|^2+(c_a+c_d)|\nabla_y\boldsymbol{w}^\dagger{}_t|^2)\mathrm{d}y\\ &-\frac{1}{2}\int_{\Omega}\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t):\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\mathrm{d}y,\end{aligned} \tag{4.53}$$

then Hölder's inequality leads to

$$\left|-\frac{1}{2}\int_{\Omega}\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t):\nabla_y\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\mathrm{d}y\right|\leq C\big\|\nabla_y\boldsymbol{w}^\dagger{}_t\big\|^2_{L^2(\Omega_0)}+C\big\|\nabla_y^2\tilde{\boldsymbol{V}}_{tt}\big\|^2_{L^2(\Omega_0)}. \tag{4.54}$$

Direct calculation gives that

$$\begin{aligned}&2\xi\int_{\Omega_0}(\boldsymbol{w}^\dagger-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\cdot(2\boldsymbol{w}^\dagger{}_t-\mathrm{curl}_y\tilde{\boldsymbol{u}}_t)\mathrm{d}y\\ &=2\xi\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}|\boldsymbol{w}^\dagger{}_t|^2\mathrm{d}y-\xi\int_{\Omega_0}\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\cdot(2\boldsymbol{w}^\dagger{}_t-\mathrm{curl}_y\tilde{\boldsymbol{u}}_t)\mathrm{d}y,\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}&\left|-\xi\int_{\Omega_0}\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\cdot(2\boldsymbol{w}^\dagger{}_t-\mathrm{curl}_y\tilde{\boldsymbol{u}}_t)\mathrm{d}y\right|\\ &\leq C\big\|\boldsymbol{w}^\dagger{}_t\big\|^2_{L^2(\Omega_0)}+C\big\|\nabla_y\tilde{\boldsymbol{u}}_t\big\|^2_{L^2(\Omega_0)}+C\big\|\nabla_y\tilde{\boldsymbol{V}}_{tt}\big\|^2_{L^2(\Omega_0)}.\end{aligned} \tag{4.56}$$

Now we estimate $\mathcal{I}_{4,i}$, $i=1,2,3$ term by term.

$$\begin{aligned}|\mathcal{I}_{4,1}|\leq&\big\|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\big\|_{L^2(\Omega_0)}(\big\|\tilde{\rho}\big\|^{\frac{1}{2}}_{L^\infty(\Omega_0)}\big\|\boldsymbol{w}^b_{tt}\big\|_{L^2(\Omega_0)}+\big\|\tilde{\rho}^{-1}\big\|^{\frac{1}{2}}_{L^\infty(\Omega_0)}\big\|\hat{\boldsymbol{F}}_{2,t}\big\|_{L^2(\Omega_0)})\\ &+\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\big\|_{L^2(\Omega_0)}(\big\|\tilde{\rho}\big\|_{L^\infty(\Omega_0)}\big\|\boldsymbol{w}^b_{tt}\big\|_{L^2(\Omega_0)}+\big\|\hat{\boldsymbol{F}}_{2,t}\big\|_{L^2(\Omega_0)})\\ \leq&\epsilon\big\|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\big\|^2_{L^2(\Omega_0)}+C(1+\big\|\tilde{\rho}\big\|^2_{L^\infty(\Omega_0)})\big\|\boldsymbol{w}^b_{tt}\big\|^2_{L^2(\Omega_0)}+C(\underline{\rho})\big\|\hat{\boldsymbol{F}}_{2,t}\big\|^2_{L^2(\Omega_0)}\\ &+C\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\big\|^2_{L^2(\Omega_0)},\end{aligned} \tag{4.57}$$

$$
\begin{aligned}
|\mathcal{I}_{4,2}| \leq & \big\|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\big\|_{L^2(\Omega_0)}\big\|\tilde{\rho}\big\|^{\frac{1}{2}}_{L^\infty(\Omega_0)}\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\big\|_{L^2(\Omega_0)} \\
\leq & \epsilon\big\|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\big\|^2_{L^2(\Omega_0)} + C\big\|\tilde{\rho}\big\|_{L^\infty(\Omega_0)}\big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\big\|^2_{L^2(\Omega_0)},
\end{aligned} \tag{4.58}
$$

and

$$
\begin{aligned}
|\mathcal{I}_{4,3}| \leq & \big\|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\big\|_{L^2(\Omega_0)}\big\|\tilde{\rho}^{-1}\big\|^{\frac{1}{2}}\big\|\tilde{\rho}_t\big\|_{L^3(\Omega_0)}(\big\|\boldsymbol{w}^\dagger{}_t\big\|_{L^6(\Omega_0)} + \big\|\boldsymbol{w}_t\big\|_{L^6(\Omega_0)}) \\
& + \big\|\mathrm{curl}_y\tilde{\boldsymbol{V}}_{tt}\big\|_{L^2(\Omega_0)}\big\|\tilde{\rho}_t\big\|_{L^3(\Omega_0)}(\big\|\boldsymbol{w}^\dagger{}_t\big\|_{L^6(\Omega_0)} + \big\|\boldsymbol{w}^b_t\big\|_{L^6(\Omega_0)}) \\
\leq & \epsilon\big\|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\big\|^2_{L^2(\Omega_0)} + C(\underline{\rho})\big\|\tilde{\rho}_t\big\|^2_{H^1(\Omega_0)}(\big\|\nabla_y\boldsymbol{w}^\dagger{}_t\big\|^2_{L^2(\Omega_0)} + \big\|\nabla_y\boldsymbol{w}^b_t\big\|^2_{L^2(\Omega_0)}) + C\big\|\nabla_y\tilde{\boldsymbol{V}}_{tt}\big\|^2_{L^2(\Omega_0)}.
\end{aligned} \tag{4.59}
$$

Plugging above estimates into (4.52), it holds that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{w}^\dagger{}_t\|^2_{L^2(\Omega_0)} + \|\nabla_y\boldsymbol{w}^\dagger{}_t\|^2_{L^2(\Omega_0)}) + \|\sqrt{\tilde{\rho}}\boldsymbol{w}^\dagger{}_{tt}\|^2_{L^2(\Omega_0)} \\
\leq & C\left(\|\boldsymbol{V}_{tt}\|^2_{H^2(\Omega_0)} + \|\hat{\boldsymbol{F}}_{2,t}\|^2_{L^2(\Omega_0)} + (1 + \big\|\tilde{\rho}\big\|^2_{H^2(\Omega_0)})\|\boldsymbol{w}^b_{tt}\|^2_{L^2(\Omega_0)}\right) + C\big\|\tilde{\rho}_t\big\|^2_{H^1(\Omega_0)}\|\nabla_y\boldsymbol{w}^\dagger{}_t\|^2_{L^2(\Omega_0)} \\
& + C(\|\boldsymbol{w}^\dagger{}_t\|^2_{L^2(\Omega_0)} + \|\nabla_y\boldsymbol{w}^\dagger{}_t\|^2_{L^2(\Omega_0)}) + C\|\nabla_y\tilde{\boldsymbol{u}}_t\|^2_{L^2(\Omega_0)} + C(\underline{\rho})\big\|\tilde{\rho}_t\big\|^2_{H^1(\Omega_0)}\|\nabla_y\boldsymbol{w}^b_t\|^2_{L^2(\Omega_0)}.
\end{aligned} \tag{4.60}
$$

Applying Grönwall's inequality, we obtain

$$
\begin{aligned}
& \|\boldsymbol{w}^\dagger{}_t\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\nabla_y\boldsymbol{w}^\dagger{}_t\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\boldsymbol{w}^\dagger{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))} \\
\leq & \phi(\underline{\rho}, \|\tilde{\rho}\|_{L^\infty(0,T;H^2(\Omega_0))}, \|\tilde{\rho}_t\|_{L^2(0,T;H^1(\Omega_0))})\left(\big\|\nabla_y\tilde{\boldsymbol{u}}_t\big\|_{L^2(0,T;L^2(\Omega_0))} + \|\tilde{\boldsymbol{V}}_{tt}\|_{L^2(0,T;H^2(\Omega_0))}\right. \\
& \left. + \|\hat{\boldsymbol{F}}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{w}^b_{tt}\|_{L^2(0,T;L^2(\Omega_0))} + \|\nabla_y\boldsymbol{w}^b_t\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{w}^\dagger{}_t(0)\|_{H^1(\Omega_0)}\right).
\end{aligned} \tag{4.61}
$$

For the term $\boldsymbol{w}^\dagger{}_t(0)$, we apply ∇_y to (4.14) and take the results in $t = 0$ to derive

$$
\|\boldsymbol{w}^\dagger{}_t(0)\|_{H^1(\Omega_0)} \leq C(\|\hat{\boldsymbol{F}}_2(0)\|_{H^1(\Omega_0)} + \big\|\tilde{\boldsymbol{u}}(0)\big\|_{H^2(\Omega_0)} + \|\boldsymbol{w}^\dagger(0)\|_{H^3(\Omega_0)}). \tag{4.62}
$$

• The estimate of $\|\boldsymbol{w}^\dagger\|_{L^2(0,T;H^2(\Omega_0))}$.

Finally, we show a bound for $\|\boldsymbol{w}^\dagger\|_{L^2(0,T;H^2(\Omega_0))}$.

Taking the derivative with respect to time of (4.14), we have

$$
\begin{cases}
-\mathrm{div}_y\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t) = 2\xi(\mathrm{curl}_y\tilde{\boldsymbol{u}}_t - 2\boldsymbol{w}^\dagger{}_t) + \hat{\boldsymbol{F}}_{2,t} - \tilde{\rho}_t\boldsymbol{w}^\dagger{}_t - \tilde{\rho}\boldsymbol{w}^\dagger{}_{tt} - \tilde{\rho}_t\boldsymbol{w}^b_t - \tilde{\rho}\boldsymbol{w}^b_{tt}, \\
\boldsymbol{w}^\dagger\cdot\boldsymbol{n}|_{\Gamma_0} = 0, \\
[\mathbb{M}(\nabla_y\boldsymbol{w}^\dagger{}_t)]_{\tan}|_{\Gamma_0} = 0.
\end{cases} \tag{4.63}
$$

The boundary condition of (4.63) is homogeneous, the classical elliptic theory yields

$$
\|\boldsymbol{w}^\dagger{}_t\|_{H^2(\Omega_0)} \leq C(\|\hat{\boldsymbol{F}}_{2,t}\|_{L^2(\Omega_0)} + \|(\nabla_y\tilde{\boldsymbol{u}}_t, \boldsymbol{w}^\dagger{}_t)\|_{L^2(\Omega_0)} + \|\tilde{\rho}_t(\boldsymbol{w}^\dagger{}_t + \boldsymbol{w}^b_t)\|_{L^2(\Omega_0)} + \|\tilde{\rho}(\boldsymbol{w}^\dagger{}_{tt} + \boldsymbol{w}^b_{tt})\|_{L^2(\Omega_0)}). \tag{4.64}
$$

Integrating (4.64) with respect to time, we have

$$
\begin{aligned}
\|\boldsymbol{w}^\dagger{}_t\|_{L^2(0,T;H^2(\Omega_0))} \leq & C\left(\|\hat{\boldsymbol{F}}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))} + \|(\tilde{\boldsymbol{u}}_t, \boldsymbol{w}^\dagger{}_t)\|_{L^2(0,T;H^1(\Omega_0))}\right. \\
& + \big\|\tilde{\rho}\big\|_{L^\infty(0,T;H^2(\Omega_0))}(\|\boldsymbol{w}^b_{tt}\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{w}^\dagger{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))}) \\
& \left. + \|\rho_t\|_{L^2(0,T;H^1(\Omega_0))}(\|\boldsymbol{w}^\dagger{}_t\|_{L^\infty(0,T;H^1(\Omega_0))} + \|\boldsymbol{w}^b_t\|_{L^\infty(0,T;H^1(\Omega_0))})\right),
\end{aligned} \tag{4.65}
$$

which together with (4.61) and (4.62) lead to

$$
\begin{aligned}
& \big\|\boldsymbol{w}^\dagger{}_{tt}\big\|_{L^2(0,T;L^2(\Omega_0))} + \big\|\boldsymbol{w}^\dagger{}_t\big\|_{L^\infty(0,T;H^1(\Omega_0))} + \|\boldsymbol{w}^\dagger{}_t\|_{L^2(0,T;H^2(\Omega_0))} \\
\leq & \phi(\underline{\rho}, \big\|\tilde{\rho}\big\|_{L^\infty(0,T;H^2(\Omega_0))}, \big\|\tilde{\rho}_t\big\|_{L^2(0,T;H^1(\Omega_0))})\left(\big\|\tilde{\boldsymbol{u}}_t\big\|_{L^2(0,T;H^1(\Omega_0))} + \|\hat{\boldsymbol{F}}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))}\right. \\
& \left. + \big\|\boldsymbol{w}^b_t\big\|_{L^2(0,T;H^1(\Omega_0))} + \big\|\tilde{\boldsymbol{V}}_{tt}\big\|_{L^2(0,T;H^2(\Omega_0))} + \big\|\boldsymbol{w}^b_{tt}\big\|_{L^2(0,T;L^2(\Omega_0))} + \big\|\boldsymbol{w}^\dagger{}_t(0)\big\|_{H^1(\Omega_0)}\right).
\end{aligned} \tag{4.66}
$$

Then combining (4.62), (4.15) and previous estimates, we complete the proof of (4.13). □

4.3. **Proof of Proposition** 4.1**.** We present the estimates on $\boldsymbol{w}^b(\tilde{\boldsymbol{w}}, \boldsymbol{V})$, this proof is quite technical. We shall construct explicit formula for $\boldsymbol{w}^b$ and leave the proof in the Appendix.

Lemma 4.2. *Let $\boldsymbol{V}$ satisfy the assumptions in Theorem 1.1. Then there exists an extension $\boldsymbol{w}^b(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{V}})$ defined by* (4.11) *satisfying the estimate*

$$\|\boldsymbol{w}^b\|_{\mathcal{Y}(T)} \le E(T)[1 + \|(\tilde{\boldsymbol{w}} - \frac{1}{2}\mathrm{curl}_y \tilde{\boldsymbol{V}})\|_{\mathcal{Y}(T)}], \tag{4.67}$$

where $E(t)$ is continuous and $E(0) = 0$.

Moreover, we list some regularity results without detailed proof which borrow from Lemma 4.5 in [23]. One can use the same method to derive it.

Lemma 4.3. *For $\boldsymbol{R}_2$ defined by* (4.7) *and denote $F(t) = \|\tilde{\boldsymbol{w}}\|_{\mathcal{Y}(t)}$, then we have for any $\eta > 0$*

$$\begin{aligned}
&\|\tilde{\rho}\boldsymbol{V} \cdot \nabla_y \tilde{\boldsymbol{w}} + \boldsymbol{R}_2\|_{L^2(0,T;H^1(\Omega_0))} + \|\partial_t[\tilde{\rho}\boldsymbol{V} \cdot \nabla_y \tilde{\boldsymbol{w}} + \boldsymbol{R}_2]\|_{L^2(0,T;L^2(\Omega_0))} \\
&\le C[(\eta + \sqrt{T}C(\eta) + E(T))F(T)] + C\sqrt{T}(\|\tilde{\boldsymbol{u}}\|_{H^2(\Omega_0)} + \|\tilde{\boldsymbol{u}}_t\|_{H^1(\Omega_0)}).
\end{aligned} \tag{4.68}$$

Combining (4.67) with (4.68), and we use the expression of the right-hand terms in (4.6) to derive that

$$\begin{aligned}
&\|\tilde{\boldsymbol{F}}_2\|_{L^2(0,T;H^1(\Omega_0))} + \|\tilde{\boldsymbol{F}}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{w}^b\|_{\mathcal{Y}(T)} \\
&\le \|\boldsymbol{F}_2\|_{L^2(0,T;H^1(\Omega_0))} + \|\boldsymbol{F}_{2,t}\|_{L^2(0,T;L^2(\Omega_0))} + CE(T)(F(T) + \|\tilde{\boldsymbol{u}}\|_{H^2(\Omega_0)} + \|\tilde{\boldsymbol{u}}_t\|_{H^1(\Omega_0)} \\
&+ \|\mathrm{curl}_y \tilde{\boldsymbol{V}}\|_{\mathcal{Y}(T)}).
\end{aligned} \tag{4.69}$$

It is known that the transformation of Lagrangian coordinates is a diffeomorphism for small time, therefore, (4.3) comes from (4.13) and (4.69).

5. Linearized momentum equations

We perform similar procedures as in the previous section to derive the local existence of strong solution to the lineariezd momentum equations (2.4)-(2.5). We shall treat the following linearized momentum equations with Navier's slip boundary conditions

$$\begin{cases}
\rho \boldsymbol{u}_t - (\mu + \lambda - \xi)\nabla_x \mathrm{div}_x \boldsymbol{u} - (\mu + \xi)\Delta_x \boldsymbol{u} - 2\xi \mathrm{curl}_y \boldsymbol{w} = \boldsymbol{F}_1, \\
(\boldsymbol{u} - \boldsymbol{V}) \cdot \boldsymbol{n}|_{\Gamma_t} = 0, \\
\boldsymbol{n} \cdot \mathbb{S} \cdot \tau_k + \kappa_1(\boldsymbol{u} - \boldsymbol{V}) \cdot \tau_k|_{\Gamma_t} = 0, \quad k = 1, 2.
\end{cases} \tag{5.1}$$

Proposition 5.1. *Let $T > 0$ be sufficiently small. Let $\boldsymbol{V}$ and $p(\rho)$ satisfy the assumptions in the Theorem 1.1. Assume that*

$$\begin{cases}
\rho \in L^\infty(0, T; H^2(\Omega_t)), \rho_t \in L^2(0, T; H^1(\Omega_t)), \\
\boldsymbol{F}_1 \in L^2(0, T; H^1(\Omega_t)), \boldsymbol{F}_{1,t} \in L^2(0, T; L^2(\Omega_t)), \\
\boldsymbol{u}_0 \in H^3(\Omega_0), \boldsymbol{w}_0 \in H^3(\Omega_0).
\end{cases} \tag{5.2}$$

Then there exists a unique solution $\boldsymbol{u}$ to the system (5.1) *such that $\boldsymbol{u} \in \mathcal{X}(T)$ and the following estimate holds*

$$\begin{aligned}
\|\boldsymbol{u}\|_{\mathcal{X}(T)} \le &\phi(\|\rho\|_{L^\infty(0,T;H^2(\Omega_t))}, \|\rho_t\|_{L^2(0,T;H^1(\Omega_t))}) \times \Big(\|\boldsymbol{w}\|_{L^2(0,T;H^1(\Omega_t))} + \|\boldsymbol{w}_t\|_{L^2(0,T;H^1(\Omega_t))} \\
&+ \|\boldsymbol{F}_1\|_{L^2(0,T;H^1(\Omega_t))} + \|\boldsymbol{F}_{1,t}\|_{L^2(0,T;L^2(\Omega_t))} + \|\boldsymbol{u}_0\|_{H^3(\Omega_0)} + \|\boldsymbol{V}_{tt}\|_{L^2(0,T;H^2(\Omega_t))} \\
&+ \|\boldsymbol{V}\|_{L^\infty(0,T;H^3(\Omega_t)) \cap L^2(0,T;H^4(\Omega_t))} + \|\boldsymbol{V}_t\|_{L^\infty(0,T;H^2(\Omega_t)) \cap L^2(0,T;H^3(\Omega_t))}\Big),
\end{aligned} \tag{5.3}$$

where ϕ is a positive increasing function of its arguments.

5.1. **Lagrangian coordinates.** We also transform (5.1) on the moving domain to a problem defined on a fixed spatial domain $(0,T)\times\Omega_0$ by using the Lagrangian coordinates determined by $\boldsymbol{V}$. We follow (4.4) and (4.5) to get

$$\frac{\partial u_i}{\partial t}=\frac{\partial \tilde{u}_i}{\partial t}+\nabla_y\tilde{u}_i\cdot\frac{\partial \boldsymbol{Y}}{\partial t}=\frac{\partial \tilde{u}_i}{\partial t}-\nabla_y\tilde{u}_i\cdot(\boldsymbol{V}\cdot\nabla_x\boldsymbol{Y}).$$

Then we can rewrite the i-th component of linearized momentum equations (5.1) as

$$\begin{aligned}&\tilde{\rho}\left(\frac{\partial \tilde{u}_i}{\partial t}-\frac{\partial \tilde{u}_i}{\partial y_j}V_k\frac{\partial Y_j}{\partial x_k}\right)-(\mu+\lambda-\xi)\frac{\partial^2\tilde{u}_p}{\partial y_k\partial y_l}\frac{\partial Y_l}{\partial x_p}\frac{\partial Y_k}{\partial x_i}-(\mu+\lambda-\xi)\frac{\partial \tilde{u}_p}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_i\partial x_p}\\&-(\mu+\xi)\frac{\partial \tilde{u}_i}{\partial y_k}\Delta_x Y_k-(\mu+\xi)\frac{\partial^2\tilde{u}_i}{\partial y_k\partial y_l}\frac{\partial Y_k}{\partial x_p}\frac{\partial Y_l}{\partial x_p}-2\xi\varepsilon_{ijk}\frac{\partial \tilde{w}_k}{\partial y_l}\frac{\partial Y_l}{\partial x_j}=F_{1,i},\end{aligned}$$

where we used the Einstein summation convention and the Levi-Civita symbol.

Finally, we obtain

$$\begin{aligned}&\tilde{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}-(\mu+\lambda-\xi)\nabla_y\mathrm{div}_y\tilde{\boldsymbol{u}}-(\mu+\xi)\Delta_y\tilde{\boldsymbol{u}}-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}}\\=&\boldsymbol{F}_1+\tilde{\rho}\boldsymbol{V}\cdot\nabla_y\tilde{u}+\boldsymbol{R}_1(\tilde{\rho},\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}}):=\tilde{\boldsymbol{F}}_1(\tilde{\rho},\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}}),\end{aligned}\tag{5.4}$$

with

$$\begin{aligned}\boldsymbol{R}_{1,i}(\tilde{\rho},\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}})=&\tilde{\rho}\frac{\partial \tilde{u}_i}{\partial y_j}V_k\left(\frac{\partial Y_j}{\partial x_k}-\delta_{jk}\right)+(\mu+\lambda-\xi)\frac{\partial^2\tilde{u}_p}{\partial y_k\partial y_l}\left(\frac{\partial Y_l}{\partial x_p}\frac{\partial Y_k}{\partial x_i}-\delta_{lp}\boldsymbol{e}_k\right)\\&+(\mu+\lambda-\xi)\frac{\partial \tilde{u}_p}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_i\partial x_p}+(\mu+\xi)\frac{\partial^2\tilde{u}_i}{\partial y_k\partial y_l}\left(\frac{\partial Y_k}{\partial x_p}\frac{\partial Y_l}{\partial x_p}-\delta_{lp}\delta_{kp}\right)\\&+(\mu+\xi)\frac{\partial \tilde{u}_i}{\partial y_k}\Delta_x Y_k+2\xi\varepsilon_{ijk}\frac{\partial \tilde{w}_k}{\partial y_l}\left(\frac{\partial Y_l}{\partial x_j}-\delta_{lj}\right),\end{aligned}\tag{5.5}$$

where $\boldsymbol{e}_j$ is the j-th unit vector.

Through these new variables (4.4), we derive from $(5.1)_2$ and $(5.1)_3$ that

$$\begin{aligned}(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})(t,y)\cdot\boldsymbol{n}(y)=&(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})(t,y)\cdot(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\\&+(\boldsymbol{V}(t,(\boldsymbol{X}(t,y)))-\tilde{\boldsymbol{V}}(t,y))\cdot\boldsymbol{n}(\boldsymbol{X}(t,y))=:\boldsymbol{d}_1(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{V}})(t,y),\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}&\left[(\mu+\xi)\nabla_y\tilde{\boldsymbol{u}}(t,y)+(\mu-\xi)\nabla_y^{\mathsf{T}}\tilde{\boldsymbol{u}}-2\xi A(\tilde{\boldsymbol{w}})\right]\boldsymbol{n}(y)\cdot\tau^k(y)+\kappa_1(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})(t,y)\cdot\tau^k(y)\\=&\left[(\mu+\xi)\nabla_y\tilde{\boldsymbol{u}}(t,y)(\mathbb{I}-\nabla_x\boldsymbol{Y})+(\mu-\xi)(\mathbb{I}-\nabla_x^{\mathsf{T}}\boldsymbol{Y})\nabla_y^{\mathsf{T}}\tilde{\boldsymbol{u}}\right]\boldsymbol{n}(\boldsymbol{X}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y))\\&+\left[(\mu+\xi)\nabla_y\tilde{\boldsymbol{u}}(t,y)+(\mu-\xi)\nabla_y^{\mathsf{T}}\tilde{\boldsymbol{u}}-2\xi A(\tilde{\boldsymbol{w}})\right](t,y)\left[(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\cdot\tau^k(\boldsymbol{X}(t,y))\right.\\&\left.+\boldsymbol{n}(y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\right]+\kappa_1(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})(t,y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\\&+\kappa_1(\boldsymbol{V}(t,\boldsymbol{X}(t,y))-\tilde{\boldsymbol{V}}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y)):=\boldsymbol{B}_1(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}},\tilde{\boldsymbol{V}})(t,y),\end{aligned}\tag{5.7}$$

where $A_{ij}(\tilde{\boldsymbol{w}})=\varepsilon_{mij}w_m$.

Finally, we formulate the linearized momentum equations in a fixed domain $(0,T)\times\Omega_0$.

$$\begin{cases}\tilde{\rho}\tilde{\boldsymbol{u}}_t-(\mu+\lambda-\xi)\nabla_y\mathrm{div}_y\tilde{\boldsymbol{u}}-(\mu+\xi)\Delta_y\tilde{\boldsymbol{u}}-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}}=\tilde{\boldsymbol{F}}_1,\\(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}|_{\Gamma_0}=\boldsymbol{d}_1,\\\boldsymbol{n}\cdot\mathbb{S}\cdot\tau_k+\kappa_1(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})\cdot\tau_k|_{\Gamma_0}=\boldsymbol{B}_1,\quad k=1,2.\end{cases}\tag{5.8}$$

5.2. **Solvability of system** (5.8)**.** In this subsection, we give the local existence of strong solution to the linearized momentum equations in a fixed domain $(0,T)\times\Omega_0$.

Lemma 5.1. *Assume that $\boldsymbol{B}_1$ and $\boldsymbol{d}_1$ admit an extension to Ω_0 given by*

$$\begin{aligned} \boldsymbol{u}^b\cdot\boldsymbol{n}|_{\Gamma_0}&=\boldsymbol{d}_1,\\ ([\mathbb{S}(\nabla_y\boldsymbol{u}^b,\boldsymbol{w}^b)\boldsymbol{n}]_{\tan}+\kappa_1[\boldsymbol{u}^b]_{\tan})|_{\Gamma_0}&=\boldsymbol{B}_1, \end{aligned}\tag{5.9}$$

such that $\boldsymbol{u}^b\in\mathcal{Y}(t)$ and $\boldsymbol{w}^b$ constructed by (4.11)*. Let p and $\boldsymbol{V}$ satisfy the assumptions in Theorem 1.1. Suppose that*

$$\begin{cases} \tilde\rho\in L^\infty(0,T;H^2(\Omega_0)),\tilde\rho_t\in L^2(0,T;H^1(\Omega_0)),\tilde\rho\ge\underline{\rho}>0,\\ \tilde{\boldsymbol{F}}_1\in L^2(0,T;H^1(\Omega_0)),\tilde{\boldsymbol{F}}_{1,t}\in L^2(0,T;L^2(\Omega_0)),\\ \tilde{\boldsymbol{w}}_0\in H^3(\Omega_0),\tilde{\boldsymbol{u}}_0\in H^3(\Omega_0). \end{cases}\tag{5.10}$$

Then there exists a unique solution $\tilde{\boldsymbol{u}}$ to equations (5.8) *such that*

$$\begin{aligned} \|\tilde{\boldsymbol{u}}\|_{\mathcal{Y}(T)}\le&\phi(\underline{\rho},\|\tilde\rho\|_{L^\infty(0,T;H^2(\Omega_0))},\|\tilde\rho_t\|_{L^2(0,T;H^1(\Omega_0))})(\left\|\tilde{\boldsymbol{w}}\right\|_{L^2(0,T;H^1(\Omega_0))}+\left\|\tilde{\boldsymbol{w}}_t\right\|_{L^2(0,T;H^1(\Omega_0))}\\ &+\|\tilde{\boldsymbol{F}}_1\|_{L^2(0,T;H^1(\Omega_0))\cap L^\infty(0,T;L^2(\Omega_0))}+\|\tilde{\boldsymbol{F}}_{1,t}\|_{L^2(0,T;L^2(\Omega_0))}+\|\boldsymbol{u}^b\|_{\mathcal{Y}(T)}\\ &+\|\tilde{\boldsymbol{V}}\|_{L^\infty(0,T;H^1(\Omega_0))\cap L^2(0,T;H^2(\Omega_0))}+\|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^2(\Omega_0))}+\|\tilde{\boldsymbol{V}}_{tt}\|_{L^2(0,T;H^2(\Omega_0))}\\ &+\|\boldsymbol{u}_0\|_{H^3(\Omega_0)}+\|\boldsymbol{w}_0\|_{H^2(\Omega_0)}), \end{aligned}\tag{5.11}$$

where ϕ denotes a positive increasing function.

Proof. Taking $\boldsymbol{u}^\dagger=\tilde{\boldsymbol{u}}-\boldsymbol{u}^b$, (5.8) yields

$$\begin{cases} \tilde\rho{\boldsymbol{u}^\dagger}_t-(\mu+\lambda-\xi)\nabla_y\mathrm{div}_y\boldsymbol{u}^\dagger-(\mu+\xi)\Delta_y\boldsymbol{u}^\dagger-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}}=\hat{\boldsymbol{F}}_1-\tilde\rho\boldsymbol{u}^b_t,\\ (\hat{\boldsymbol{u}}-\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}|_{\Gamma_0}=0,\\ \boldsymbol{n}\cdot\mathbb{S}\cdot\tau_k+\kappa_1(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\cdot\tau_k|_{\Gamma_0}=0,\quad k=1,2, \end{cases}\tag{5.12}$$

where

$$\hat{\boldsymbol{F}}_1=\tilde{\boldsymbol{F}}_1+(\mu+\lambda-\xi)\nabla_y\mathrm{div}_y\boldsymbol{u}^b+(\mu+\xi)\Delta_y\boldsymbol{u}^b.\tag{5.13}$$

The positive frictions contribute to lower-order terms which are easy to estimate, so we assume that $\kappa_1=0$.

The proof of (5.11) is divided into several parts.

• The estimate of $\left\|\boldsymbol{u}^\dagger\right\|_{L^\infty(0,T;L^2(\Omega_0))}$.

Multiplying (5.12) by $\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}}$ and integrating resulting equation over Ω_0, we obtain

$$\begin{aligned} &\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\tilde\rho|\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}}|^2\mathrm{d}y+\int_{\Omega_0}(T_1\boldsymbol{u}^\dagger-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}})\cdot(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y\\ =&\frac12\int_{\Omega_0}\tilde\rho_t|\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}}|^2\mathrm{d}y-\int_{\Omega_0}\tilde\rho\boldsymbol{V}_t(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y+\int_{\Omega_0}\hat{\boldsymbol{F}}_1\cdot(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y-\int_{\Omega_0}\tilde\rho\boldsymbol{u}^b_t\cdot(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y, \end{aligned}\tag{5.14}$$

where we denote $T_1\boldsymbol{u}^\dagger=-(\mu+\lambda-\xi)\nabla_y\mathrm{div}_y\boldsymbol{u}^\dagger-(\mu+\xi)\Delta_y\boldsymbol{u}^\dagger$.

We only pick up some typical terms in (5.14) to give detailed proof, others terms are derived from direct calculation and Hölder's inequality which are similar to Lemma 4.1.

We investigate the second term on the (5.14). We integrate it to derive

$$\begin{aligned}
&\int_{\Omega_0}(T_1\boldsymbol{u}^\dagger-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}})\cdot(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y=-\int_{\Omega}\mathrm{div}_y\mathbb{S}\cdot(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y\\
&=\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger,\tilde{\boldsymbol{w}}):\nabla_y(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\mathrm{d}y-\int_{\partial\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger,\tilde{\boldsymbol{w}})\boldsymbol{n}\cdot(\boldsymbol{u}-\tilde{\boldsymbol{V}})\mathrm{d}S\\
&=\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger,\tilde{\boldsymbol{w}}):\nabla_y\boldsymbol{u}^\dagger\mathrm{d}y-\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger,\tilde{\boldsymbol{w}}):\nabla_y\tilde{\boldsymbol{V}}\mathrm{d}y,
\end{aligned}\tag{5.15}$$

where in the last identity we have used the boundary conditions.

Notice the fact that

$$\begin{aligned}
\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger,\tilde{\boldsymbol{w}}):\nabla_y\boldsymbol{u}^\dagger\mathrm{d}y=&\mu\|\nabla_y\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}+(\mu+\lambda)\|\mathrm{div}_y\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}\\
&+\xi\|\mathrm{curl}_y\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}-2\xi\int_{\Omega_0}\tilde{\boldsymbol{w}}\cdot\mathrm{curl}_y\boldsymbol{u}^\dagger\mathrm{d}y,
\end{aligned}\tag{5.16}$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\tilde{\rho}}(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\|^2_{L^2(\Omega_0)}+\|\nabla_y\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}\\
\leq&C(\underline{\rho})(\|\tilde{\rho}\|_{H^2(\Omega_0)}+\|\tilde{\rho}_t\|^2_{L^3(\Omega_0)})\|\sqrt{\tilde{\rho}}(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})\|^2_{L^2(\Omega_0)}\\
&+C(\|\tilde{\boldsymbol{w}}\|^2_{L^2(\Omega_0)}+\|\hat{\boldsymbol{F}}_1\|^2_{L^2(\Omega_0)}+\|\boldsymbol{u}_t^b\|^2_{L^2(\Omega_0)}+\|\tilde{\boldsymbol{V}}\|^2_{H^2(\Omega_0)}+\|\tilde{\boldsymbol{V}}_t\|^2_{H^1(\Omega_0)}).
\end{aligned}\tag{5.17}$$

Applying Grönwall's inequality, we get

$$\begin{aligned}
&\|\boldsymbol{u}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}+\|\nabla_y\boldsymbol{u}^\dagger\|_{L^2(0,T;L^2(\Omega_0))}\\
\leq&\phi(\underline{\rho},\|\tilde{\rho}\|_{L^\infty(0,T;H^2(\Omega_0)},\|\tilde{\rho}_t\|_{L^2(0,T;L^2(\Omega_0))})(\|\tilde{\boldsymbol{w}}\|_{L^2(0,T;H^1(\Omega_0))}+\|\hat{\boldsymbol{F}}_1\|_{L^2(0,T;L^2(\Omega_0))}\\
&+\|\boldsymbol{u}_t^b\|_{L^2(0,T;L^2(\Omega_0))}+\|\tilde{\boldsymbol{V}}\|_{L^2(0,T;H^2(\Omega_0))}+\|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^1(\Omega_0))}+\|\tilde{\boldsymbol{V}}\|_{L^\infty(0,T;H^1(\Omega_0))}).
\end{aligned}\tag{5.18}$$

- The estimate of $\|\nabla_y\boldsymbol{u}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))}$.

We multiply (5.12) by $(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})_t+\epsilon T_1\boldsymbol{u}^\dagger$, where ϵ is sufficiently small and integrate the results over Ω_0 to derive

$$\begin{aligned}
&\int_{\Omega_0}\tilde{\rho}|\partial_t\boldsymbol{u}^\dagger|^2\mathrm{d}y+\int_{\Omega_0}(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})_t\cdot(T_1\boldsymbol{u}^\dagger-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}})\mathrm{d}y+\epsilon\int_{\Omega_0}|T_1\boldsymbol{u}^\dagger|^2\mathrm{d}y\\
=&-\epsilon\int_{\Omega_0}\tilde{\rho}\boldsymbol{u}^\dagger{}_tT_1\boldsymbol{u}^\dagger\mathrm{d}y+\int_{\Omega_0}\tilde{\rho}\boldsymbol{u}^\dagger{}_t\tilde{\boldsymbol{V}}_t\mathrm{d}y+2\xi\epsilon\int_{\Omega_0}T_1\boldsymbol{u}^\dagger\cdot\mathrm{curl}_y\tilde{\boldsymbol{w}}\mathrm{d}y\\
&+\int_{\Omega_0}(\hat{\boldsymbol{F}}_1-\tilde{\rho}\partial_t\boldsymbol{u}^b)\cdot[(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})_t+\epsilon T_1\boldsymbol{u}^\dagger]\mathrm{d}y.
\end{aligned}\tag{5.19}$$

Using the fact that $(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})_t\cdot\boldsymbol{n}=0$, we use integration by parts to obtain

$$\int_{\Omega_0}(\boldsymbol{u}^\dagger-\tilde{\boldsymbol{V}})_t\cdot(T_1\boldsymbol{u}^\dagger-2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}})\mathrm{d}y=\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger{}_t,\tilde{\boldsymbol{w}}):\nabla_y\boldsymbol{u}^\dagger{}_t\mathrm{d}y-\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger{}_t,\tilde{\boldsymbol{w}}):\nabla_y\tilde{\boldsymbol{V}}_t\mathrm{d}y\tag{5.20}$$

and

$$\begin{aligned}
\int_{\Omega_0}\mathbb{S}:\nabla_y\boldsymbol{u}^\dagger{}_t\mathrm{d}y=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu\|\nabla_y\boldsymbol{u}\|^2_{L^2(\Omega_0)}+(\mu+\lambda)\|\mathrm{div}_y\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}+\xi\|\mathrm{curl}_y\boldsymbol{u}^\dagger\|^2_{L^2}\right]\\
&-2\xi\int_{\Omega}\tilde{\boldsymbol{w}}\cdot\mathrm{curl}_y\boldsymbol{u}^\dagger{}_t\mathrm{d}y.
\end{aligned}\tag{5.21}$$

Combining the above estimates, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_y \boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)} + \|\sqrt{\tilde\rho}\boldsymbol{u}^\dagger{}_t\|^2_{L^2(\Omega_0)} + \epsilon\|\nabla_y^2\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}\\ \le& C\left(\left\|\nabla_y \boldsymbol{w}^\dagger\right\|^2_{L^2(\Omega_0)} + \|\tilde{\boldsymbol{V}}_t\|^2_{H^2(\Omega_0)} + \|\hat{\boldsymbol{F}}_1\|^2_{L^2(\Omega_0)} + \left\|\tilde\rho\right\|_{H^2(\Omega_0)}\|\boldsymbol{u}^b_t\|^2_{L^2(\Omega_0)}\right) + C\|\nabla_y\boldsymbol{u}^\dagger\|^2_{L^2(\Omega_0)}.\end{aligned} \tag{5.22}$$

Applying Grönwall's inequality, we obtain

$$\begin{aligned}&\|\nabla_y \boldsymbol{u}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^\dagger{}_t\|_{L^2(0,T;L^2(\Omega_0))} + \|\nabla_y^2\boldsymbol{u}^\dagger\|_{L^2(0,T;L^2(\Omega_0))}\\ \le&\phi(\underline{\rho}, \|\rho\|_{L^\infty(0,T;H^1(\Omega_0))})\left(\left\|\tilde{\boldsymbol{w}}\right\|_{L^2(0,T;H^1(\Omega_0))} + \|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^2(\Omega_0))} + \|(\hat{\boldsymbol{F}}_1, \boldsymbol{u}^b_t)\|_{L^2(0,T;L^2(\Omega_0))}\right).\end{aligned} \tag{5.23}$$

- The estimates of $\left\|\boldsymbol{u}^\dagger{}_t\right\|_{L^\infty(0,T;L^2(\Omega_0))}$.

Next we apply ∂_t to $(5.12)_1$ and multiply the resultant by $(\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_t$ to get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\tilde\rho|\boldsymbol{u}^\dagger{}_t|^2\mathrm{d}y + \int_{\Omega_0}(T_1\boldsymbol{u}^\dagger{}_t - 2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}}_t)\cdot(\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_t\mathrm{d}y\\ =&-\frac{1}{2}\int_{\Omega_0}\tilde\rho_t|\boldsymbol{u}^\dagger{}_t|^2\mathrm{d}y + \int_{\Omega_0}\tilde\rho\boldsymbol{u}^\dagger{}_{tt}\cdot\tilde{\boldsymbol{V}}_t\mathrm{d}y + \int_{\Omega}\tilde\rho_t\boldsymbol{u}^\dagger{}_t\cdot\tilde{\boldsymbol{V}}_t\mathrm{d}y\\ &+\int_{\Omega}(\hat{\boldsymbol{F}}_{1,t} - \tilde\rho_t\boldsymbol{u}^b_t)\cdot(\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_t\mathrm{d}y - \int_{\Omega_0}\tilde\rho\boldsymbol{u}^b_{tt}\cdot(\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_t\mathrm{d}y.\end{aligned} \tag{5.24}$$

According to the boundary conditions $(5.12)_2$ and $(5.12)_3$, we use integration by parts to derive

$$\begin{aligned}&\int_{\Omega_0}(\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_t\cdot(T_1\boldsymbol{u}^\dagger{}_t - 2\xi\mathrm{curl}_y\tilde{\boldsymbol{w}}_t)\mathrm{d}y\\ =&\int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger{}_t, \tilde{\boldsymbol{w}}_t):\nabla_y\boldsymbol{u}^\dagger{}_t\mathrm{d}x - \int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger{}_t, \tilde{\boldsymbol{w}}_t):\nabla_y\tilde{\boldsymbol{V}}_t\mathrm{d}y\\ =&\mu\left\|\nabla_y\boldsymbol{u}^\dagger{}_t\right\|^2_{L^2(\Omega_0)} + (\mu+\lambda)\left\|\mathrm{div}_y\boldsymbol{u}^\dagger{}_t\right\|^2_{L^2(\Omega_0)} + \xi\left\|\mathrm{curl}_y\boldsymbol{u}^\dagger{}_t\right\|^2_{L^2(\Omega_0)}\\ &-\int_{\Omega_0}\tilde{\boldsymbol{w}}_t\cdot\mathrm{curl}_y\boldsymbol{u}^\dagger{}_t\mathrm{d}y - \int_{\Omega_0}\mathbb{S}(\nabla_y\boldsymbol{u}^\dagger{}_t, \tilde{\boldsymbol{w}}_t):\nabla_y\tilde{\boldsymbol{V}}_t\mathrm{d}y.\end{aligned} \tag{5.25}$$

Combining (5.24) with (5.25), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\tilde\rho}\boldsymbol{u}^\dagger{}_t\|^2_{L^2(\Omega_0)} + \|\nabla_y\boldsymbol{u}^\dagger{}_t\|^2_{L^2(\Omega_0)}\\ \le& C(\underline{\rho})(1 + \|\tilde\rho_t\|^2_{H^1(\Omega_0)})\|\sqrt{\tilde\rho}\boldsymbol{u}^\dagger{}_t\|^2_{L^2(\Omega_0)} + \delta\|\boldsymbol{u}^\dagger{}_{tt}\|^2_{L^2(\Omega_0)}\\ &+ C(\left\|\nabla_y\tilde{\boldsymbol{w}}_t\right\|^2_{L^2(\Omega_0)} + \|\hat{\boldsymbol{F}}_{1,t}\|^2_{L^2(\Omega_0)} + \left\|\tilde\rho_t\right\|^2_{H^1(\Omega_0)}\|\boldsymbol{u}^b_t\|^2_{L^2(\Omega_0)} + \left\|\tilde\rho\right\|_{H^2(\Omega_0)}\|\boldsymbol{u}^b_{tt}\|^2_{L^2(\Omega_0)})\\ &+ C(1 + \left\|\tilde\rho_t\right\|^2_{H^1(\Omega_0)} + \left\|\tilde\rho\right\|^2_{H^2(\Omega_0)})\|\tilde{\boldsymbol{V}}_t\|^2_{H^1}.\end{aligned} \tag{5.26}$$

By Grönwall's inequality, we obtain

$$\begin{aligned}&\|\boldsymbol{u}^\dagger{}_t\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\nabla_y\boldsymbol{u}^\dagger{}_t\|_{L^2(0,T;L^2(\Omega_0))}\\ \le&\phi(\underline{\rho}, \|\tilde\rho\|_{L^\infty(0,T;H^2(\Omega_0))}, \|\tilde\rho_t\|_{L^2(0,T;H^2(\Omega_0))})(\left\|\tilde{\boldsymbol{w}}_t\right\|_{L^2(0,T;H^1(\Omega_0))} + \|\hat{\boldsymbol{F}}_{1,t}\|_{L^2(0,T;L^2(\Omega_0))}\\ &+ \|\boldsymbol{u}^b_t\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^b_{tt}\|_{L^2(0,T;L^2(\Omega_0))} + \|\tilde{\boldsymbol{V}}_t\|_{L^2(0,T;H^1(\Omega_0))}) + \delta\|\boldsymbol{u}^\dagger{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))} := \Phi_2.\end{aligned} \tag{5.27}$$

- The estimate of $\left\|\nabla_y^2\boldsymbol{u}^\dagger\right\|_{L^\infty(0,T;L^(\Omega_0))}$.

Applying the classical elliptic theory, we derive that

$$\begin{aligned}\|\nabla_y^2\boldsymbol{u}^\dagger\|_{L^\infty(0,T;L^2(\Omega_0))} \le&\left\|\nabla_y\tilde{\boldsymbol{w}}\right\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\hat{\boldsymbol{F}}_1\|_{L^\infty(0,T;L^2(\Omega_0))}\\ &+ \|\tilde\rho\boldsymbol{u}^b_t\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\tilde\rho\|_{L^\infty(0,T;L^\infty(\Omega_0))}\Phi_2.\end{aligned} \tag{5.28}$$

• The estimate of $\left\|\nabla_y^3 \boldsymbol{u}^\dagger\right\|_{L^\infty(0,T;L^(\Omega_0))}$.
Applying gradient operator ∇_y on $(5.12)_1$,

$$\begin{aligned}\|\nabla_y^3 \boldsymbol{u}^\dagger\|_{L^2(0,T;L^2(\Omega_0))} \leq & \|\nabla_y^2 \tilde{\boldsymbol{w}}\|_{L^2(0,T;L^2(\Omega_0))} + \|\nabla \hat{\boldsymbol{F}}_1\|_{L^2(0,T;L^2(\Omega_0))} \\ & + \phi(\|\rho\|_{L^\infty(0,T;H^2(\Omega_0))})\|\boldsymbol{u}_t + \boldsymbol{u}_t^b\|_{L^2(0,T;H^1(\Omega_0))}.\end{aligned} \tag{5.29}$$

Lastly, it leaves us to derive a bound on $\|\boldsymbol{u}^\dagger{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))}$.
• The estimates of $\left\|\boldsymbol{u}^\dagger{}_{tt}\right\|_{L^2(0,T;L^2(\Omega_0))}$.
We apply ∂_t to $(5.12)_1$, multiply the results by $(\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_{tt}$, and integrate over Ω_0 to get

$$\begin{aligned}&\int_{\Omega_0} \tilde{\rho}|\boldsymbol{u}^\dagger{}_{tt}|^2 \mathrm{d}y + \int_{\Omega_0} (\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_{tt} \cdot (T_1 \boldsymbol{u}^\dagger{}_t - 2\xi \mathrm{curl}_y \tilde{\boldsymbol{w}}_t) \mathrm{d}y \\ =& \int_{\Omega_0} (\hat{\boldsymbol{F}}_{1,t} - \tilde{\rho}\boldsymbol{u}_{tt}^b) \cdot (\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_{tt} \mathrm{d}y + \int_{\Omega_0} \tilde{\rho}\boldsymbol{u}^\dagger{}_{tt} \cdot \tilde{\boldsymbol{V}}_{tt} \mathrm{d}y + \int_{\Omega_0} \tilde{\rho}_t(\boldsymbol{u}^\dagger{}_t + \boldsymbol{u}_t^b) \cdot (\tilde{\boldsymbol{V}} - \boldsymbol{u}^\dagger)_{tt} \mathrm{d}y.\end{aligned} \tag{5.30}$$

Then we apply the boundary condition to get

$$\begin{aligned}&\int_{\Omega_0} (\boldsymbol{u}^\dagger - \tilde{\boldsymbol{V}})_{tt} \cdot (T_1 \boldsymbol{u}^\dagger{}_t - 2\xi \mathrm{curl}_y \tilde{\boldsymbol{w}}_t) \mathrm{d}y \\ =& \int_{\Omega_0} \mathbb{S}(\nabla_y \boldsymbol{u}^\dagger{}_t, \tilde{\boldsymbol{w}}_t) : \nabla_y \boldsymbol{u}^\dagger{}_{tt} \mathrm{d}y - \int_{\Omega_0} \mathbb{S}(\nabla_y \boldsymbol{u}^\dagger{}_t, \tilde{\boldsymbol{w}}_t) : \nabla_y \tilde{\boldsymbol{V}}_{tt} \mathrm{d}y.\end{aligned} \tag{5.31}$$

Hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_y \boldsymbol{u}^\dagger{}_t\|_{L^2(\Omega_0)}^2 + \|\sqrt{\tilde{\rho}}\boldsymbol{u}_{tt}\|_{L^2(\Omega_0)}^2 \\ \leq & C\left(\|\tilde{\boldsymbol{V}}_{tt}\|_{H^2(\Omega_0)}^2 + \|\hat{\boldsymbol{F}}_{1,t}\|_{L^2(\Omega_0)}^2 + (1 + \left\|\tilde{\rho}\right\|_{H^2(\Omega_0)}^2)\|\boldsymbol{u}_{tt}^b\|_{L^2(\Omega_0)}^2\right) \\ & + C(1 + \left\|\tilde{\rho}_t\right\|_{H^1(\Omega_0)}^2)\left\|\nabla_y \boldsymbol{w}^\dagger{}_t\right\|_{L^2(\Omega_0)}^2 + C\left\|\nabla_y \boldsymbol{u}_t^b\right\|_{L^2(\Omega_0)}^2 + C\left\|\nabla_y \tilde{\boldsymbol{w}}_t\right\|_{L^2(\Omega_0)}^2.\end{aligned} \tag{5.32}$$

Applying Grönwall's inequality, we obtain

$$\begin{aligned}&\|\nabla_y \boldsymbol{u}^\dagger{}_t\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^\dagger{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))} \\ \leq & \phi(\left\|\tilde{\rho}\right\|_{L^\infty(0,T;H^2(\Omega_0))}, \|\rho_t\|_{L^2(0,T;H^1(\Omega_0))})\left(\left\|\tilde{\boldsymbol{w}}_t\right\|_{L^2(0,T;L^2(\Omega_0))} + \|\tilde{\boldsymbol{V}}_{tt}\|_{L^2(0,T;H^2(\Omega_0))}\right. \\ & \left. + \|\hat{\boldsymbol{F}}_{1,t}\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}_{tt}^b\|_{L^2(0,T;L^2(\Omega_0))} + \left\|\nabla_y \boldsymbol{u}_t^b\right\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^\dagger{}_t(0)\|_{H^1(\Omega_0)}\right).\end{aligned} \tag{5.33}$$

For the term $\boldsymbol{u}_t(0)$ in above estimate, we apply ∇_x to $(5.12)_1$ and take the results in $t = 0$ to derive

$$\|\boldsymbol{u}^\dagger{}_t(0)\|_{H^1(\Omega_0)} \leq C(\|\hat{\boldsymbol{F}}_1(0)\|_{H^1(\Omega_0)} + \|\tilde{\boldsymbol{w}}_0\|_{H^3(\Omega_0)} + \left\|\boldsymbol{u}^\dagger{}_0\right\|_{H^3(\Omega_0)}). \tag{5.34}$$

Finally, we show a bound for $\|\boldsymbol{u}^\dagger\|_{L^2(0,T;H^3(\Omega_0))}$. Taking the derivative with respect to time of $(5.12)_1$, we have

$$\begin{cases} T_1(\boldsymbol{u}^\dagger{}_t) = 2\xi \mathrm{curl}_y \tilde{\boldsymbol{w}}_t + \hat{\boldsymbol{F}}_{1,t} - \tilde{\rho}_t \boldsymbol{u}^\dagger{}_t - \tilde{\rho}\boldsymbol{u}^\dagger{}_{tt} - \tilde{\rho}_t \boldsymbol{u}_t^b - \tilde{\rho}\boldsymbol{u}_{tt}^b, \\ (\boldsymbol{u}^\dagger - \boldsymbol{V}) \cdot \boldsymbol{n}|_\Gamma = 0, \\ [\mathbb{S}(\nabla_y \boldsymbol{u}^\dagger{}_t, \tilde{\boldsymbol{w}}_t)\boldsymbol{n}]_{\tan}|_\Gamma = 0, \end{cases} \tag{5.35}$$

where $\mathbb{S}(\nabla \boldsymbol{u}_t, \boldsymbol{w}_t) = T_1(\boldsymbol{u}^\dagger{}_t) - 2\xi \mathrm{curl}_y \tilde{\boldsymbol{w}}_t$, recall that we have set $\kappa_1 = 0$.

For the inhomogeneous boundary condition of the elliptic problem for $\boldsymbol{u}^\dagger{}_t$, we derive from (5.35) and classical elliptic theory that

$$\|\boldsymbol{u}^\dagger{}_t\|_{H^2(\Omega_0)} \leq C(\|\hat{\boldsymbol{F}}_{1,t}\|_{L^2(\Omega_0)} + \|\tilde{\boldsymbol{w}}_t\|_{H^1(\Omega_0)} + \|\tilde{\rho}_t(\boldsymbol{u}^\dagger{}_t + \boldsymbol{u}_t^b)\|_{L^2(\Omega_0)} + \|\tilde{\rho}(\boldsymbol{u}^\dagger{}_{tt} + \boldsymbol{u}_{tt}^b)\|_{L^2(\Omega_0)}). \tag{5.36}$$

Integrating (5.36) with respect to time we have

$$\begin{aligned}\|\boldsymbol{u}^{\dagger}{}_t\|_{L^2(0,T;H^2(\Omega_0))} \le & C\Big(\|\hat{\boldsymbol{F}}_{1,t}\|_{L^2(0,T;L^2(\Omega_0))} + \|\tilde{\boldsymbol{w}}_t\|_{L^2(0,T;H^1(\Omega_0))} \\ & + \|\tilde{\rho}\|_{L^\infty(0,T;H^2(\Omega_0))}(\|\boldsymbol{u}^b_{tt}\|_{L^2(\Omega_0)} + \|\boldsymbol{u}^{\dagger}{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))}) \\ & + \|\tilde{\rho}_t\|_{L^2(0,T;H^1(\Omega_0))}(\|\boldsymbol{u}^{\dagger}{}_t\|_{L^2(0,T;H^1(\Omega_0))} + \|\boldsymbol{u}^b_t\|_{L^\infty(0,T;H^1(\Omega_0))})\Big),\end{aligned} \tag{5.37}$$

together with (5.33) and (5.34), it holds that

$$\begin{aligned}&\|\nabla_y \boldsymbol{u}^{\dagger}{}_t\|_{L^\infty(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^{\dagger}{}_{tt}\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^{\dagger}{}_t\|_{L^2(0,T;H^2(\Omega_0))} \\ \le& \phi(\underline{\rho}, \|\tilde{\rho}\|_{L^\infty(0,T;H^2(\Omega_0))}, \|\tilde{\rho}_t\|_{L^2(0,T;H^1(\Omega_0))})(\|\tilde{\boldsymbol{w}}_t\|_{L^2(0,T;H^1(\Omega_0))} + \|\hat{\boldsymbol{F}}_{1,t}\|_{L^2(0,T;L^2(\Omega_0))} \\ & + \|\hat{\boldsymbol{F}}_1(0)\|_{H^1(\Omega_0)} + \|\boldsymbol{u}^b_t\|_{L^2(0,T;H^1(\Omega_0))} + \|\tilde{\boldsymbol{V}}_{tt}\|_{L^2(0,T;H^2(\Omega_0))} \\ & + \|\boldsymbol{u}^b_{tt}\|_{L^2(0,T;L^2(\Omega_0))} + \|\boldsymbol{u}^{\dagger}{}_t(0)\|_{H^1(\Omega_0)}).\end{aligned} \tag{5.38}$$

Furthermore, combining (5.13), (5.34) and previous estimates, we finish the proof of (5.11). □

5.3. **Proof of Proposition** 5.1. We directly give the estimate of $\|\boldsymbol{u}^b\|_{\mathcal{Y}(T)}$, the detailed construction is given in the Appendix.

Lemma 5.2. *Let $\boldsymbol{V}$ satisfy the assumptions of Theorem 1.1. Then there exists extension $\boldsymbol{u}^b(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{V}})$ defined by* (5.9) *satisfying*

$$\|\boldsymbol{u}^b\|_{\mathcal{Y}(T)} \le E(T)[1 + \|(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{V}})\|_{\mathcal{Y}(T)} + \|\tilde{\boldsymbol{w}}\|_{\mathcal{Y}(T)}], \tag{5.39}$$

where $E(t)$ is continuous and $E(0) = 0$.

By using the similar method in Lemma 4.3, we can get following estimates.

Lemma 5.3. *For $\boldsymbol{R}_1$ defined by* (5.5) *and denote $G(t) = \|\boldsymbol{u}^{\dagger}\|_{\mathcal{Y}(t)}$, we have for any $\eta > 0$*

$$\begin{aligned}&\|\tilde{\rho}\boldsymbol{V} \cdot \nabla_y \tilde{\boldsymbol{u}} + \boldsymbol{R}_1\|_{L^2(0,T;H^1(\Omega_0))} + \|\partial_t[\tilde{\rho}\boldsymbol{V} \cdot \nabla_y \boldsymbol{u} + \boldsymbol{R}_1]\|_{L^2(0,T;L^2(\Omega_0))} \\ \le& C[(\eta + \sqrt{T}C(\eta) + E(T))G(T)] + C\sqrt{T}(\|\tilde{\boldsymbol{u}}\|_{H^2(\Omega_0)} + \|\tilde{\boldsymbol{u}}_t\|_{H^1(\Omega_0)}).\end{aligned} \tag{5.40}$$

6. Proof of Theorem 1.1

In this section, by using the estimates from the linearized system (5.1) and (4.1), we prove the convergence of the sequence $(\rho_n, \boldsymbol{u}_n, \boldsymbol{w}_n)$ defined in (2.1)-(2.3) which contributes to the existence of the solution.

Firstly, we denote the space where we are seeking the solution,

$$F_n(t) = \|\boldsymbol{w}_n\|_{\mathcal{X}(t)}, \quad G_n(t) = \|\boldsymbol{u}_n\|_{\mathcal{X}(t)}. \tag{6.1}$$

Lemma 6.1. *For any $\eta > 0$,*

$$\begin{aligned}&\|(\boldsymbol{F}_1^n, \boldsymbol{F}_2^n)\|_{L^2(0,T;H^1(\Omega_t))} + \|\partial_t(\boldsymbol{F}_1^n, \boldsymbol{F}_2^n)\|_{L^2(0,T;L^2(\Omega_t))} \\ &\le \phi(\sqrt{T}F_n(T))[(\eta + \sqrt{T}C(\eta) + E(T))(F_n(T) + G_n(T)) + E(T)(F_{n+1}(T) + G_{n+1}(T))],\end{aligned} \tag{6.2}$$

where $E(t)$ is small for small time.

Proof. Here we only give precise estimates for some typical terms, other terms can be treated similarly.

Due to

$$\begin{aligned}&\nabla_x^2 p(\rho) \sim \rho(|\nabla_x \rho|^2 + \nabla_x^2 \rho), \\ &\partial_t \nabla_x p(\rho) \sim \rho(\rho_t \nabla_x \rho + \nabla_x \rho_t),\end{aligned}$$

hence, it follows from (3.7) and (3.8) that

$$\begin{aligned}
&\|\nabla_x^2 p(\rho_{n+1})\|_{L^2(0,T;L^2(\Omega_t))}\\
\le&\big\|\rho_{n+1}\big\|_{L^\infty(0,T;L^2(\Omega_t))}\sqrt{T}\|\rho_{n+1}\|_{L^\infty(0,T;H^3(\Omega_t))}(1+\big\|\rho_{n+1}\big\|_{L^\infty(0,T;H^2(\Omega_t))})\\
\le&C\big\|\rho_{n+1}(0)\big\|_{H^2(\Omega_0)}\sqrt{T}\phi(\sqrt{T}\|\boldsymbol{u}_n\|_{L^2(0,T;H^3(\Omega_t))}),
\end{aligned}\tag{6.3}$$

$$\begin{aligned}
\|\partial_t\nabla_x p(\rho_{n+1})\|_{L^2(0,T;L^2(\Omega_t))}\le&(1+\big\|\rho_{n+1}\big\|^2_{L^\infty(0,T;H^2(\Omega_t))})\big\|\rho_{n+1,t}\big\|_{L^2(0,T;H^1(\Omega_t))}\\
\le&C\big\|\rho_{n+1}(0)\big\|_{H^2(\Omega_0)}\sqrt{T}\phi(\sqrt{T}\big\|\boldsymbol{u}_n\big\|_{L^2(0,T;H^3(\Omega_t))})\big\|\boldsymbol{u}_n\big\|_{L^2(0,T;H^3(\Omega_t))},
\end{aligned}\tag{6.4}$$

and

$$\begin{aligned}
&\big\|\rho_{n+1}\boldsymbol{u}_n\cdot\nabla_x^2\boldsymbol{u}_n\big\|_{L^2(0,T;L^2(\Omega_t))}\\
\le&\ \big\|\rho_{n+1}\big\|_{L^\infty(0,T;L^8(\Omega_t))}\big\|\boldsymbol{u}_n\big\|_{L^\infty(0,T;L^8(\Omega_t))}\big\|\nabla_x^2\boldsymbol{u}_n\big\|_{L^2(0,T;L^4(\Omega_t))}\\
\le&\ \big\|\rho_{n+1}\big\|_{L^\infty(0,T;L^8(\Omega_t))}\big\|\boldsymbol{u}_n\big\|_{L^\infty(0,T;H^1(\Omega_t))}\left(\int_0^T\big\|\nabla_x^2\boldsymbol{u}_n\big\|^{\frac12}_{L^2(\Omega_t)}\big\|\nabla_x^3\boldsymbol{u}_n\big\|^{\frac32}_{L^2(\Omega_t)}\right)^{\frac12}\\
\le&\ \big\|\rho_{n+1}\big\|_{L^\infty(0,T;L^8(\Omega_t))}\big\|\boldsymbol{u}_n\big\|_{L^\infty(0,T;H^1(\Omega_t))}(\eta\big\|\boldsymbol{u}_n\big\|_{L^2(0,T;H^3(\Omega_t))}+\sqrt{T}C(\eta)\big\|\boldsymbol{u}_n\big\|_{L^\infty(0,T;H^2(\Omega_t))}).
\end{aligned}\tag{6.5}$$

Collecting above estimates, we use (3.7), (4.68) and (5.40) to derive that

$$\begin{aligned}
&\|(\boldsymbol{F}_1^n,\boldsymbol{F}_2^n)\|_{L^2(0,T;H^1(\Omega_t))}+\|\partial_t(\boldsymbol{F}_1^n,\boldsymbol{F}_2^n)\|_{L^2(0,T;L^2(\Omega_t))}\\
&\le\phi(\sqrt{T}F_n(T))[(\eta+\sqrt{T}C(\eta)+E(T))(F_n(T)+G_n(T))]+E(T)(F_{n+1}(T)+G_{n+1}(T)).
\end{aligned}\tag{6.6}$$

Hence, we finish the proof. ☐

According to Lemma 6.1, we show in the next proposition that the sequence $(\rho_n,\boldsymbol{u}_n,\boldsymbol{w}_n)$ is bounded for small time in the space defined in (6.1).

Proposition 6.1. *Let $F_n(t)$, $G_n(t)$ be defined in* (6.1)*. Then there exists a sufficiently large $N>0$ and a sufficiently small $T_*>0$ such that for any $t\in[0,T_*]$,*

$$F_n(t)+G_n(t)\le N. \tag{6.7}$$

Proof. By the estimate derived in Lemma 5.2,

$$\|(\boldsymbol{u}_{tt}^b,\boldsymbol{w}_{tt}^b)\|_{\mathcal{X}(t)}\le E(t)(F_n(t)+G_n(t)), \tag{6.8}$$

and Proposition 5.1, we have

$$\begin{aligned}
F_{n+1}(t)+G_{n+1}(t)\le&\ \phi(\sqrt{t}F_n(t))[(\eta+tC(\eta)+E(t))(F_n(t)+G_n(t))\\
&+\|(\boldsymbol{u}_0,\boldsymbol{w}_0)\|_{H^3(\Omega_0)}+\|\rho_0\|_{H^2(\Omega_0)}+\|\boldsymbol{V},\mathrm{curl}_x\boldsymbol{V}\|_{\mathcal{X}(t)}.
\end{aligned}$$

Hence, we can choose a sufficiently large positive constant $N=N(\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{V})$ and a small time $T*>0$ such that (6.7) holds. ☐

6.1. **Convergence of the sequence of approximations.** In this subsection, we show the convergence of the sequence. Firstly, we denote

$$\varrho_{n+1}=\rho_{n+1}-\rho_n,\quad \boldsymbol{\sigma}_{n+1}=\boldsymbol{w}_{n+1}-\boldsymbol{w}_n,\quad \boldsymbol{v}_{n+1}=\boldsymbol{u}_{n+1}-\boldsymbol{u}_n.$$

For the continuity equation,

$$\partial_t\varrho_{n+1}+\boldsymbol{u}_n\cdot\nabla_x\varrho_{n+1}+\varrho_{n+1}\mathrm{div}_x\boldsymbol{u}_n=-\rho_n\mathrm{div}_x\varrho_n-\varrho_n\cdot\nabla_x\rho_n,\quad \varrho_{n+1}(0,\cdot)=0. \tag{6.9}$$

By (2.2), $\boldsymbol{\sigma}_{n+1}$ belongs to

$$\begin{aligned}
&\rho_n\partial_t\boldsymbol{\sigma}_{n+1}-(c_0+c_d-c_a)\nabla_x\mathrm{div}_x\boldsymbol{\sigma}_{n+1}-(c_a+c_d)\Delta_x\boldsymbol{\sigma}_{n+1}-2\xi(\mathrm{curl}_x\boldsymbol{v}_n-2\boldsymbol{\sigma}_{n+1})\\
&=-\varrho_{n+1}\partial_t\boldsymbol{w}_{n+1}-\varrho_{n+1}\boldsymbol{u}_{n-1}\cdot\nabla_x\boldsymbol{w}_{n-1}-\rho_{n+1}(\boldsymbol{v}_n\cdot\nabla_x\boldsymbol{w}_{n-1}+\boldsymbol{u}_n\cdot\nabla_x\boldsymbol{\sigma}_n),
\end{aligned}\tag{6.10}$$

with slip boundary conditions

$$\boldsymbol{\sigma}_{n+1}\cdot\boldsymbol{n}|_{\Gamma_t}=0,\quad [\mathbb{M}(\nabla_x\boldsymbol{\sigma}_{n+1})\cdot\boldsymbol{n}]_{\tan}+\kappa_2[\boldsymbol{\sigma}_{n+1}]_{\tan}|_{\Gamma_t}=0. \tag{6.11}$$

Moreover, according to (2.4), $\boldsymbol{v}_{n+1}$ satisfies

$$\begin{aligned}&\rho_n\partial_t\boldsymbol{v}_{n+1}-(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\boldsymbol{v}_{n+1}-(\mu+\xi)\Delta_x\boldsymbol{v}_{n+1}-2\xi\mathrm{curl}_x\boldsymbol{\sigma}_n\\ =&-\varrho_{n+1}\partial_t\boldsymbol{u}_{n+1}-\varrho_{n+1}\boldsymbol{u}_{n-1}\cdot\nabla_x\boldsymbol{u}_{n-1}-\rho_{n+1}(\boldsymbol{v}_n\cdot\nabla_x\boldsymbol{u}_{n-1}+\boldsymbol{u}_n\cdot\nabla_x\boldsymbol{v}_n)\\ &-p'(\rho_n)\nabla_x\varrho_n-\nabla_x\rho_n(p'(\rho_{n+1})-p'(\rho_n)),\end{aligned} \tag{6.12}$$

with slip boundary conditions

$$\boldsymbol{v}_{n+1}\cdot\boldsymbol{n}|_{\Gamma_t}=0,\quad [\mathbb{S}(\nabla_x\boldsymbol{v}_{n+1},\boldsymbol{\sigma}_n)\cdot\boldsymbol{n}]_{\tan}+\kappa_1[\boldsymbol{v}_{n+1}]_{\tan}|_{\Gamma_t}=0. \tag{6.13}$$

We observe that (6.10)-(6.11) and (6.12)-(6.13) are linear problems with slip boundary conditions, having a similar structure with (2.2)-(2.3) and (2.4)-(2.5). To utilize the previous results on the linearized problem, we also translate the (6.10)-(6.11) and (6.12)-(6.13) into a fixed domain,

$$\begin{aligned}&\rho_n\partial_t\boldsymbol{\sigma}_{n+1}-(c_0+c_d-c_a)\nabla_y\mathrm{div}_y\boldsymbol{\sigma}_{n+1}-(c_a+c_d)\Delta_y\boldsymbol{\sigma}_{n+1}-2\xi\mathrm{curl}_y(\boldsymbol{v}_n-2\boldsymbol{\sigma}_{n+1})\\ =&\rho_n\boldsymbol{V}\cdot\nabla_y\boldsymbol{v}_{n+1}+\boldsymbol{R}_2(\rho_n,\boldsymbol{v}_{n+1},\boldsymbol{\sigma}_{n+1})-\varrho_{n+1}\partial_t\boldsymbol{w}_{n+1}\\ &-\varrho_{n+1}\boldsymbol{u}_{n-1}\cdot\nabla_x\boldsymbol{w}_{n-1}-\rho_{n+1}(\boldsymbol{v}_n\cdot\nabla_x\boldsymbol{w}_{n-1}+\boldsymbol{u}_n\cdot\nabla_x\boldsymbol{\sigma}_n):=\tilde{\boldsymbol{R}}_2^n,\end{aligned} \tag{6.14}$$

with boundary conditions

$$\begin{aligned}&\boldsymbol{\sigma}_{n+1}\cdot\boldsymbol{n}|_{\Gamma_0}=\boldsymbol{\sigma}_{n+1}\cdot(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y))=:\tilde{\boldsymbol{d}}_2(\boldsymbol{\sigma}_{n+1}),\\ &[\mathbb{M}(\nabla_y\boldsymbol{\sigma}_{n+1})\cdot\boldsymbol{n}]_{\tan}+\kappa_2[\boldsymbol{\sigma}_{n+1}]_{\tan}|_{\Gamma_0}\\ =&\left[(c_a+c_d)\nabla_y\boldsymbol{\sigma}_{n+1}(t,y)(\mathbb{I}-\nabla_x\boldsymbol{Y})+(c_d-c_a)(\mathbb{I}-\nabla_x^{\mathsf T}\boldsymbol{Y})\nabla_y^{\mathsf T}\boldsymbol{\sigma}_{n+1}\right]\boldsymbol{n}(\boldsymbol{X}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y))\\ &+\left[(c_a+c_d)\nabla_y\boldsymbol{\sigma}_{n+1}(t,y)+(c_d-c_a)\nabla_y^{\mathsf T}\boldsymbol{\sigma}_{n+1}\right](t,y)\left[(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\cdot\tau^k(\boldsymbol{X}(t,y))\right.\\ &\left.+\boldsymbol{n}(y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\right]+\kappa_2\boldsymbol{\sigma}_{n+1}(t,y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y))):=\tilde{\boldsymbol{B}}_2(\boldsymbol{\sigma}_{n+1})(t,y),\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}&\rho_n\partial_t\boldsymbol{v}_{n+1}-(\mu+\lambda-\xi)\nabla_y\mathrm{div}_y\boldsymbol{v}_{n+1}-(\mu+\xi)\Delta_y\boldsymbol{v}_{n+1}-2\xi\mathrm{curl}_y\boldsymbol{\sigma}_n=\rho_n\boldsymbol{V}\cdot\nabla_y\boldsymbol{v}_{n+1}\\ &+\boldsymbol{R}_1(\rho_n,\boldsymbol{v}_{n+1},\boldsymbol{\sigma}_{n+1})-\varrho_{n+1}\partial_t\boldsymbol{u}_{n+1}-\varrho_{n+1}\boldsymbol{u}_{n-1}\cdot\nabla_x\boldsymbol{u}_{n-1}-\rho_{n+1}(\boldsymbol{v}_n\cdot\nabla_x\boldsymbol{u}_{n-1}+\boldsymbol{u}_n\cdot\nabla_x\boldsymbol{v}_n)\\ &-p'(\rho_n)\nabla_x\varrho_n-\nabla_x\rho_n(p'(\rho_{n+1})-p'(\rho_n))):=\tilde{\boldsymbol{R}}_1^n,\end{aligned} \tag{6.16}$$

with boundary conditions

$$\begin{aligned}&\boldsymbol{v}_{n+1}\cdot\boldsymbol{n}|_{\Gamma_0}=\boldsymbol{v}_{n+1}\cdot(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y))=:\tilde{\boldsymbol{d}}_1(\boldsymbol{v}_{n+1}),\\ &[\mathbb{S}(\nabla\boldsymbol{v}_{n+1},\boldsymbol{\sigma}_n)\cdot\boldsymbol{n}]_{\tan}+\kappa_1[\boldsymbol{v}_{n+1}]_{\tan}|_{\Gamma_0}\\ =&\left[(\mu+\xi)\nabla_y\boldsymbol{v}_{n+1}(t,y)(\mathbb{I}-\nabla_x\boldsymbol{Y})+(\mu-\xi)(\mathbb{I}-\nabla_x^{\mathsf T}\boldsymbol{Y})\nabla_y^{\mathsf T}\boldsymbol{v}_{n+1}\right]\boldsymbol{n}(\boldsymbol{X}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y))\\ &+\left[(\mu+\xi)\nabla_y\boldsymbol{v}_{n+1}(t,y)+(\mu-\xi)\nabla_y^{\mathsf T}\boldsymbol{v}_{n+1}-2\xi A(\boldsymbol{\sigma}_n)\right](t,y)\left[(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\cdot\tau^k(\boldsymbol{X}(t,y))\right.\\ &\left.+\boldsymbol{n}(t,y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\right]+\kappa_1\boldsymbol{v}_{n+1}(t,y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y))):=\tilde{\boldsymbol{B}}_1(\boldsymbol{v}_{n+1},\boldsymbol{\sigma}_n)(t,y).\end{aligned} \tag{6.17}$$

We denote the convergence space as follows

$$H_n(T)=\|\boldsymbol{v}_n\|_{L^\infty(0,T;H^1(\Omega_t))}+\|\boldsymbol{v}_n\|_{L^2(0,T;H^2(\Omega_t))}+\|\boldsymbol{v}_{n,t}\|_{L^2(0,T;L^2(\Omega_t))}, \tag{6.18}$$

$$J_n(T)=\|\boldsymbol{\sigma}_n\|_{L^\infty(0,T;H^1(\Omega_t))}+\|\boldsymbol{\sigma}_n\|_{L^2(0,T;H^2(\Omega_t))}+\|\boldsymbol{\sigma}_{n,t}\|_{L^2(0,T;L^2(\Omega_t))}. \tag{6.19}$$

Then, we can derive that

$$\begin{aligned}&H_{n+1}(T)+J_{n+1}(T)\\ &\le C[\|(\tilde{\boldsymbol{R}}_1^n,\tilde{\boldsymbol{R}}_2^n)\|_{L^2(0,T;L^2(\Omega_t))}+\|(\tilde{\boldsymbol{B}}_1,\tilde{\boldsymbol{B}}_2)\|_{L^2(0,T;H^{\frac12}(\partial\Omega_t))}+\|(\tilde{\boldsymbol{d}}_1,\tilde{\boldsymbol{d}}_2)\|_{L^2(0,T;H^{\frac32}(\partial\Omega_t))}\\ &+\|\nabla_x\boldsymbol{v}_n,\nabla_x\boldsymbol{\sigma}_n\|_{L^2(0,T;L^2(\Omega_0))}].\end{aligned} \tag{6.20}$$

Moreover, the boundary data can be estimated

$$\|(\tilde{\boldsymbol{B}}_1, \tilde{\boldsymbol{B}}_2)\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega_t))} + \|(\tilde{\boldsymbol{d}}_1, \tilde{\boldsymbol{d}}_2)\|_{L^2(0,T;H^{\frac{3}{2}}(\partial\Omega_t))} \leq E(T)\|(\boldsymbol{v}_{n+1}, \boldsymbol{\sigma}_{n+1})\|_{L^2(0,T;H^2(\Omega_t))}. \quad (6.21)$$

Next, we give the bounds of $\tilde{\boldsymbol{R}}_1^n$ and $\tilde{\boldsymbol{R}}_2^n$.

Lemma 6.2. *For any $\eta > 0$, $\tilde{\boldsymbol{R}}_1^n$ and $\tilde{\boldsymbol{R}}_2^n$ defined in* (6.16) *and* (6.14) *satisfy*

$$\begin{aligned}\|(\tilde{\boldsymbol{R}}_1^n, \tilde{\boldsymbol{R}}_2^n)\|_{L^2(0,T;L^2(\Omega_t))} \leq & C[\|\varrho_{n+1}\|_{L^\infty(0,T;L^2(\Omega_t))} + \|\varrho_n\|_{L^\infty(0,T;L^2(\Omega_t))} \\ & + (\eta + TC(\eta) + E(T))(\|\nabla\varrho_n\|_{L^\infty(0,T;L^2(\Omega_t))} + H_n(T) + J_n(T))],\end{aligned} \quad (6.22)$$

where $E(t)$ is small for the small time.

Proof. By the same procedure as Lemma 6.1 and Proposition 6.1, we have

$$\|(\boldsymbol{R}_1(\rho_n, \boldsymbol{v}_{n+1}, \boldsymbol{\sigma}_{n+1}), \boldsymbol{R}_2(\rho_n, \boldsymbol{v}_{n+1}, \boldsymbol{\sigma}_{n+1}))\|_{L^2(0,T;H^1(\Omega_t))} \leq E(t)\|(\boldsymbol{v}_{n+1}, \boldsymbol{\sigma}_{n+1})\|_{L^2(0,T;H^3(\Omega_t))}.$$

For rest terms in $\tilde{\boldsymbol{R}}_1^n$ and $\tilde{\boldsymbol{R}}_2^n$, we only treat some typical terms, other terms can be estimated similarly. Simple calculation directly gives that

$$\begin{aligned}\|\varrho_{n+1}\partial_t\boldsymbol{w}_{n+1}\|_{L^2(0,T;L^2(\Omega_t))} \leq & \|\varrho_{n+1}\|_{L^\infty(0,T;L^2(\Omega_t))}\|\partial_t\boldsymbol{w}_{n+1}\|_{L^2(0,T;H^2(\Omega_t))} \\ \leq & C\|\varrho_{n+1}\|_{L^\infty(0,T;L^2(\Omega_t))},\end{aligned}$$

and

$$\begin{aligned}&\big\|\varrho_{n+1}\boldsymbol{u}_{n+1} \cdot \nabla_x\boldsymbol{w}_{n-1}\big\|_{L^2(0,T;L^2(\Omega_t))} \\ &\leq \big\|\varrho_{n+1}\big\|_{L^\infty(0,T;L^2)}\big\|\boldsymbol{u}_{n+1}\big\|_{L^\infty(0,T;H^2(\Omega_t))}\big\|\nabla_x\boldsymbol{w}_{n-1}\big\|_{L^2(0,T;H^2(\Omega_t))} \\ &\leq \big\|\varrho_{n+1}\big\|_{L^\infty(0,T;L^2)}\big\|\boldsymbol{u}_{n+1}\big\|_{L^\infty(0,T;H^2(\Omega_t))}(\epsilon\big\|\boldsymbol{w}_{n-1}\big\|_{L^2(0,T;H^3(\Omega_t))} + \sqrt{t}\big\|\boldsymbol{w}_{n-1}\big\|_{L^\infty(0,T;H^2(\Omega_t))}).\end{aligned}$$

Similarly, we derive

$$\begin{aligned}&\big\|\rho_{n+1}\boldsymbol{v}_n \cdot \nabla_x\boldsymbol{u}_{n-1}\big\|_{L^2(0,T;L^2(\Omega_t))} + \big\|\rho_{n+1}\boldsymbol{u}_n \cdot \nabla_x\boldsymbol{v}_n\big\|_{L^2(0,T;L^2(\Omega_t))} \\ &\leq \epsilon\big\|\boldsymbol{v}_n\big\|_{L^2(0,T;L^2(\Omega_t))} + C\sqrt{t}\big\|\boldsymbol{v}_n\big\|_{L^\infty(0,T;L^2(\Omega_t))}.\end{aligned}$$

For the pressure, we have

$$\|p'(\rho_n)\nabla_x\varrho_n\|_{L^2(0,T;L^2(\Omega_t))} \leq C\|\varrho_{n+1}\|_{L^\infty(0,T;H^1(\Omega_t))}[\eta\|\rho_n\|_{L^2(0,T;H^2(\Omega_t))} + tC(\eta)\|\rho_n\|_{L^\infty(0,T;L^2(\Omega_t))}],$$

and

$$\begin{aligned}&\big\|\nabla_x\rho_{n-1}(p'(\rho_n)) - p'(\rho_{n-1})\big\|_{L^2(0,T;L^2(\Omega_t))} \\ &\leq \|\nabla_x\rho_{n-1}\varrho_n\int_0^1 p''(s\rho_n + (1-s)\rho_{n-1})\mathrm{d}s\|_{L^2(0,T;L^2(\Omega_t))} \\ &\leq C\|\varrho_n\|_{L^\infty(0,T;L^2(\Omega_t))}\|\nabla_x\rho_{n-1}\|_{L^2(0,T;L^\infty(\Omega_t))}.\end{aligned}$$

Combining the above inequalities, we finish the proof. □

Now, we only should derive the bound of ϱ_n. The corresponding lemma is given in the below section, whose proof can be found in [25].

Lemma 6.3. *For any $\eta > 0$, the ϱ_{n+1} which solves* (6.9) *satisfy*

$$\|\varrho_{n+1}\|_{L^\infty(0,T;L^2(\Omega_t))} \leq (\eta + \sqrt{T}C(\eta))H_n(T), \quad \|\nabla_x\varrho_{n+1}\|_{L^\infty(0,T;L^2(\Omega_t))} \leq CH_n(T). \quad (6.23)$$

Finally, we show that $(\boldsymbol{u}_n, \boldsymbol{w}_n)$ are the Cauchy sequences.

Lemma 6.4. *There exists $T^* > 0$ and $0 < K < 1$ such that, for any $t \leq T^*$,*

$$H_n(t) + G_n(t) \leq K(H_{n-1}(t) + G_{n-1}(t)). \quad (6.24)$$

Proof. According to (6.20), (6.22) and (6.23), we have

$$H_n(t) + G_n(t) \leq (\eta + \sqrt{t}(\eta) + E(t))(H_n(t) + G_n(t) + H_{n-1}(t) + G_{n-1}(t)),$$

then we choose a small positive time $T^*(\leq T_*)$ to complete the proof. □

In this position, we can complete the proof of Theorem 1.1. The sequence of $\boldsymbol{u}_n$ and $\boldsymbol{w}_n$ strongly converge in the spaces of (6.18) and (6.19). Moreover, (6.7) implies that $(\boldsymbol{u}_n, \boldsymbol{w}_n)$ up to a subsequence weakly converges in $\mathcal{X}(T^*)$. Furthermore, (6.23), (3.7) and (3.8) show the convergence of the density ρ_n. Finally, we can conclude that the limit $(\rho, \boldsymbol{u}, \boldsymbol{w})$ satisfies (1.11).

7. Weak Formulation

In this section, we aim to establish the global existence of weak solutions to (1.1)-(1.9) for any finite energy initial data.

Due to the wide application of the penalization method in a series of papers [11, 12, 23, 22], we also use it to deal with system (1.1) with slip boundary conditions.

To utilize the existence of finite energy weak solutions in [2] and results of nonconstant viscosity coefficients in [9], we confine the system (1.1) to a fixed domain with singular forcing terms (7.1), and these terms yield the boundary conditions (1.8). Thanks to the derivations of (8.9) and (8.10), there are some new cancellations such that it is possible to derive the modified energy inequality (8.12).

Now we design a detailed scheme to obtain the global existence of weak solutions to system (1.1) with slip boundary conditions.

- To begin with, we introduce two singular forcing terms to deal with the slip boundary conditions
$$\frac{1}{\epsilon}\int_0^T\int_{\Gamma_t}(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}\boldsymbol{\varphi}\cdot\boldsymbol{n}\mathrm{d}\sigma\mathrm{d}t, \quad \frac{1}{\epsilon}\int_0^T\int_{\Gamma_t}(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\cdot\boldsymbol{n}\boldsymbol{\eta}\cdot\boldsymbol{n}\mathrm{d}\sigma\mathrm{d}t \tag{7.1}$$
for equations of momentum and micro-angular, respectively. Passing to the limit $\epsilon \to 0$, we derive the boundary conditions (1.8) and the fixed domain is divided by impermeable interface to a fluid domain and a solid domain.
- We also introduce a series of viscosity coefficients μ_ω, λ_ω, $c_{0,\omega}$, $c_{a,\omega}$, $c_{d,\omega}$ and ξ_ω as (8.2). Moreover, the artificial pressure is introduced by $p(\rho) + \delta\rho^\beta$, with $\beta > \{4, \gamma\}$ and $\delta > 0$, in (8.11).
- Furthermore, when parameters ϵ, δ, ω are fixed, we solve the penalized problem (8.1)-(8.7) in a fixed domain $B \in \mathbb{R}^3$ such that $\bar{\Omega}_\tau \in B$, $\forall \tau \in [0, T]$. Here, we use the existing results in [2] as long as we artificially choose magnetization to be zero and the conclusion in [2] concerning the nonconstant viscosity coefficients.
- It is similar to the Lemma 4.1 in [11] that when the initial density ρ_0 vanishes outside Ω_0, passage to the limit $\epsilon \to 0$ gives a two-phase system where the density vanishes in $((0,T)\times B)\backslash(\cup_{t\in(0,T)}\{t\}\times\Omega_t)$ under assumption of the L^2 regularity of the density.
- In the end, we perform the limit $\omega \to 0$ to show that the extra stresses vanish in the solid part. Eventually, the weak formulations of (1.1) are derived by the limit process $\delta \to 0$. It means that we deduce (7.2)-(7.6) from (8.4)-(8.6). We mention that the method of these limit processes become standard. We only make the effort to deduce some crucial estimates, sketch the outline of the proof, and neglect some tedious details.

Weak formulation of the continuity equation:

In the weak formulation, we assume that the density ρ is extended by zero outside the fluid domain, and $(1.1)_1$ is supposed to hold that

$$\int_{\Omega_\tau}(\rho\phi)(\tau,\cdot)\mathrm{d}x - \int_{\Omega_0}(\rho\phi)(0,\cdot)\mathrm{d}x = \int_0^\tau\int_{\Omega_t}(\rho\partial_t\phi + \rho\boldsymbol{u}\cdot\nabla_x\phi)\mathrm{d}x\mathrm{d}t, \tag{7.2}$$

$\forall \tau \in [0,T], \forall \phi \in C_c^\infty([0,T]\times\mathbb{R}^3)$.

Furthermore, $(1.1)_1$ is also fulfilled in the sense of renormalized

$$\begin{aligned}&\int_{\Omega_\tau}(b(\rho)\phi)(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}(b(\rho)\phi)(0,\cdot)\mathrm{d}x\\&=\int_0^\tau\int_{\Omega_t}(b(\rho)\partial_t\phi+b(\rho)\boldsymbol{u}\cdot\nabla_x\phi+(b(\rho)-b'(\rho)\rho)\mathrm{div}_x\boldsymbol{u}\phi)\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{7.3}$$

$\forall\tau\in[0,T],\forall\phi\in C_c^\infty([0,T]\times\mathbb{R}^3)$, function $b\in C^1[0,\infty), b(0)=0, b'(r)=0$ for large r.

Weak formulation of momentum equations:

The momentum equations $(1.1)_2$ are formed by an integrated identity

$$\begin{aligned}&\int_{\Omega_\tau}(\rho\boldsymbol{u}\cdot\boldsymbol{\varphi})(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}(\rho\boldsymbol{u}\cdot\boldsymbol{\varphi})(0,\cdot)\mathrm{d}x\\&=\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi}+\rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{\varphi}+p(\rho)\mathrm{div}_x\boldsymbol{\varphi}\\&-\mu\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{\varphi}-(\lambda+\mu)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{\varphi}-\xi\mathrm{curl}_x\boldsymbol{u}:\mathrm{curl}_x\boldsymbol{\varphi}+2\xi\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{7.4}$$

$\forall\boldsymbol{\varphi}\in C_c^\infty([0,T]\times\mathbb{R}^3)$ satisfying $\boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\Gamma_\tau}=0,\forall\tau\in[0,T]$.

Weak formulation of micro-rotation velocity equations:

The micro-rotation velocity equation $(1.1)_2$ are also given by an integrated identity

$$\begin{aligned}&\int_{\Omega_\tau}(\rho\boldsymbol{w}\cdot\boldsymbol{\eta})(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}(\rho\boldsymbol{w}\cdot\boldsymbol{\eta})(0,\cdot)\mathrm{d}x\\&=\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{w}\cdot\partial_t\boldsymbol{\eta}+\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\boldsymbol{\eta}-4\xi\boldsymbol{w}\cdot\boldsymbol{\eta}\\&-(c_a+c_d)\nabla_x\boldsymbol{w}:\nabla_x\boldsymbol{\eta}-(c_0+c_d-c_a)\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\boldsymbol{\eta}+2\xi\mathrm{curl}_x\boldsymbol{u}\cdot\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{7.5}$$

$\forall\boldsymbol{\eta}\in C_c^\infty([0,T]\times\mathbb{R}^3)$ satisfying $\boldsymbol{\eta}\cdot\boldsymbol{n}|_{\Gamma_\tau}=0,\forall\tau\in[0,T]$.

The impermeability conditions are satisfied in the sense of traces

$$\begin{aligned}&\boldsymbol{u}\in L^2(0,T;\mathbb{R}^3),\quad(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}|_{\Gamma_\tau}=0,\\&\boldsymbol{w}\in L^2(0,T;\mathbb{R}^3),\quad(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\cdot\boldsymbol{n}|_{\Gamma_\tau}=0,\quad\forall\tau\in[0,T].\end{aligned}\tag{7.6}$$

8. Penalization

8.1. **Weak formulation of penalized problem.** We choose a sufficiently large $R>0$ such that

$$\boldsymbol{V}|_{[0,T]\times\{|x|>R\}}=0,\quad\bar{\Omega}_0\in\{|x|<R\},\tag{8.1}$$

and the reference domain $B=\{|x|<2R\}$.

Then we extend the definition of viscosities on the whole reference domain

$$\begin{aligned}&(\mu_\omega,\lambda_\omega,c_{0,\omega},c_{a,\omega},c_{d,\omega},\xi_\omega)\in C_c^\infty([0,T]\times\mathbb{R}^3),\\&0<\underline{\mu}\le\mu_\omega(t,x)\le\mu,\quad\underline{\lambda}\le\lambda_\omega(t,x)\le\lambda,\quad0\le3\lambda_\omega+2\mu_\omega\le3\lambda+2\mu,\\&\underline{c_0}\le c_{0,\omega}(t,x)\le c_0,\quad0<\underline{c_a}\le c_{a,\omega}(t,x)\le c_a,\quad0<\underline{c_d}\le c_{d,\omega}(t,x)\le c_d,\\&0\le3c_{0,\omega}+2c_{d,\omega}\le3c_0+2c_d,\quad0<\underline{\xi}\le\xi_\omega(t,x)\le\xi,\quad\forall(t,x)\in[0,T]\times B,\\&\mu_\omega(\tau,\cdot)|_{\Omega_\tau}=\mu,\quad\lambda_\omega(\tau,\cdot)|_{\Omega_\tau}=\lambda,\\&c_{0,\omega}(\tau,\cdot)|_{\Omega_\tau}=c_0,\quad c_{a,\omega}(\tau,\cdot)|_{\Omega_\tau}=c_a,\\&c_{d,\omega}(\tau,\cdot)|_{\Omega_\tau}=c_d,\quad\xi_\omega(\tau,\cdot)|_{\Omega_\tau}=\xi,\quad\forall\tau\in[0,T].\end{aligned}\tag{8.2}$$

The initial data satisfy that

$$\begin{aligned}&\rho_0=\rho_{0,\delta},\quad \rho_{0,\delta}\geq 0,\quad \rho_{0.\delta}\not\equiv 0,\quad \rho_{0,\delta}|_{\mathbb{R}^3\setminus\Omega_0}=0,\quad \int_B(\rho_{0,\delta}^{\gamma}+\delta\rho_{0,\delta}^{\beta})\leq C,\\&(\rho\boldsymbol{u})_0=(\rho\boldsymbol{u})_{0,\delta},\quad (\rho\boldsymbol{u})_{0,\delta}=0,\quad a.e.\quad on\{x|\rho_{0,\delta}(x)=0\},\quad \int_{\Omega_0}\frac{1}{\rho_{0,\delta}}|(\rho\boldsymbol{u})_{0,\delta}|^2\mathrm{d}x\leq C,\\&(\rho\boldsymbol{w})_0=(\rho\boldsymbol{w})_{0,\delta},\quad (\rho\boldsymbol{w})_{0,\delta}=0,\quad a.e.\quad on\{x|\rho_{0,\delta}(x)=0\},\quad \int_{\Omega_0}\frac{1}{\rho_{0,\delta}}|(\rho\boldsymbol{w})_{0,\delta}|^2\mathrm{d}x\leq C.\end{aligned}\tag{8.3}$$

Notice that δ,ϵ,ω are positive parameters.

In this stage, we present the weak formulation of the penalized problem.

The continuity equation is formed by following identity,

$$\int_B(\rho\phi)(t,\cdot)\mathrm{d}x-\int_B\rho_{0,\delta}\phi(0,\cdot)\mathrm{d}x=\int_0^\tau\int_B(\rho\partial_t\phi+\rho\boldsymbol{u}\cdot\nabla_x\phi)\mathrm{d}x\mathrm{d}t,\tag{8.4}$$

$\forall\tau\in[0,T],\forall\phi\in C_c^\infty([0,T]\times\mathbb{R}^3)$.

Moreover, the momentum equations satisfy an integrated identity

$$\begin{aligned}&\int_B(\rho\boldsymbol{u}\cdot\boldsymbol{\varphi})(\tau,\cdot)\mathrm{d}x-\int_B(\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\mathrm{d}x-\frac{1}{\epsilon}\int_0^\tau\int_{\Gamma_t}((\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\boldsymbol{\varphi}\cdot\boldsymbol{n})\mathrm{d}\sigma\mathrm{d}t\\&=\int_0^\tau\int_B(\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi}+\rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{\varphi}+p(\rho)\mathrm{div}_x\boldsymbol{\varphi}+\delta\rho^\beta\mathrm{div}_x\boldsymbol{\varphi}\\&\quad-\mu_\omega\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{\varphi}-(\lambda_\omega+\mu_\omega)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{\varphi}-\xi_\omega\mathrm{curl}_x\boldsymbol{u}:\mathrm{curl}_x\boldsymbol{\varphi}+2\xi_\omega\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{8.5}$$

$\forall\tau\in[0,T],\forall\boldsymbol{\varphi}\in C_c^\infty([0,T]\times\mathbb{R}^3)$, and $\boldsymbol{u}\in L^2(0,T;W_0^{1,2}(B))$.

The micro-rotation velocity equation are given by an integrated identity

$$\begin{aligned}&\int_B(\rho\boldsymbol{w}\cdot\boldsymbol{\eta})(\tau,\cdot)\mathrm{d}x-\int_B(\rho\boldsymbol{w})_{0,\delta}\cdot\boldsymbol{\eta}(0,\cdot)\mathrm{d}x-\frac{1}{\epsilon}\int_0^\tau\int_{\Gamma_t}((\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}-\boldsymbol{w})\cdot\boldsymbol{n}\boldsymbol{\eta}\cdot\boldsymbol{n})\mathrm{d}\sigma\mathrm{d}t\\&=\int_0^\tau\int_B(\rho\boldsymbol{w}\cdot\partial_t\boldsymbol{\eta}+\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\boldsymbol{\eta}-4\xi_\omega\boldsymbol{w}\cdot\boldsymbol{\eta}\\&\quad-(c_a+c_d)\nabla_x\boldsymbol{w}:\nabla_x\boldsymbol{\eta}-(c_0+c_d-c_a)\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\boldsymbol{\eta}+2\xi_\omega\mathrm{curl}_x\boldsymbol{u}\cdot\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{8.6}$$

$\forall\tau\in[0,T],\forall\boldsymbol{\eta}\in C_c^\infty([0,T]\times\mathbb{R}^3)$, and $\boldsymbol{w}\in L^2(0,T;W_0^{1,2}(B))$.

These weak solutions satisfy the following energy inequality

$$\begin{aligned}&\int_B\left(\frac{1}{2}\rho|\boldsymbol{u}|^2+\frac{1}{2}\rho|\boldsymbol{w}|^2+\frac{1}{\gamma-1}\rho^\gamma+\frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot)\mathrm{d}x\\&+\int_0^\tau\int_B\big[(\lambda_\omega+\mu_\omega)(\mathrm{div}_x\boldsymbol{u})^2+\mu_\omega|\nabla_x\boldsymbol{u}|^2+(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})(\mathrm{div}_x\boldsymbol{w})^2\\&+(c_{a,\omega}+c_{d,\omega})|\nabla_x\boldsymbol{w}|^2+\xi_\omega|2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}|^2\big]\mathrm{d}x\mathrm{d}t\\&+\frac{1}{\epsilon}\int_0^\tau\int_{\Gamma_t}\left[(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}\boldsymbol{u}\cdot\boldsymbol{n}+(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\cdot\boldsymbol{n}\boldsymbol{w}\cdot\boldsymbol{n}\right]\mathrm{d}\sigma\mathrm{d}t\\&\leq\int_B\left(\frac{1}{2\rho_{0,\delta}}(|(\rho\boldsymbol{u})_{0,\delta}|^2+|(\rho\boldsymbol{w})_{0,\delta}|^2)+\frac{1}{\gamma-1}\rho_{0,\delta}^\gamma+\frac{\delta}{\beta-1}\rho_{0,\delta}^\beta\right)\mathrm{d}x.\end{aligned}\tag{8.7}$$

Definition 8.1. *We say that $[\rho,\boldsymbol{u},\boldsymbol{w}]$ is a finite energy weak solution to the penalized problem with* (8.2)-(8.3)*, if the following items hold.*

- $\rho>0$, *and* $\rho\in L^\infty(0,T;L^\gamma)\cap L^\infty(0,T;L^\beta)$,
- $(\boldsymbol{u},\nabla\boldsymbol{u},\boldsymbol{w},\nabla\boldsymbol{w})\in L^2((0,T)\times B)$, *and* $(\rho\boldsymbol{u},\rho\boldsymbol{w})\in L^\infty(0,T;L^1)$,
- *Integral forms* (8.4)-(8.6), *and* (8.7) *are satisfied.*

For this penalized problem, the existence of finite energy weak solutions can be established by the existence of weak solutions in [2] *and results of nonconstant viscosity coefficents in* [9]*. We only state the existence theorem without detailed proof.*

Theorem 8.1. *Suppose that $\boldsymbol{V} \in C^1(0,T;C_c^4(\mathbb{R}^3))$, and $\beta > \{4,\gamma\}$. Moreover, assume that initial data satisfies* (8.3)*. Then there exists a weak solution with finite energy on any time interval to the penalized problem in the sense of Definition* 8.1*.*

Our goal is to get the desired results from the limit processes $\epsilon \to 0$, $\omega \to 0$ and $\delta \to 0$.

8.2. **A modified energy inequality and uniform bounds.** Firstly, the mass of fluids is conserved. Precisely, it holds that

$$\int_B \rho(\tau,\cdot)\mathrm{d}x = \int_B \rho_{0,\delta}\mathrm{d}x = \int_{\Omega_0} \rho_{0,\delta}\mathrm{d}x, \tag{8.8}$$

$\forall \tau \in [0,T]$.

Due to the fact that $\boldsymbol{V}$ vanishes on the boundary ∂B, we choose it as a test function in (8.5). Then we derive that

$$\begin{aligned} &\int_B (\rho\boldsymbol{u}\cdot\boldsymbol{V})(\tau,\cdot)\mathrm{d}x - \int_B (\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{V}(0,\cdot)\mathrm{d}x - \frac{1}{\epsilon}\int_0^\tau\int_{\Gamma_t}((\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\boldsymbol{V}\cdot\boldsymbol{n})\mathrm{d}\sigma\mathrm{d}t \\ &= \int_0^\tau\int_B (\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{V} + \rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{V} + p(\rho)\mathrm{div}_x\boldsymbol{V} + \delta\rho^\beta\mathrm{div}_x\boldsymbol{V} \\ &\quad - \mu_\omega\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{V} - (\lambda_\omega+\mu_\omega)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{V} - \xi_\omega\mathrm{curl}_x\boldsymbol{u}:\mathrm{curl}_x\boldsymbol{V} + 2\xi_\omega\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{V})\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{8.9}$$

Moreover, we choose $\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}$ as a test function in (8.6) to deduce that

$$\begin{aligned} &\frac{1}{2}\int_B (\rho\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{V})(\tau,\cdot)\mathrm{d}x - \frac{1}{2}\int_B (\rho\boldsymbol{w})_{0,\delta}\cdot\mathrm{curl}_x\boldsymbol{V}(0,\cdot)\mathrm{d}x \\ &\quad - \frac{1}{2\epsilon}\int_0^\tau\int_{\Gamma_t}((\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}-\boldsymbol{w})\cdot\boldsymbol{n}\mathrm{curl}_x\boldsymbol{V}\cdot\boldsymbol{n})\mathrm{d}\sigma\mathrm{d}t \\ &= \int_0^\tau\int_B (\frac{1}{2}\rho\boldsymbol{w}\cdot\partial_t\mathrm{curl}_x\boldsymbol{V} + \frac{1}{2}\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\mathrm{curl}_x\boldsymbol{V} - 2\xi_\omega\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{V} \\ &\quad - \frac{1}{2}(c_{a,\omega}+c_{d,\omega})\nabla_x\boldsymbol{w}:\nabla_x\mathrm{curl}_x\boldsymbol{V} - \frac{1}{2}(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\mathrm{curl}_x\boldsymbol{V} \\ &\quad + \xi_\omega\mathrm{curl}_x\boldsymbol{u}\cdot\mathrm{curl}_x\boldsymbol{V})\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{8.10}$$

Combining (8.9) with (8.10), some new cancellations give that

$$\begin{aligned} &\int_B (\rho\boldsymbol{u}\cdot\boldsymbol{V})(\tau,\cdot)\mathrm{d}x + \frac{1}{2}\int_B (\rho\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{V})(\tau,\cdot)\mathrm{d}x \\ &\quad - \int_B (\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{V}(0,\cdot)\mathrm{d}x - \frac{1}{2}\int_B (\rho\boldsymbol{w})_{0,\delta}\cdot\mathrm{curl}_x\boldsymbol{V}(0,\cdot)\mathrm{d}x \\ &\quad - \frac{1}{\epsilon}\int_0^\tau\int_{\Gamma_t}((\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\boldsymbol{V}\cdot\boldsymbol{n})\mathrm{d}\sigma\mathrm{d}t - \frac{1}{2\epsilon}\int_0^\tau\int_{\Gamma_t}((\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}-\boldsymbol{w})\cdot\boldsymbol{n}\mathrm{curl}_x\boldsymbol{V}\cdot\boldsymbol{n})\mathrm{d}\sigma\mathrm{d}t \\ &= \int_0^\tau\int_B (\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{V} + \frac{1}{2}\rho\boldsymbol{w}\cdot\partial_t\mathrm{curl}_x\boldsymbol{V} + \rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{V} + \frac{1}{2}\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\mathrm{curl}_x\boldsymbol{V} \\ &\quad + p(\rho)\mathrm{div}_x\boldsymbol{V} + \delta\rho^\beta\mathrm{div}_x\boldsymbol{V} - \mu_\omega\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{V} - (\lambda_\omega+\mu_\omega)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{V} \\ &\quad - \frac{1}{2}(c_{a,\omega}+c_{d,\omega})\nabla_x\boldsymbol{w}:\nabla_x\mathrm{curl}_x\boldsymbol{V} - \frac{1}{2}(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\mathrm{curl}_x\boldsymbol{V})\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{8.11}$$

We put (8.7) and (8.11) together to get a modified energy inequality as follows

$$\begin{aligned}
&\int_B \left(\frac{1}{2}\rho|\boldsymbol{u}|^2+\frac{1}{2}\rho|\boldsymbol{w}|^2+\frac{1}{\gamma-1}\rho^\gamma+\frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot)\mathrm{d}x\\
&+\int_0^\tau\int_B \big[(\lambda_\omega+\mu_\omega)(\mathrm{div}_x\boldsymbol{u})^2+\mu_\omega|\nabla_x\boldsymbol{u}|^2\\
&+(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})(\mathrm{div}_x\boldsymbol{w})^2+(c_{a,\omega}+c_{d,\omega})|\nabla_x\boldsymbol{w}|^2+\xi_\omega|2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}|^2\big]\,\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{\epsilon}\int_0^\tau\int_{\Gamma_t}\left[|(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}|^2+|(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\cdot\boldsymbol{n}|^2\right]\mathrm{d}\sigma\mathrm{d}t\\
&\le\int_B\left(\frac{1}{2\rho_{0,\delta}}(|(\rho\boldsymbol{u})_{0,\delta}|^2+|(\rho\boldsymbol{w})_{0,\delta}|^2)+\frac{1}{\gamma-1}\rho_{0,\delta}^\gamma+\frac{\delta}{\beta-1}\rho_{0,\delta}^\beta\right)\mathrm{d}x\\
&\quad+\int_B(\rho\boldsymbol{u}\cdot\boldsymbol{V})(\tau,\cdot)\mathrm{d}x+\frac{1}{2}\int_B(\rho\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{V})(\tau,\cdot)\mathrm{d}x\\
&\quad-\int_B(\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{V}(0,\cdot)\mathrm{d}x-\frac{1}{2}\int_B(\rho\boldsymbol{w})_{0,\delta}\cdot\mathrm{curl}_x\boldsymbol{V}(0,\cdot)\mathrm{d}x\\
&\quad-\int_0^\tau\int_B(\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{V}+\frac{1}{2}\rho\boldsymbol{w}\cdot\partial_t\mathrm{curl}_x\boldsymbol{V}+\rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{V}+\frac{1}{2}\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\mathrm{curl}_x\boldsymbol{V}\\
&\quad+p(\rho)\mathrm{div}_x\boldsymbol{V}+\delta\rho^\beta\mathrm{div}_x\boldsymbol{V}-\mu_\omega\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{V}-(\lambda_\omega+\mu_\omega)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{V}\\
&\quad-\frac{1}{2}(c_{a,\omega}+c_{d,\omega})\nabla_x\boldsymbol{w}:\nabla_x\mathrm{curl}_x\boldsymbol{V}-\frac{1}{2}(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\mathrm{curl}_x\boldsymbol{V})\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{8.12}$$

Based on the regularity of $\boldsymbol{V}$, Hölder's and Young's inequalities, (8.8), then simple calculation gives the following estimates

$$\int_B(\rho\boldsymbol{u}\cdot\boldsymbol{V})(\tau,\cdot)\mathrm{d}x\le\|\boldsymbol{V}\|_{L^\infty(B)}\int_B\rho^{\frac12}\rho^{\frac12}|\boldsymbol{u}|\mathrm{d}x\le C\|\boldsymbol{V}\|^2_{L^\infty(B)}+\frac14\int_B\rho|\boldsymbol{u}|^2\mathrm{d}x,$$

$$\frac12\int_B(\rho\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{V})(\tau,\cdot)\mathrm{d}x\le\|\mathrm{curl}_x\boldsymbol{V}\|_{L^\infty(B)}\int_B\rho^{\frac12}\rho^{\frac12}|\boldsymbol{w}|\mathrm{d}x\le C\|\nabla_x\boldsymbol{V}\|^2_{L^\infty(B)}+\frac14\int_B\rho|\boldsymbol{w}|^2\mathrm{d}x,$$

$$\begin{aligned}
&\int_0^\tau\int_B|\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{V}|\mathrm{d}x\mathrm{d}t+\frac12\int_0^\tau\int_B|\rho\boldsymbol{w}\cdot\partial_t\mathrm{curl}_x\boldsymbol{V}|\mathrm{d}x\mathrm{d}t\\
&\le\|\partial_t\boldsymbol{V}\|_{L^\infty(0,\tau;L^\infty(B))}\int_0^\tau\int_B\rho^{\frac12}\rho^{\frac12}|\boldsymbol{u}|\mathrm{d}x\mathrm{d}t+\frac12\|\partial_t\mathrm{curl}_x\boldsymbol{V}\|_{L^\infty(0,\tau;L^\infty(B))}\int_0^\tau\int_B\rho^{\frac12}\rho^{\frac12}|\boldsymbol{w}|\mathrm{d}x\mathrm{d}t\\
&\le C\|\boldsymbol{V}_t\|^2_{L^\infty(0,\tau;L^\infty(B))}+C\|\nabla_x\boldsymbol{V}_t\|^2_{L^\infty(0,\tau;L^\infty(B))}+\int_0^\tau\int_B\rho|\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_B\rho|\boldsymbol{w}|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

$$\begin{aligned}
&\int_0^\tau\int_B(\rho|(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{V}|+\frac12\rho|(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\mathrm{curl}_x\boldsymbol{V}|)\mathrm{d}x\mathrm{d}t\\
&\le C(\|\nabla_x\boldsymbol{V}\|_{L^\infty(0,\tau;L^\infty(B))}+\|\nabla_x^2\boldsymbol{V}\|_{L^\infty(0,\tau;L^\infty(B))})(\int_0^\tau\int_B\rho|\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_B\rho|\boldsymbol{w}|^2\mathrm{d}x\mathrm{d}t),
\end{aligned}$$

$$\begin{aligned}
&\int_0^\tau\int_B(|p(\rho)\mathrm{div}_x\boldsymbol{V}|+|\delta\rho^\beta\mathrm{div}_x\boldsymbol{V}|)\mathrm{d}x\mathrm{d}t\\
&\le C\|\boldsymbol{V}\|_{L^\infty(0,\tau;L^\infty(B))}\int_0^\tau\int_B(P(\rho)+\delta\rho^\beta)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

$$
\begin{aligned}
&\int_0^\tau\int_B(\mu_\omega|\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{V}|+(\lambda_\omega+\mu_\omega)|\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{V}|\\
&+\frac{1}{2}(c_{a,\omega}+c_{d,\omega})|\nabla_x\boldsymbol{w}:\nabla_x\mathrm{curl}_x\boldsymbol{V}|+\frac{1}{2}(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})|\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\mathrm{curl}_x\boldsymbol{V}|)\mathrm{d}x\mathrm{d}t\\
&\leq\frac{1}{2}\int_0^\tau\int_B\left[(\lambda_\omega+\mu_\omega)(\mathrm{div}_x\boldsymbol{u})^2+\mu_\omega|\nabla_x\boldsymbol{u}|^2+(c_{a,\omega}+c_{d,\omega})(\mathrm{div}_x\boldsymbol{w})^2\right.\\
&\left.+(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})|\nabla_x\boldsymbol{w}|^2\right]\mathrm{d}x\mathrm{d}t+C(\left\|\nabla_x\boldsymbol{V}\right\|^2_{L^2(0,\tau;L^2(B))}+\left\|\nabla_x^2\boldsymbol{V}\right\|^2_{L^2(0,\tau;L^2(B))}).
\end{aligned}
$$

Plugging above estimates into (8.12), and using Grönwall's inequality, we derive that

$$
\begin{aligned}
&\sup_{t\in(0,T)}(\left\|\rho^{\frac{1}{2}}\boldsymbol{u}\right\|_{L^2(B)}+\left\|\rho^{\frac{1}{2}}\boldsymbol{w}\right\|_{L^2(B)})\leq C,\\
&\sup_{t\in(0,T)}(\left\|\rho\right\|^\gamma_{L^\gamma(B)}+\delta\left\|\rho\right\|^\beta_{L^\beta(B)})\leq C,\\
&\int_0^T\int_B\left[(\lambda_\omega+\mu_\omega)(\mathrm{div}_x\boldsymbol{u})^2+\mu_\omega|\nabla_x\boldsymbol{u}|^2\right]\mathrm{d}x\mathrm{d}t\leq C,\\
&\int_0^T\int_B\left[(c_{a,\omega}+c_{d,\omega})(\mathrm{div}_x\boldsymbol{w})^2+(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})|\nabla_x\boldsymbol{w}|^2\right]\mathrm{d}x\mathrm{d}t\leq C,\\
&\int_0^T\int_B\xi_\omega|2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t\leq C,\\
&\int_0^T\int_{\Gamma_t}\left(|(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}|^2+|(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\cdot\boldsymbol{n}|^2\right)\leq\epsilon C.
\end{aligned}
\tag{8.13}
$$

Notice that the above constant C is independent of parameters ϵ, δ and ω, so we call above estimates as uniform bounds which play a crucial role in singular limits.

8.3. **Pressure estimates.** We use Bogovskii operator $\mathcal{B}$ to improve the estimates of the density. Firstly, we present some properties borrowed from [13].

$\mathcal{B}:\left\{f\in L^p(\tilde{K})\right\}\mapsto W_0^{1,p}(\tilde{K})$ is a bounded linear operator, and satisfies $\|\mathcal{B}[f]\|_{W_0^{1,p}(\tilde{K})}\leq C(p)\|f\|_{L^p(\tilde{K})}$ for any $1<p<+\infty$. Moreover, if $f\in L^p(\tilde{K})$ can be written in the form $f=\mathrm{div}_x g$ for a certain $g\in L^r(\tilde{K}), g\cdot n|_{\partial\tilde{K}}=0$, then it holds that $\|\mathcal{B}[f]\|_{L^r(\tilde{K})}\leq C(r)\|g\|_{L^r(\tilde{K})}$.

Lemma 8.1. *There is a constant $C(\delta,K)$, independent of ϵ, such that*

$$
\int\int_K(\rho^{\gamma+1}+\delta\rho^{\beta+1})\mathrm{d}x\mathrm{d}t\leq C(\delta,K). \tag{8.14}
$$

Proof. We choose $\psi\in\mathcal{D}(0,T),\bar{m}=\frac{1}{|\tilde{K}|}\int_{\tilde{K}}\rho\mathrm{d}x$ and $\phi(t,x)=\psi(t)\mathcal{B}(\rho-\bar{m})$, then take $\phi(t,x)$ as a test function for (7.4) to obtain

$$
\begin{aligned}
&\int\int_K \psi(t)(\rho^{\gamma+1}+\delta\rho^{\beta+1})\mathrm{d}x\mathrm{d}t \\
&=\bar{m}\int\int_K \psi(t)(\rho^{\gamma}+\delta\rho^{\beta})\mathrm{d}x\mathrm{d}t+(\mu_\omega+\lambda_\omega)\int\int_K \psi(t)\rho\mathrm{div}_x\boldsymbol{u}\mathrm{d}x\mathrm{d}t \\
&-\int\int_K \psi_t(t)\rho\boldsymbol{u}\cdot\mathcal{B}[\rho-\bar{m}]\mathrm{d}x\mathrm{d}t+\mu_\omega\int\int_K \psi(t)\nabla_x\boldsymbol{u}\cdot\nabla_x\mathcal{B}[\rho-\bar{m}]\mathrm{d}x\mathrm{d}t \\
&-\int\int_K \psi(t)\rho\boldsymbol{u}\otimes\boldsymbol{u}\cdot\nabla_x\mathcal{B}[\rho-\bar{m}]]\mathrm{d}x\mathrm{d}t+\int\int_K \psi(t)\rho\boldsymbol{u}\cdot\mathcal{B}[\mathrm{div}_x(\rho\boldsymbol{u})]\mathrm{d}x\mathrm{d}t \\
&-\xi_\omega\int\int_K \psi(t)\mathrm{curl}_x\boldsymbol{u}:\mathrm{curl}_x\mathcal{B}[\rho-\bar{m}]\mathrm{d}x\mathrm{d}t+2\xi_\omega\int\int_K \psi(t)\boldsymbol{w}\cdot\mathrm{curl}_x\mathcal{B}[\rho-\bar{m}]\mathrm{d}x\mathrm{d}t \\
&:=\sum_{i=1}^{8}\mathcal{J}_i,
\end{aligned}
\tag{8.15}
$$

for any compact $K=I\times\tilde{K}\subset[0,T]\times B$ such that $K\cap\left(\cup_{\tau\in[0,T]}(\{\tau\}\times\Gamma_\tau)\right)=\emptyset$.

By using the same method in [13], we can estimate terms $\sum_{i=1}^{6}\mathcal{J}_i$. To avoid tedious calculations, the details are omitted.

For term $\mathcal{J}_7$, it holds that

$$
\begin{aligned}
|\mathcal{J}_7| &\le \xi_\omega\|\psi(t)\|_{L^\infty(I)}\|\mathrm{curl}_x\boldsymbol{u}\|_{L^2(I;L^2(\bar{K}))}\|\mathrm{curl}_x\mathcal{B}[\rho-\bar{m}]\|_{L^2(I;L^2(\bar{K}))} \\
&\le C\|\psi(t)\|_{L^\infty(I)}\|\boldsymbol{u}\|_{L^2(I;H^1(\bar{K}))}\|\rho\|_{L^2(I;L^2(\bar{K}))} \\
&\le C.
\end{aligned}
\tag{8.16}
$$

The term $\mathcal{J}_8$ has following estimate

$$
\begin{aligned}
|\mathcal{J}_8| &\le \xi_\omega\|\psi(t)\|_{L^\infty(I)}\|\boldsymbol{w}\|_{L^2(I;L^2(\bar{K}))}\|\mathcal{B}[\rho-\bar{m}]\|_{L^2(I;L^2(\bar{K}))} \\
&\le C\|\psi(t)\|_{L^\infty(I)}\|\boldsymbol{u}\|_{L^2(I;L^2(\bar{K}))}\|\rho\|_{L^2(I;L^{\frac{6}{5}}(\bar{K}))} \\
&\le C.
\end{aligned}
\tag{8.17}
$$

Obviously, we sum up above estimates to get the estimate (8.14).

□

8.4. **Singular limits.** We perform sucessively the singular limits $\epsilon\to 0$, $\omega\to 0$ and $\delta\to 0$.

• Penalization limit ($\epsilon\to 0$).

When parameters ω and δ remain fixed, we consider the limit $\epsilon\to 0$. $\{\rho_\epsilon,\boldsymbol{u}_\epsilon,\boldsymbol{w}_\epsilon\}$ are the corresponding weak solutions to the perturbed problem obtained in the Section 8. We use estimates $(8.13)_2$, $(8.13)_3$, $(8.13)_4$ and continuity equation (8.4) to get

$$
\rho_\epsilon\to\rho \quad in \quad C_{weak}([0,T];L^\gamma(B)),
\tag{8.18}
$$

$$
\boldsymbol{u}_\epsilon\to\boldsymbol{u} \quad weakly \quad in \quad L^2([0,T];W_0^{1,2}(B)),
\tag{8.19}
$$

and

$$
\boldsymbol{w}_\epsilon\to\boldsymbol{w} \quad weakly \quad in \quad L^2([0,T];W^{1,2}(B)).
\tag{8.20}
$$

Moverover, we use $(8.13)_6$ to derive that

$$
\begin{aligned}
&(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}(\tau,\cdot)|_{\Gamma_\tau}=0, \\
&(\boldsymbol{w}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\cdot\boldsymbol{n}(\tau,\cdot)|_{\Gamma_\tau}=0, \quad for \quad a.a. \quad \tau\in[0,T].
\end{aligned}
\tag{8.21}
$$

Then, $(8.13)_1$, $(8.13)_2$ and the compact embedding $L^\gamma(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$ give that

$$\begin{aligned} &\rho_\epsilon \boldsymbol{u}_\epsilon \to \rho \boldsymbol{u} \quad weakly-(*) \quad in \quad L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(B)),\\ &\rho_\epsilon \boldsymbol{w}_\epsilon \to \rho \boldsymbol{w} \quad weakly-(*) \quad in \quad L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(B)), \end{aligned} \tag{8.22}$$

together with the embedding $W_0^{1,2}(B) \hookrightarrow L^6(B)$,

$$\begin{aligned} &\rho_\epsilon \boldsymbol{u}_\epsilon \otimes \boldsymbol{u}_\epsilon \to \overline{\rho \boldsymbol{u} \otimes \boldsymbol{u}} \quad weakly \quad in \quad L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(B)),\\ &\rho_\epsilon \boldsymbol{w}_\epsilon \otimes \boldsymbol{u}_\epsilon \to \overline{\rho \boldsymbol{w} \otimes \boldsymbol{u}} \quad weakly \quad in \quad L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(B)). \end{aligned} \tag{8.23}$$

The momentum equations (8.5) and micro-rotation velocity equations (8.6) give that

$$\begin{aligned} &\rho_\epsilon \boldsymbol{u}_\epsilon \to \rho \boldsymbol{u} \quad in \quad C_{weak}([T_1,T_2];L^{\frac{2\gamma}{\gamma+1}}(O)),\\ &\rho_\epsilon \boldsymbol{w}_\epsilon \to \rho \boldsymbol{w} \quad in \quad C_{weak}([T_1,T_2];L^{\frac{2\gamma}{\gamma+1}}(O)), \end{aligned} \tag{8.24}$$

for any

$$(T_1,T_2)\times O \subset [0,T]\times B, \quad [T_1,T_2]\times \bar{O} \cap \big(\cup_{\tau\in[0,T]}(\{\tau\}\times\Gamma_\tau)\big) = \emptyset. \tag{8.25}$$

Lastly, with $L^{\frac{2\gamma}{\gamma+1}}(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$ in hand, we obtain that

$$\begin{aligned} &\overline{\rho \boldsymbol{u}\otimes \boldsymbol{u}} = \rho \boldsymbol{u}\otimes \boldsymbol{u},\\ &\overline{\rho \boldsymbol{w}\otimes \boldsymbol{u}} = \rho \boldsymbol{w}\otimes \boldsymbol{u}, \quad a.a. \quad in \quad (0,T)\times B. \end{aligned} \tag{8.26}$$

By the same method in [9], we can establish following identity

$$\overline{p_\delta(\rho)T_k(\rho)} - \overline{p_\delta(\rho)}\;\overline{T_k(\rho)} = (\lambda_\omega + 2\mu_\omega)(\overline{T_k(\rho)\mathrm{div}_x \boldsymbol{u}} - \overline{T_k(\rho)}\mathrm{div}_x \boldsymbol{u}), \tag{8.27}$$

where $p_\delta(\rho) = p(\rho) + \delta\rho^\beta$ and $T_k(\rho) = \min\{\rho, k\}$. Precisely, (8.27) holds on compact set $K \subset [0,T]\times B$ satisfying $K \cap \big(\cup_{\tau\in[0,T]}(\{\tau\}\times\Gamma_\tau\big) = \emptyset$.

Furthermore, with the oscillations defect measure defined in [13]

$$\mathbf{osc}_q[\rho_\epsilon \to \rho](K) = \sup_{k\geq 0}\left(\limsup_{\epsilon\to 0}\int_K |T_k(\rho_\epsilon) - T_k(\rho)|^q \mathrm{d}x\mathrm{d}t\right), \tag{8.28}$$

we use (8.27) to derive that

$$\mathbf{osc}_{\gamma+1}[\rho_\epsilon \to \rho](K) \leq C(\omega) < +\infty, \tag{8.29}$$

where the consant $C(\omega)$ is independent of K. So it holds that

$$\mathbf{osc}_{\gamma+1}[\rho_\epsilon \to \rho]([0,T]\times B) \leq C(\omega) < +\infty, \tag{8.30}$$

together with the same procedure in [8], we get

$$\rho_\epsilon \to \rho \quad a.a. \quad in \quad (0,T)\times B. \tag{8.31}$$

Passing to the limit in (8.4), we get

$$\int_B (\rho\phi)(t,\cdot)\mathrm{d}x - \int_B \rho_{0,\delta}\phi(0,\cdot)\mathrm{d}x = \int_0^\tau \int_B (\rho\partial_t\phi + \rho\boldsymbol{u}\cdot\nabla_x\phi)\mathrm{d}x\mathrm{d}t, \tag{8.32}$$

$\forall \tau \in [0,T], \forall \phi \in C_c^\infty([0,T]\times\mathbb{R}^3)$.

Passing to the limit in (8.6), we obtain

$$\begin{aligned} &\int_B (\rho\boldsymbol{w}\cdot\boldsymbol{\eta})(\tau,\cdot)\mathrm{d}x - \int_B (\rho\boldsymbol{w})_{0,\delta}\cdot\boldsymbol{\eta}(0,\cdot)\mathrm{d}x\\ &= \int_0^\tau \int_B (\rho\boldsymbol{w}\cdot\partial_t\boldsymbol{\eta} + \rho(\boldsymbol{u}\otimes\boldsymbol{w}) : \nabla_x\boldsymbol{\eta} - 4\xi_\omega \boldsymbol{w}\cdot\boldsymbol{\eta}\\ &\quad - (c_a + c_d)\nabla_x\boldsymbol{w} : \nabla_x\boldsymbol{\eta} - (c_0 + c_d - c_a)\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\boldsymbol{\eta} + 2\xi_\omega \mathrm{curl}_x\boldsymbol{u}\cdot\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{8.33}$$

$\forall \tau \in [0,T], \forall \boldsymbol{\eta} \in C_c^\infty([0,T]\times\mathbb{R}^3)$.

With only local estimate (8.14) in hand, we shall choose a specific class of test functions which were construted in [11] as follows

$$\boldsymbol{\varphi} \in C^1(([0,T]; W_0^{1,\infty}(B)), \operatorname{supp}[\operatorname{div}_x \boldsymbol{\varphi}(\tau,\cdot)] \cap \Gamma_\tau = \emptyset, \boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\Gamma_\tau} = 0, \forall \tau \in [0,T], \tag{8.34}$$

then the limit of the momentum equations holds that

$$\begin{aligned}
&\int_B (\rho\boldsymbol{u}\cdot\boldsymbol{\varphi})(\tau,\cdot)\mathrm{d}x - \int_B (\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\mathrm{d}x\\
&= \int_0^\tau \int_B (\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \rho(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\boldsymbol{\varphi} + p(\rho)\operatorname{div}_x\boldsymbol{\varphi} + \delta\rho^\beta \operatorname{div}_x\boldsymbol{\varphi}\\
&\quad - \mu_\omega \nabla_x \boldsymbol{u} : \nabla_x\boldsymbol{\varphi} - (\lambda_\omega + \mu_\omega)\operatorname{div}_x\boldsymbol{u}\operatorname{div}_x\boldsymbol{\varphi} - \xi_\omega \operatorname{curl}_x\boldsymbol{u} : \operatorname{curl}_x\boldsymbol{\varphi} + 2\xi_\omega\boldsymbol{w}\cdot\operatorname{curl}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{8.35}$$

Additionally, the limit solution $(\rho, \boldsymbol{u})$ also satisfies the renormalized equation (7.3).

To get rid of the density-dependent terms in (8.33) and (8.35) supported by the "solid" part $((0,T)\times B)\backslash Q_T$, we shall choose a specific intial data $\rho_{0,\delta}$ as Lemma 4.1 in [11]. We emphasize that Lemma 4.1 allows us to get rid the terms with the density on the part $B\backslash\Omega_t$, and the complete Lemma 4.1 is presented in the Appendix. We mention that test functions in (8.34) could also be extended by the density argument, for details see Section 4.3.1, [11].

Finally, (8.33) and (8.35) reduce to

$$\begin{aligned}
&\int_{\Omega_\tau} (\rho\boldsymbol{w}\cdot\boldsymbol{\eta})(\tau,\cdot)\mathrm{d}x - \int_{\Omega_0} (\rho\boldsymbol{w})_{0,\delta}\cdot\boldsymbol{\eta}(0,\cdot)\mathrm{d}x\\
&= \int_0^\tau \int_{\Omega_t} (\rho\boldsymbol{w}\cdot\partial_t\boldsymbol{\eta} + \rho(\boldsymbol{u}\otimes\boldsymbol{w}) : \nabla_x\boldsymbol{\eta} - 4\xi_\omega\boldsymbol{w}\cdot\boldsymbol{\eta}\\
&\quad - (c_a+c_d)\nabla_x\boldsymbol{w} : \nabla_x\boldsymbol{\eta} - (c_0+c_d-c_a)\operatorname{div}_x\boldsymbol{w}\operatorname{div}_x\boldsymbol{\eta} + 2\xi_\omega\operatorname{curl}_x\boldsymbol{u}\cdot\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t\\
&\quad + \int_0^\tau \int_{B\backslash\Omega_t} (-4\xi_\omega\boldsymbol{w}\cdot\boldsymbol{\eta} - (c_a+c_d)\nabla_x\boldsymbol{w} : \nabla_x\boldsymbol{\eta} - (c_0+c_d-c_a)\operatorname{div}_x\boldsymbol{w}\operatorname{div}_x\boldsymbol{\eta} + 2\xi_\omega\operatorname{curl}_x\boldsymbol{u}\cdot\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{8.36}$$

$\forall\tau\in[0,T], \forall\boldsymbol{\eta}\in C_c^\infty([0,T]\times\mathbb{R}^3)$,

and

$$\begin{aligned}
&\int_{\Omega_\tau} (\rho\boldsymbol{u}\cdot\boldsymbol{\varphi})(\tau,\cdot)\mathrm{d}x - \int_{\Omega_0} (\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\mathrm{d}x\\
&= \int_0^\tau \int_{\Omega_t} (\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \rho(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\boldsymbol{\varphi} + p(\rho)\operatorname{div}_x\boldsymbol{\varphi} + \delta\rho^\beta\operatorname{div}_x\boldsymbol{\varphi}\\
&\quad - \mu_\omega\nabla_x\boldsymbol{u} : \nabla_x\boldsymbol{\varphi} - (\lambda_\omega+\mu_\omega)\operatorname{div}_x\boldsymbol{u}\operatorname{div}_x\boldsymbol{\varphi} - \xi_\omega\operatorname{curl}_x\boldsymbol{u} : \operatorname{curl}_x\boldsymbol{\varphi} + 2\xi_\omega\boldsymbol{w}\cdot\operatorname{curl}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t\\
&\quad + \int_0^\tau \int_{B\backslash\Omega_t} (-\mu_\omega\nabla_x\boldsymbol{u} : \nabla_x\boldsymbol{\varphi} - (\lambda_\omega+\mu_\omega)\operatorname{div}_x\boldsymbol{u}\operatorname{div}_x\boldsymbol{\varphi} - \xi_\omega\operatorname{curl}_x\boldsymbol{u} : \operatorname{curl}_x\boldsymbol{\varphi} + 2\xi_\omega\boldsymbol{w}\cdot\operatorname{curl}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{8.37}$$

$\forall\tau\in[0,T], \forall\boldsymbol{\varphi}\in C_c^\infty([0,T]\times\mathbb{R}^3)$.

• Vanishing viscosity limit ($\omega\to 0$).

To get rid of the last integral terms in (8.47) and (8.46), we choose the viscosity coefficients

$$\mu_\omega = \begin{cases} \mu = constant > 0, & in \quad Q_T,\\ \mu_\omega \to 0, \quad a.a. & in \quad ((0,T)\times B)\backslash Q_T, \end{cases} \tag{8.38}$$

$$\lambda_\omega = \begin{cases} \lambda = constant, & in \quad Q_T,\\ \lambda_\omega \to 0, \quad a.a. & in \quad ((0,T)\times B)\backslash Q_T, \end{cases} \tag{8.39}$$

$$\xi_\omega = \begin{cases} \xi = constant > 0, & in \quad Q_T,\\ \xi_\omega \to 0, \quad a.a. & in \quad ((0,T)\times B)\backslash Q_T, \end{cases} \tag{8.40}$$

$$c_{0,\omega}=\begin{cases}c_0=constant, & in \quad Q_T,\\ c_{0,\omega}\to 0, \quad a.a. & in \quad ((0,T)\times B)\backslash Q_T,\end{cases} \tag{8.41}$$

$$c_{a,\omega}=\begin{cases}c_a=constant>0, & in \quad Q_T,\\ c_{a,\omega}\to 0, \quad a.a. & in \quad ((0,T)\times B)\backslash Q_T,\end{cases} \tag{8.42}$$

$$c_{d,\omega}=\begin{cases}c_d=constant>0, & in \quad Q_T,\\ c_{d,\omega}\to 0, \quad a.a. & in \quad ((0,T)\times B)\backslash Q_T.\end{cases} \tag{8.43}$$

According to $(8.13)_2$, $(8.13)_3$, and $(8.13)_4$, we derive that

$$\begin{aligned}&\int_0^T\int_{\Omega_t}\left[(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u})^2+\mu|\nabla_x\boldsymbol{u}|^2\right]\mathrm{d}x\mathrm{d}t\le C,\\ &\int_0^T\int_{\Omega_t}\left[(c_a+c_d)(\mathrm{div}_x\boldsymbol{w})^2+(c_0+c_d-c_a)|\nabla_x\boldsymbol{w}|^2\right]\mathrm{d}x\mathrm{d}t\le C,\\ &\int_0^T\int_{\Omega_t}\xi|2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t\le C,\end{aligned} \tag{8.44}$$

and

$$\begin{aligned}&\int_0^T\int_{B\backslash\Omega_t}\left[(\lambda_\omega+\mu_\omega)(\mathrm{div}_x\boldsymbol{u})^2+\mu_\omega|\nabla_x\boldsymbol{u}|^2\right]\mathrm{d}x\mathrm{d}t\le C,\\ &\int_0^T\int_{B\backslash\Omega_t}\left[(c_{a,\omega}+c_{d,\omega})(\mathrm{div}_x\boldsymbol{w})^2+(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})|\nabla_x\boldsymbol{w}|^2\right]\mathrm{d}x\mathrm{d}t\le C,\\ &\int_0^T\int_{B\backslash\Omega_t}\xi_\omega|2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t\le C.\end{aligned} \tag{8.45}$$

With above estimates (8.45) and (8.40) in hand, we can obtain that

$$\begin{aligned}&\left|-\int_0^\tau\int_{B\backslash\Omega_t}2\xi_\omega(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})\cdot\boldsymbol{\eta}\mathrm{d}x\mathrm{d}t\right|\\ &\le C\left(\int_0^T\int_{B\backslash\Omega_t}\xi_\omega|2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}|^2\mathrm{d}x\mathrm{d}t\right)^{\frac12}\left(\int_0^T\int_{B\backslash\Omega_t}\xi_\omega\boldsymbol{\eta}^2\mathrm{d}x\mathrm{d}t\right)^{\frac12}\\ &\to 0, \quad as \quad \omega\to 0.\end{aligned}$$

By the same spirit, it holds that

$$-\int_0^\tau\int_{B\backslash\Omega_t}((c_{a,\omega}+c_{d,\omega})\nabla_x\boldsymbol{w}:\nabla_x\boldsymbol{\eta}+(c_{0,\omega}+c_{d,\omega}-c_{a,\omega})\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t\to 0, \quad as \quad \omega\to 0,$$

$$-\int_0^\tau\int_{B\backslash\Omega_t}(\mu_\omega\nabla_x\boldsymbol{u}:\nabla_x\boldsymbol{\varphi}+(\lambda_\omega+\mu_\omega)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t\to 0, \quad as \quad \omega\to 0.$$

Let $\omega\to 0$, we repeat the arguments of the previous section. We derive that the continuity equation still satisfies (8.32). The micro-rotation velocity equations satisfy the following integral form,

$$\begin{aligned}&\int_{\Omega_\tau}(\rho\boldsymbol{w}\cdot\boldsymbol{\eta})(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}(\rho\boldsymbol{w})_{0,\delta}\cdot\boldsymbol{\eta}(0,\cdot)\mathrm{d}x\\ &=\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{w}\cdot\partial_t\boldsymbol{\eta}+\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\boldsymbol{\eta}-4\xi\boldsymbol{w}\cdot\boldsymbol{\eta}\\ &\quad-(c_a+c_d)\nabla_x\boldsymbol{w}:\nabla_x\boldsymbol{\eta}-(c_0+c_d-c_a)\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x\boldsymbol{\eta}+2\xi\mathrm{curl}_x\boldsymbol{u}\cdot\boldsymbol{\eta})\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{8.46}$$

$\forall \tau \in [0,T], \forall \boldsymbol{\eta} \in C_c^\infty([0,T]\times\mathbb{R}^3)$.

The momentum equations satisfy the following integral equation

$$\begin{aligned}
&\int_{\Omega_\tau} (\rho\boldsymbol{u}\cdot\boldsymbol{\varphi})(\tau,\cdot)\mathrm{d}x - \int_{\Omega_0} (\rho\boldsymbol{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\mathrm{d}x \\
&= \int_0^\tau \int_{\Omega_t} (\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \rho(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\boldsymbol{\varphi} + p(\rho)\mathrm{div}_x\boldsymbol{\varphi} + \delta\rho^\beta\mathrm{div}_x\boldsymbol{\varphi} \\
&\quad - \mu\nabla_x\boldsymbol{u} : \nabla_x\boldsymbol{\varphi} - (\lambda+\mu)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x\boldsymbol{\varphi} - \xi\mathrm{curl}_x\boldsymbol{u} : \mathrm{curl}_x\boldsymbol{\varphi} + 2\xi\boldsymbol{w}\cdot\mathrm{curl}_x\boldsymbol{\varphi})\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{8.47}$$

$\forall \tau \in [0,T], \forall \boldsymbol{\varphi} \in C_c^\infty([0,T]\times\mathbb{R}^3)$.

- Vanishing artificial pressure limit ($\delta \to 0$)

(We state that the method used here is the same as the way in [8].)

In this part, we pursue the final limit procedure $\delta \to 0$ to get rid of the artificial term $\delta\rho^\beta$ in the weak form of the momentum equations (8.47). The critical step is to establish the strong convergence of density. This idea in the existence theory of finite energy weak solutions to compressible Navier-Stoke equations is standard, and we can follow the same procedures in [8]. Eventually, we can proceed in the same steps to establish Theorem 1.2.

9. Weak-strong uniqueness

9.1. **Relative energy inequality.** In this subsection we construct the relative energy inequality to system (1.1). Firstly, we define relative energy $\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|r,\boldsymbol{U},\boldsymbol{W})$ as

$$\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|r,\boldsymbol{U},\boldsymbol{W}) = \int_{\Omega_\tau} \left[\frac{1}{2}\rho|\boldsymbol{u}-\boldsymbol{U}|^2 + \frac{1}{2}\rho|\boldsymbol{w}-\boldsymbol{W}|^2 + \frac{1}{\gamma-1}(\rho^\gamma - \gamma r^{\gamma-1}(\rho-r) - r^\gamma)\right]\mathrm{d}x, \tag{9.1}$$

where $(\rho,\boldsymbol{u},\boldsymbol{w})$ is a weak solution to system (1.1) and $(r,\boldsymbol{U},\boldsymbol{W})$ is a test function.

Then we derive the following lemma.

Lemma 9.1. *Assume that $(\rho,\boldsymbol{u},\boldsymbol{w})$ is a weak solution to the system (1.1), and for any test functions $(r,\boldsymbol{U},\boldsymbol{W})$ satisfying $(r>0,\boldsymbol{U},\boldsymbol{W}) \in C_c^\infty(\overline{Q_T})$ with $\boldsymbol{U}\cdot\boldsymbol{n} = \boldsymbol{V}\cdot\boldsymbol{n}$, $\boldsymbol{W}\cdot\boldsymbol{n} = \frac{1}{2}\mathrm{curl}_x\boldsymbol{V}\cdot\boldsymbol{n}$ on Γ_t for $t\in[0,T]$. Then the relative energy inequality for $(\rho,\boldsymbol{u},\boldsymbol{w})$ is established as follows*

$$\begin{aligned}
&\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|r,\boldsymbol{U},\boldsymbol{W})(\tau) \\
&+\int_0^\tau\int_{\Omega_t} [\mu(\nabla_x\boldsymbol{u}-\nabla_x\boldsymbol{U}) : (\nabla_x\boldsymbol{u}-\nabla_x\boldsymbol{U}) + (\lambda+\mu)(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\boldsymbol{U})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\boldsymbol{U}) \\
&+(c_a+c_d)(\nabla_x\boldsymbol{w}-\nabla_x\boldsymbol{W}) : (\nabla_x\boldsymbol{w}-\nabla_x\boldsymbol{W}) + (c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\boldsymbol{W})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\boldsymbol{W}) \\
&+\xi((2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}) - (2\boldsymbol{W}-\mathrm{curl}_x\boldsymbol{U}))^2]\,\mathrm{d}x\mathrm{d}t \\
&\le \mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|r,\boldsymbol{U},\boldsymbol{W})(\tau) + \int_0^\tau Re(t)\mathrm{d}t,
\end{aligned} \tag{9.2}$$

where the remainder term Re is defined by

$$\begin{aligned}
Re = &\int_{\Omega_t} [(\rho\boldsymbol{U}_t + \rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{U})\cdot(\boldsymbol{U}-\boldsymbol{u}) + \mu\nabla_x\boldsymbol{U} : (\nabla_x\boldsymbol{U}-\nabla_x\boldsymbol{u}) + (\lambda+\mu)\mathrm{div}_x\boldsymbol{U}(\mathrm{div}_x\boldsymbol{U}-\mathrm{div}_x\boldsymbol{u}) \\
&+ (\rho\boldsymbol{W}_t + \rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{W})\cdot(\boldsymbol{W}-\boldsymbol{w}) + (c_a+c_d)\nabla_x\boldsymbol{W} : (\nabla_x\boldsymbol{W}-\nabla_x\boldsymbol{w}) \\
&+ (c_0+c_d-c_a)\mathrm{div}_x\boldsymbol{W}(\mathrm{div}_x\boldsymbol{W}-\mathrm{div}_x\boldsymbol{w}) \\
&+\xi(2\boldsymbol{W}-\mathrm{curl}_x\boldsymbol{U})\cdot((2\boldsymbol{W}-\mathrm{curl}_x\boldsymbol{U}) - (2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}))]\,\mathrm{d}x \\
&+ \int_{\Omega_t} (\gamma(r-\rho)r^{\gamma-2}r_t + \mathrm{div}_x\boldsymbol{U}(r^\gamma-\rho^\gamma) + \gamma r^{\gamma-2}(r\boldsymbol{U}-\rho\boldsymbol{u})\cdot\nabla_x r)\mathrm{d}x.
\end{aligned} \tag{9.3}$$

Proof. We choose $\boldsymbol{\varphi}=\boldsymbol{U}-\boldsymbol{V}$ as a test function in (7.4) to derive that

$$
\begin{aligned}
&\int_{\Omega_\tau}(\rho\boldsymbol{u}\cdot(\boldsymbol{U}-\boldsymbol{V}))(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}(\rho\boldsymbol{u}\cdot(\boldsymbol{U}-\boldsymbol{V}))(0,\cdot)\mathrm{d}x\\
&=\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{u}\cdot\partial_t(\boldsymbol{U}-\boldsymbol{V})+\rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x(\boldsymbol{U}-\boldsymbol{V})+p(\rho)\mathrm{div}_x(\boldsymbol{U}-\boldsymbol{V})\\
&\quad-\mu\nabla_x\boldsymbol{u}:\nabla_x(\boldsymbol{U}-\boldsymbol{V})-(\lambda+\mu)\mathrm{div}_x\boldsymbol{u}\mathrm{div}_x(\boldsymbol{U}-\boldsymbol{V})-\xi\mathrm{curl}_x\boldsymbol{u}:\mathrm{curl}_x(\boldsymbol{U}-\boldsymbol{V})\\
&\quad+2\xi\boldsymbol{w}\cdot\mathrm{curl}_x(\boldsymbol{U}-\boldsymbol{V}))\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{9.4}
$$

Then we choose $\eta=\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}$ in (7.5) to obtain that

$$
\begin{aligned}
&\int_{\Omega_\tau}(\rho\boldsymbol{w}\cdot(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}))(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}(\rho\boldsymbol{w}\cdot(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}))(0,\cdot)\mathrm{d}x\\
&=\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{w}\cdot\partial_t(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})+\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})-4\xi\boldsymbol{w}\cdot(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\\
&-(c_a+c_d)\nabla_x\boldsymbol{w}:\nabla_x(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})-(c_0+c_d-c_a)\mathrm{div}_x\boldsymbol{w}\mathrm{div}_x(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V})\\
&+2\xi\mathrm{curl}_x\boldsymbol{u}(\boldsymbol{W}-\frac{1}{2}\mathrm{curl}_x\boldsymbol{V}))\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{9.5}
$$

We subtract (9.4) and (9.5) from the energy inequality (8.7) to get

$$
\begin{aligned}
&\int_{\Omega_\tau}\left(\frac{1}{2}\rho|\boldsymbol{u}|^2+\frac{1}{2}\rho|\boldsymbol{w}|^2+\frac{1}{\gamma-1}\rho^\gamma-\rho\boldsymbol{u}\cdot\boldsymbol{U}-\rho\boldsymbol{w}\cdot\boldsymbol{W}\right)(\tau,\cdot)\mathrm{d}x\\
&-\int_{\Omega_0}\left(\frac{1}{2\rho_0}(|(\rho\boldsymbol{u})_0|^2+|(\rho\boldsymbol{w})_0|^2)+\frac{1}{\gamma-1}\rho_0^\gamma-(\rho\boldsymbol{u})_0\cdot\boldsymbol{U}_0-(\rho\boldsymbol{w})_0\cdot\boldsymbol{W}_0\right)\mathrm{d}x\\
&+\int_0^\tau\int_{\Omega_t}[(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\boldsymbol{U})+\mu(\nabla_x\boldsymbol{u}):(\nabla_x\boldsymbol{u}-\nabla_x\boldsymbol{U})\\
&\quad+(c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\boldsymbol{W})+(c_a+c_d)(\nabla_x\boldsymbol{w}):(\nabla_x\boldsymbol{w}-\nabla_x\boldsymbol{W})\\
&\quad+\xi(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})\cdot(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}-2\boldsymbol{W}+\mathrm{curl}_x\boldsymbol{U})]\,\mathrm{d}x\mathrm{d}t\\
&\le-\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{U}+\rho\boldsymbol{w}\cdot\partial_t\boldsymbol{W}+\rho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla_x\boldsymbol{U}\\
&\quad+\rho(\boldsymbol{u}\otimes\boldsymbol{w}):\nabla_x\boldsymbol{W}+p(\rho)\mathrm{div}_x\boldsymbol{U})\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{9.6}
$$

When we choose $\phi=\frac{1}{2}(|\boldsymbol{U}|^2+|\boldsymbol{W}|^2)$ and $\phi=\frac{\gamma}{\gamma-1}r^{\gamma-1}$ as test functions in (7.2), it gives that

$$
\begin{aligned}
&\int_{\Omega_\tau}\frac{1}{2}\rho(|\boldsymbol{U}|^2+|\boldsymbol{W}|^2)(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}\frac{1}{2}\rho_0(|\boldsymbol{U}|^2+|\boldsymbol{W}|^2)(0,\cdot)\mathrm{d}x\\
&=\int_0^\tau\int_{\Omega_t}(\rho\boldsymbol{U}\cdot\partial_t\boldsymbol{U}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{U}\cdot\boldsymbol{U}+\rho\boldsymbol{W}\cdot\partial_t\boldsymbol{W}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{W}\cdot\boldsymbol{W})\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{9.7}
$$

and

$$
\begin{aligned}
&\int_{\Omega_\tau}\frac{\gamma}{\gamma-1}\rho r^{\gamma-1}\mathrm{d}x-\int_{\Omega_0}\frac{\gamma}{\gamma-1}\rho_0r_0^{\gamma-1}\mathrm{d}x\\
&=\int_0^\tau\int_{\Omega_t}(\gamma\rho r^{\gamma-2}r_t+\gamma\rho r^{\gamma-2}\boldsymbol{u}\cdot\nabla_x r)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{9.8}
$$

We combine (9.6) with (9.7), then subtract (9.8) from the resulting equation to get

$$\begin{aligned}
&\int_{\Omega_\tau}\left(\frac{1}{2}\rho|\boldsymbol{u}-\boldsymbol{U}|^2+\frac{1}{2}\rho|\boldsymbol{w}-\boldsymbol{W}|^2+\frac{1}{\gamma-1}\rho^\gamma-\frac{\gamma}{\gamma-1}r^{\gamma-1}\rho\right)(\tau,\cdot)\mathrm{d}x\\
&-\int_{\Omega_0}\left(\frac{1}{2\rho_0}(|(\rho\boldsymbol{u})_0-\rho_0\boldsymbol{U}(0,\cdot)|^2+|(\rho\boldsymbol{w})_0-\rho_0\boldsymbol{W}(0,\cdot)|^2)+\frac{1}{\gamma-1}\rho_0^\gamma-\frac{\gamma}{\gamma-1}r_0^{\gamma-1}\rho_0\right)\mathrm{d}x\\
&+\int_0^\tau\int_{\Omega_t}[(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\boldsymbol{U})+\mu(\nabla_x\boldsymbol{u}):(\nabla_x\boldsymbol{u}-\nabla_x\boldsymbol{U})\\
&\quad+(c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\boldsymbol{W})+(c_a+c_d)(\nabla_x\boldsymbol{w}):(\nabla_x\boldsymbol{w}-\nabla_x\boldsymbol{W})\\
&\quad+\xi(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})\cdot(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}-2\boldsymbol{W}+\mathrm{curl}_x\boldsymbol{U})]\,\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\tau\int_{\Omega_t}((\rho\partial_t\boldsymbol{U}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{U})\cdot(\boldsymbol{U}-\boldsymbol{u})\\
&\quad+(\rho\partial_t\boldsymbol{W}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{W})\cdot(\boldsymbol{W}-\boldsymbol{w})-p(\rho)\mathrm{div}_x\boldsymbol{U})\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^\tau\int_{\Omega_t}(\gamma\rho r^{\gamma-2}r_t+\gamma\rho r^{\gamma-2}\boldsymbol{u}\cdot\nabla_x r)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{9.9}$$

Due to the fact that $\int_0^\tau\int_{\Omega_t}\partial_t p(r)\mathrm{d}x\mathrm{d}t=\int_0^\tau\int_{\Omega_t}\gamma r^{\gamma-1}r_t\mathrm{d}x\mathrm{d}t$, we add this equality on the both sides in (9.9) to get

$$\begin{aligned}
&\int_{\Omega_\tau}\left(\frac{1}{2}\rho|\boldsymbol{u}-\boldsymbol{U}|^2+\frac{1}{2}\rho|\boldsymbol{w}-\boldsymbol{W}|^2+\frac{1}{\gamma-1}\rho^\gamma-\frac{\gamma}{\gamma-1}r^{\gamma-1}\rho\right)(\tau,\cdot)\mathrm{d}x\\
&-\int_{\Omega_0}\left(\frac{1}{2\rho_0}(|(\rho\boldsymbol{u})_0-\rho_0\boldsymbol{U}(0,\cdot)|^2+|(\rho\boldsymbol{w})_0-\rho_0\boldsymbol{W}(0,\cdot)|^2)+\frac{1}{\gamma-1}\rho_0^\gamma-\frac{\gamma}{\gamma-1}r_0^{\gamma-1}\rho_0\right)\mathrm{d}x\\
&+\int_0^\tau\int_{\Omega_t}[(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\boldsymbol{U})+\mu(\nabla_x\boldsymbol{u}):(\nabla_x\boldsymbol{u}-\nabla_x\boldsymbol{U})\\
&\quad+(c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\boldsymbol{W})+(c_a+c_d)(\nabla_x\boldsymbol{w}):(\nabla_x\boldsymbol{w}-\nabla_x\boldsymbol{W})\\
&\quad+\xi(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})\cdot(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}-2\boldsymbol{W}+\mathrm{curl}_x\boldsymbol{U})]\,\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_{\Omega_t}\partial_t p(r)\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\tau\int_{\Omega_t}((\rho\partial_t\boldsymbol{U}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{U})\cdot(\boldsymbol{U}-\boldsymbol{u})+(\rho\partial_t\boldsymbol{W}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{W})\cdot(\boldsymbol{W}-\boldsymbol{w}))\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\Omega_t}((r-\rho)r^{\gamma-2}r_t-p(\rho)\mathrm{div}_x\boldsymbol{U}+\gamma\rho r^{\gamma-2}\boldsymbol{u}\cdot\nabla_x r)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{9.10}$$

The boundary condition $\boldsymbol{U}\cdot\boldsymbol{n}=\boldsymbol{V}\cdot\boldsymbol{n}$ and integration by parts give that

$$\begin{aligned}
\int_{\Omega_t}p(r)\mathrm{div}_x\boldsymbol{U}\mathrm{d}x\mathrm{d}t&=\int_{\Omega_t}p(r)\mathrm{div}_x(\boldsymbol{U}-\boldsymbol{V})\mathrm{d}x\mathrm{d}t+\int_{\Omega_t}p(r)\mathrm{div}_x\boldsymbol{V}\mathrm{d}x\mathrm{d}t\\
&=-\int_{\Omega_t}\boldsymbol{U}\cdot\nabla_x p(r)\mathrm{d}x\mathrm{d}t+\int_{\Omega_t}\mathrm{div}_x(\boldsymbol{V}p(r))\mathrm{d}x\mathrm{d}t\\
&=-\int_{\Omega_t}\gamma r^{\gamma-1}\boldsymbol{U}\cdot\nabla_x r\mathrm{d}x\mathrm{d}t+\int_{\Omega_t}\mathrm{div}_x(\boldsymbol{V}p(r))\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{9.11}$$

The standard transport theorem gives that

$$\begin{aligned}
\int_0^\tau\int_{\Omega_t}(\partial_t p(r)+\mathrm{div}_x\boldsymbol{V}p(r)))\mathrm{d}x\mathrm{d}t&=\int_0^\tau\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}p(r)\mathrm{d}x\mathrm{d}t\\
&=\int_{\Omega_\tau}p(r)(\tau,\cdot)\mathrm{d}x-\int_{\Omega_0}p(r)(0,\cdot)\mathrm{d}x.
\end{aligned}\tag{9.12}$$

We put (9.10) and (9.11) together to get

$$
\begin{aligned}
&\int_{\Omega_\tau}\left(\frac{1}{2}\rho|\boldsymbol{u}-\boldsymbol{U}|^2+\frac{1}{2}\rho|\boldsymbol{w}-\boldsymbol{W}|^2+\frac{1}{\gamma-1}\rho^\gamma-\frac{\gamma}{\gamma-1}r^{\gamma-1}\rho\right)(\tau,\cdot)\mathrm{d}x\\
&-\int_{\Omega_0}\left(\frac{1}{2\rho_0}(|(\rho\boldsymbol{u})_0-\rho_0\boldsymbol{U}(0,\cdot)|^2+|(\rho\boldsymbol{w})_0-\rho_0\boldsymbol{W}(0,\cdot)|^2)+\frac{1}{\gamma-1}\rho_0^\gamma-\frac{\gamma}{\gamma-1}r_0^{\gamma-1}\rho_0\right)\mathrm{d}x\\
&+\int_0^\tau\int_{\Omega_t}[(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\boldsymbol{U})+\mu(\nabla_x\boldsymbol{u}):(\nabla_x\boldsymbol{u}-\nabla_x\boldsymbol{U})\\
&\quad+(c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\boldsymbol{W})+(c_a+c_d)(\nabla_x\boldsymbol{w}):(\nabla_x\boldsymbol{w}-\nabla_x\boldsymbol{W})\\
&\quad+\xi(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})\cdot(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}-2\boldsymbol{W}+\mathrm{curl}_x\boldsymbol{U})]\,\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_{\Omega_t}(\partial_t p(r)+\mathrm{div}_x(\boldsymbol{V}p(r)))\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\tau\int_{\Omega_t}((\rho\partial_t\boldsymbol{U}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{U})\cdot(\boldsymbol{U}-\boldsymbol{u})+(\rho\partial_t\boldsymbol{W}+\rho\boldsymbol{u}\cdot\nabla_x\boldsymbol{W})\cdot(\boldsymbol{W}-\boldsymbol{w}))\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\Omega_t}((r-\rho)r^{\gamma-2}r_t+(p(r)-p(\rho))\mathrm{div}_x\boldsymbol{U}+\gamma r^{\gamma-2}(r\boldsymbol{U}-\rho\boldsymbol{u})\cdot\nabla_x r)\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{9.13}
$$

then we use (9.12) to finish the proof of (9.2). □

9.2. **Proof of Theorem** 1.3. For strong solution $(\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})$ to system (1.1), we substitute $(r,\boldsymbol{U},\boldsymbol{W})$ with $(\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})$ in (9.2) to get

$$
\begin{aligned}
&\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(\tau)\\
&+\int_0^\tau\int_{\Omega_t}[\mu(\nabla_x\boldsymbol{u}-\nabla_x\widehat{\boldsymbol{u}}):(\nabla_x\boldsymbol{u}-\nabla_x\widehat{\boldsymbol{u}})+(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\widehat{\boldsymbol{u}})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\widehat{\boldsymbol{u}})\\
&+(c_a+c_d)(\nabla_x\boldsymbol{w}-\nabla_x\widehat{\boldsymbol{w}}):(\nabla_x\boldsymbol{w}-\nabla_x\widehat{\boldsymbol{w}})+(c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\widehat{\boldsymbol{w}})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\widehat{\boldsymbol{w}})\\
&+\xi((2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})-(2\widehat{\boldsymbol{w}}-\mathrm{curl}_x\widehat{\boldsymbol{u}}))^2\big]\,\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\tau\int_{\Omega_t}[(\rho\widehat{\boldsymbol{u}}_t+\rho\boldsymbol{u}\cdot\nabla_x\widehat{\boldsymbol{u}})\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})+\mu\nabla_x\widehat{\boldsymbol{u}}:(\nabla_x\widehat{\boldsymbol{u}}-\nabla_x\boldsymbol{u})+(\lambda+\mu)\mathrm{div}_x\widehat{\boldsymbol{u}}(\mathrm{div}_x\widehat{\boldsymbol{u}}-\mathrm{div}_x\boldsymbol{u})\\
&+(\rho\widehat{\boldsymbol{w}}_t+\rho\boldsymbol{u}\cdot\nabla_x\widehat{\boldsymbol{w}})\cdot(\widehat{\boldsymbol{w}}-\boldsymbol{w})+\mu'\nabla_x\widehat{\boldsymbol{w}}:(\nabla_x\widehat{\boldsymbol{w}}-\nabla_x\boldsymbol{w})+(\lambda'+\mu')\mathrm{div}_x\widehat{\boldsymbol{w}}(\mathrm{div}_x\widehat{\boldsymbol{w}}-\mathrm{div}_x\boldsymbol{w})\\
&+\xi(2\widehat{\boldsymbol{w}}-\mathrm{curl}_x\widehat{\boldsymbol{u}})\cdot((2\widehat{\boldsymbol{w}}-\mathrm{curl}_x\widehat{\boldsymbol{u}})-(2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u}))]\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\Omega_t}(\gamma(r-\rho)r^{\gamma-2}r_t+\mathrm{div}_x\widehat{\boldsymbol{u}}(r^\gamma-\rho^\gamma)+\gamma r^{\gamma-2}(r\widehat{\boldsymbol{u}}-\rho\boldsymbol{u})\cdot\nabla_x r)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{9.14}
$$

Thanks to the strong solution $(\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})$ of (1.1), it holds that

$$
\partial_t\widehat{\boldsymbol{u}}+\widehat{\boldsymbol{u}}\cdot\nabla_x\widehat{\boldsymbol{u}}=\frac{1}{\widehat{\rho}}[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]-\gamma\widehat{\rho}^{\gamma-2}\nabla_x\widehat{\rho},\tag{9.15}
$$

$$
\partial_t\widehat{\boldsymbol{w}}+\widehat{\boldsymbol{u}}\cdot\nabla_x\widehat{\boldsymbol{w}}=\frac{1}{\widehat{\rho}}[(c_a+c_d)\Delta_x\widehat{\boldsymbol{w}}+(c_0+c_d-c_a)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{w}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}-4\xi\widehat{\boldsymbol{w}}],\tag{9.16}
$$

and

$$
\gamma\widehat{\rho}^{\gamma-2}\widehat{\rho}_t+\gamma\widehat{\rho}^{\gamma-2}\widehat{\boldsymbol{u}}\cdot\nabla_x\widehat{\rho}=-\gamma\widehat{\rho}^{\gamma-1}\mathrm{div}_x\widehat{\boldsymbol{u}}.\tag{9.17}
$$

We use integration by parts and combine (9.14), (9.15), (9.16) and (9.17) to derive that

$$\begin{aligned}
&\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(\tau)\\
&+\int_0^\tau\int_{\Omega_t}[\mu(\nabla_x\boldsymbol{u}-\nabla_x\widehat{\boldsymbol{u}}):(\nabla_x\boldsymbol{u}-\nabla_x\widehat{\boldsymbol{u}})+(\lambda+\mu)(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\widehat{\boldsymbol{u}})(\mathrm{div}_x\boldsymbol{u}-\mathrm{div}_x\widehat{\boldsymbol{u}})\\
&+(c_a+c_d)(\nabla_x\boldsymbol{w}-\nabla_x\widehat{\boldsymbol{w}}):(\nabla_x\boldsymbol{w}-\nabla_x\widehat{\boldsymbol{w}})+(c_0+c_d-c_a)(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\widehat{\boldsymbol{w}})(\mathrm{div}_x\boldsymbol{w}-\mathrm{div}_x\widehat{\boldsymbol{w}})\\
&+\xi((2\boldsymbol{w}-\mathrm{curl}_x\boldsymbol{u})-(2\widehat{\boldsymbol{w}}-\mathrm{curl}_x\widehat{\boldsymbol{u}}))^2\Big]\,\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\tau\int_{\Omega_t}[\rho(\boldsymbol{u}-\widehat{\boldsymbol{u}})\cdot\nabla_x\widehat{\boldsymbol{u}}\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})+\rho(\boldsymbol{w}-\widehat{\boldsymbol{w}})\cdot\nabla_x\widehat{\boldsymbol{w}}\cdot(\widehat{\boldsymbol{w}}-\boldsymbol{w})\\
&\quad-\mathrm{div}_x\widehat{\boldsymbol{u}}(\rho^\gamma-\gamma\widehat{\rho}^{\gamma-1}(\rho-\widehat{\rho})-\widehat{\rho}^\gamma)]\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\Omega_t}\left(\frac{\rho}{\widehat{\rho}}-1\right)[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\Omega_t}\left(\frac{\rho}{\widehat{\rho}}-1\right)[(c_a+c_d)\Delta_x\widehat{\boldsymbol{w}}+(c_0+c_d-c_a)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{w}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}-4\xi\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{w}}-\boldsymbol{w})\mathrm{d}x\mathrm{d}t\\
&:=\sum_{k=1}^3\mathcal{I}_k.
\end{aligned}\tag{9.18}$$

Firstly, for $\frac{\widehat{\rho}}{2}\le\rho\le2\widehat{\rho}$, we deduce that

$$\frac{1}{\gamma-1}\rho^\gamma-\frac{\gamma}{\gamma-1}r^{\gamma-1}(\rho-\widehat{\rho})-\frac{1}{\gamma-1}\widehat{\rho}^\gamma\ge C(\widehat{\rho})(\rho-\widehat{\rho})^2.$$

For $\rho\le\frac{\widehat{\rho}}{2}$ and $\rho\ge2\widehat{\rho}$, we have that

$$\frac{1}{\gamma-1}\rho^\gamma-\frac{\gamma}{\gamma-1}r^{\gamma-1}(\rho-\widehat{\rho})-\frac{1}{\gamma-1}\widehat{\rho}^\gamma\ge C(\widehat{\rho})(1+\rho^\gamma),$$

then it is easy to deduce that

$$\mathcal{I}_1\le C\int_0^\tau(\|\nabla_x\widehat{\boldsymbol{u}}\|_{L^\infty(\Omega_t)}+\|\nabla_x\widehat{\boldsymbol{w}}\|_{L^\infty(\Omega_t)})\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(t)\mathrm{d}t.$$

We divide $\mathcal{I}_2$ into three parts, and the standard Hölder's inequality leads to

$$\begin{aligned}
&\int_{\{\frac{\widehat{\rho}}{2}\le\rho\le2\widehat{\rho}\}}\left(\frac{\rho}{\widehat{\rho}}-1\right)[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})\mathrm{d}x\\
&\le\epsilon\|\widehat{\boldsymbol{u}}-\boldsymbol{u}\|^2_{L^6(\Omega_t)}+C\Big\|\frac{1}{\widehat{\rho}}\Big\|^2_{L^\infty(\Omega_t)}\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^3(\Omega_t)}\int_{\{\frac{\widehat{\rho}}{2}\le\rho\le2\widehat{\rho}\}}(\rho-\widehat{\rho})^2\mathrm{d}x\\
&\le\epsilon\|\nabla_x\widehat{\boldsymbol{u}}-\nabla_x\boldsymbol{u}\|^2_{L^2(\Omega_t)}+C\Big\|\frac{1}{\widehat{\rho}}\Big\|^2_{L^\infty(\Omega_t)}\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^3(\Omega_t)}\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(t),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{0\le\rho<\frac{\widehat{\rho}}{2}\}}\left(\frac{\rho}{\widehat{\rho}}-1\right)[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})\mathrm{d}x\\
&\le\int_{\{0\le\rho<\frac{\widehat{\rho}}{2}\}}|[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})|\mathrm{d}x\\
&\le\epsilon\|\widehat{\boldsymbol{u}}-\boldsymbol{u}\|^2_{L^6(\Omega_t)}+C\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^3(\Omega_t)}\int_{\{\frac{\widehat{\rho}}{2}\le\rho\le2\widehat{\rho}\}}1\mathrm{d}x\\
&\le\epsilon\|\nabla_x\widehat{\boldsymbol{u}}-\nabla_x\boldsymbol{u}\|^2_{L^2(\Omega_t)}+C\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^3(\Omega_t)}\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(t).
\end{aligned}$$

Moreover, we treat the integration on the set $\{x|\rho>2\widehat{\rho}\}$.

When $\gamma \le 2$, it holds that

$$
\begin{aligned}
&\int_{\{\rho>2\widehat{\rho}\}} \left(\frac{\rho}{\widehat{\rho}}-1\right)[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})\mathrm{d}x\\
&\le \int_{\{\rho>2\widehat{\rho}\}} |\frac{\rho}{\widehat{\rho}}[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})|\mathrm{d}x\\
&\le \epsilon\|\widehat{\boldsymbol{u}}-\boldsymbol{u}\|^2_{L^6(\Omega_t)}+C\|\frac{1}{\widehat{\rho}}\|^2_{L^\infty(\Omega_t)}\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_t)}\left(\int_{\{\rho>2\widehat{\rho}\}}\rho^\gamma\mathrm{d}x\right)^{\frac{2}{\gamma}}\\
&\le \epsilon\|\nabla_x\widehat{\boldsymbol{u}}-\nabla_x\boldsymbol{u}\|^2_{L^2(\Omega_t)}+C\|\frac{1}{\widehat{\rho}}\|^2_{L^\infty(\Omega_t)}\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_t)}\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(t)^{\frac{2}{\gamma}-1+1}.
\end{aligned}
$$

When $\gamma > 2$, it gives that

$$
\begin{aligned}
&\int_{\{\rho>2\widehat{\rho}\}} \left(\frac{\rho}{\widehat{\rho}}-1\right)[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})\mathrm{d}x\\
&\le \int_{\{\rho>2\widehat{\rho}\}} |\frac{\rho^{\frac{\gamma}{2}}}{\widehat{\rho}}[(\mu+\xi)\Delta_x\widehat{\boldsymbol{u}}+(\mu+\lambda-\xi)\nabla_x\mathrm{div}_x\widehat{\boldsymbol{u}}+2\xi\mathrm{curl}_x\widehat{\boldsymbol{w}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u})|\mathrm{d}x\\
&\le \epsilon\|\widehat{\boldsymbol{u}}-\boldsymbol{u}\|^2_{L^6(\Omega_t)}+C\|\frac{1}{\widehat{\rho}}\|^2_{L^\infty(\Omega_t)}\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^3(\Omega_t)}\left(\int_{\{\rho>2\widehat{\rho}\}}\rho^\gamma\mathrm{d}x\right)\\
&\le \epsilon\|\nabla_x\widehat{\boldsymbol{u}}-\nabla_x\boldsymbol{u}\|^2_{L^2(\Omega_t)}+C\|\frac{1}{\widehat{\rho}}\|^2_{L^\infty(\Omega_t)}\|(\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}})\|^2_{L^3(\Omega_t)}\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(t).
\end{aligned}
$$

By the same spirit, we can derive that

$$
\mathcal{I}_3 \le \epsilon\int_0^\tau \|\nabla_x\widehat{\boldsymbol{w}}-\nabla_x\boldsymbol{w}\|^2_{L^2(\Omega_t)}\mathrm{d}t+\int_0^\tau h(t)\mathcal{E}(\rho,\boldsymbol{u},\boldsymbol{w}|\widehat{\rho},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})(t)\mathrm{d}t,
$$

where $h(t)\in L^1(0,\tau)$ depends on $\|\frac{1}{\widehat{\rho}}\|_{L^\infty(\Omega_t)}$, $\|(\nabla_x^2\widehat{\boldsymbol{w}},\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}},\widehat{\boldsymbol{w}})\|_{L^3(\Omega_t)}$ and $\|(\nabla_x^2\widehat{\boldsymbol{w}},\nabla_x^2\widehat{\boldsymbol{u}},\nabla_x\widehat{\boldsymbol{w}},\widehat{\boldsymbol{w}})\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega_t)}$ for $\gamma > 2$.

Putting above estimates together and using Grönwall's inequality, we conclude the proof of Theorem 1.3.

10. Appendix

Proof of Lemma 4.2 and Lemma 5.2. We observe that the stress tension for the velocity is coupled with micro-rotational velocity, however, the stress tension of angular-rotational velocity is decoupled. Hence, we should first construct the micro-rotational velocity, then we use it to construct the velocity of fluids. We look for the extension of the boundary data satisfying the following conditions:

For $\boldsymbol{w}^b$, it should hold that

$$
\begin{aligned}
\boldsymbol{w}^b(t,y)\cdot\boldsymbol{n}(y) =&(\tilde{\boldsymbol{w}}(t,y)-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\cdot(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\\
&+\frac{1}{2}[((\nabla_x\boldsymbol{Y}-\mathbb{I})\nabla_y)\wedge\tilde{\boldsymbol{V}}]\cdot\boldsymbol{n}(\boldsymbol{X}(t,y)),\\
\boldsymbol{w}^b(t,y)\cdot\tau^k(y) =&(\tilde{\boldsymbol{w}}(t,y)-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\\
&+\frac{1}{2}[((\nabla_x\boldsymbol{Y}-\mathbb{I})\nabla_y)\wedge\tilde{\boldsymbol{V}}]\cdot\tau^k(y),
\end{aligned}
\tag{10.1}
$$

and

$$\begin{aligned}
&\left[(c_a+c_d)\nabla_y \boldsymbol{w}^b(t,y)+(c_d-c_a)\nabla_y^{\mathsf T}\boldsymbol{w}^b\right]\boldsymbol{n}(y)\cdot\tau^k(y)\\
=&\left[(c_a+c_d)\nabla_y\tilde{\boldsymbol{w}}(t,y)(\mathbb{I}-\nabla_x\boldsymbol{Y})+(c_d-c_a)(\mathbb{I}-\nabla_x^{\mathsf T}\boldsymbol{Y})\nabla_y^{\mathsf T}\tilde{\boldsymbol{w}}\right]\boldsymbol{n}(\boldsymbol{X}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y))\\
&+\left[(c_a+c_d)\nabla_y\tilde{\boldsymbol{w}}(t,y)+(c_d-c_a)\nabla_y^{\mathsf T}\tilde{\boldsymbol{w}}\right](t,y)\left[(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\cdot\tau^k(\boldsymbol{X}(t,y))\right.\\
&\left.+\boldsymbol{n}(y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\right].
\end{aligned}\tag{10.2}$$

For $\boldsymbol{u}^b$, it should hold that

$$\begin{aligned}
\boldsymbol{u}^b(t,y)\cdot\boldsymbol{n}(y)=&(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})(t,y)\cdot(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\\
&+(\boldsymbol{V}(t,(\boldsymbol{X}(t,y)))-\tilde{\boldsymbol{V}}(t,y))\cdot\boldsymbol{n}(\boldsymbol{X}(t,y)),\\
\boldsymbol{u}^b(t,y)\cdot\tau^k(y)=&(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})(t,y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\\
&+(\boldsymbol{V}(t,(\boldsymbol{X}(t,y)))-\tilde{\boldsymbol{V}}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y)),
\end{aligned}\tag{10.3}$$

and

$$\begin{aligned}
&\left[(\mu+\xi)\nabla_y\boldsymbol{u}^b(t,y)+(\mu-\xi)\nabla_y^{\mathsf T}\boldsymbol{u}^b-2\xi A(\boldsymbol{w}^b)\right]\boldsymbol{n}(y)\cdot\tau^k(y)\\
=&\left[(\mu+\xi)\nabla_y\tilde{\boldsymbol{u}}(t,y)(\mathbb{I}-\nabla_x\boldsymbol{Y})+(\mu-\xi)(\mathbb{I}-\nabla_x^{\mathsf T}\boldsymbol{Y})\nabla_y^{\mathsf T}\tilde{\boldsymbol{u}}\right]\boldsymbol{n}(\boldsymbol{X}(t,y))\cdot\tau^k(\boldsymbol{X}(t,y))\\
&+\left[(\mu+\xi)\nabla_y\tilde{\boldsymbol{u}}(t,y)+(\mu-\xi)\nabla_y^{\mathsf T}\tilde{\boldsymbol{u}}-2\xi A(\tilde{\boldsymbol{w}})\right](t,y)\left[(\boldsymbol{n}(y)-\boldsymbol{n}(\boldsymbol{X}(t,y)))\cdot\tau^k(\boldsymbol{X}(t,y))\right.\\
&\left.+\boldsymbol{n}(y)\cdot(\tau^k(y)-\tau^k(\boldsymbol{X}(t,y)))\right].
\end{aligned}\tag{10.4}$$

Firstly, we flatten the boundary and choose a proper smooth cut-off function which enjoy the regularity of the boundary. The detail can be found in [25]. Therefore, we can choose

$$\boldsymbol{n}(y_1,y_2,0)=(0,0,1),\quad \tau^1(y_1,y_2,0)=(1,0,0),\quad \tau^2(y_1,y_2,0)=(0,1,0).$$

Then we construct $\tilde{\boldsymbol{u}}^b$ and $\tilde{\boldsymbol{w}}^b$ satisfying (10.3) and (10.1), respectively. Moreover, in the following step we use it to define $\boldsymbol{w}^b$. Lastly, with $\boldsymbol{w}^b$ in hand, we define $\boldsymbol{u}^b$.

The construction of $\tilde{\boldsymbol{u}}^b$ is directly borrowed from Appendix A in [25]. We list the result as follows

$$\|\tilde{\boldsymbol{u}}^b_{tt}\|_{L^2(0,T;L^2(\Omega_0))}\le E(T)[1+\|(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}})_{tt}\|_{L^2(0,T;L^2(\Omega_0))}+\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{V}}\|_{W^{1,\infty}(0,T;L^2(\Omega_0))}].\tag{10.5}$$

Now, we consider $\tilde{\boldsymbol{w}}^b$. Precisely, for any $(y_1,y_2,y_3)\in\Omega_0$, we define

$$\delta\boldsymbol{n}(t,y)=\boldsymbol{n}(t,(y_1,y_2,0))-\boldsymbol{n}(\boldsymbol{X}(t,(y_1,y_2,0))).$$

Moreover, the normal component of $\tilde{\boldsymbol{w}}^b$ can be defined as

$$\tilde{\boldsymbol{w}}^b\cdot\boldsymbol{n}=(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})(t,y)\cdot\delta\boldsymbol{n}(t,y)+\frac{1}{2}[((\nabla_x\tilde{\boldsymbol{Y}}-\mathbb{I})\nabla_y)\wedge\tilde{\boldsymbol{V}}]\cdot\boldsymbol{n}(\boldsymbol{X}(t,y)).\tag{10.6}$$

For the first term of right-hand side in (10.6), we have

$$\begin{aligned}
&[(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})(t,y)\cdot\delta\boldsymbol{n}(t,y)]_{tt}\\
=&(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\cdot\delta\boldsymbol{n}+2(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_t\cdot(\delta\boldsymbol{n})_t+(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})(t,y)\cdot(\delta\boldsymbol{n})_t.
\end{aligned}\tag{10.7}$$

Thanks to (1.6), we get

$$\delta\boldsymbol{n}\sim\int_0^T\tilde{\boldsymbol{V}},\quad(\delta\boldsymbol{n})_t\sim\tilde{\boldsymbol{V}},\quad(\delta\boldsymbol{n})_{tt}\sim\tilde{\boldsymbol{V}}_t.\tag{10.8}$$

We only estimate the first term of right-hand side in (10.7), the rest terms can be done in the same way,

$$\begin{aligned}
\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\delta\boldsymbol{n}\|_{L^2(0,T;L^2(\Omega_0))}\le&\|\delta\boldsymbol{n}\|_{L^\infty(0,T;L^\infty(\Omega_0))}\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\|_{L^2(0,T;L^2(\Omega_0))}\\
\le&T\|\tilde{\boldsymbol{V}}\|_{L^\infty(0,T;L^(\Omega_0))}\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\|_{L^2(0,T;L^2(\Omega_0))}.
\end{aligned}$$

Finally, we get

$$
\begin{aligned}
&\|[(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\cdot\delta\boldsymbol{n}(t,y)]_{tt}\|_{L^2(L^2)}\\
\le& C(T+\sqrt{T})\|\tilde{\boldsymbol{V}}\|_{W^{1,\infty}(0,T;L^\infty(\Omega_0))}[\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\|_{L^2(0,T;L^2(\Omega_0))}\\
&+\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\|_{W^{1,\infty}(0,T;L^2(\Omega_0))}].
\end{aligned}
\tag{10.9}
$$

For the second term of right-hand side in (10.6), we denote $E\boldsymbol{Y}=\nabla_x\boldsymbol{Y}-\mathbb{I}$, then it holds that

$$
\begin{aligned}
&[((E\boldsymbol{Y}\nabla_y)\wedge\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}(\boldsymbol{X}(t,y))]_{tt}\\
=&((E\boldsymbol{Y}_{tt}\nabla_y)\wedge\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}+2((E\boldsymbol{Y}_t\nabla_y)\wedge\tilde{\boldsymbol{V}}_t)\cdot\boldsymbol{n}+((E\boldsymbol{Y}_t\nabla_y)\wedge\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}_t\\
&+((E\boldsymbol{Y}\nabla_y)\wedge\tilde{\boldsymbol{V}}_{tt})\cdot\boldsymbol{n}+2((E\boldsymbol{Y}\nabla_y)\wedge\tilde{\boldsymbol{V}}_t)\cdot\boldsymbol{n}_t+((E\boldsymbol{Y}\nabla_y)\wedge\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}_{tt}.
\end{aligned}
$$

It is easy to observe that

$$
\|E\boldsymbol{Y}\|_{L^\infty(0,T;L^\infty(\Omega_0))}\le E(T),\quad \|(E\boldsymbol{Y}_t,E\boldsymbol{Y}_{tt})\|_{L^\infty(0,T;L^\infty(\Omega_0))}\le C,
\tag{10.10}
$$

where E is continuous, $E(0)=0$. Hence, we have

$$
\begin{aligned}
&\|[((E\boldsymbol{Y}\nabla_y)\wedge\tilde{\boldsymbol{V}})\cdot\boldsymbol{n}(\boldsymbol{X}(t,y))]_{tt}\|_{L^2(0,T;L^2(\Omega_0))}\\
\le& E(T)(\|\nabla_y\tilde{\boldsymbol{V}}\|_{W^{1,\infty}(0,T;L^2(\Omega_0))}+\|\nabla_y\tilde{\boldsymbol{V}}_{tt}\|_{L^2(0,T;L^2(\Omega_0))}).
\end{aligned}
\tag{10.11}
$$

Combining (10.9) and (10.11), we obtain

$$
\begin{aligned}
&\|(\tilde{\boldsymbol{w}}^b\cdot\boldsymbol{n})_{tt}\|_{L^2(0,T;L^2(\Omega_0))}\\
\le& E(t)[1+\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})_{tt}\|_{L^2(0,T;L^2(\Omega_0))}+\|(\tilde{\boldsymbol{w}}-\frac{1}{2}\mathrm{curl}_y\tilde{\boldsymbol{V}})\|_{W^{1,\infty}(0,T;L^2(\Omega_0))}].
\end{aligned}
\tag{10.12}
$$

The tangential parts have similar bounds.

Now we construct $\boldsymbol{w}^b$ and rewrite (10.1) in a compact form

$$
\tilde{\boldsymbol{w}}^b=E^1\tilde{\boldsymbol{w}}+E^2,
\tag{10.13}
$$

where E^1 and E^2 are sufficiently regular matrix and vector functions. Meanwhile, we rewrite (10.2) as

$$
\begin{aligned}
(c_a+c_d)w^b_{1,y_3}+(c_d-c_a)w^b_{3,y_1}&=\sum_{i,j=1}^3 B^1_{ij}(t,x)\tilde{w}_{i,y_j},\\
(c_a+c_d)w^b_{2,y_3}+(c_d-c_a)w^b_{3,y_2}&=\sum_{i,j=1}^3 C^1_{ij}(t,x)\tilde{w}_{i,y_j}.
\end{aligned}
\tag{10.14}
$$

First, we take $w^b_3=\tilde{w}^b_3$. Next we construct w^b_1, since w^b_2 can be constructed in the same way. In particular,

$$
w^b_{3,y_1}=\tilde{w}^b_{3,y_1}=\sum_{i=1}^3 E^1_{3i}w_{i,y_1}+\sum_{i=1}^3 E^1_{3i,y_1}w_i+E^2_{3,y_1}.
\tag{10.15}
$$

Substituting the above identity into (10.14), we have

$$
\begin{aligned}
(c_a+c_d)w^b_{1,y_3}&=\sum_{i,j=1}^3 B^1_{ij}(t,x)w_{i,y_j}-(c_d-c_a)\sum_{i=1}^3 E^1_{3i}w_{i,y_1}-(c_d-c_a)\sum_{i=1}^3 E^1_{3i,y_1}w_i-(c_d-c_a)E^2_{3,y_1}\\
&=\sum_{i,j=1}^3 \bar{B}^1_{ij}(t,x)w_{i,y_j}-(c_d-c_a)\sum_{i=1}^3 E^1_{3i,y_1}w_i-(c_d-c_a)E^2_{3,y_1}.
\end{aligned}
\tag{10.16}
$$

Then we divide w_1^b into two parts such that $w_1^b = w_1^{b1} + w_1^{b2}$ where

$$\begin{aligned}(c_a + c_d)w_1^{b1}(y) = &\left(\sum_{i=1}^{3} E_{1i}^1 w_i + E_1^2\right)(y) + 2\sum_{i,j} \tilde{B}_{ij}^1 w_i((y_1, y_2, 0) + y_3 \boldsymbol{e}_j) \\ &- 2\sum_{i,j} \tilde{B}_{ij}^1 w_i \left((y_1, y_2, 0) + \frac{y_3}{2}\boldsymbol{e}_j\right),\end{aligned} \tag{10.17}$$

here

$$\begin{aligned}\tilde{B}_{ij}^1 =& \bar{B}_{ij}^1, \quad j \neq 3, \\ \tilde{B}_{i3}^1 =& \bar{B}_{i3}^1 - E_{1i}^1, \quad i = 1, 2, 3.\end{aligned} \tag{10.18}$$

Applying ∂_{y_3} to (10.17), we get

$$\begin{aligned}(c_a + c_d)w_{1,y_3}^{b1}(y) &= \sum_{i=1}^{3} E_{1i}^1 w_{i,y_3}(y) + \sum_{i=1}^{3} E_{1i,y_3}^1 w_i(y) + E_{1,y_3}^2 + \sum_{i,j} \tilde{B}_{ij}^1 w_{i,y_j}(y) \\ &= \sum_{i,j} \bar{B}_{ij}^1 w_{i,y_j} + \sum_{i} E_{1i,y_3}^1 w_i + E_{1,y_3}^2.\end{aligned} \tag{10.19}$$

So, subtracting (10.19) from (10.16) gives that

$$w_{1,y_3}^{b2} = -\frac{1}{c_a + c_d}\left(\sum_i E_{1i}^1 w_{i,y_3} + (c_d - c_a)\sum_i E_{3i,y_1}^1 w_i + (c_d - c_a)E_{3,y_1}^2 + E_{1,y_3}^2\right) := P_w(y). \tag{10.20}$$

Hence, we can define w_1^{b2} as follows

$$w_1^{b2} = \int_0^{y_3} P_w(y_1, y_2, s)ds. \tag{10.21}$$

Finally, $w_1^b = w_1^{b1} + w_1^{b2}$ is constructed as required relations. With $\boldsymbol{w}^b$ in hand, we can construct $\boldsymbol{u}^b$ in the same way which depends on $\boldsymbol{w}^b$. We also rewrite (10.3) in a compact way

$$\tilde{\boldsymbol{u}}^b = E^3\tilde{\boldsymbol{u}} + E^4, \tag{10.22}$$

where E^3 and E^4 are regular matrix and vector functions. And, (10.4) gives that

$$\begin{aligned}(\mu + \xi)u_{1,y_3}^b + (\mu - \xi)u_{3,y_1}^b + w_2^b &= \sum_{i,j=1}^{3} B_{ij}^2(t,x)\tilde{u}_{i,y_j} + D^1(\tilde{\boldsymbol{w}}), \\ (\mu + \xi)u_{2,y_3}^b + (\mu - \xi)u_{3,y_2}^b - w_1^b &= \sum_{i,j=1}^{3} C_{ij}^2(t,x)\tilde{u}_{i,y_j} + D^2(\tilde{\boldsymbol{w}}).\end{aligned} \tag{10.23}$$

Also, we take $u_3^b = \tilde{u}_3^b$, then

$$\begin{aligned}(\mu + \xi)u_{1,y_3}^b &= \sum_{i,j=1}^{3} B_{ij}^2(t,x)\tilde{u}_{i,y_j} - (\mu - \xi)\tilde{u}_{3,y_1}^b - w_2^b + D^1(\tilde{\boldsymbol{w}}) \\ &= \sum_{i,j=1}^{3} B_{ij}^2(t,x)\tilde{u}_{i,y_j} - (\mu - \xi)\left(\sum_{i=1}^{3} E_{3i}^3\tilde{u}_{i,y_1} + \sum_{i=1}^{3} E_{3i,y_1}^3\tilde{u}_i + E_{3,y_1}^4\right) - w_2^b + D^1(\tilde{\boldsymbol{w}}) \\ &= \sum_{i,j=1}^{3} \bar{B}_{ij}^2(t,x)\tilde{u}_{i,y_j} - (\mu - \xi)\sum_{i=1}^{3} E_{3i,y_1}^3\tilde{u}_i - (\mu - \xi)E_{3,y_1}^4 - w_2^b + D^1(\tilde{\boldsymbol{w}}).\end{aligned} \tag{10.24}$$

We define $u_1^b = u_1^{b1} + u_1^{b2}$ where

$$\begin{aligned}(\mu+\xi)u_1^{b1}(y) = &\left(\sum_{i=1}^{3} E_{1i}^3 \tilde{u}_i + E_1^4\right)(y) + 2\sum_{i,j} \tilde{B}_{ij}^2 \tilde{u}_i((y_1,y_2,0)+y_3\boldsymbol{e}_j) \\ &- 2\sum_{i,j} \tilde{B}_{ij}^2 \tilde{u}_i\left((y_1,y_2,0)+\frac{y_3}{2}\boldsymbol{e}_j\right),\end{aligned} \tag{10.25}$$

where

$$\begin{aligned}\tilde{B}_{ij}^2 =& \bar{B}_{ij}^2, \quad j \neq 3, \\ \tilde{B}_{i3}^2 =& \bar{B}_{i3}^2 - E_{1i}^3, \quad i = 1,2,3.\end{aligned} \tag{10.26}$$

Hence, we have

$$\begin{aligned}u_{1,y_3}^{b2} =& -\frac{1}{\mu+\xi}\left(\sum_i E_{1i}^3 \tilde{u}_{i,y_3} + E_{1,y_3}^4 + (\mu-\xi)\sum_i E_{3i,y_1}^3 u_i + (\mu-\xi)E_{3,y_1}^4 + w_2^b + D^1(\tilde{\boldsymbol{w}})\right) \\ :=& P_u(y).\end{aligned} \tag{10.27}$$

Finally, we can define u_1^{b2} as follows

$$u_1^{b2} = \int_0^{y_3} P_u(y_1,y_2,s)\mathrm{d}s. \tag{10.28}$$

We notice that $u_1^b = u_1^{b1} + u_1^{b2}$ satisfies (10.3) and (10.4).

Note that B_{ij}^1, B_{ij}^2, C_{ij}^1 and C_{ij}^2 are made of $\tau(y) - \tau(\boldsymbol{X}(y))$, $\boldsymbol{n}(y) - \boldsymbol{n}(\boldsymbol{X}(y))$ and $\nabla_x \boldsymbol{Y} - \mathbb{I}$. Thanks to (10.8) and (10.10), we obtain the estimate for w_1^{b1} and u_1^{b1}. When estimates comes to w_1^{b2} and u_1^{b2}, we use the fact

$$\nabla_y E^i \sim \int_0^T \nabla_y \tilde{\boldsymbol{V}}, \quad i = 1,2,3,4.$$

Therefore, repeating all the argument as before, we can derive the bound for $\|\boldsymbol{u}_{tt}^b\|_{L^2(0,T;L^2(\Omega_0))}$ and $\|\boldsymbol{w}_{tt}^b\|_{L^2(0,T;L^2(\Omega_0))}$. Other components in $\mathcal{Y}(T)$ are obtained in a similar way, we omit the details.

Lemma 4.1. [11] *Let $\rho \in L^\infty(0,T;L^2(B))$, $\rho \geq 0$, $\boldsymbol{u} \in L^2(0,T;W_0^{1,2}(B;\mathbb{R}^3))$ be a weak solution of the equation of continuity,*

sepcifically

$$\int_B (\rho(\tau,\cdot)\phi(\tau,\cdot) - \rho_0\phi(0,\cdot))\mathrm{d}x = \int_0^\tau \int_B (\rho\phi_t + \rho\boldsymbol{u}\cdot\nabla_x\phi)\mathrm{d}x\mathrm{d}t$$

for any $\tau \in [0,T]$ and any test function $\phi \in C_c^1([0,T]\times\mathbb{R}^3)$.

In addition, assume that $(\boldsymbol{u}-\boldsymbol{V})(\tau,\cdot)\cdot\boldsymbol{n}|_{\Gamma_\tau} = 0$ for a.a.$\tau \in (0,T)$, and that $\rho_0 \in L^2(\mathbb{R}^3)$, $\rho_0 \geq 0$, $\rho_0|_{B\setminus\Omega_0} = 0$. Then

$$\rho(\tau,\cdot)|_{B\setminus\Omega_0} = 0, \quad \forall \tau \in [0,T].$$

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(Bingkang Huang)
School of Mathematics, Hefei University of Technology, Hefei 230009, China
Email address: bkhuang92@hotmail.com, bkhuang@whu.edu.cn

(Šárka Nečasová)
Institute of Mathematics Czech Academy of Sciences, Žitná 25, Czech Republic
Email address: matus@math.cas.cz

(Lan Zhang)
Center for Mathematical Sciences and Department of Mathematics, Wuhan University of Technology, Wuhan 430070, China
Email address: zhang_lan@whu.edu.cn