INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Factorisation in stopping-time Banach spaces: Identifying unique maximal ideals

Tomasz Kania

Richard Lechner

Preprint No. 82-2021

PRAHA 2021

FACTORISATION IN STOPPING-TIME BANACH SPACES: IDENTIFYING UNIQUE MAXIMAL IDEALS

TOMASZ KANIA AND RICHARD LECHNER

Abstract. Stopping-time Banach spaces is a collective term for the class of spaces of eventually null integrable processes that are defined in terms of the behaviour of the stopping times with respect to some fixed filtration. From the point of view of Banach space theory, these spaces in many regards resemble the classical spaces such as L^1 or $C(\Delta)$, but unlike these, they do have unconditional bases.

In the present paper, we study the canonical bases in the stopping-time spaces in relation to factorising the identity operator thereon. Since we work exclusively with the dyadic-tree filtration, this setup enables us to work with tree-indexed bases rather than directly with stochastic processes. *En route* to the factorisation results, we develop general criteria that allow one to deduce the uniqueness of the maximal ideal in the algebra of operators on a Banach space. These criteria are applicable to many classical Banach spaces such as (mixed-norm) L^p-spaces, BMO, SL^∞, and others.

1. Introduction and main results

The familiar Banach spaces $L^1[0,1]$ and $C[0,1]$ are fundamental examples of classical Banach spaces, which do not have an unconditional Schauder basis; actually neither space embeds into a space with such a basis. Rosenthal introduced in an unpublished manuscript an analogue of $L^1[0,1]$, denoted S^1, that admits an unconditional Schauder basis. (Strictly speaking, S^1 is an analogue of $L^1(\Delta)$, the space of functions on the Cantor group Δ, *i.e.*, the product $\{-1,1\}^{\mathbb{N}}$, with respect to the normalised Haar measure, which is the product of infinitely many copies of the coin-toss measure on $\{-1,1\}$; this space is isometric to $L^1[0,1]$, though.) The space S^1 naturally comes in tandem with a space denoted by B, which is a space with an unconditional Schauder basis that resembles in many ways the space $C(\Delta)$ of continuous functions on Δ; the analogies between S^1 and B and their classical counterparts go deeper and will be delineated in subsequent paragraphs.

Buehler in her Ph.D. thesis ([5, Section 3.1]) considered the space S^2 that may be viewed as a certain convexification of S^1 and proved that ℓ^p does not embed therein for $p \in [1,2)$. The space S^2 is a member of a broader scale of spaces S^p parametrised by $p \in [1,\infty)$, whose left end-point is S^1 indeed. Schechtman proved in an unpublished manuscript that S^1 contains isometric copies of ℓ^p for all $p \in [1,\infty)$ and more generally, for every $p \in [1,\infty)$, S^p contains isometric copies of ℓ^q for $q \geqslant p$. The space S^1 was studied further by Dew in his Ph.D. thesis [9] who proved that S^1 contains isomorphic copies Orlicz sequence spaces ℓ^M for a rather wide class of Orlicz functions M.

The reader is owed an explanation of how the spaces S^p and B are constructed and why we call them *stopping-time Banach spaces*. The following construction to an extent

Date: December 23, 2021.

2010 *Mathematics Subject Classification.* 47L20, 46A35, 46B25, 60G46, 46B26.

Key words and phrases. Factorisation, stopping-time Banach spaces, unique maximal operator ideals, primarity.

The first-named author acknowledges with thanks funding received from SONATA 15 2019/35/D/ST1/01734. The second-named author was supported by the Austrian Science Foundation (FWF) Pr.Nr. P32728.

follows [9] (who considered only the case $p=1$), however alternative description based on martingale differences may be found in [4].

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n)_{n=0}^\infty$ be a fixed filtration in $\mathcal{F}$. A stochastic process $(X_n)_{n=0}^\infty$ on $(\Omega, \mathcal{F}, \mathsf{P})$ is $\mathbb{F}$-*adapted*, whenever X_n is $\mathcal{F}_n$ measurable $(n \in \mathbb{N})$. A *stopping time* is an $\mathbb{N}\cup\{\infty\}$-valued random variable T on $(\Omega, \mathcal{F}, \mathsf{P})$ such that for each n, we have $[T=n]\in\mathcal{F}_n$. Let $\mathcal{T}$ denote the family of all stopping times on $(\Omega, \mathcal{F}, \mathsf{P})$ with respect to $\mathbb{F}$. The space $S^p_{\mathbb{F}}$ $(p\in[1,\infty))$ is the completion of the space of all eventually null p-integrable processes $X=(X_n)_{n=0}^\infty$ with respect to the norm

$$\|X\|_{S^p_{\mathbb{F}}} = (\sup_{T\in\mathcal{T}} \mathsf{E}|X_T|^p)^{1/p}. \tag{1.1}$$

In this paper, we shall be exclusively interested in the 'dyadic-tree filtration' in Borel σ-algebra of $[0,1]$. Let us denote by $\mathcal{F}_n$ the σ-algebra generated by the dyadic intervals $\{[\frac{j-1}{2^n}, \frac{j}{2^n}]\colon 1\leqslant j\leqslant 2^n\}$ (in particular, each algebra $\mathcal{F}_n$ is finite so $\mathcal{F}_n$-measurable random variables assume only finitely many values almost surely) and put $\mathbb{F}_d = (\mathcal{F}_n)_{n=0}^\infty$. Among the stopping-time spaces, in the present paper, we shall be exclusively interested in the spaces $S^p_{\mathbb{F}_d} = S^p$ for $p\in[1,\infty)$ together with the space B we are about to define.

Let $2^{<\omega}$ denote the binary tree, which indexes the normalised Haar basis $(h_t)_{t\in 2^{<\omega}}$ of $C(\Delta)$ (for details, see [36, Definition 2.2.2]). The space B may be viewed as the space with a minimal 1-unconditional Schauder basis that dominates the Haar basis in $C(\Delta)$. More specifically, B is the completion of the space $c_{00}(2^{<\omega})$ (or the linear span of $\{h_t\colon t\in 2^{<\omega}\}$ in $C(\Delta)$) with respect to the norm

$$\|(a_t)_{t\in 2^{<\omega}}\|_B = \sup\Big\{\Big\|\sum_{t\in 2^{<\omega}} a_t\varepsilon_t h_t\Big\|_{C(\Delta)}\colon t\in 2^{<\omega}, |\varepsilon_t|=1\Big\} \quad \Big((a_t)_{t\in 2^{<\omega}}\in c_{00}(2^{<\omega})\Big).$$

As observed in [4, Proposition 1], the norm in B may be isometrically realised as

$$\|(a_t)_{t\in 2^{<\omega}}\|_B = \sup\Big\{\sum_{t\in\mathcal{A}} |a_t|\colon \mathcal{A} \text{ is a branch of } 2^{<\omega}\Big\} \quad \Big((a_t)_{t\in 2^{<\omega}}\in c_{00}(2^{<\omega})\Big).$$

In a similar fashion, the norm in S^p $(p\in[1,\infty))$ can be seen as arising from the completion of

$$\|((a_t)_{t\in 2^{<\omega}}\|_{S^p} = \sup\Big\{(\sum_{s\in\mathcal{A}} |a_s|^p)^{1/p}\colon \mathcal{A}\subset 2^{<\omega} \text{ is an antichain}\Big\} \quad \Big((a_t)_{t\in 2^{<\omega}}\in c_{00}(2^{<\omega})\Big).$$

The standard unit vector basis $(e_t)_{t\in 2^{<\omega}}$ of c_{00} is then a 1-unconditional Schauder basis of S^p. Furthermore, if $(e_t^*)_{t\in 2^{<\omega}}$ denotes the coordinate functionals associated to $(e_t)_{t\in 2^{<\omega}}$, then B may be viewed as a closed subspace of $D=(S^1)^*$ spanned by $(e_t^*)_{t\in 2^{<\omega}}$ ([4, Proposition 1]).

In the present paper we extend the scale S^p defined in terms of ℓ^p-spaces $(p\in[1,\infty))$ to spaces, that we denote by S^E, parametrised by spaces having a 1-subsymmetric Schauder basis (with respect to the standard linear order of the dyadic tree). Even though the definition is not directly probabilistic, we still call S^E stopping-time Banach spaces. (It is still possible to define these spaces in the spirit of (1.1), however we shall not require it here.)

For a subset $\mathcal{A}\subseteq 2^{<\omega}$, let $P_{\mathcal{A}}$ denote the standard basis projection on $\mathcal{A}$, which is well defined by unconditionality of $(e_t)_{t\in 2^{<\omega}}$. We define the spaces S^E and B^E as completions of c_{00} with respect to the norms of $(a_t)_{t\in 2^{<\omega}}\in c_{00}(2^{<\omega})$:

$$\begin{aligned}
\|(a_t)_{t\in 2^{<\omega}}\|_{S^E} &= \sup\Big\{\|P_{\mathcal{A}}(a_t)_{t\in 2^{<\omega}}\|_E\colon \mathcal{A}\subset 2^{<\omega} \text{ is an antichain}\Big\},\\
\|(a_t)_{t\in 2^{<\omega}}\|_{B^E} &= \sup\Big\{\|P_{\mathcal{A}}(a_t)_{t\in 2^{<\omega}}\|_E\colon \mathcal{A}\subset 2^{<\omega} \text{ is a branch}\Big\},
\end{aligned}$$

respectively. Of course, B^{ℓ^1} corresponds to the space B defined above. Moreover, we put $D^E = (S^E)^*$.

Our motivation for introducing and studying the stopping-time spaces is twofold. First of all we observe that their canonical unit vector bases constitute natural examples of the so-called strategically reproducible bases introduced by Motakis, Müller, Schlumprecht, and the second-named author in [22] (all unexplained terminology will be discussed in subsequent sections), where it was proved that the Haar basis of L^1 is strategically reproducible. Strategic reproducibility is intimately related to the factorisation property of a Banach space with a Schauder basis, which in turn often implies primarity of the space or uniqueness of the maximal ideal of the algebra of operators on such a space.

For a space E with a 1-subsymmetric Schauder basis we denote by

- ▷ $(e_t)_{t\in 2^{<\omega}}$ the standard Schauder basis in S^E or B^E,
- ▷ $(f_t)_{t\in 2^{<\omega}}$ the associated biorthogonal functionals in X^* for $X = S^E$ or $X = B^E$.

We are now ready to present the first main result.

Theorem A. *Let E be a space with a 1-subsymmetric Schauder basis and let X denote either of the spaces S^E or B^E. Let $1 \leqslant p, p' \leqslant \infty$ with $1/p + 1/p' = 1$. If*

- ▷ $(e_{k,t} \colon k \in \mathbb{N},\ t \in 2^{<\omega})$ *denotes the standard Schauder basis of $\ell^p(X)$, and*
- ▷ $(f_{k,t} \colon k \in \mathbb{N},\ t \in 2^{<\omega})$ *denotes the corresponding biorthogonal functionals in $\ell^{p'}(X^*)$,*

then the following assertions hold true:

(i) $(e_t)_{t\in 2^{<\omega}}$ *is strategically reproducible.*
(ii) $((e_t, f_t))_{t\in 2^{<\omega}}$ *is strategically supporting and has the factorisation property in $X \times X^*$.*
(iii) $((e_{k,t}, f_{k,t}) \colon k \in \mathbb{N},\ t \in 2^{<\omega})$ *is strategically supporting and has the factorisation property in $\ell^p(X) \times \ell^{p'}(X^*)$.*
(iv) *If the 1-subsymmetric Schauder basis for E is incomparably non-c_0 on antichains, then the system $((f_t, e_t))_{t\in 2^{<\omega}}$ is strategically supporting and has the factorisation property in $D^E \times S^E$.*

Proof. Assertion (i) follows from Theorem 4.7 and (ii) from Corollary 4.8. Corollary 4.9 proves (iii) and Corollary 5.5 shows (iv). □

Apatsidis [3] studied (bounded, linear) operators from a space X with an unconditional Schauder basis into S^1 and proved, among other things, that:

- ▷ the space S^1 is *complementably homogeneous* in the sense that every isomorphic copy of S^1 in S^1 contains a further copy of S^1 that is, moreover complemented in S^1;
- ▷ more generally, if an operator $T \colon X \to S^1$ fixes a copy of S^1, then I_{S^1}, the identity operator on S^1, factors through T, which means that $I_{S_1} = ATB$ for some operators $A \colon S^1 \to X$ and $B \colon X \to S^1$;
- ▷ if S^1 is isomorphic to the ℓ^1-sum of a sequence of Banach spaces, then at least one summand therein is isomorphic to S^1—in particular, S^1 is primary.

The above results provide yet another example of resemblance between S^1 and L^1 as Enflo and Starbird [11] proved that the identity operator on L^1 factors through every operator $T \colon L^1 \to L^1$ that fixes a copy of L^1.

For a general Banach space X, Johnson and Dosev [10] considered the following subset of $\mathcal{B}(X)$, the algebra of all operators on X:

$$\mathcal{M}_X = \Big\{T \in \mathcal{B}(X) \colon I_X \neq ATB \ \ \big(A, B \in \mathcal{B}(X)\big)\Big\}. \tag{1.2}$$

Clearly, $\mathcal{M}_X$ is closed under multiplication from left and right, and it is (the unique maximal) ideal of $\mathcal{B}(X)$ if and only if it is closed under addition. It follows from the above-mentioned results by Apatsidis that $\mathcal{M}_X$ coincides with the set $\mathcal{S}_{S^1}(S^1)$ comprising all S^1-singular operators, that is operators that do not fix any copies of S^1. As S^1 is complementably homogeneous $\mathcal{S}_{S^1}(S^1)$ (hence $\mathcal{M}_{S^1}$) is the unique maximal ideal of $\mathcal{B}(S^1)$ (see [14, Corollary 2.3]).

The case where $\mathcal{M}_X$ is indeed the unique maximal ideal of $\mathcal{B}(X)$ is of particular interest from the point of view the theory of operator ideals; a list of spaces (containing many classical ones) for which $\mathcal{M}_X$ is closed under addition may be found in [17]. Usually the proof of the fact that for a given space X the set $\mathcal{M}_X$ is (or is not) closed under addition are specific to idiosyncratic properties of the space. In the present paper we propose a rather general criterion for a space with a strategically reproducible Schauder basis that allows for concluding that $\mathcal{M}_X$ is indeed closed under addition. In order to state it, we require to introduce auxiliary definitions.

Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. We say that a system $((e_\gamma, f_\gamma))_{\gamma\in\Gamma}$ in $X \times Y$:

- ▷ is *biorthogonal*, whenever $\langle e_{\gamma_1}, f_{\gamma_2}\rangle = \delta_{\gamma_1,\gamma_2}$ $(\gamma_1, \gamma_2 \in \Gamma)$;
- ▷ has the *positive factorisation property (in $X \times Y$)*, whenever for every $T \in \mathcal{B}(X)$ with $\inf_{\gamma\in\Gamma}\langle Te_\gamma, f_\gamma\rangle > 0$ one has $T \notin \mathcal{M}_X$.

Remark 1.1. In [22], the notion of the factorisation property for $((e_\gamma, f_\gamma))_{\gamma\in\Gamma}$ was introduced in the following way: for every operator $T \in \mathcal{B}(X)$ with $\inf_{\gamma\in\Gamma} |\langle Te_\gamma, f_\gamma\rangle| > 0$ one has $T \notin \mathcal{M}_X$. Clearly, the factorisation property implies the positive factorisation property. For unconditional Schauder bases $(e_\gamma)_{\gamma\in\Gamma}$ and the associated coordinate functionals $(f_\gamma)_{\gamma\in\Gamma}$, the factorisation property is equivalent to the positive factorisation property.

We are now ready to state the general criterion for $\mathcal{M}_X$ being closed under addition (hence being the unique maximal of $\mathcal{B}(X)$).

Theorem B. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. Suppose that*

- ▷ $(e_n)_{n=1}^\infty$ *is a basis of X with respect to the topology $\sigma(X, Y)$,*
- ▷ $(f_n)_{n=1}^\infty$ *is a basis of Y with respect to the topology $\sigma(Y, X)$,*
- ▷ *the system $((e_n, f_n))_{n=1}^\infty$ is biorthogonal,*
- ▷ *there exists $c > 0$ such that*

$$c\|x\| \leqslant \sup_{\|y\|\leqslant 1}\langle x, y\rangle \leqslant \|x\| \qquad (x \in X), \tag{1.3}$$

- ▷ $((e_n, f_n))_{n=1}^\infty$ *is strategically supporting,*
- ▷ $((e_n, f_n))_{n=1}^\infty$ *has the positive factorisation property.*

Then $\mathcal{M}_X$ is the unique closed proper maximal ideal of $\mathcal{B}(X)$.

A proof for Theorem B is provided in Section 3.

The framework of Theorem B is general enough to encompass dual spaces that have weak* Schauder bases such as the space ℓ^∞. With the aid of this result, we shall prove that the stopping-time spaces, their duals, and various other related spaces X have the property that $\mathcal{M}_X$ is closed under addition (and hence the unique maximal ideal of $\mathcal{B}(X)$).

Theorem C. *Let X be any Banach space and let E denote a space with a normalised 1-subsymmetric Schauder basis. Then $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$ in the following cases:*

- ▷ $X = S^E$, $X = B^E$;
- ▷ $X = \ell^p(S^E)$ *or* $X = \ell^p(B^E)$, $1 \leqslant p \leqslant \infty$;
- ▷ $X = D^E = (S^E)^*$, *whenever the 1-subsymmetric Schauder basis of E is incomparably non-c_0 on antichains.*

Consequently, the spaces $\ell^p(S^E)$, $\ell^p(B^E)$ $(1 \leqslant p \leqslant \infty)$ are primary.

Proof. The first two assertions follow immediately from combining the results Corollary 4.8, Corollary 5.5, Corollary 4.9, and Theorem B.

To see the last claim holds true, first note that for any Banach space X, we have that $\ell^p(\ell^p(X))$ is isomorphic to $\ell^p(X)$. Now, let $X = \ell^p(S^E)$ or $X = \ell^p(B^E)$. Since $\mathcal{M}_X$ is closed under addition, for a given operator $T \in \mathcal{B}(X)$ it is impossible that both T and $I - T$ are in $\mathcal{M}_X$ (otherwise we would have $I \in \mathcal{M}_X$). Thus, whenever P is a projection on X, the identity I_X on X either factors through P or $I_X - P$. This implies that either the range of X or the range of $I_X - P$ contains a complemented copy of X. Finally, by using Pełczyński's decomposition method ([34]; see also [38, II.B.24]) the proof is complete. □

The above-described techniques find their applications beyond the class of stopping-time spaces. More specifically, Corollary 6.2, Corollary 6.5, and Corollary 6.9 expand the list of spaces X collected in [17] for which $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$; we record these results jointly below.

Theorem D. *Let X be one of the spaces:*

- ▷ L^p $(1 \leqslant p < \infty)$, H^1, BMO, SL^∞
- ▷ $\ell^p(\ell^q)$ *for* $1 \leqslant p \leqslant \infty$ *and* $1 < q < \infty$, *or*
- ▷ $H^1(H^1)$ *or* $L^p(L^q)$ *for* $1 < p, q < \infty$.

Then $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$.

2. Preliminaries

We use standard Banach space terminology that is mostly in-line with [26]. We consider real Banach spaces, however the results, with some effort, may be extended to complex scalars too. By an *operator* we understand a bounded linear map acting between normed spaces. A Banach space X is *primary* as long as whenever $X = W \oplus V$, then at least one of the subspaces W, V is isomorphic to X.

Let X be a Banach space and $1 \leqslant p \leqslant \infty$. We define

$$\ell^p(X) = \Big\{(x_n)_{n=1}^\infty \colon x_n \in X,\ \|(x_n)_{n=1}^\infty\|_{\ell^p(X)} < \infty\Big\},$$

where the norm $\|\cdot\|_{\ell^p(X)}$ is given by

$$\|(x_n)_{n=1}^\infty\|_{\ell^p(X)} = \Big\|\Big(\|x_n\|_X\Big)_{n=1}^\infty\Big\|_{\ell^p}.$$

Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces and let $\sigma(X, Y)$ denote the weakest topology such that all the maps $\langle\cdot, y\rangle$ $(y \in Y)$ are continuous. A sequence $(e_n)_{n=1}^\infty$ in X is a *basis of X with respect to the topology $\sigma(X, Y)$*, whenever for every $x \in X$ there exists a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that

$$x = \sum_{n=1}^\infty a_n e_n, \tag{2.1}$$

where the above series converges in the $\sigma(X, Y)$ topology.

- ▷ If $Y = X^*$, that is when $\sigma(X, X^*)$ is the usual weak topology, we then call bases *Schauder bases* (by the Orlicz–Pettis theorem, the series (2.1) converges in the norm topology).
- ▷ When the weak* topology $\sigma(X^*, X)$ is considered, we call the corresponing bases *weak* Schauder bases.*
- ▷ If $\|e_n\|_X = 1$ $(n \in \mathbb{N})$, the basis $(e_n)_{n=1}^\infty$ is called *normalised.*

- ▷ Let $(x_n)_{n=1}^\infty$ and $(\xi_n)_{n=1}^\infty$ be sequences in X. We say that $(x_n)_{n=1}^\infty$ is C-*dominated* by $(\xi_n)_{n=1}^\infty$, whenever for all scalar sequences $(a_n)_{n=1}^\infty$

$$\sum_{n=1}^\infty a_n x_n \text{ converges whenever } \sum_{n=1}^\infty a_n \xi_n \text{ converges}$$

(both series convergence is with respect to the $\sigma(X,Y)$ topology) and for all sequences $(a_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty a_n\xi_n$ converges in the $\sigma(X,Y)$ topology we have

$$\Big\|\sum_{n=1}^\infty a_n x_n\Big\| \leqslant C\Big\|\sum_{n=1}^\infty a_n \xi_n\Big\|.$$

If $(x_n)_{n=1}^\infty$ is C-dominated by $(\xi_n)_{n=1}^\infty$ for some $C>0$, then we say that $(x_n)_{n=1}^\infty$ *is dominated by* $(\xi_n)_{n=1}^\infty$.

- ▷ If $(x_n)_{n=1}^\infty$ is C-dominated by $(\xi_n)_{n=1}^\infty$ and $(\xi_n)_{n=1}^\infty$ is C-dominated by $(x_n)_{n=1}^\infty$, then we say that $(x_n)_{n=1}^\infty$ *is C-equivalent to* $(\xi_n)_{n=1}^\infty$. If $(x_n)_{n=1}^\infty$ is C-equivalent to $(\xi_n)_{n=1}^\infty$ for some $C>0$, we simply say that $(x_n)_{n=1}^\infty$ *is equivalent to* $(\xi_n)_{n=1}^\infty$.

2.1. **The dyadic tree.** Let $2^{<\omega}$ denote the rooted dyadic tree, that is, the tree comprising all finite sequences of 0s and 1s, with the root denoted by $\varnothing$. Thus, $2^{<\omega} = \bigcup_{n=0}^\infty \{0,1\}^n$ with the convention $\{0,1\}^0 = \{\varnothing\}$. Moreover, we write $2^{\leqslant n} = \bigcup_{j=1}^n \{0,1\}^n$. For $s\in 2^{\leqslant n}$ and $m\leqslant n$, we write $s|_m = (s(1),\ldots,s(m))$ when $m>0$ and $s|_0=\varnothing$. As every $s\in 2^{<\omega}$ belongs to a uniquely determined set $2^{\leqslant n}$, in such a case we call n the *length* of s and denote it by $|s|$. The set $2^{<\omega}$ is naturally ordered by the initial-segment partial ordering: $s\sqsubseteq t$ whenever $|s|\leqslant|t|$ and $s=t|_{|s|}$. For $A,B\subseteq 2^{<\omega}$, we write $A\sqsubseteq B$ whenever $s\sqsubseteq t$ for all $s\in A$ and $t\in B$. For $t,s\in 2^{<\omega}$ we denote by $t^\frown s$ the *concatenation* of t and s: $t^\frown s = (t(1),\ldots,t(|t|),s(1),\ldots,s(|s|))$.

A *branch* of $2^{<\omega}$ is a maximal linearly ordered (with respect to '$\sqsubseteq$') subset of $2^{<\omega}$. Let us denote by β the set of all branches in $2^{<\omega}$. An *antichain* in $2^{<\omega}$ is any subset comprising pairwise $\sqsubseteq$-incomparable elements. Given a node $t\in 2^{<\omega}\setminus\{\varnothing\}$, let $\tilde t$ denote the unique node such that $t=\tilde t^\frown\alpha$ for some $\alpha\in\{0,1\}$. We say that $\mathcal{T}\subset 2^{<\omega}$ is a *subtree*, if $\mathcal{T}$ is order isomorphic to $2^{<\omega}$ with respect to the partial order '$\sqsubseteq$'.

For $s\neq t\in 2^{<\omega}\setminus\{\varnothing\}$, we say that s *is left of* t, if n is the unique maximal integer such that $s(n)=t(n)$ (with the agreement that $r(0)=\varnothing$ for all $r\in 2^{<\omega}$) and $s(n+1)=0$ (which of course implies that $t(n+1)=1$). Otherwise, we say that s *is right of* t. Given $A,B\subseteq 2^{<\omega}\setminus\{\varnothing\}$, we say that A *is left (right) of* B, whenever all elements of A are left (right) to all elements of B. By '$<$', we denote the *standard linear order on* $2^{<\omega}$, *i.e.*, $s<t$ if and only if $|s|<|t|$ or if $|s|=|t|$ and s is left of t. For $A,B\subset 2^{<\omega}$, we write $A<B$ whenever $s<t$ for all $s\in A$ and $t\in B$. Let $\mathcal{O}\colon 2^{<\omega}\to\mathbb{N}$ denote the bijective order preserving map with respect to the standard linear order '$<$' on the tree $2^{<\omega}$ and the natural order on $\mathbb{N}$. If $t\in 2^{<\omega}$ and $k\in\mathbb{Z}$, we write $t+k$ for the unique $t'\in 2^{<\omega}$ such that $\mathcal{O}(t')=\mathcal{O}(t)+k$, if it exists. We say that a subtree $\mathcal{T}$ of $2^{<\omega}$ is *linearly order isomorphic* to $2^{<\omega}$, if $\mathcal{T}$ is order isomorphic with respect to the standard linear order.

Let E be a space with a normalised Schauder basis indexed by $2^{<\omega}$, say $(e_t)_{t\in 2^{<\omega}}$. We say that $(e_t)_{t\in 2^{<\omega}}$ is 1-*unconditional*, whenever for all finitely supported sequences of scalars $(a_t)_{t\in 2^{<\omega}}$ and $(\gamma_t)_{t\in 2^{<\omega}}$ one has

$$\Big\|\sum_{t\in 2^{<\omega}}\gamma_t a_t e_t\Big\|_E \leqslant \sup_{t\in 2^{<\omega}}|\gamma_t|\Big\|\sum_{t\in 2^{<\omega}} a_t e_t\Big\|_E.$$

Moreover, we say that $(e_t)_{t\in 2^{<\omega}}$ is 1-*spreading*, if $(e_t)_{t\in 2^{<\omega}}$ is 1-equivalent to each of its increasing subsequences (with respect to the standard linear order), *i.e.*, if for all finitely supported sequences of scalars $(a_t)_{t\in 2^{<\omega}}$

$$\Big\|\sum_{t\in 2^{<\omega}} a_t e_t\Big\|_E = \Big\|\sum_{t\in 2^{<\omega}} a_t e_{s_t}\Big\|_E,$$

whenever $(s_t)_{t\in 2^{<\omega}}$ is such that $s_{t_1} < s_{t_2}$ if $t_1 < t_2$. We say that the sequence $(e_t)_{t\in 2^{<\omega}}$ is 1-*subsymmetric* if it is 1-unconditional as well as 1-spreading.

3. The factorisation property and the uniqueness of maximal ideals

Let E be a Banach space with a normalised Schauder basis $(e_n)_{n=1}^\infty$. It is tempting to speculate that for an operator $T \in \mathcal{B}(E)$, the condition $\inf_n |\langle Te_n, e_n^*\rangle| > 0$ is a sufficient condition for factoring the identity, *i.e.*, for generating the whole $\mathcal{B}(E)$ as an ideal by T. Using Gowers' space with an unconditional Schauder basis, one can construct a counterexample ([18, Theorem 2.1]). However, for many classical spaces such as BMO, SL^∞, $\ell^p(\ell^q)$, and other, the condition $\inf_n |\langle Te_n, e_n^*\rangle| > 0$ is indeed sufficient (see Section 6 for details).

Definition 3.1. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. Suppose that $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ are sequences in X and Y, respectively. We say that $((e_n, f_n))_{n=1}^\infty$ *is almost annihilating for a set* $\mathcal{A} \subset \mathcal{B}(X)$, whenever for every $T \in \mathcal{A}$ and $\eta > 0$ there exist sequences $(x_n)_{n=1}^\infty$ in X and $(y_n)_{n=1}^\infty$ in Y, respectively, such that

(i) $(x_n)_{n=1}^\infty$ is dominated by $(e_n)_{n=1}^\infty$;
(ii) $(y_n)_{n=1}^\infty$ is dominated by $(f_n)_{n=1}^\infty$;
(iii) $\inf_{n\in\mathbb{N}}\langle x_n, y_n\rangle \geqslant 1$;
(iv) $\sup_{n\in\mathbb{N}}\langle Tx_n, y_n\rangle \leqslant \eta$.

The notion of almost annihilation is a property of that in tandem with the factorisation property yields the uniqueness of $\mathcal{M}_X$ as a maximal ideal.

Theorem 3.2. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. Let $(e_n)_{n=1}^\infty$ be a basis for X with respect to the $\sigma(X, Y)$ topology, let $(f_n)_{n=1}^\infty$ be a basis for space Y with respect to the $\sigma(Y, X)$ topology and assume there exists a constant $c > 0$ such that*

$$c\|x\| \leqslant \sup_{\|y\|\leqslant 1}\langle x, y\rangle \leqslant \|x\| \qquad (x \in X). \tag{3.1}$$

If $((e_n, f_n))_{n=1}^\infty$ is almost annihilating $\mathcal{M}_X$ and has the positive factorisation property, then $\mathcal{M}_X$ is the unique closed proper maximal ideal of $\mathcal{B}(X)$.

Proof. Let $0 < \eta < 1$, $S \in \mathcal{M}_X$, $T \in \mathcal{B}(X)$ and $S + T \notin \mathcal{M}_X$. Then there exist operators $A, B \in \mathcal{B}(X)$ such that $I_X = B(S + T)A$. Since BSA must be in $\mathcal{M}_X$ and $((e_n, f_n))_{n=1}^\infty$ is almost annihilating for $\mathcal{M}_X$, we can find sequences $(x_n)_{n=1}^\infty$ in X dominated by $(e_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in Y dominated by $(f_n)_{n=1}^\infty$ such that

$$\langle x_n, y_n\rangle \geqslant 1 \quad\text{and}\quad \langle BSAx_n, y_n\rangle \leqslant \eta \qquad (n \in \mathbb{N}).$$

Hence,

$$1 \leqslant \langle x_n, y_n\rangle = \langle B(S + T)Ax_n, y_n\rangle \leqslant \eta + \langle BTAx_n y_n\rangle \qquad (n \in \mathbb{N}). \tag{3.2}$$

Define the operators $L\colon X \to X$ and $R\colon Y \to Y$ as the linear extensions of

$$Le_n = x_n \quad\text{and}\quad Rf_n = y_n \qquad (n \in \mathbb{N}).$$

By (i) and (ii) in Definition 3.1, the operators L and R are well defined and bounded. Next, put $U = R^*BTAL$, where R^* denotes the unique operator such that $\langle R^*x, y\rangle = \langle x, Ry\rangle$ for all $x \in X$, $y \in Y$. Note that (3.1) yields $R^* \in \mathcal{B}(X)$ and by (3.2) we obtain

$$\inf_n\langle Ue_n, f_n\rangle = \inf_n\langle BTAx_n, y_n\rangle \geqslant 1 - \eta > 0.$$

Thus, since $((e_n, f_n))_{n=1}^\infty$ has the positive factorisation property, we obtain $U \notin \mathcal{M}_X$ and consequently $T \notin \mathcal{M}_X$. We showed that $\mathcal{M}_X$ is closed under addition, and so it is the unique closed proper maximal ideal of $\mathcal{B}(X)$; see [10, Section 5] for details. ☐

Definition 3.3. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. Let $(e_n)_{n=1}^\infty$ be a basis for X with respect to the $\sigma(X, Y)$ topology and let $(f_n)_{n=1}^\infty$ be a basis for Y with respect to the $\sigma(Y, X)$ topology. We say that $((e_n, f_n))_{n=1}^\infty$ *is strategically supporting (in $X \times Y$)* if for all $\eta > 0$ and all partitions N_1, N_2 of $\mathbb{N}$ there exists $i \in \{1, 2\}$ and

$$\begin{aligned}
&\exists \text{ finite } E_1 \subset N_i\ \exists(\lambda_j^1)_j, (\mu_j^1)_j \in \mathbb{R}^{E_1}\ \forall(\varepsilon_j^1)_j \in \{\pm1\}^{E_1}\\
&\exists \text{ finite } E_2 \subset N_i\ \exists(\lambda_j^2)_j, (\mu_j^2)_j \in \mathbb{R}^{E_2}\ \forall(\varepsilon_j^2)_j \in \{\pm1\}^{E_2}\\
&\quad\vdots\\
&\exists \text{ finite } E_k \subset N_i\ \exists(\lambda_j^k)_j, (\mu_j^k)_j \in \mathbb{R}^{E_k}\ \forall(\varepsilon_j^k)_j \in \{\pm1\}^{E_k}\\
&\quad\vdots
\end{aligned}$$

such that if we define

$$x_k = \sum_{j\in E_k} \varepsilon_j^k \lambda_j^k e_j \quad \text{and} \quad y_k = \sum_{j\in E_k} \varepsilon_j^k \mu_j^k f_j \qquad (k \in \mathbb{N}),$$

we have that

(i) $(x_k)_{k=1}^\infty$ is dominated by $(e_k)_{k=1}^\infty$;
(ii) $(y_k)_{k=1}^\infty$ is dominated by $(f_k)_{k=1}^\infty$;
(iii) $1 \leqslant \langle x_k, y_k\rangle \leqslant 1 + \eta$, $k \in \mathbb{N}$;
(iv) $\lambda_j^k \mu_j^k \geqslant 0$, $k \in \mathbb{N}$, $j \in E^k$.

We are now prepared to prove Theorem B.

Proof of Theorem B. Let $\eta > 0$ and $T \in \mathcal{M}_X$ and define $N_1 = \{n \in \mathbb{N} \colon \langle Te_n, f_n\rangle \leqslant \eta/(1+\eta)\}$ and $N_2 = \mathbb{N} \setminus N_1$. Since $((e_n, f_n))_{n=1}^\infty$ is strategically supporting, there exist sequences $(x_k)_{k=1}^\infty$ in X and $(y_k)_{k=1}^\infty$ in Y given by

$$x_k = \sum_{j\in E_k} \varepsilon_j^k \lambda_j^k e_j \quad \text{and} \quad y_k = \sum_{j\in E_k} \varepsilon_j^k \mu_j^k f_j \qquad (k \in \mathbb{N})$$

that satisfy (i)–(iv) in Definition 3.3, and there exists $i \in \{1, 2\}$ such that $E_k \subset N_i$ for all $k \in \mathbb{N}$. Additionally, we note that in the k^{th} step of Definition 3.3, we are free to choose the signs $(\varepsilon_j^k)_j \in \{\pm1\}^{E_k}$. Let E_ε denote the average over all those possible choices of signs and observe that

$$\begin{aligned}
\mathsf{E}_\varepsilon\langle Tx_k, y_k\rangle &= \mathsf{E}_\varepsilon\Big\langle T\sum_{j\in E_k} \varepsilon_j\lambda_j^k e_j, \sum_{l\in E_k} \varepsilon_l\mu_l^k f_l\Big\rangle = \mathsf{E}_\varepsilon \sum_{j,l\in E_k} \varepsilon_j\varepsilon_l\lambda_j^k\mu_l^k\langle Te_j, f_l\rangle\\
&= \sum_{j\in E_k} \lambda_j^k\mu_j^k\langle Te_j, f_j\rangle.
\end{aligned} \tag{3.3}$$

Note that we only need to verify Definition 3.1 (iv).

Now suppose to the contrary that $i = 2$. In this case, using (3.3) we would have picked signs $(\varepsilon_j^k)_j \in \{\pm1\}^{E_k}$ in the k^{th} step of Definition 3.3 so that

$$\langle Tx_k, y_k\rangle \geqslant \sum_{j\in E_k} \lambda_j^k\mu_j^k\langle Te_j, f_j\rangle \qquad (k \in \mathbb{N}).$$

Since $E_k \subset N_2$, $\lambda_j^k\mu_j^k \geqslant 0$, $j \in E_k$, by the biorthogonality of $((e_n, f_n))_{n=1}^\infty$ and $\langle x_k, y_k\rangle \geqslant 1$, it follows that

$$\langle Tx_k, y_k\rangle \geqslant \eta/(1+\eta)\sum_{j\in E_k} \lambda_j^k\mu_j^k = \eta/(1+\eta)\langle x_k, y_k\rangle \geqslant \eta/(1+\eta) \qquad (k \in \mathbb{N}). \tag{3.4}$$

We define $A\colon X\to X$ and $B\colon Y\to Y$ as the linear extensions of

$$Ae_k = x_k \quad\text{and}\quad Bf_k = y_k \qquad (k\in\mathbb{N})$$

and note that since $(x_k)_{k=1}^\infty$ is dominated by $(e_k)_{k=1}^\infty$ and $(y_k)_{k=1}^\infty$ is dominated by $(f_k)_{k=1}^\infty$, A and B are well defined and bounded. Let B^* denote the unique operator such that $\langle B^*x, y\rangle = \langle x, By\rangle$ for all $x\in X$, $y\in Y$, and note that B^* is bounded by (1.3). Put $S = B^*TA$ and observe that by (3.4), we obtain $\langle Se_k, f_k\rangle = \langle Tx_k, y_k\rangle \geqslant \eta/(1+\eta) > 0$ for all $k\in\mathbb{N}$. Since $((e_n, f_n))_{n=1}^\infty$ also has the factorisation property, we have $S\notin \mathcal{M}_X$; hence, $T\notin\mathcal{M}_X$, which contradicts our assumption $T\in\mathcal{M}_X$.

Thus $i=1$. In this case, using (3.3) we would have picked signs $(\varepsilon_j^k)_j\in\{\pm1\}^{E_k}$ in the k^{th} step of Definition 3.3 such that

$$\langle Tx_k, y_k\rangle \leqslant \sum_{j\in E_k}\lambda_j^k\mu_j^k\langle Te_j, f_j\rangle \qquad (k\in\mathbb{N}).$$

Since $E_k\subset N_1$, $\lambda_j^k\mu_j^k\geqslant 0$, $j\in E_k$, as $((e_n,f_n))_{n=1}^\infty$ is biorthogonal and $\langle x_k, y_k\rangle\leqslant 1+\eta$, it follows that

$$\langle Tx_k, y_k\rangle \leqslant \eta/(1+\eta)\sum_{j\in E_k}\lambda_j^k\mu_j^k = \eta/(1+\eta)\langle x_k, y_k\rangle\leqslant\eta \qquad (k\in\mathbb{N}).$$ □

4. Factorisation of the identity in S^E and B^E

The present section specialises to Banach spaces of the form S^E and B^E to which we shall apply the just-established results. For this, let us fix a Banach space E with a normalised 1-subsymmetric Schauder basis $(e_t)_{t\in 2^{<\omega}}$; we denote by $(e_t^*)_{t\in 2^{<\omega}}$ the associated biorthogonal functionals.

4.1. **Linearly order-isomorphic subtrees.** We begin the present section with a result that is known to the experts in Ramsey theory, however for the sake of completeness, we include its proof.

Lemma 4.1. *Let $\mathcal{S}$ be a subset of $2^{<\omega}$. Then either $\mathcal{S}$ or $2^{<\omega}\setminus\mathcal{S}$ contains a subtree $\mathcal{T}$ which is linearly order-isomorphic to $2^{<\omega}$.*

Proof. Given a subset $\mathcal{S}\subset 2^{<\omega}$, either $\mathcal{S}$ or $2^{<\omega}\setminus\mathcal{S}$ contains a subtree $\mathcal{T}'$ [2, Proposition 4] (see [12, Proposition 3.a.9] for a proof). By inductively replacing nodes in $\mathcal{T}'$ which violate the standard linear order with conforming successors, we can find a subtree $\mathcal{T}\subset\mathcal{T}'$ as claimed. □

Lemma 4.2. *Let X denote either of the spaces S^E or B^E. Given a linearly order isomorphic subtree $\mathcal{T} = \{s_t\colon t\in 2^{<\omega}\}$, we define pointwise the operators $B, Q\colon X\to X$ by*

$$Bx = \sum_{t\in 2^{<\omega}}\langle x, e_t^*\rangle e_{s_t} \qquad\text{and}\qquad Qx = \sum_{t\in 2^{<\omega}}\langle x, e_{s_t}^*\rangle e_t \qquad (x\in X). \tag{4.1}$$

Then we have

$$QB = I_X \qquad\text{and}\qquad \|B\| = \|Q\| = 1, \tag{4.2}$$

where I_X denotes the identity operator on X.

Proof. Firstly, we note that

$$QBe_t = e_t \qquad (t\in 2^{<\omega}) \tag{4.3}$$

and hence $QB = I_X$.

Secondly, for every finite sequence of scalars $(a_t)_{t\in 2^{<\omega}}$, we have that

$$\Big\|B\sum_t a_te_t\Big\|_X = \Big\|\sum_t a_te_{s_t}\Big\|_X = \sup_{\mathcal{C}}\Big\|\sum_{t: s_t\in\mathcal{C}} a_te_{s_t}\Big\|_E. \tag{4.4}$$

If $X = S^E$, the supremum above extends over all antichains $\mathcal{C} \subset 2^{<\omega}$, and if $X = B^E$, the supremum extends over all branches $\mathcal{C} \subset 2^{<\omega}$. Since our subtree is linearly order isomorphic to $2^{<\omega}$, *i.e.*, $s_{t_1} < s_{t_2}$ whenever $t_1 < t_2$, we obtain by the 1-subsymmetry of $(e_t)_{t\in 2^{<\omega}}$ that

$$\Big\| \sum_{t:s_t\in\mathcal{C}} a_t e_{s_t} \Big\|_E \leqslant \Big\| \sum_{t:s_t\in\mathcal{C}} a_t e_t \Big\|_E.$$

Combining the latter estimate with (4.4) yields

$$\Big\| B \sum_t a_t e_t \Big\|_X \leqslant \sup_{\mathcal{C}} \Big\| \sum_{t:s_t\in\mathcal{C}} a_t e_t \Big\|_E.$$

Note that since the set $\{t\colon s_t \in \mathcal{C}\}$ is either an antichain or a branch (depending on whether $X = S^E$ or $X = B^E$), the latter estimate together with the definition of the norm in X yield $\|B\| \leqslant 1$, as claimed.

Thirdly, observe that

$$\Big\| Q \sum_t a_t e_t \Big\|_X = \Big\| \sum_t a_{s_t} e_t \Big\|_X = \sup_{\mathcal{C}} \Big\| \sum_{t\in\mathcal{C}} a_{s_t} e_t \Big\|_E, \tag{4.5}$$

where in case $X = S^E$, the supremum above extends over all antichains $\mathcal{C} \subset 2^{<\omega}$, and if $X = B^E$, the supremum extends over all branches $\mathcal{C} \subset 2^{<\omega}$. Since $\mathfrak{T}$ is linearly order isomorphic to $2^{<\omega}$, we obtain by the 1-subsymmetry of $(e_t)_{t\in 2^{<\omega}}$ that

$$\Big\| \sum_{t\in\mathcal{C}} a_{s_t} e_t \Big\|_E \leqslant \Big\| \sum_{t\in\mathcal{C}} a_{s_t} e_{s_t} \Big\|_E.$$

So far, together with (4.5), we showed that

$$\Big\| Q \sum_t a_t e_t \Big\|_X \leqslant \sup_{\mathcal{C}} \Big\| \sum_{t\in\mathcal{C}} a_{s_t} e_{s_t} \Big\|_E,$$

If $X = S^E$, then $\{t\colon s_t \in \mathcal{C}\}$ is an antichain, and if $X = B^E$ it is a branch. Hence, the latter estimate together with the definition of the norm in X gives us the desired estimate $\|Q\| \leqslant 1$. □

Remark 4.3. Dualising Lemma 4.2 yields $I_{X^*} = B^* Q^*$. Moreover, observe that for finitely supported $x \in S^E$ and all $x^* \in X^*$ we have

$$\langle Bx, x^*\rangle = \sum_{t\in 2^{<\omega}} \langle x, e_t^*\rangle \langle e_{s_t}, x^*\rangle = \Big\langle x, \sum_{t\in 2^{<\omega}} \langle e_{s_t}, x^*\rangle e_t^* \Big\rangle = \langle x, B^* x^*\rangle.$$

as well as

$$\langle Qx, x^*\rangle = \sum_{t\in 2^{<\omega}} \langle x, e_{s_t}^*\rangle \langle e_t, x^*\rangle = \Big\langle x, \sum_{t\in 2^{<\omega}} \langle e_t, x^*\rangle e_{s_t}^* \Big\rangle = \langle x, Q^* x^*\rangle,$$

We record that

$$B^* x^* = \sum_{t\in 2^{<\omega}} \langle e_{s_t}, x^*\rangle e_t^* \qquad \text{and} \qquad Q^* x^* = \sum_{t\in 2^{<\omega}} \langle e_t, x^*\rangle e_{s_t}^* \qquad (x^* \in X^*).$$

By appealing to Lemma 4.2, we have thus established that:

- ▷ in either space, S^E or B^E, $(e_{s_t})_{t\in 2^{<\omega}}$ is 1-equivalent to $(e_t)_{t\in 2^{<\omega}}$,
- ▷ in either space, D^E or $(B^E)^*$, $(e_{s_t}^*)_{t\in 2^{<\omega}}$ is 1-equivalent to $(e_t^*)_{t\in 2^{<\omega}}$.

4.2. **Subspace annihilation.** The present short section collects two lemmata describing the weakly null nature of branches/antichains in D^E and the dual space of B^E, respectively.

Lemma 4.4. *Let $x^* \in D^E$, $t \in 2^{<\omega}$, and let $\Gamma \subset 2^{<\omega}$ be a branch. Then*

$$\lim_{n\to\infty} \sup_{s\in\Gamma, |s|\geqslant n} |\langle e_{t^\frown s}, x^*\rangle| = 0.$$

Proof. We can assume that $\|x^*\| = 1$. Let $\Gamma = \{t_j : j \in \mathbb{N}\}$ with $\varnothing = t_1 \sqsubset t_2 \sqsubset t_3 \cdots$ and put $s_j = t^\frown t_j$, $j \in \mathbb{N}$. Let $N \in \mathbb{N}$, $(\omega_j) \subset \mathbb{R}^N$ and observe that since $|\mathcal{A} \cap \{s_j : j \in \mathbb{N}\}| \leqslant 1$ for every antichain $\mathcal{A}$, we have

$$\Big\langle \sum_{j=1}^N \omega_j e_{s_j}, x^*\Big\rangle \leqslant \Big\|\sum_{j=1}^N \omega_j e_{s_j}\Big\|_{S^E} = \sup_{\mathcal{A}} \Big\|\sum_{s\in\mathcal{A}} \Big\langle \sum_{j=1}^N \omega_j e_{s_j}, e_s^*\Big\rangle e_s\Big\|_E \leqslant \max_{1\leqslant j\leqslant N} |\omega_j|.$$

Defining $\omega_j = \operatorname{sign}(\langle e_{s_j}, x^*\rangle)$, the latter estimate yields

$$\sum_{j=1}^N |\langle e_{s_j}, x^*\rangle| \leqslant 1 \qquad (N \in \mathbb{N}).$$

Thus, $\lim_j |\langle e_{s_j}, x^*\rangle| = 0$ as claimed. □

Lemma 4.5. *Let $x^* \in (B^E)^*$, $t \in 2^{<\omega}$, and let $\mathcal{A} = \{t_j : j \in \mathbb{N}\}$ be an antichain. Suppose that $t_i < t_j$, whenever $i < j$. Then*

$$\lim_{j\to\infty} \langle e_{t_j}, x^*\rangle = 0.$$

Proof. We can assume that $\|x^*\|_{(B^E)^*} = 1$. Let $N \in \mathbb{N}$, $(\omega_j) \in \mathbb{R}^N$ and note that for each branch $\Gamma \in \beta$, we have $|\Gamma \cap \mathcal{A}| \leqslant 1$. Hence,

$$\Big\langle \sum_{j=1}^N \omega_j e_{t_j}, x^*\Big\rangle \leqslant \Big\|\sum_{j=1}^N \omega_j e_{t_j}\Big\|_{B^E} = \sup_{\Gamma\in\beta} \Big\|\sum_{s\in\Gamma} \Big\langle \sum_{j=1}^N \omega_j e_{t_j}, e_s^*\Big\rangle e_s\Big\|_E \leqslant \max_{1\leqslant j\leqslant N} |\omega_j|.$$

Next, we define $\omega_j = \operatorname{sign}(\langle e_{t_j}, x^*\rangle)$ and obtain from the latter estimate

$$\sum_{j=1}^N |\langle e_{t_j}, x^*\rangle| \leqslant 1 \qquad (N \in \mathbb{N}).$$

Thus, $\lim_j |\langle e_{t_j}, x^*\rangle| = 0$ as claimed. □

4.3. **Strategic reproducibility and factorisation.** For a Banach space X we denote by $\operatorname{cof}(X)$ the set of cofinite dimensional subspaces of X, while $\operatorname{cof}_{w^*}(X^*)$ denotes the set of cofinite dimensionl w^*-closed subspaces of X^*. Hereinafter, unless otherwise stated,

- ▷ X is either S^E or B^E,
- ▷ $(e_s)_{s\in 2^{<\omega}}$ denotes the standard Schauder basis in X, and
- ▷ $(e_s^*)_{s\in 2^{<\omega}}$ are the associated biorthogonal functionals.

Since in either case $(e_s)_{s\in 2^{<\omega}}$ is 1-unconditional, strategic reproducibility (strategic reproducibility was conceived in [22] and investigated further in [24]) for $(e_s)_{s\in 2^{<\omega}}$ is reads as follows.

Definition 4.6. Let $C \geqslant 1$ and consider the following two-player game between player (I) and player (II). For $t \in 2^{<\omega}$, turn t is played out in three steps.

Step 1: Player (I) chooses $\eta_t > 0$, $W_t \in \operatorname{cof}(X)$, and $G_t \in \operatorname{cof}_{w^*}(X^*)$,

Step 2: Player (II) chooses a finite subset E_t of $2^{<\omega}$ and sequences of non-negative real numbers $(\lambda_s^{(t)})_{s\in E_t}$, $(\mu_s^{(t)})_{s\in E_t}$ satisfying

$$\sum_{s\in E_t} \lambda_s^{(t)} \mu_s^{(t)} = 1.$$

Step 3: Player (I) chooses $(\varepsilon_s^{(t)})_{s\in E_t}$ in $\{-1,1\}^{E_t}$.

We say that player (II) has a winning strategy in the game $\mathrm{Rep}_{(X,(e_s))}(C)$ if he can force the following properties on the result:

For all $t \in 2^{<\omega}$ we set

$$b_t = \sum_{s\in E_t} \varepsilon_s^{(t)} \lambda_s^{(t)} e_s \qquad \text{and} \qquad b_t^* = \sum_{s\in E_t} \varepsilon_s^{(t)} \mu_s^{(t)} e_s^*$$

and demand:

(i) the sequences $(b_t)_{t\in 2^{<\omega}}$ and $(e_t)_{t\in 2^{<\omega}}$ are impartially C-equivalent,
(ii) the sequences $(b_t^*)_{t\in 2^{<\omega}}$ and $(e_t^*)_{t\in 2^{<\omega}}$ are impartially C-equivalent,
(iii) for all $t \in \mathbb{N}$ we have $\mathrm{dist}(b_t, W_t) < \eta_t$, and
(iv) for all $t \in \mathbb{N}$ we have $\mathrm{dist}(b_t^*, G_t) < \eta_t$.

We say that $(e_s)_{s\in 2^{<\omega}}$ is *C-strategically reproducible in X* if for every $\eta > 0$ player II has a winning strategy in the game $\mathrm{Rep}_{(X,(e_s))}(C + \eta)$.

Theorem 4.7. *The Schauder basis $(e_t)_{t\in 2^{<\omega}}$ is 1-strategically reproducible both in S^E and B^E.*

Corollary 4.8. *The system $((e_t, e_t^*) \colon t \in 2^{<\omega})$ is strategically supporting and has the factorisation property in both $S^E \times D^E$ and $B^E \times (B^E)^*$.*

Corollary 4.9. *Suppose that $1 \leqslant p, p' \leqslant \infty$ with $1/p + 1/p' = 1$. Let*

$$\langle \cdot, \cdot \rangle \colon \ell^p(X) \times \ell^{p'}(X^*) \to \mathbb{R}$$

be the duality bracket given by

$$\langle (x_n)_{n=1}^\infty, (x_n^*)_{n=1}^\infty \rangle = \sum_{n=1}^\infty \langle x_n^*, x_n \rangle.$$

For each $k \in \mathbb{N}$ let $(e_{k,t})_{t\in 2^{<\omega}}$ denote a copy of $(e_t)_{t\in 2^{<\omega}}$ in the k^{th} coordinate of $\ell^p(X)$, and let $(f_{k,t})_{t\in 2^{<\omega}}$ denote the functionals given by $\langle (x_n)_{n=1}^\infty, f_{k,t} \rangle = \langle e_t^*, x_k \rangle$. Then the following assertions hold true:

(i) *$(\ell^p(X), \ell^{p'}(X^*), \langle \cdot, \cdot \rangle)$ is a dual pair which satisfies (1.3) with $c = 1$;*
(ii) *$((e_{k,t}, f_{k,t}) \colon k \in \mathbb{N},\ t \in 2^{<\omega})$ is strategically supporting in $\ell^p(X) \times \ell^{p'}(X^*)$;*
(iii) *$((e_{k,t}, f_{k,t}) \colon k \in \mathbb{N},\ t \in 2^{<\omega})$ has the factorisation property in $\ell^p(X) \times \ell^{p'}(X^*)$.*

Proof of Theorem 4.7. We will only present the proof for S^E, since the proof for B^E is similar (we use Lemma 4.5 instead of Lemma 4.4).

Let $t \in 2^{<\omega}$ and assume we have already played out the turns before t (or none yet if $t = \varnothing$). Let $\tilde{t}$ denote the dyadic predecessor of t and assume that $t = \tilde{t}^\frown \alpha$ for some $\alpha \in \{0, 1\}$. We will now describe turn t.

Step 1. Player I chooses $\eta_t > 0$ and the subspaces $W_t \in \mathrm{cof}(S^E)$ and $G_t \in \mathrm{cof}_{w*}(D^E)$. Thus, there exist finite sets $V_t \subset D^E$ and $F_t \subset S^E$ such that $W_t = (V_t)_\perp$ and $G_t = F_t^\perp$.

Step 2. We assume that $E_u = \{s_u\}$ $(u < t)$ and pick any branch Γ through α, *i.e.*, $s(1) = \alpha$ $(s \in \Gamma)$.

First we remark that by Lemma 4.4, e_s converges weakly to 0, whenever $|s| \to \infty$ along the branch Γ. Moreover, since $(e_s)_{s\in 2^{<\omega}}$ is a Schauder basis for S^E, e_s^* converges to 0 in the weak* topology, whenever $|s| \to \infty$. Thus, we can pick $s_t \in \Gamma$ with $s_t \sqsupset s_{\tilde{t}}$ and $s_t > s_u$ for all $u < t$ such that

$$\mathrm{dist}(e_{s_t}, W_t) \leqslant \eta_t \qquad \text{and} \qquad \mathrm{dist}(e_{s_t}^*, G_t) \leqslant \eta_t.$$

Player II chooses $E_t = \{s_t\}$ and $\lambda^{(t)}_{s_t} = \mu^{(t)}_{s_t} = 1$.

Step 3. At the end of turn t, player I selects a sign $\varepsilon_{s_t} \in \{\pm 1\}$.

Having completed the game, we *claim* that we have the postulated properties. By the very construction, the properties (iii) and (iv) of Definition 4.6 are satisfied. By Remark 4.3, the properties (i) and (ii) is satisfied for $C = 1$ as well. □

Proof of Corollary 4.8. By the remark between Definitions 3.9–3.10 in [22] and Theorem 4.7, using [22, Theorem 3.12] yields that $((e_t, e^*_t) : t \in 2^{<\omega})$ has the factorisation property in S^E and in B^E. Let $\mathcal{S} \subset 2^{<\omega}$ be any subset. By Lemma 4.1, either $\mathcal{S}$ or $2^{<\omega} \setminus \mathcal{S}$ contains a linearly order-isomorphic subtree $\mathcal{T}$ of $2^{<\omega}$. Thus, by Remark 4.3, we obtain that $(e_s)_{s\in\mathcal{T}}$ is 1-equivalent to $(e_s)_{s\in 2^{<\omega}}$ in both S^E and B^E and $(e^*_s)_{s\in\mathcal{T}}$ is 1-equivalent to $(e^*_s)_{s\in 2^{<\omega}}$ (in D^E and $(B^E)^*$). □

In what follows, we shall repeatedly quote the relevant results, in particular [24, Corollary 3.8], which involve the uniform diagonal factorisation property. Since the uniform diagonal factorisation property is implied by unconditionality and is not directly discussed elsewhere in this paper, we recapitulate the relevant facts required for in Remark 4.10 below.

Remark 4.10. Note that by the paragraph between Definition 3.9 and Definition 3.10 in [22], a 1-unconditional Schauder basis $(e_n)_{n=1}^\infty$ has the uniform diagonal factorisation property; see [22, Definition 3.9] for a definition of the uniform diagonal factorisation property.

Proof of Corollary 4.9. Assertion (i) is obvious, so we skip the proof. Since the basis $(e_t : t \in 2^{<\omega})$ is 1-unconditional, Remark 4.10 yields that $(e_t : t \in 2^{<\omega})$ has the uniform diagonal factorisation property. In Theorem 4.7 we already proved that $(e_t : t \in 2^{<\omega})$ is also 1-strategically reproducible in X. Hence, using [22, Theorem 7.6] for $1 \leqslant p < \infty$ and [24, Corollary 3.8] for $p = \infty$, we obtain (iii).

Now we only need to show that (ii) is true as well. To this end, let $\mathcal{S} \subset \mathbb{N} \times 2^{<\omega}$ be any subset. For each $k \in \mathbb{N}$, define $\mathcal{S}_k = \{t \in 2^{<\omega} : (k,t) \in \mathcal{S}\}$ and note that by Lemma 4.1, either $\mathcal{S}_k$ or $2^{<\omega} \setminus \mathcal{S}_k$ contains a linearly order-isomorphic subtree. Thus, if we define

$$\mathcal{K} = \{k \in \mathbb{N} : \mathcal{S}_k \text{ contains a linearly order-isomorphic subtree}\},$$

then either $\mathcal{K}$ or $\mathbb{N} \setminus \mathcal{K}$ is infinite. Let us assume without restriction that $\mathcal{K}$ is infinite and for each $k \in \mathcal{K}$ let $\mathcal{T}_k$ denote a linearly order-isomorphic subtree. For fixed $k \in \mathbb{N}$, Remark 4.3 asserts that $(e_{k,t} : t \in \mathcal{T}_k)$ is 1-equivalent to $(e_{k,t} : t \in 2^{<\omega})$ in X as well as that $(f_{k,t} : t \in \mathcal{T}_k)$ is 1-equivalent to $(f_{k,t} : t \in 2^{<\omega})$ in X^*. Hence,

- ▷ $(e_{k,t} : k \in \mathcal{K},\ t \in \mathcal{T}_k)$ is 1-equivalent to $(e_{k,t} : k \in \mathbb{N},\ t \in 2^{<\omega})$ in $\ell^p(X)$ and
- ▷ $(f_{k,t} : k \in \mathcal{K},\ t \in \mathcal{T}_k)$ is 1-equivalent to $(f_{k,t} : k \in \mathbb{N},\ t \in 2^{<\omega})$ in $\ell^{p'}(X^*)$.

This shows that $((e_{k,t}, f_{k,t}) : k \in \mathbb{N},\ t \in 2^{<\omega})$ is strategically supporting in $\ell^p(X) \times \ell^{p'}(X^*)$ as claimed. □

5. Factorisation of the identity in D^E

The present section is a step towards the proof of Theorem C as it specialises to the problem of factorisation of operators on the space D^E, when E is a Banach space with a 1-subsymmetric Schauder basis that is incomparably non-c_0 on antichains.

If not stated otherwise, $(e_s)_{s\in 2^{<\omega}}$ denotes the standard Schauder basis for S^E and $(f_s)_{s\in 2^{<\omega}}$ the biorthogonal functionals. Note that $(f_s)_{s\in 2^{<\omega}}$ forms a weak* Schauder basis for D^E.

5.1. **Subspace annihilation.** Suppose that $(e_s)_{s\in 2^{<\omega}}$ is a 1-subsymmetric Schauder basis for the Banach space E. We say that $(e_s)_{s\in 2^{<\omega}}$ is *incomparably non-c_0 on antichains*, whenever for all sets $\sigma_j \subset 2^{<\omega}$ with $\sigma_j \perp \sigma_k$, $j \neq k \in \mathbb{N}$, we have that

$$\inf_{\substack{N\in\mathbb{N}\\ \|(a_j)\|_{\ell^1_N}=1}} \sup\Big\{\sum_{j=1}^N |a_j|\|z_j\|_E \colon \mathcal{A} \text{ antichain, } \operatorname{supp}(z_j) \subset \sigma_j \cap \mathcal{A},\ \Big\|\sum_{j=1}^N z_j\Big\|_E = 1\Big\} = 0. \quad (5.1)$$

Remark 5.1. Note that $(e_s)_{s\in 2^{<\omega}}$ is incomparably non-c_0 on antichains if for some $r < \infty$, every sequence of incomparable sets $(\sigma_j)_{j=1}^\infty$ and every finite sequence of vectors $(z_j)_{j=1}^N$ with $z_j \in E$, $\operatorname{supp}(z_j) \subset \sigma_j$ satisfies a lower r-estimate, *i.e.*,

$$\Big\|\sum_{j=1}^N z_j\Big\|_E \geqslant c_r \Big(\sum_{j=1}^N \|z_j\|_E^r\Big)^{1/r}, \quad (5.2)$$

where the constant c_r neither does depend on N nor on $(z_j)_{j=1}^N$.

To see this, define $a_j = N^{-1}$ and let z_j with $\operatorname{supp}(z_j) \subset \sigma_j$, and $\Big\|\sum_{j=1}^N z_j\Big\|_E = 1$. Then by (5.2), we obtain

$$\Big(\sum_{j=1}^N \|z_j\|_E^r\Big)^{1/r} \leqslant c_r^{-1},$$

and thus,

$$\sum_{j=1}^N |a_j|\|z_j\|_E \leqslant N^{-1+1/r'} \Big(\sum_{j=1}^N \|z_j\|_E^r\Big)^{1/r} \leqslant c_r^{-1} N^{-1+1/r'}.$$

Clearly, the right hand side tends to 0 if $N \to \infty$.

Lemma 5.2. *Let $T_1, \dots, T_K$ be subtrees of $2^{<\omega}$ with $T_j \perp T_k$, $j \neq k$ and suppose that the 1-subsymmetric Schauder basis of E is incomparably non-c_0 on antichains. Then for all $\eta > 0$ and every operator $A \colon D^E \to D^E$ and $y \in S^E$, there exist subtrees $S_1 \dots, S_K$ with $S_k \subset T_k$, $1 \leqslant k \leqslant K$ such that*

$$\sup_{\|x\|_{D^E}\leqslant 1} \Big|\Big\langle y, A(x|_S)\Big\rangle\Big| \leqslant \eta,$$

where $S = \bigcup_{k=1}^K S_k$, $x|_S = \sum_{s\in S}\langle e_s, x\rangle f_s$ and the series converges in the weak topology of D^E.*

Proof. Now let $\eta > 0$, $A\colon D^E \to D^E$, $y \in S^E$, and assume the assertion is false, i.e., for all subtrees $S_1, \dots S_K$ with $S_k \subset T_k$, $1 \leqslant k \leqslant K$ we have that

$$\sup_{\|x\|_{D^E}\leqslant 1} \Big|\Big\langle y, A(x|_S)\Big\rangle\Big| > \eta, \quad (5.3)$$

where $S = \bigcup_{k=1}^K S_k$.

Pick infinite antichains $\mathcal{A}_k$ in T_k, $1 \leqslant k \leqslant K$ and let $\mathcal{A}_k = \{t_k^j\}_{j=1}^\infty$. Define the subtrees $S_k^j = \{s \in T_k \colon s \sqsupseteq t_k^j\}$, $j \in \mathbb{N}$ of T_k, $1 \leqslant k \leqslant K$ and note that $S_k^j \perp S_{k'}^{j'}$ for all $(j,k) \neq (j',k')$. In particular, if we set $S^j = \bigcup_{k=1}^K S_k^j$, $j \in \mathbb{N}$, then $S^j \perp S^{j'}$, $j \neq j' \in \mathbb{N}$. Now pick $N \in \mathbb{N}$ and $(a_j)_{j=1}^N$ according to (5.1) such that

$$\sup\Big\{\sum_{j=1}^N |a_j|\|z_j\|_E \colon \mathcal{A} \text{ antichain, } \operatorname{supp}(z_j) \subset S^j \cap \mathcal{A},\ \Big\|\sum_{j=1}^N z_j\Big\|_E = 1\Big\} \leqslant \frac{\eta}{2\|A\|\|y\|_{S^E}}. \quad (5.4)$$

Now, for each $1 \leqslant j \leqslant N$, we use (5.3) on $S_1^j, \dots S_K^j$ to find a vector $x_j \in D^E$ with $\|x_j\| = 1$, $\operatorname{supp}(x_j) \subset S^j$ such that $\langle Ax_j, y\rangle > \eta$.

Multiplying with $|a_j|$ and summing over $1 \leqslant j \leqslant N$ yields

$$\eta < \Big\langle A \sum_{j=1}^N |a_j| x_j, y \Big\rangle \leqslant \|A\| \Big\| \sum_{j=1}^N |a_j| x_j \Big\|_{D^E} \|y\|_{S^E}. \tag{5.5}$$

We will now estimate $\big\| \sum_{j=1}^N |a_j| x_j \big\|_{D^E}$. Observe that

$$\begin{aligned}\Big\| \sum_{j=1}^N |a_j| x_j \Big\|_{D^E} &= \sup_{(c_s)_{s\in 2^{<\omega}}} \frac{\sum_{j=1}^N |a_j| \langle \sum_{s\in S^j} c_s e_s, x_j\rangle}{\big\|\sum_{j=1}^N \sum_{s\in S^j} c_s e_s\big\|_{S^E}} \leqslant \sup_{(c_s)_{s\in 2^{<\omega}}} \frac{\sum_{j=1}^N |a_j| \big\|\sum_{s\in S^j} c_s e_s\big\|_{S^E}}{\big\|\sum_{j=1}^N \sum_{s\in S^j} c_s e_s\big\|_{S^E}}\\ &= \sup_{(c_s)_{s\in 2^{<\omega}}} \frac{\sum_{j=1}^N |a_j| \sup_{\mathcal{A}} \big\|\sum_{s\in S^j\cap\mathcal{A}} c_s e_s\big\|_{E}}{\big\|\sum_{j=1}^N \sum_{s\in S^j} c_s e_s\big\|_{S^E}},\end{aligned}$$

where the supremum over the $(c_s)_{s\in 2^{<\omega}}$ is restricted to $\big\|\sum_{j=1}^N \sum_{s\in S^j} c_s e_s\big\|_E \neq 0$ and the other supremum is taken over all antichains $\mathcal{A}$. Since $S^j \perp S^{j'}$, $j \neq j'$, the antichains $\mathcal{A}$ which depend on j have a common antichain, *i.e.*,

$$\sum_{j=1}^N |a_j| \sup_{\mathcal{A}} \Big\| \sum_{s\in S^j\cap\mathcal{A}} c_s e_s \Big\|_E = \sup_{\mathcal{A}} \sum_{j=1}^N |a_j| \Big\| \sum_{s\in S^j\cap\mathcal{A}} c_s e_s \Big\|_E.$$

With this observation and the definition of the norm in S^E, we obtain

$$\begin{aligned}\Big\| \sum_{j=1}^N |a_j| x_j \Big\|_{D^E} &\leqslant \sup_{(c_s)_{s\in 2^{<\omega}}} \sup_{\mathcal{A}} \frac{\sum_{j=1}^N |a_j| \big\|\sum_{s\in S^j\cap\mathcal{A}} c_s e_s\big\|_E}{\big\|\sum_{j=1}^N \sum_{s\in S^j} c_s e_s\big\|_{S^E}}\\ &= \sup_{(c_s)_{s\in 2^{<\omega}}} \sup_{\mathcal{A}} \inf_{\mathcal{B}} \frac{\sum_{j=1}^N |a_j| \big\|\sum_{s\in S^j\cap\mathcal{A}} c_s e_s\big\|_E}{\big\|\sum_{j=1}^N \sum_{s\in S^j\cap\mathcal{B}} c_s e_s\big\|_{E}},\end{aligned}$$

where the infimum is taken over all antichains $\mathcal{B}$. In particular, choosing $\mathcal{B} = \mathcal{A}$ and using (5.4) yields

$$\begin{aligned}\Big\| \sum_{j=1}^N |a_j| x_j \Big\|_{D^E} &\leqslant \sup_{(c_s)_{s\in 2^{<\omega}}} \sup_{\mathcal{A}} \frac{\sum_{j=1}^N |a_j| \big\|\sum_{s\in S^j\cap\mathcal{A}} c_s e_s\big\|_E}{\big\|\sum_{j=1}^N \sum_{s\in S^j\cap\mathcal{A}} c_s e_s\big\|_{E}}\\ &= \sup\Big\{ \sum_{j=1}^N |a_j| \|z_j\| : \mathcal{A} \text{ antichain}, \operatorname{supp}(z_j) \subset S^j \cap \mathcal{A}, \Big\|\sum_{j=1}^N z_j\Big\|_E = 1 \Big\}\\ &\leqslant \frac{\eta}{2\|A\| \|y\|_{S^E}}.\end{aligned}$$

Inserting the latter estimate into (5.5) leads to a contradiction. ☐

Lemma 5.3. *Let $x_1, \dots, x_n \in D^E$, let $\mathcal{T}$ denote a subtree of $2^{<\omega}$ and let $\eta > 0$. Then there exists an $s \in \mathcal{T}$ such that*

$$\max_{1\leqslant j\leqslant n} |\langle e_s, x_j\rangle| \leqslant \eta.$$

Proof. For fixed $1 \leqslant j \leqslant n$ and $\Gamma \in \beta$, define $y = \sum_{s\in\Gamma} \varepsilon_s e_s$, where $\varepsilon_s = \operatorname{sign}(\langle e_s, x_j\rangle)$, $s \in \Gamma$, and observe that since every antichain intersects the branch Γ at most once, we have

$$\|y\|_{S^E} = \sup_{\mathcal{A}} \Big\| \sum_{s\in\mathcal{A}\cap\Gamma} \langle y, f_s\rangle e_s \Big\|_E = \sup_{\mathcal{A}} \Big\| \sum_{s\in\mathcal{A}\cap\Gamma} \varepsilon_s e_s \Big\|_E = 1.$$

Thus, we obtain

$$\infty > \|x_j\|_{D^E} \geqslant \langle y, x_j\rangle = \sum_{s\in\Gamma} |\langle e_s, x_j\rangle| \geqslant \sum_{s\in\Gamma\cap\mathcal{T}} |\langle e_s, x_j\rangle|,$$

hence $|\langle e_s, x_j\rangle| \to 0$ as s tends to infinity along the branch $\Gamma\cap\mathcal{T}$ of the subtree $\mathcal{T}$. Since there are only finitely many x_j, the assertion follows. □

5.2. **Factorisation of the identity.** In the present section we fix a Banach space E with a 1-subsymmetric Schauder basis for the Banach space. Let $(e_s)_{s\in 2^{<\omega}}$ denote the standard unit vector basis in S^E and let $(f_s)_{s\in 2^{<\omega}}$ denote the associated biorthogonal functionals in $D^E = (S^E)^*$.

Theorem 5.4. *Suppose that E is incomparably non-c_0 on antichains. Let $T\colon D^E \to D^E$ be an operator having large diagonal with respect to $(f_s)_{s\in 2^{<\omega}}$, i.e.,*

$$\delta = \inf_{s\in 2^{<\omega}} |\langle e_s, Tf_s\rangle| > 0. \tag{5.6}$$

Then for each $\eta > 0$ there exist operators $A, B\colon D^E \to D^E$ such that $ATB = I_{D^E}$ and $\|A\|\|B\| \leqslant \frac{1+\eta}{\delta}$.

Corollary 5.5. *Suppose that $(e_s)_{s\in 2^{<\omega}}$ is incomparably non-c_0 on antichains. Then:*

- ▷ *$((f_s, e_s)\colon s \in 2^{<\omega})$ is strategically supporting and*
- ▷ *$((f_s, e_s)\colon s \in 2^{<\omega})$ has the factorisation property in $D^E \times S^E$.*

Proof of Corollary 5.5. By Theorem 5.4, $((f_s, e_s)\colon s \in 2^{<\omega})$ has the factorisation property. To see that $((f_s, e_s)\colon s \in 2^{<\omega})$ is also strategically supporting, we reason as in the proof of Corollary 4.8. □

Proof of Theorem 5.4. We define the constant $\eta_0 = \eta_0(\delta, \eta)$ such that

$$\frac{\eta_0}{3\delta} < 1 \qquad\text{and}\qquad \frac{1}{1-\frac{\eta_0}{3\delta}} \leqslant 1+\eta. \tag{5.7}$$

First, note that since the weak* Schauder basis $(f_s)_{s\in 2^{<\omega}}$ is 1-unconditional, we may assume that $\delta = \inf_{s\in 2^{<\omega}} \langle Tf_s, e_s\rangle > 0$. In this proof, we will regularly identify b_t and b_t^* with b_n and b_n^*, where n is the index of the node s in the standard linear order of the tree $2^{<\omega}$.

Diagonalisation of the operator. We will now inductively define biorthogonal subsequences $(b_t)_{t\in 2^{<\omega}}$ and $(b_t^*)_{t\in 2^{<\omega}}$ of $(f_s)_{s\in 2^{<\omega}}$ and $(e_s)_{s\in 2^{<\omega}}$. We begin our construction by putting $b_\varnothing = f_\varnothing$ and $b_\varnothing^* = e_\varnothing$. We use Lemma 5.2 on the subtrees $S^\alpha = \{s \in 2^{<\omega}\colon s \sqsupseteq \alpha\}$, $\alpha \in \{0,1\}$ and find subtrees $S_\varnothing^\alpha \subset S^\alpha$, $\alpha \in \{0,1\}$ such that

$$\sup_{\|x\|_{D^E}\leqslant 1} |\langle b_\varnothing^*, T(x|_{S_\varnothing})\rangle| \leqslant \eta_0/4, \tag{5.8}$$

where $S_\varnothing = S_\varnothing^0 \cup S_\varnothing^1$.

Let $t_0 \in 2^{<\omega}$ with $t_0 > \varnothing$ and assume that we have selected finite unions of pairwise incomparable subtrees S_t, $t < t_0$ with $S_t \supset S_{t+1}$ and constructed b_t, b_t^*, $t < t_0$ such that

$$b_t = f_{s_t}, \quad b_t^* = e_{s_t} \qquad (t < t_0), \tag{5.9a}$$

where

$$s_t \in S_{t-1} \quad (\varnothing < t < t_0), \qquad\qquad s_{t^\frown\alpha} \sqsubset s_t, \quad (\alpha\in\{0,1\},\ t < t_0 - 1), \tag{5.9b}$$

$s_{t^\frown\alpha}$ is left of s_t if $\alpha = 0$ and right of s_t if $\alpha = 1$,

$$\{s \in S_{t_0-1}\colon s \sqsupset s_t\} \text{ is a union of at least two incomparable subtrees} \tag{5.9c}$$

whenever $t < t_0$, as well as

$$\sum_{t_1<t} |\langle b_t^*, Tb_{t_1}\rangle| \leqslant \eta_0 4^{-\mathcal{O}(t)} \qquad (t < t_0), \tag{5.9d}$$

$$\sup_{\|x\|_{D^E}\leqslant 1} |\langle b_t^*, T(x|_{S_t})\rangle| \leqslant \eta_0 4^{-\mathcal{O}(t)} \qquad (t < t_0). \tag{5.9e}$$

We will now construct a finite union of pairwise incomparable subtrees $S_{t_0} \subset S_{t_0-1}$ and b_{t_0}, $b_{t_0}^*$ such that (5.9) is satisfied for all $t \leqslant t_0$. Pick $\alpha \in \{0,1\}$ such that $(\widetilde{t_0})^\frown\alpha = t_0$. Since $S = \{s \in S_{t_0-1} \colon s \sqsupset s_{\widetilde{t_0}}\}$ is a union of at least two pairwise incomparable subtrees $S^0, S^1 \subset S$ such that S^0 is left of S^1. We use Lemma 5.3 with $x_t = Tb_t$, $t < t_0$ and the subtree S^α to find a node $s_{t_0} \in S^\alpha$ such that

$$\sum_{t<t_0} |\langle e_{s_{t_0}}, Tb_t\rangle| \leqslant \eta_0 4^{-\mathcal{O}(t_0)}. \tag{5.10}$$

We put $b_{t_0} = f_{s_{t_0}}$ and $b_{t_0}^* = e_{s_{t_0}}$ and note that (5.9a) and (5.9b) are both satisfied for all $t \leqslant t_0$. Next, we use Lemma 5.2 on the finite union of pairwise incomparable subtrees $S' := S_{t_0-1} \setminus S^\alpha \cup \{s \in S^\alpha \colon s \sqsupset s_{t_0}\} \subset S_{t_0-1}$ to obtain a union of pairwise incomparable subtrees $S_{t_0} \subset S'$ such that

$$\sup_{\|x\|_{D^E}\leqslant 1} |\langle b_{t_0}^*, T(x|_{S_{t_0}})\rangle| \leqslant \eta_0 4^{\mathcal{O}(t_0)}, \tag{5.11}$$

Combining (5.10) with (5.11) shows that (5.9d) and (5.9e) both hold true for all $t \leqslant t_0$. Finally, we observe that $\{s \in S^\alpha \colon s \sqsupset s_{t_0}\}$ is the union of two incomparable subtrees and that the application of Lemma 5.2 replaced each subtree in S' with another subtree; hence, (5.9c) is satisfied for all $t \leqslant t_0$, as well. This concludes the inductive construction.

Conclusion of the proof. It is clear from the principle of our construction that $\{s_t \colon t \in 2^{<\omega}\}$ is a subtree. By Remark 4.3, the operators $B, Q \colon D^E \to D^E$ given by

$$Bx = \sum_{t\in 2^{<\omega}} \langle e_t, x\rangle b_t \qquad \text{and} \qquad Qx = \sum_{t\in 2^{<\omega}} \langle b_t^*, x\rangle f_t \qquad (x \in D^E)$$

satisfy $\|B\| = \|Q\| = 1$. Define the norm-one projection $P \colon D^E \to D^E$ by $P = BQ$ and put $Z = P(D^E)$, *i.e.*,

$$Z = \Big\{ z = \sum_{t\in 2^{<\omega}} a_t b_t \colon a_t \in \mathbb{R}, \|z\|_{D^E} < \infty \Big\}, \tag{5.12}$$

where the series converges in the weak* topology of D^E. The following diagram commutes:

$$\begin{array}{ccc} D^E & \xrightarrow{I_{D^E}} & D^E \\ {\scriptstyle B}\downarrow & & \uparrow{\scriptstyle Q} \\ Z & \xrightarrow[I_{D^E}]{} & Z \end{array} \qquad \|B\|, \|Q\| = 1. \tag{5.13}$$

Next, define $U \colon D^E \to Z$ by

$$Ux = \sum_{t\in 2^{<\omega}} \frac{\langle b_t^*, x\rangle}{\langle b_t^*, Tb_t\rangle} b_t \tag{5.14}$$

and note that by (5.6) and 1-unconditionality of $(f_t)_{t\in 2^{<\omega}}$, we have that

$$\|Ux\|_{D^E} \leqslant \frac{1}{\delta} \Big\| \sum_{t\in 2^{<\omega}} \langle b_t^*, x\rangle b_t \Big\|_{D^E} = \frac{1}{\delta}\|BQx\|_{D^E} = \frac{1}{\delta}\|Px\|_{D^E} \leqslant \frac{1}{\delta}\|x\|_{D^E} \qquad (x \in D^E). \tag{5.15}$$

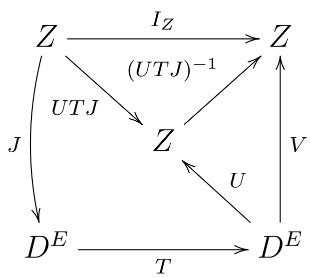
Let $z = \sum_{i=1}^{\infty} a_i b_i \in Z$ (recall that we identify a node with its index in the standard linear order of the tree) and observe that

$$UTz - z = \sum_{i=1}^{\infty}\Big(\sum_{j\colon j<i} a_j \frac{\langle b_i^*, Tb_j\rangle}{\langle b_i^*, Tb_i\rangle} + \frac{\big\langle b_i^*, T\sum_{j\colon j>i} a_j b_j\big\rangle}{\langle b_i^*, Tb_i\rangle}\Big) b_i. \tag{5.16}$$

Using (5.9), $|a_j| \leqslant \|z\|$, $j \in \mathbb{N}$ and (5.6), we obtain

$$\begin{aligned}\|UTz - z\|_{D^E} &\leqslant \sum_{i=1}^{\infty}\sum_{j\colon j<i} |a_j| \frac{|\langle b_i^*, Tb_j\rangle|}{|\langle b_i^*, Tb_i\rangle|} + \frac{|\big\langle b_i^*, T\sum_{j\colon j>i} a_j b_j\big\rangle|}{|\langle b_i^*, Tb_i\rangle|}\\ &\leqslant \frac{\eta_0}{3\delta}\|z\|_{D^E}.\end{aligned}$$

Let $J\colon Z \to D^E$ be the formal inclusion map, *i.e.*, $Jz = z$ $(z \in Z)$. Let us define the operator $V\colon D^E \to Z$ by $V = (UTJ)^{-1}U$. By (5.7), V is well defined and the following diagram commutes:



$$\|J\|\|V\| \leqslant (1+\eta)/\delta. \tag{5.17}$$

Merging the diagrams (5.13) and (5.17) concludes the proof. ☐

6. Further applications in classical Banach spaces

Even though we have been primarily interested in the class of stopping-time space, the methods developed along the way to prove that for various spaces X in this class, the set described by (1.2) is the unique maximal ideal of $\mathcal{B}(X)$, are general enough and apply to other Banach sequence/function spaces. In the present section for a given class of spaces, having verified that Theorem B applies, we derive the uniqueness of the maximal ideal of $\mathcal{B}(X)$.

6.1. **Application to L^p, Hardy spaces, BMO, and SL$^\infty$.** The collection of *dyadic intervals* $\mathcal{D}$ is given by

$$\mathcal{D} = \{[(k-1)2^{-n}, k2^{-n})\colon 1 \leqslant k \leqslant 2^n, n \geqslant 0\}.$$

Given any $\mathcal{C} \subset \mathcal{D}$, we define

$$\limsup \mathcal{C} = \{t \in [0,1]\colon t \text{ is contained in infinitely many } I \in \mathcal{C}\}.$$

For $I \in \mathcal{D}$, the L^∞-normalised *Haar function* h_I is given by $h_I = \mathbb{1}_{I_0} - \mathbb{1}_{I_1}$, where $I_0 \in \mathcal{D}$ denotes the left half of I, $I_1 \in \mathcal{D}$ denotes the right half of I and $\mathbb{1}_A$ is the indicator function of the set A. The sequence $(h_I\colon I \in \mathcal{D})$ is called *Haar system*. Since the dyadic intervals form a dyadic tree in the sense of Section 2.1, the notion of the standard linear order of a dyadic tree applies to $\mathcal{D}$, which ultimately linearly orders the Haar system (the standard linear order of the Haar system). In this standard linear order, the Haar system is a Schauder basis in L^p, $1 \leqslant p < \infty$; for the parameters $1 < p < \infty$, the Haar system is even unconditional (see [33] and [27]).

The dyadic *Hardy space* H^1 is given by

$$\Big\{f \in L^1\colon \int_0^1 f(t)\,\mathrm{d}t = 0,\ \|f\|_{H^1} < \infty\Big\},$$

where the square function norm $\|\cdot\|_{H^1}$ is given by

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_{H^1} = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_{L^1} = \int_0^1\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2(t)\Big)^{1/2}\,\mathrm{d}t.$$

We note that the Haar system is a 1-unconditional Schauder basis for H^1 (see [37, Section 6]). The (non-separable) dual of H^1 is denoted by BMO. Hence, the Haar system forms a 1-unconditional weak* Schauder basis in BMO. A closely related space is the also non-separable Banach space SL^∞ (see [16]), which is given by

$$\mathrm{SL}^\infty = \Big\{f\in L^2\colon \int_0^1 f(t)\,\mathrm{d}t = 0,\ \|f\|_{\mathrm{SL}^\infty}<\infty\Big\},$$

where the norm $\|\cdot\|_{\mathrm{SL}^\infty}$ is given by

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_{\mathrm{SL}^\infty} = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_{L^\infty} = \operatorname*{ess\,sup}_{t\in[0,1]}\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2(t)\Big)^{1/2}.$$

Theorem 6.1. *The following systems are strategically supporting and have the positive factorisation property:*

(i) $((h_I/|I|^{1/p}, h_I/|I|^{1/p'})\colon I\in\mathcal{D})$ *in* $L^p\times L^{p'}$*, where* $1\leqslant p<\infty$ *and* $1/p+1/p'=1$.

(ii) $((h_I/|I|, h_I)\colon I\in\mathcal{D})$ *in* $H^1\times \mathrm{BMO}$.

(iii) $((h_I, h_I^*)\colon I\in\mathcal{D})$ *in* $H^1\times Y$*, where* Y *is the norm-closure of the biorthogonal functionals* h_I^* *in* $(\mathrm{SL}^\infty)^*$.

Moreover, (1.3) *is satisfied with* $c=1$ *in all above cases.*

Before we prove Theorem 6.1, we record an immediate consequence of Theorem 6.1 and Theorem B.

Corollary 6.2. *Let* X *denote one of the spaces* L^p $(1\leqslant p<\infty)$*,* H^1*,* BMO*, or* SL^∞*. Then* $\mathcal{M}_X$ *is the unique closed proper maximal ideal of* $\mathcal{B}(X)$*.*

We would like to point out that the results stated in Corollary 6.2 also follow by combining [10, Section 5] with [28] for L^p, with [30] for H^1 and BMO and with [19] for SL^∞.

Proof of Theorem 6.1. Note that only in the case $X=\mathrm{SL}^\infty$, we need to be diligent in regards to (1.3). In the subsequent proofs, we will several times assert that a pair of bases has the '(positive) factorisation property'. By that we wish to indicate that within the respective paper the assertion is not stated explicitly for the positive factorisation property, but for the factorisation property. By Remark 1.1, the factorisation property implies the positive factorisation property.

Case $X=L^p$ $(1<p<\infty)$. For the Hölder conjugates $1<p,p'<\infty$, Andrew [1] (we remark that the result is not stated explicitly but directly contained within the proof; see also [22] for an exposition) showed that $((h_I/|I|^{1/p}, h_I/|I|^{1/p'})\colon I\in\mathcal{D})$ has the (positive) factorisation property in $L^p\times L^{p'}$. Gamlen and Gaudet [13] proved that for any collection $\mathcal{C}\subset\mathcal{D}$ with $|\limsup\mathcal{C}|>0$ (here, $|\cdot|$ denotes the Lebesgue-measure), there is a block basis $(b_I\colon I\in\mathcal{D})$ of $(h_I\colon I\in\mathcal{C})$ that is equivalent to $(h_I\colon I\in\mathcal{D})$ in L^p (and in $L^{p'}$). Reviewing their proof, we find that $(b_I\colon I\in\mathcal{D})$ can be constructed so that it also satisfies (iii) and (iv) in Definition 3.3. Since for any collection $\mathcal{C}\subset\mathcal{D}$ we either have that $|\limsup\mathcal{C}|\geqslant 1/2$ or $|\limsup(\mathcal{D}\setminus\mathcal{C})|\geqslant 1/2$, we obtain that $((h_I/|I|^{1/p}, h_I/|I|^{1/p'})\colon I\in\mathcal{D})$ is strategically supporting in $L^p\times L^{p'}$.

Case $X=L^1$. Corollary 6.3 in [22] asserts that $((h_I/|I|, h_I)\colon I\in\mathcal{D})$ has the (positive) factorisation property in $L^1\times L^\infty$. By the theorem of Gamlen and Gaudet [13], we obtain that for any collection $\mathcal{C} \subset \mathcal{D}$ with $|\limsup \mathcal{C}| > 0$, there exists a block basis $(b_I : I \in \mathcal{D})$ of $(h_I : I \in \mathcal{C})$ that is equivalent to $(h_I : I \in \mathcal{D})$ in both L^1 and L^∞ (see also [32, p. 176 ff.] or the proof of Theorem 6.1 in [22]) such that $(b_I : I \in \mathcal{D})$ also satisfies (iii) and (iv) in Definition 3.3. Consequently, $((h_I/|I|, h_I) : I \in \mathcal{D})$ is strategically supporting in $L^1 \times L^\infty$.

Case $X = H^1$. By [22, Theorem 5.2], the Haar system is strategically reproducible (see also [29, 32]). Since the Haar system is an unconditional Schauder basis in H^1, it has the uniform diagonal factorisation property (see Remark 4.10). Thus, by [22, Theorem 3.12], the system $((h_I/|I|, h_I) : I \in \mathcal{D})$ has the (positive) factorisation property in $H^1 \times \mathrm{BMO}$. By [29, Theorem 1(c)] and reviewing the proof, we find that (by the same mechanisms that were described in more detail above), $((h_I/|I|, h_I) : I \in \mathcal{D})$ is strategically supporting in $H^1 \times \mathrm{BMO}$.

Case $X = \mathrm{SL}^\infty$. We define $h_I^* : \mathrm{SL}^\infty \to \mathbb{R}$ by

$$h_I^*\Big(\sum_{J \in \mathcal{D}} a_J h_J\Big) = a_I,$$

note that h_I^* is linear and satisfies

$$h_I^*(h_I) = 1, \qquad h_I^*(h_J) = 0 \qquad \text{and} \qquad \|h_I^*\|_{(\mathrm{SL}^\infty)^*} = 1,$$

for all $I \neq J \in \mathcal{D}$. Let Y denote the closed linear span of $\{h_I^* : I \in \mathcal{D}\}$ in $(\mathrm{SL}^\infty)^*$. We define a bilinear form $\langle \cdot, \cdot \rangle : \mathrm{SL}^\infty \times Y \to \mathbb{R}$ by

$$\langle f, g \rangle = g(f), \qquad (f \in \mathrm{SL}^\infty,\ g \in Y).$$

Given $f = \sum_{I \in \mathcal{D}} a_I h_I \in \mathrm{SL}^\infty$ and $\varepsilon > 0$, there exist $t \in [0, 1]$ and $n \in \mathbb{N}$ such that

$$\Big(\sum_{\substack{I \ni t\\ |I| \geqslant 2^{-n}}} a_I^2\Big)^{1/2} \geqslant (1 - \varepsilon)\|f\|_{\mathrm{SL}^\infty}. \tag{6.1}$$

Next, we define

$$g = \sum_{\substack{I \ni t\\ |I| \geqslant 2^{-n}}} a_I h_I^* \Big/ \Big(\sum_{\substack{J \ni t\\ |J| \geqslant 2^{-n}}} a_J^2\Big)^{1/2},$$

and note that $g \in \operatorname{span}\{h_I^*\} \subset Y$. The Cauchy–Schwarz inequality yields

$$\Big|g\Big(\sum_{I \in \mathcal{D}} c_I h_I\Big)\Big| = \Big|\sum_{\substack{I \ni t\\ |I| \geqslant 2^{-n}}} a_I c_I\Big| \Big/ \Big(\sum_{\substack{J \ni t\\ |J| \geqslant 2^{-n}}} a_J^2\Big)^{1/2} \leqslant \Big(\sum_{\substack{I \ni t\\ |I| \geqslant 2^{-n}}} c_I^2\Big)^{1/2} \leqslant \Big\|\sum_{I \in \mathcal{D}} c_I h_I\Big\|_{\mathrm{SL}^\infty},$$

for all $\sum_{I \in \mathcal{D}} c_I h_I \in \mathrm{SL}^\infty$, thus, $\|g\|_{(\mathrm{SL}^\infty)^*} \leqslant 1$. Moreover, by the definition of g and (6.1), we obtain

$$\langle f, g \rangle = \Big(\sum_{\substack{I \ni t\\ |I| \geqslant 2^{-n}}} a_I^2\Big)^{1/2} \geqslant (1 - \varepsilon)\|f\|_{\mathrm{SL}^\infty}.$$

Altogether, we proved that (1.3) holds for $c = 1$, *i.e.*,

$$\sup_{\|g\|_Y \leqslant 1} \langle f, g \rangle = \|f\|_{\mathrm{SL}^\infty} \qquad (f \in \mathrm{SL}^\infty). \tag{6.2}$$

The second-named author proved in [19, Theorem 2.1] that the Haar system has the (positive) factorisation property in SL^∞. Moreover, contained in the proof of [19, Theorem 2.2] is the assertion that the Haar system is strategically supporting in SL^∞, which we will now elucidate.

Basically, we rerun the proof of [19, Theorem 2.2] with $T = I_{\mathrm{SL}^\infty}$. First, we skip Step 1 (which amounts to $B_I = I$, $I \in \mathcal{D}$). In Step 2, we split the dyadic intervals into two collections $\mathcal{E}$ and $\mathcal{F}$ determined by the partition of $\mathbb{N}$ into N_1, N_2 as demanded by Definition 3.3. The output of Step 2 are two normalised block bases $(\widetilde{h}_I : I \in \mathcal{D})$, $(\widetilde{h}^*_I : I \in \mathcal{D})$ which satisfy (i)–(iv) in Definition 3.3 and for which

$$\text{either} \qquad \bigcup_I \widetilde{h}_I \subset \mathcal{E} \qquad \text{or} \qquad \bigcup_I \widetilde{h}_I \subset \mathcal{F}.$$

The equivalence between $(\widetilde{h}_I : I \in \mathcal{D})$ and $(h_I : I \in \mathcal{D})$ in SL^∞ is obtained directly from the boundedness of the operators $B, Q\colon \mathrm{SL}^\infty \to \mathrm{SL}^\infty$ given by

$$Bf = \sum_{I\in\mathcal{D}} \langle f, h^*_I\rangle \widetilde{h}_I, \qquad Qf = \sum_{I\in\mathcal{D}} \langle f, \widetilde{h}^*_I\rangle h_I \qquad (f \in \mathrm{SL}^\infty)$$

and the fact that $QB = I_{\mathrm{SL}^\infty}$. The equivalence between $(\widetilde{h}^*_I : I \in \mathcal{D})$ and $(h^*_I : I \in \mathcal{D})$ in Y is established by takin the adjoints of the operators (with respect to the dual pairing $(\mathrm{SL}^\infty, Y, \langle\cdot,\cdot\rangle)$) and observing their boundedness (as maps from Y to itself) using (6.2).

Case $X = \mathrm{BMO}$. The quickest way to describe why $((h_I, h_I/|I|) : I \in \mathcal{D})$ has the positive factorisation property and is strategically supporting in $\mathrm{BMO} \times H^1$, is to modify the above argument given for SL^∞. We *claim* that the same proof (beginning after (6.2)) and given for SL^∞ works if we replace SL^∞ with BMO. Since Jones' compatibility conditions for SL^∞, (see [19, Section 3.1]) demand less than Jones' compatibility conditions for BMO (see [15]; see also [32, page 105 in Section 1.5]) the corresponding operators B, Q (defined in the previous case) as mappings from BMO to itself might be unbounded. However with some minor (yet important) modifications in constructing the block bases (which define the operators B, Q) we can also achieve Jones' compatibility conditions for BMO, and thereby guarantee the boundedness of $B, Q\colon \mathrm{BMO} \to \mathrm{BMO}$. We refer to [15, 29] and also to [32, Section 4.2]. □

6.2. **Application to $\boldsymbol{\ell^p(\ell^q)}$.** Let $\leqslant$ denote the linear order on $\mathbb{N}^2$ defined by the property that $(i_0, j_0) \leqslant (i_1, j_1)$ whenever $i_0 < i_1$ or if $i_0 = i_1$ and $j_0 \leqslant j_1$. For $k \in \mathbb{N}$ let $(i(k), j(k))$ denote the k-th largest element in $\mathbb{N}^2$ with respect to that linear order. Vice versa, let $k(m,n)$ denote the unique $r \in \mathbb{N}$ such that $(i(r), j(r)) = (m, n)$.

For $1 \leqslant p, q \leqslant \infty$, p' and q' denote their respective Hölder conjugates, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The space $\ell^p(\ell^q)$ is defined as

$$\Big\{(a_{ij} : i,j \in \mathbb{N}) : \|(a_{ij})_{i,j}\|_{\ell^p(\ell^q)} < \infty\Big\},$$

where the norm $\|\cdot\|_{\ell^p(\ell^q)}$ is given by

$$\|(a_{ij})_{i,j}\|_{\ell^p(\ell^q)} = \Big\|(\|(a_{ij})_j\|_{\ell^q})_i\Big\|_{\ell^p} = \bigg(\sum_{i=1}^\infty \Big(\sum_{j=1}^\infty |a_{ij}|^q\Big)^{p/q}\bigg)^{1/p}.$$

Let $(e_{ij} : i,j \in \mathbb{N})$ denote the standard Schauder basis in $\ell^p(\ell^q)$ and $(f_{ij} : i,j \in \mathbb{N})$ the standard Schauder basis in $\ell^{p'}(\ell^{q'})$.

Next, we define a bilinear form $\langle\cdot,\cdot\rangle\colon \ell^p(\ell^q) \times \ell^{p'}(\ell^{q'}) \to \mathbb{R}$ by

$$\Big\langle \sum_{i,j} a_{ij} e_{ij}, \sum_{k,l} b_{kl} f_{kl} \Big\rangle = \sum_{k,l} a_{kl} b_{kl} \quad \Big(\sum_{i,j} a_{ij} e_{ij} \in \ell^p(\ell^q),\ \sum_{k,l} b_{kl} f_{kl} \in \ell^{p'}(\ell^{q'})\Big)$$

making $(\ell^p(\ell^q), \ell^{p'}(\ell^{q'}), \langle\cdot,\cdot\rangle)$ a dual pair. One can check that $(e_{ij} : i,j \in \mathbb{N})$ is a Schauder basis for $\ell^p(\ell^q)$ in the topology $\sigma(\ell^p(\ell^q), \ell^{p'}(\ell^{q'}))$, and $(f_{ij} : i,j \in \mathbb{N})$ is a basis for $\ell^{p'}(\ell^{q'})$ in the topology $\sigma(\ell^{p'}(\ell^{q'}), \ell^p(\ell^q))$. Moreover, we note that (1.3) is satisfied with $c = 1$, *i.e.*,

$$\sup_{\|y\|_{\ell^{p'}(\ell^{q'})} \leqslant 1} \langle x, y\rangle = \|x\|_{\ell^p(\ell^q)} \qquad (x \in \ell^p(\ell^q)).$$

The upcoming Theorem 6.3 and 6.4, serve the purpose to verify the rest of the hypotheses in Theorem B: $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ has the positive factorisation property and is strategically supporting.

Theorem 6.3. *Suppose that* $1\leqslant p,p'\leqslant\infty$, $1<q,q'<\infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$. *Then* $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ *has the positive factorisation property in* $\ell^p(\ell^q)\times\ell^{p'}(\ell^{q'})$.

Proof. It is straightforward to show that the standard Schauder basis $(e_n)_{n=1}^\infty$ of ℓ^q is 1-strategically reproducible, whenever $1<q<\infty$; we therefore omit the proof. Since $(e_n)_{n=1}^\infty$ is 1-unconditional, Remark 4.10 asserts that $(e_n)_{n=1}^\infty$ has the uniform diagonal factorisation property; to be more precise, one can quickly check that $(e_n)_{n=1}^\infty$ has in fact the $1/\delta$-diagonal factorisation property, $\delta>0$. Thus, [22, Theorem 7.6] yields that $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ has the $1/\delta$-factorisation property in $\ell^p(\ell^q)\times\ell^{p'}(\ell^{q'})$ for all $1\leqslant p<\infty$, $1<q<\infty$. In particular, by Remark 1.1 we obtain that $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ has the positive factorisation property in $\ell^p(\ell^q)\times\ell^{p'}(\ell^{q'})$ for $1\leqslant p<\infty$, $1<q<\infty$. Moreover, by [21, Corollary 3.8], $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ has the $1/\delta$-factorisation property in $\ell^\infty(\ell^q)\times\ell^1(\ell^{q'})$ for all $1<q<\infty$. Invoking Remark 1.1 yields that $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ has the positive factorisation property in $\ell^\infty(\ell^q)\times\ell^1(\ell^{q'})$, $1<q<\infty$. □

Theorem 6.4. *Let* $1\leqslant p,q\leqslant\infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$. *Then* $((e_{ij}, f_{ij})\colon i,j\in\mathbb{N})$ *is strategically supporting in* $\ell^p(\ell^q)\times\ell^{p'}(\ell^{q'})$.

Proof. Let N_1, N_2 be a partition of $\mathbb{N}$. For each $i\in\mathbb{N}$, we define the sets

$$\mathcal{A}_i^m=\{j\in\mathbb{N}\colon k(i,j)\in N_m\},\qquad m=1,2.$$

Observe that for each $i\in\mathbb{N}$, either of the two sets $\mathcal{A}_i^m$, $m=1,2$ is infinite. Next, we define $\mathcal{B}^m=\{i\in\mathbb{N}\colon |\mathcal{A}_i^m|=\infty\}$, $m=1,2$ and note that either $\mathcal{B}^1$ or $\mathcal{B}^2$ must be infinite. Pick $m_0\in\{1,2\}$ such that $\mathcal{B}^{m_0}$ is infinite. When properly relabelled, $(e_{ij}\colon i\in\mathcal{B}^{m_0},\ j\in\mathcal{A}_i^{m_0})$ is then equivalent to $(e_{ij}\colon i,j\in\mathbb{N})$ in $\ell^p(\ell^q)$ and $(f_{ij}\colon i\in\mathcal{B}^{m_0},\ j\in\mathcal{A}_i^{m_0})$ is equivalent to $(f_{ij}\colon i,j\in\mathbb{N})$ in $\ell^{p'}(\ell^{q'})$. □

Combining Theorem 6.3 with Theorem 6.4, the remarks at the beginning of Section 6.2 and Theorem B yield the following conclusion that had been known to the experts, yet to the best of our knowledge it has not been recorded in the literature. Indeed, primarity of various infinite direct sums of Banach spaces has been assessed in the literature (see, *e.g.*, [6], [8], [35], and more recently in [20]), however not in relation to ideals of operators acting thereon. A closer inspection of the proofs often allows one to deduce closedness under addition of the set defined in (1.2). Corollary 6.5 puts these results on a more systematic footing.

Corollary 6.5. *Let* $1\leqslant p\leqslant\infty$ *and* $1<q<\infty$. *Then* $\mathcal{M}_{\ell^p(\ell^q)}$ *is the unique closed proper maximal ideal of* $\mathcal{B}(\ell^p(\ell^q))$.

6.3. **Application to mixed-norm Lebesgue spaces.** The collection of dyadic rectangles $\mathcal{R}$ is given by

$$\mathcal{R}=\{I\times J\colon I,J\in\mathcal{D}\}.$$

The biparameter Haar system $(h_{I,J}\colon I,J\in\mathcal{D})$ is given by

$$h_{I,J}(x,y)=h_I(x)h_J(y)\quad (x,y\in[0,1], I,J\in\mathcal{D}).$$

For $1\leqslant p,q<\infty$, the mixed-norm Lebesgue space $L^p(L^q)$ is the completion of

$$\mathcal{H}:=\operatorname{span}\Big\{h_{I,J}\colon I,J\in\mathcal{D}\Big\}$$

under the norm $\|\cdot\|_{L^p(L^q)}$ given by

$$\|f\|_{L^p(L^q)} = \left(\int_0^1 \left(\int_0^1 |f(x,y)|^q \mathrm{d}y\right)^{p/q} \mathrm{d}x\right)^{1/p}.$$

The mixed-norm dyadic Hardy space $H^p(H^q)$ is defined as the completion of $\mathcal{H}$ under the square function norm $\|\cdot\|_{H^p(H^q)}$ given by

$$\Big\|\sum_{I,J\in\mathcal{D}} a_{I,J} h_{I,J}\Big\|_{H^p(H^q)} = \Big\|\Big(\sum_{I,J\in\mathcal{D}} a_{I,J}^2 h_{I,J}^2\Big)^{1/2}\Big\|_{L^p(L^q)}.$$

M. Capon [7, Proposition I.1] shows that the identity operator is an isomorphism between $L^p(L^q)$ and $H^p(H^q)$, whenever $1 < p,q < \infty$ and that the biparameter Haar system $(h_{I,J} \colon I,J\in\mathcal{D})$ is an unconditional Schauder basis for $L^p(L^q)$, $1 < p,q < \infty$.

The following crucial theorem for $L^p(L^q)$, $1 < p,q < \infty$ is due to M. Capon [7, Theorem 1.3], and for $H^1(H^1)$ it is due to P.F.X. Müller [31, Theorem 4].

Theorem 6.6. *Let X denote either of the spaces $L^p(L^q)$, $1 < p,q < \infty$ or $H^1(H^1)$. Given $\mathcal{C}\subset\mathcal{R}$, $I\in\mathcal{D}$ and $t\in[0,1]$, we define*

$$\mathcal{C}_I = \{J\in\mathcal{D}\colon I\times J\in\mathcal{C}\} \quad \textit{and} \quad C_t = \limsup\{I\in\mathcal{D}\colon t\in\limsup\mathcal{C}_I\}.$$

If $|\{t\in[0,1]\colon |C_t|\geqslant 1/2\}| > 0$, then one may find a block basis $(\widetilde{h}_{I,J}\colon I\times J\in\mathcal{D})$ of $(h_{I,J}\colon I\times J\in\mathcal{C})$ which is equivalent to $(h_{I,J}\colon I,J\in\mathcal{D})$ in X and in X^*.

The result for $L^p(L^q)$ is due to M. Capon [7] (note that the dual of $L^p(L^q)$ is $L^{p'}(L^{q'})$, where $1/p+1/p'=1$ and $1/q+1/q'=1$), the result for $H^1(H^1)$ is due to P.F.X. Müller [31] (dualising the natural projection onto the block basis yields the equivalence of the block basis in $(H^1(H^1))^*$; also see Remark 4.3). We refer to [31, 32, 25, 23] for related works.

Corollary 6.7. *Let $1 < p,p',q,q' < \infty$ with $1/p+1/p'=1$ and $1/q+1/q'=1$. Then the following systems are strategically supporting:*

(i) $((h_{I,J}/(|I|^{1/p}|J|^{1/q}), h_{I,J}/(|I|^{1/p'}|J|^{1/q'}))\colon I,J\in\mathcal{D})$ *in* $L^p(L^q)\times L^{p'}(L^{q'})$,
(ii) $((h_{I,J}/(|I||J|), h_{I,J})\colon I,J\in\mathcal{D})$ *in* $H^1(H^1)\times(H^1(H^1))^*$.

Proof of Corollary 6.7. Additionally to the definitions $\mathcal{C}_I$ and C_t as in Theorem 6.6, we define

$$\mathcal{C}_I' = \{J\in\mathcal{D}\colon I\times J\in\mathcal{R}\setminus\mathcal{C}\} \quad \text{and} \quad C_t' = \limsup\{I\in\mathcal{D}\colon t\in\limsup\mathcal{C}_I'\}.$$

Certainly, for each $t\in[0,1]$ and $I\in\mathcal{D}$, either $t\in\limsup\mathcal{C}_I$ or $t\in\limsup\mathcal{C}_I'$, and thus $C_t\cup C_t' = [0,1]$. Consequently, either

$$|\{t\in[0,1]\colon |C_t|\geqslant 1/2\}|\geqslant 1/2 \qquad \text{or} \qquad |\{t\in[0,1]\colon |C_t'|\geqslant 1/2\}|\geqslant 1/2.$$

Applying Theorem 6.6, yields a block basis of either $(h_{I,J}\colon I\times J\in\mathcal{C})$ or $(h_{I,J}\colon I\times J\in\mathcal{R}\setminus\mathcal{C})$ which is equivalent to $(h_{I,J}\colon I\times J\in\mathcal{D})$. Thus, we showed Definition 3.3 (i) and (ii) are satisfied. The other two properties of Definition 3.3 are obvious. □

Theorem 6.8. *Let $1 < p,q < \infty$. Then the following systems have the positive factorisation property:*

(i) $((h_{I,J}/(|I|^{1/p}|J|^{1/q}), h_{I,J}/(|I|^{1/p'}|J|^{1/q'}))\colon I,J\in\mathcal{D})$ *in* $L^p(L^q)\times L^{p'}(L^{q'})$,
(ii) $((h_{I,J}/(|I||J|), h_{I,J})\colon I,J\in\mathcal{D})$ *in* $H^1(H^1)\times(H^1(H^1))^*$.

Proof. For $1\leqslant p,q<\infty$, [18, Theorem 3.1] yields that the identity operator factors through every operator $T\colon H^p(H^q)\to H^p(H^q)$ satisfying

$$\inf_{I,J}\frac{\langle Th_{I,J}, h_{I,J}\rangle}{|I||J|} > 0.$$

By [7, Proposition I.1], the identity operator is an isomorphism between $L^p(L^q)$ and $H^p(H^q)$, whenever $1 < p, q < \infty$. □

Using Corollary 6.7 together with Theorem 6.8 and subsequently applying Theorem B yields Corollary 6.9, below.

Corollary 6.9. *Let X denote either of the spaces $H^1(H^1)$ or $L^p(L^q)$, $1 < p, q < \infty$. Then $\mathcal{M}_X$ is the unique closed proper maximal ideal of $\mathcal{B}(X)$.*

The results in Corollary 6.9 also follow by combining [10, Section 5] with [31] for $H^1(H^1)$ and with [7] for $L^p(L^q)$.

References

[1] A. D. Andrew. Perturbations of Schauder bases in the spaces $C(K)$ and L^p, $p > 1$. *Studia Math.*, 65(3):287–298, 1979.

[2] A. D. Andrew. The Banach space JT is primary. *Pacific J. Math.*, 108(1):9–17, 1983.

[3] D. Apatsidis. Operators on the stopping time space. *Stud. Math.*, 228(3):235–258, 2015.

[4] H. Bang and E. Odell. On the stopping time Banach space. *Q. J. Math., Oxf. II. Ser.*, 40(159):257–273, 1989.

[5] S. G. Buehler. *The James function space.* ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)–The University of Texas at Austin.

[6] M. Capon. Primarité de certains espaces de Banach. *Proc. London Math. Soc. (3)*, 45(1):113–130, 1982.

[7] M. Capon. Primarité de $L^p(L^r)$, $1 < p$, $r < \infty$. *Israel J. Math.*, 42(1-2):87–98, 1982.

[8] P. G. Casazza, C. A. Kottman, and B. L. Lin. On some classes of primary Banach spaces. *Canad. J. Math.*, 29(4):856–873, 1977.

[9] N. Dew. Asymptotic structure of banach spaces. *Ph.D. Thesis, University of Oxford*, 2003.

[10] D. Dosev and W. B. Johnson. Commutators on ℓ_∞. *Bull. Lond. Math. Soc.*, 42(1):155–169, 2010.

[11] P. Enflo and T. W. Starbird. Subspaces of L^1 containing L^1. *Studia Math.*, 65(2):203–225, 1979.

[12] H. Fetter and B. Gamboa de Buen. *The James forest*, volume 236 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1997. With a foreword by Robert C. James and a prologue by Bernard Beauzamy.

[13] J. L. B. Gamlen and R. J. Gaudet. On subsequences of the Haar system in L_p $[1, 1](1 \leq p \leq \infty)$. *Israel J. Math.*, 15:404–413, 1973.

[14] B. Horváth and T. Kania. Surjective homomorphisms from algebras of operators on long sequence spaces are automatically injective. *Quarterly Journal of Mathematics*, 72(4):1167–1189, 2021.

[15] P. W. Jones. BMO and the Banach space approximation problem. *Amer. J. Math.*, 107(4):853–893, 1985.

[16] P. W. Jones and P. F. X. Müller. Conditioned Brownian motion and multipliers into SL^∞. *Geom. Funct. Anal.*, 14(2):319–379, 2004.

[17] T. Kania and N. J. Laustsen. Uniqueness of the maximal ideal of the Banach algebra of bounded operators on $C([0, \omega_1])$. *J. Funct. Anal.*, 262(11):4831–4850, 2012.

[18] N. J. Laustsen, R. Lechner, and P. F. X. Müller. Factorization of the identity through operators with large diagonal. *J. Funct. Anal.*, 275(11):3169–3207, 2018.

[19] R. Lechner. Factorization in SL^∞. *Israel J. Math.*, 226(2):957–991, 2018.

[20] R. Lechner. Subsymmetric weak* Schauder bases and factorization of the identity. *Studia Math.*, 248(3):295–319, 2019.

[21] R. Lechner, P. Motakis, P. F. X. Müller, and T. Schlumprecht. The factorization property of $\ell^\infty(X_k)$. *arXiv e-prints*, page arXiv:1910.11188, Oct. 2019.

[22] R. Lechner, P. Motakis, P. F. X. Müller, and T. Schlumprecht. Strategically reproducible bases and the factorization property. *Israel J. Math.*, 238(1):13–60, 2020.

[23] R. Lechner, P. Motakis, P. F. X. Müller, and T. Schlumprecht. The space $L_1(L_p)$ is primary for $1 < p < \infty$. *arXiv e-prints*, page arXiv:2102.10088, Feb. 2021.

[24] R. Lechner, P. Motakis, P. F. Müller, and T. Schlumprecht. The factorisation property of $\ell^\infty(X_k)$. *Mathematical Proceedings of the Cambridge Philosophical Society*, 171(2):421–448, 2021.

[25] R. Lechner and P. F. X. Müller. Localization and projections on bi-parameter BMO. *Q. J. Math.*, 66(4):1069–1101, 2015.

[26] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. I.* Springer-Verlag, Berlin-New York, 1977. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[27] J. Marcinkiewicz. Quelques theorèmes sur les séries orthogonales. *Ann. Soc. Polon. Math.*, 16:85–96, 1937.

[28] B. Maurey. Sous-espaces complémentés de L^p, d'après P. Enflo. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. III*, pages 15 pp. (erratum, p. 1). Centre Math., École Polytech., Paris, 1975.

[29] P. F. X. Müller. On subsequences of the Haar basis in $H^1(\delta)$ and isomorphism between H^1-spaces. *Studia Math.*, 85(1):73–90 (1987), 1986.

[30] P. F. X. Müller. On projections in H^1 and BMO. *Studia Math.*, 89(2):145–158, 1988.

[31] P. F. X. Müller. Orthogonal projections on martingale H^1 spaces of two parameters. *Illinois J. Math.*, 38(4):554–573, 1994.

[32] P. F. X. Müller. *Isomorphisms between H^1 spaces*, volume 66 of *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)]*. Birkhäuser Verlag, Basel, 2005.

[33] R. E. A. C. Paley. A Remarkable Series of Orthogonal Functions. *Proc. London Math. Soc. (2)*, 34(4):241–264, 1932.

[34] A. Pełczyński. Projections in certain Banach spaces. *Studia Math.*, 19:209–228, 1960.

[35] C. Samuel. Primarité des produits d'espaces de suites. *Colloq. Math.*, 39(1):123–132, 1978.

[36] Z. Semadeni. *Schauder bases in Banach spaces of continuous functions*, volume 918. Springer, Cham, 1982.

[37] L. Tzafriri. Uniqueness of structure in Banach spaces. In *Handbook of the geometry of Banach spaces. Volume 2*, pages 1635–1669. Amsterdam: North-Holland, 2003.

[38] P. Wojtaszczyk. *Banach spaces for analysts*, volume 25 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.

Tomasz Kania, Mathematical Institute, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic and Institute of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

Email address: kania@math.cas.cz, tomasz.marcin.kania@gmail.com

Richard Lechner, Institute of Analysis, Johannes Kepler University Linz, Altenberger Strasse 69, A-4040 Linz, Austria

Email address: richard.lechner@jku.at