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A CATEGORY OF V-STRUCTURES FOR FOLIATIONS

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Abstract. For a foliation $\mathcal{F}$ of a smooth complex manifold we introduce the category $\mathcal{C}$ of V-structures associated to a vertex operator algebra V and the category of its modules. The main result consists of the construction of V-structures and canonicity proof of $\mathcal{C}$ on $\mathcal{F}$.

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1. Introduction: results of the paper

There exist a few main approaches to the theory of foliations [6, 3, 7, 8, 12, 13, 21]. Vertex operator algebras, generalizations of ordinary Lie algebras, constitute an essential part of conformal field theory. The idea of studies of cohomology of a foliation of smooth foliated manifold M and establishing connections to cohomology of M has first appeared in [3]. The main motivation for studies in this paper is to develop vertex operator algebra approach to cohomology of foliations. In this approach we use structural and computation properties of vertex operator algebra objects considered on transversal basis to a foliation. In order to understand deep structure of foliation leaves, and taking into account power of vertex operator algebra approach, it would be useful to formulate non-commutative approach to cohomology of foliations. In this paper we apply techniques of Lie algebra of vector fields on a manifold in the cosimplicial setup. Taking into account the standard methods of defining canonical (i.e., independent of the choice of covering domains) cosimplicial object [10, 23] as well as the Čech-de Rham cohomology construction [8] for a foliation, we develop the vertex operator algebra approach to description of a foliation $\mathcal{F}$ by consideration of local picture arising from distribution of points on a smooth manifold M and, in particular, on transversal sections to $\mathcal{F}$, with attaching vertex operator algebra states at points. In applications, one would be interested in relating techniques of this paper to studies of foliations of codimension one [1, 2, 15, 16]. In particular, the problem of finding non-vanishing cohomological invariants, as well as the problem of distinguishing of compact and non-compact leaves for the Reeb foliation of the full torus, are also of high importance. Based on V-structures introduced in this paper, we plan to approach consideration of a characteristic classes theory for arbitrary codimension regular and singular foliations. It would be interesting also to develope intrinsic (i.e., purely coordinate independent) theory of a smooth manifold foliation cohomology involving vertex operator algebra bundles [4]. It would be important to

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establish connection to chiral de-Rham complex on a smooth manifold introduced in [22].

The plan of the paper is the following. Section 2 contains preliminaries needed for understanding of the construction described in this paper and proof of the main proposition. In Subsection 2.1 we recall the notions of holonomy and basis of transversal sections for a foliation of a smooth manifold. In Subsection 2.2 definition and properties of vertex operator algebras and their modules are discussed. Section 3 contains the main construction of the category $\mathcal{C}$ of V-structures for a foliation. Section 1 is devoted to the proof of the main proposition. Let us briefly describe the construction of the category of V-structures. In order to give a local description of leaves of a foliation $\mathcal{F}$ of n-dimensional smooth manifold M we use the classical approach of transversal sections as well as algebraic and analytic properties of vertex operator algebras. We chose two sets of points on M and on a basis $\mathcal{U}$ of transversal sections with corresponding domains of local coordinates. Points on M and $\mathcal{U}$ are then endowed with sets of a V vertex operator algebra elements. By taking algebraic completion of elements of the category $\mathcal{W}$ of V-modules, we formulate the definition of spaces $\widehat{W}^q_r$, q, $r \geq 0$ of vectors of $\overline{W}$-valued rational forms combined with sets of vertex operators. To keep elements of $\widehat{W}^q_r$ coherent with respect to actions of the operators Δ^q_r shifting indexes l and m, we apply certain analytic restriction on their characteristics provided by values of non-degenerate bilinear forms of $\widehat{W}^q_r$-elements. The spaces $\widehat{W}^q_r$ are defined on cosimplicial domains chosen on transversal section of $\mathcal{F}$. We formulate the definition of the category $\mathcal{C}$ of canonical V-structures of $\mathcal{F}$ associated with a vertex operator algebra V. Its objects consitst of sets of points $\mathfrak{P}$ and vertex operator algebra elements $\mathfrak{V}$, and the spaces $\widehat{W}^q_r$ associated to the category $\mathcal{W}$ of admissible V-modules defined in Section 2. We denote by $\widehat{W}$ the spaces $\widehat{W}^q_r$ for all q, r. In Section 4 we prove the main result of this paper:

Proposition 1. *The category $\mathcal{W}$ of admissible V-modules induces the category $\mathcal{C}$ of canonical V-structures on $\mathcal{F}$.*

In the proof we show that elements of the spaces $\widehat{W}$ are invariant torsors with respect to the group of foliation preserving changes of transversal basis and local coordinates. Though a V-structure does not depend neither on the choice of transversal basis nor on the choice of coordinates on M, it does depend on the choice of a set of points from the set $\mathfrak{P}$, the choice of vertex operator algebra elements from the set $\mathfrak{V}$ as well as on a particular element of the category $\mathcal{W}$ of admissible V-modules. In Section 1 we also show that characteristic functions of elements of $\widehat{W}^q_r$ spaces form a chain-cochain complexes (C^q_r, δ^q_r) with the coboundary operators δ^q_r induced by the shift operators Δ^q_r. The vertex operator algebra cohomology of a foliation on a smooth complex curve is introduced via such complexes.

2. Preliminaries

In this Section we recall properties of a foliation $\mathcal{F}$ transversal basis $\mathcal{U}$ on a smooth n-dimensional complex manifold M [8], vertex operator algebras, and related topics.

2.1. **Holonomy and transversal basis for a foliation.** Let us first recall [8] definitions of holonomy embeddings and transversal basis for a foliation $\mathcal{F}$. Suppose we are given a path α between two points p_1 and p_2 which belong to the same leaf of $\mathcal{F}$. For two transversal sections U_1 and U_2 passing through p_1 and p_2 one defines a transport α along the leaves from a neighborhood of $p_1 \in U_1$ to a neighborhood of $p_2 \in U_2$. Then it is assumed that there exists a germ of a diffeomorphism $\mathrm{hol}(\alpha) : (U_1, p_1) \longrightarrow (U_2, p_2)$ called the holonomy of α. In the case when the transport α is defined in all of U_1 and embeds into U_2 then $h : U_1 \hookrightarrow U_2$ is denoted by $\mathrm{hol}(\alpha) : U_1 \hookrightarrow U_2$ and called a holonomy embedding. A composition of paths induces a composition of corresponding holonomy embeddings. Now recall the definition of the transversal basis for a foliation $\mathcal{F}$. Transversal sections U_i passing through p_i, $i \geq 0$, are neighborhoods of the leaves through p_i in the leaf space $M/\mathcal{F}$. A transversal basis $\mathcal{U}$ for $\mathcal{F}$ is a set of transversal sections $U_i \subset M$ such that for a section U_i passing through a point p_i, and for any transversal section U_j passing through $p_j \in M$, one can find a holonomy embedding $h : U_i \hookrightarrow U_j$ with $U_i \in \mathcal{U}$ and $p_j \in h(U_i)$.

2.2. **Vertex operator algebras and their modules.** In this section, we recall the definition and basic properties of vertex operator algebras over the base field $\mathbb{C}$ of complex numbers and their generalized modules. A vertex operator algebra $(V, Y_V, \mathbf{1}_V, c)$, [20] of Virasoro algebra central charge c, consists of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{r\in\mathbb{Z}} V_{(r)}$, with finite-dimensional grading subspaces $V_{(r)}$ $\dim V_{(r)} < \infty$ for each $r \in \mathbb{Z}$, equipped with a linear map $Y_V : V \to \mathrm{End}(V)[[z, z^{-1}]]$, for a formal complex parameter z and a distinguished vector $\mathbf{1}_V \in V$. The vertex operator for $v \in V$ is given by $Y_V(v, z) = \sum_{r\in\mathbb{Z}} v(r) z^{-r-1}$, with components $(Y_V(v))_r = v(r) \in \mathrm{End}\,(V)$, with the property $Y_V(v, z)\mathbf{1}_V = v + O(z)$. In this paper we apply the following restrictions on a grading of a vertex operator algebra V or its module W. A vertex operator algebra V-module W is a vector space W equipped with a vertex operator map o $Y_W : V \otimes W \to W[[z, z^{-1}]]$, and $u \otimes w \mapsto Y_W(u, z)w = \sum_{r\in\mathbb{Z}} (Y_W)_r(u, w) z^{-r-1}$. W is also subject of actions of and linear operators $L_W(0)$ and $L_W(-1)$ (0 and -1 Virasoro modes) satisfying the following conditions. One assumes that $V_{(r)} = 0$ for $r \ll 0$. The vector space W is $\mathbb{C}$-graded, that is, $W = \bigoplus_{\alpha\in\mathbb{C}} W_{(\alpha)}$, such that $W_{(\alpha)} = 0$, when the real part of α is sufficiently negative. The result of a vertex operator $Y_{V,W}(u, z)(v, w)$, u, $v \in V$, $w \in W$, contains only finitely many negative power terms, that is $Y_{V,W}(u, z)(v, w) \in (V, W)((z))$, i.e., belongs to the space of formal Laurent series in z with coefficients in (V, W). Here (V, W) and subscript ${}_{V,W}$ mean corresponding expression either for vertex operator algebra V elements or its module W. Let $\mathrm{Id}_{V,W}$ be the identity operator on (V, W). Then $Y_{V,W}(\mathbf{1}_V, z) = \mathrm{Id}_{V,W}$. For $u \in V$, $Y_V(u, z)\mathbf{1}_V \in V[[z]]$ and $\lim_{z\to 0} Y_V(u, z)\mathbf{1}_V = u$. We assume that there exist non-degenerate bilinear pairings $(.,.)_V$, $V' \otimes V \to \mathbb{C}$, and $(.,.)_W$ for W. For $v' \in V' = \coprod_{r\in\mathbb{Z}} V^*_{(r)}$, where $V^*_{(r)}$ denotes the dual vector space to $V_{(r)}$ with respect to the pairing $(.,.)_V$, and similarly for $w' \in W' = \coprod_{r\in\mathbb{Z}} W^*_{(r)}$. Here W' denotes the dual V-module to W. In case the meromorphic function $\Omega(A) = (v', A)_W$ for an operator $A(v_1, z_1; \ldots; v_r, z_r)$ of several complex variables $(z_1, \ldots, z_r)$ on a domain extends to a rational function on a larger domain we call it the characteristic of A. We assume that the characteristics $\Omega\left(Y_{V,W}(u_1, z_1) Y_{V,W}(u_2, z_2) v\right)$, $\Omega\left(Y_{V,W}(u_2, z_2) Y_{V,W}(u_1, z_1)(v, w)\right)$, and $\Omega\left(Y_{V,W}(Y_V(u_1, z_1 - z_2)u_2, z_2)(v, w)\right)$, converge absolutely in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1, z_2. Poles of these characteristics are only allowed at $z_1 = 0 = z_2$, and $z_1 = z_2$. The role of a grading operator for V is played by the zero Virasoro mode with $L_V(0)v = rv$ for $v \in V_{(r)}$. Then for $v \in V$ one has $[L_{V,W}(0), Y_{V,W}(v, z)] = Y_{V,w}(L_V(0)v, z) + z\frac{d}{dz}Y_{V,W}(v, z)$. For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that $(L_W(0) - \alpha)^N w = 0$. For $v \in V$ the operator $L_V(-1)$ is given by $L_V(-1)v = \mathrm{Res}_z z^{-2}Y_V(v, z)\mathbf{1}_V = Y_{(-2)}(v)\mathbf{1}_V$. Then $\frac{d}{dz}Y_V(u, z) = Y_{V,W}(L_V(-1)u, z) = [L_{V,W}(-1), Y_V(u, z)]$. We denote $\mathrm{wt}(v) = k$ the weight for $v \in V_{(k)}$. The translation property for vertex operators can be written as $Y_W(u, z) = e^{-z'L_W(-1)}Y_W(u, z + z')e^{z'L_W(-1)}$, $z' \in \mathbb{C}$. A vertex operator algebra V satisfying conditions above is called conformal of central charge $c \in \mathbb{C}$, if there exists a non-zero conformal vector $\omega \in V_2$ such that the Fourier coefficients $L_V(r)$ of the corresponding vertex operator $Y(\omega, z) = \sum_{r\in\mathbb{Z}} L_V(k)z^{-r-2}$, is determined by Virasoro modes $L_V(r) : V \to V$ subject to the commutation relations $[L_V(s), L_V(r)] = (s-r)L_V(s+r)+\frac{c}{12}(s^3-s)\delta_{s,-r}\,\mathrm{Id}_V$. For $u \in V$, one gets $\frac{d}{dz}Y(u, z) = Y(L_V(-1)u, z)$. As in [9], we describe here properties of intertwining operators. An intertwining operator $Y^{W^{(i)}}_{W^{(j)},W^{(k)}}(v, z)$ a linear map from $W^{(j)} \to W^{(k)}$ to the vector space of formal series in rational powers of z with coefficients in $W^{(i)}$, or, equivalently, a linear map $W^{(j)} \to \left(\mathrm{Hom}\left(\mathrm{W^{(k)}}, \mathrm{W^{(i)}}\right)\right)(z)$, $w \mapsto Y^{W^{(i)}}_{W^{(j)},W^{(k)}}(v, z) = \sum\limits_{r\in Q} w_r z^{-r-1}$, where $w_r \in \mathrm{Hom}\left(\mathrm{W^{(k)}}, \mathrm{W^{(i)}}\right)(\mathrm{w_1}, \mathrm{z_2})$. Introduce the formal generating function $\delta(z) = \sum_{r\in\mathbb{Z}} z^r$. The following generalized Jacobi identity holds on $W^{(k)}$ for the operators $Y(v, z_1)$ and $Y^{W^{(i)}}_{W^{(j)},W^{(k)}}$:

$$\begin{aligned}
&z_0^{-1}\chi^{\kappa}(z_1, z_2, z_0)\ \delta\chi(z_1, z_2, z_0)Y_{W^{(i)}}(v, z_1)\ Y^{W^{(i)}}_{W^{(j)},W^{(k)}}(w_1, z_2)\\
&-\kappa z_0^{-1}\chi^{\kappa}(z_2, z_1, z_0); \delta\chi(z_2, z_1, z_0)Y^{W^{(i)}}_{W^{(j)},W^{(k)}}(w_1, z_2)\ Y_{W^{(k)}}(v, z_1)\\
&\quad = z_2^{-1}\delta\chi(z_2, z_1, z_0)\ Y^{W^{(i)}}_{W^{(j)},W^{(k)}}(Y_{W^{(j)}}(v, z_0)w_1, z_2)\chi^{-\kappa}(z_1, z_0, z_2),
\end{aligned}$$

where $v \in V$, $w_1 \in W^{(j)}$, $\chi(z_1, z_2, z_3) = (z_1 - z_2)/z_0$, $\kappa \in \mathbb{Q}$ corresponds to an element of the V-automorphism group G. Note that $\frac{d}{dz}Y^{W^{(i)}}_{W^{(j)},W^{(k)}}(w_1, z) = Y^{W^{(i)}}_{W^{(j)},W^{(k)}}$ $(L_W(-1)\ w_1, z)$, where $L_W(-1)$ is the operator acting on $W^{(j)}$. Note that V-vertex operator is $Y(., z) = Y^V_{V,V}(., z)$, and a vertex operator acting on V-module W is $Y_W(., z) = Y^V_{V,W}(., z)$. For $v \in V$, $w \in W$, the intertwining operator Y^W_{WV} : $V \to W$, $v \mapsto Y^W_{WV}(w, z)v$, is defined by $Y^W_{WV}(w, z)v = e^{zL_W(-1)}Y_W(v, -z)w$. For $a \in \mathbb{C}$, the conjugation property with respect to the grading operator $L_W(0)$ is given by $a^{L_W(0)}\ Y_W(v, z)\ a^{-L_W(0)} = Y_W\left(a^{L_W(0)}v, az\right)$. In [4], $v \in V$, the following formula was derived $[L_W(r), Y_W(v, z)] = \sum_{r\geq -1}\frac{1}{(r+1)!}\partial_z^{r+1}z^{r+1}\ Y_W(L_V(r)v, z)$, of a Virasoro generator commutation with a vertex operator. Given a vector field $\beta(z)\partial_z = \sum_{r\geq -1}\beta_r z^{r+1}\partial_z$, $\beta(z)\partial_z \in \mathrm{Der}\mathcal{O}^{(1)}$, which belongs to the local Lie algebra of Aut $\mathcal{O}^{(1)}$, introduce the operator $\overline{\beta} = -\sum_{r\geq -1}\beta_r L_W(r)$. One concludes with the following formula [4] $\left[\overline{\beta}, Y_W(v, z)\right] = \sum_{r\geq -1}\frac{1}{(r+1)!}\left(\partial_z^{r+1}\beta(z)\right)\ Y_W(L_V(r)v, z)$. We call a vertex operator algebra V-module W quasi-conformal if it carries an action of Der $\mathcal{O}^{(n)}$ on n-dimensional smooth manifold M such that commutation formula above holds for any $v \in V$, and $z = z_j$, $1 \leq j \leq n$, the element $L_W(-1) = -\partial_z$ acts as the translation operator $L_W(0) = -z\partial_z$, acts semi-simply with integral eigenvalues, and the Lie subalgebra $\mathrm{Der}_+ \, \mathcal{O}^{(n)}$ acts locally nilpotently on M. A vector w which belongs to a module W of a quasi-conformal vertex operator algebra V is called primary of conformal dimension $\nu \in \mathbb{Z}_+$ if $L_W(k)w = 0$, $k > 0$, $L_W(0) \cdot w = \nu w$. Let $\overline{W}$ denote the algebraic completion of W, $\overline{W} = \prod_{r \in \mathbb{C}} W_{(r)} = (W')^*$. For $r \in \mathbb{Z}_+$, denote by $F_r\mathbb{C}$ the configuration space of r ordered points in $\mathbb{C}$, $F_r\mathbb{C} = \{(z_1, \ldots, z_r) \in \mathbb{C}^r, z_i \neq z_j, i \neq j\}$. A $\overline{W}$-valued rational function in variables $(z_1, \ldots, z_r)$ is called a map $f : F_r\mathbb{C} \to \overline{W}$, $(z_1, \ldots, z_r) \mapsto f(z_1, \ldots, z_r)$, such that the characteristic $\Omega(f) = \Omega\left(f(z_1, \ldots, z_r)\right)$, is a rational function in $(z_1, \ldots, z_r)$ that admits only poles at $z_i = z_j$, $i \neq j$. We denote by $\mathcal{W}$ the category of V-module admissibles that satisfy all related properties of this Subsection.

Recall now the definition of shuffles. For $l \in \mathbb{N}$ and $1 \leq s \leq l-1$, let $J_{l;s}$ be the set of elements of S_l which preserve the order of the first s numbers and the order of the last $l-s$ numbers, that is, $J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \ldots < \sigma(s),\ \sigma(s+1) < \ldots < \sigma(l)\}$. The elements of $J_{l;s}$ are called shuffles. We will use the notation $J_{l;s}^{-1} = \{\sigma \mid \sigma \in J_{l;s}\}$ for them.

Finally, let us remind the definition of a torsor [4]. Let $\mathfrak{G}$ be a group, and S a non-empty set. Then S is called a $\mathfrak{G}$-torsor if it is equipped with a simply transitive right action of $\mathfrak{G}$. For s_1, $s_2 \in S$, there exists a unique $\mu \in \mathfrak{G}$ such that $s_1 \cdot \mu = s_2$, where the right action is given by $s_1 \cdot (\mu\mu') = (s_1 \cdot \mu) \cdot \mu'$. The choice of any $s_1 \in S$ allows us to identify S with $\mathfrak{G}$ by sending $s_1 \cdot \mu$ to μ.

3. Category of V-structures for a foliation

In this Section we introduce explicitly V-structures associated to a vertex operator algebra V module W for a codimension p foliation $\mathcal{F}$ of a smooth n-dimensional manifold M. Let M be endowed with a coordinate chart $\mathcal{V} = \{V_r, r \in \mathbb{Z}\}$. Consider a (possibly infinite) arbitrary set p_l, $l \geq 0$, of l distinct points and corresponding domains $V_l \subset M$. Let $\mathcal{U} = \{U_k\}$, $k \geq 0$, be a transversal basis of $\mathcal{F}$. We chose a (possibly infinite) set $(p_{l+1}, \ldots, p_{l+m})$ of arbitrary distinct $m \geq 0$ points on a set of sections $U_{i',b}$, $1 \leq i' \leq m$, $1 \leq b \leq k$ of $\mathcal{U}$. Denote by $\mathfrak{P}$ the set of all choices of l and m points on M and $\mathcal{U}$ correspondingly. Let us associate to each point of $(p_1, \ldots, p_l)$ and $(p_{l+1}, \ldots, p_{l+m})$ vertex operator algebra elements $(v_{i,1}, \ldots, v_{i,n})$, $1 \leq i \leq l$, and $(v_{j,1}, \ldots, v_{j,p})$, $l+1 \leq j \leq l+m$ correspondingly. Denote by $\mathfrak{V}$ the set of all choices of $ln + mp$ vertex operator algebra V elements. For $\widetilde{n} = ln + mp$, it is convenient to renotate the vertex operator algebra elements as $(\widetilde{v}_1, \ldots, \widetilde{v}_{\widetilde{n}}) = (v_{1,1}, \ldots, v_{l,1}, \ldots, v_{1,n}, \ldots, v_{l,n}, v_{l+1}, \ldots, v_{l+1,p}, \ldots, v_{l+m,1}, \ldots, v_{l+m,p})$. Endow each of points among $(p_1, \ldots, p_l)$ and $(p_{l+1}, \ldots, p_{l+m})$ with sets $(z_{i,1}, \ldots, z_{i,n})$, $1 \leq i \leq l$ and $(z_{j,1}, \ldots, z_{j,p})$, $l+1 \leq j \leq l+m$, of local coordinates on domains V_l and V_{l+m}. Denote also $(\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}) = (z_{1,1}, \ldots, z_{l,1}, \ldots, z_{1,n}, \ldots, z_{l,n}, z_{l+1}, \ldots, z_{l+1,p}, \ldots, z_{l+m,1}, \ldots, z_{l+m,p})$.

For sets p_j, $1 \le j \le l$ of l points on M we consider a map $X(v_{i,1}, z_{i,1}; \ldots; v_{i,n}, z_{i,n}) : V^{\otimes ln} \in \overline{W}[[z_1, \ldots, z_r]]$, $1 \le i \le l$, combined with a set of mp vertex operators at points p_{l+k}, $1 \le k \le m$, such that its characteristic $\Omega(X)$ satisfy the following properties. We require that characteristics are coherent with vertex operator algebra definition of $\overline{W}$ [19]. The characteristics of both sides of the equations $\partial_{z_i} X(v_1, z_1; \ldots; v_s, z_s) = X(v_1, z_1; \ldots; v_{i-1}, z_{i-1}; L_V(-1)v_i, z_i; v_{i+1}, z_{i+1}; \ldots; v_s, z_s)$, $(\partial_{z_1} + \cdots + \partial_{z_s}) X(v_1, z_1; \ldots; v_s, z_s) = L_W(-1)X(v_1, z_1; \ldots; v_s, z_s)$. for $i = 1, \ldots, s$, should coincide. Since $L_W(-1)$ is a weight-one operator on W, $e^{zL_W(-1)}$ is a linear operator on $\overline{W}$ for any $z \in \mathbb{C}$. For a linear map X with $(v_1, \ldots, v_s) \in V^{\otimes s}$, $(z_1, \ldots, z_s) \in F_s\mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1 + z, \ldots, z_s + z) \in F_s\mathbb{C}$, the characteristics of the elements $e^{zL_W(-1)} X(v_1, z_1; \ldots; v_s, z_s) = X(v_1, z_1 + z; \ldots; v_s, z_s + z)$, and for $(v_1, \ldots, v_s) \in V^{\otimes s}$, $(z_1, \ldots, z_s) \in F_s\mathbb{C}$, $z \in \mathbb{C}$ and $1 \le i \le s$ such that $(z_1, \ldots, z_{i-1}, z_i + z, z_{i+1}, \ldots, z_s) \in F_s\mathbb{C}$, the power series expansion of the characteristic of the element $X(v_1, z_1; \ldots; v_{i-1}, z_{i-1}; v_i, z_i + z; v_{i+1}, z_{i+1}; \ldots; v_s, z_s)$, in z are equal to the power series expansion of the characteristic of $X(v_1, z_1; \ldots; v_{i-1}, z_{i-1}; e^{zL_V(-1)}v_i, z_i; v_{i+1}, z_{i+1}; \ldots; v_s, z_s)$, in z. In particular, the power series in z is absolutely convergent on the open disk $|z| < \min_{i \ne j}\{|z_i - z_j|\}$. Assume also that the space $\overline{W}$ is endowed with a grading $\overline{W} = \bigcup_{s \in \mathbb{Z}, s > s_0} \overline{W}_{(s)}$ bounded from below with respect to the grading operator $L_W(0)$. Denote by $P_s : \overline{W} \to \overline{W}_{(s)}$, the projection of $\overline{W}$ on $\overline{W}_{(s)}$. In addition to that, for $(v_1, \ldots, v_l) \in V^{\otimes s}$, $(z_1, \ldots, z_s) \in F_s\mathbb{C}$ and $z \in \mathbb{C}^\times$ so that $(zz_1, \ldots, zz_s) \in F_s\mathbb{C}$, a linear map $X : V^{\otimes s} \to \overline{W}$ the characteristic of elements in both sides of the equation $z^{L_W(0)} X(v_1, z_1; \ldots; v_s, z_s) = X(z^{L_V(0)}v_1, zz_1; \ldots; z^{L_V(0)}v_s, zz_s)$. should coincide.

Denote by $(l_i, \ldots, l_{\widetilde{n}})$ a partition of $\widetilde{n}$ of $\widetilde{n} = \sum_{i \ge 1} l_i$, $k_i = l_1 + \cdots + l_{i-1}$, and $\zeta_i \in \mathbb{C}$. Consider the local coordinates $(\widetilde{z}_{ln+1}, \ldots, \widetilde{z}_{\widetilde{n}})$ of points $(p_{l+1}, \ldots, p_{l+m})$ bounded in the domains $|\widetilde{z}_{k_i+k'} - \zeta_i| + |\widetilde{z}_{k_j+k''} - \zeta_j| < |\zeta_i - \zeta_j|$, for i, $j = 1, \ldots, n$, $i \ne j$, and for $k' = 1, \ldots, l_i$, $k'' = 1, \ldots, l_j$. For $\widetilde{k}_i = k_i + l_i$, define $f_i = X\left(\mathcal{Y}_W(\widetilde{v}_1, \widetilde{z}_1 - \zeta_i) \ldots \mathcal{Y}_W(\widetilde{v}_{\widetilde{k}_i}, \widetilde{z}_{\widetilde{k}_i} - \zeta_i)\right)$, for $i = 1, \ldots, ln$. Assume that there exist positive integers $\beta(\widetilde{v}_{l',i}, \widetilde{v}_{l",j})$ depending only on $\widetilde{v}_{l',i}$ and $\widetilde{v}_{l'',j}$ for $i, j = 1, \ldots, \widetilde{n}$, $i \ne j$, $1 \le l', l'' \le \widetilde{n}$, such that the characteristic of the element $\sum_{r_1, \ldots, r_{\widetilde{n}} \in \mathbb{Z}} X\left(P_{r_1} f_1, \zeta_1; \ldots; P_{r_{\widetilde{n}}} f_{\widetilde{n}}, \zeta_{\widetilde{n}}\right)$ is absolutely convergent in the domains defined above to an analytic extension in $(\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}})$ independently of complex parameters $(\zeta_1, \ldots, \zeta_{\widetilde{n}})$, with poles of order less than or equal to $\beta(\widetilde{v}_{l',i}, \widetilde{v}_{l'',j})$ allowed only on the diagonal of $(\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}})$. For the domains $|\widetilde{z}_i| > |\widetilde{z}_s| > 0$, for $i = 1, \ldots, m$, and $s = m+1, \ldots, m+l$, we assume that for $(\widetilde{v}_1, \ldots, \widetilde{v}_{\widetilde{n}})$, the characteristic of the element $\sum_{q \in \mathbb{C}} \mathcal{Y}_W(v_{l+1,1}, z_{l+1,1}) \ldots \mathcal{Y}_W(v_{l+m,1}, z_{l+m,p}) P_q\left(X\left(v_{1,1}, z_{1,1}; \ldots; v_{l,1}, z_{l,n}\right)\right)$, (incorporating local coordinates on M and transversal sections) is absolutely convergent in the domain above when $\widetilde{z}_i \ne \widetilde{z}_j$, $i \ne j$ and the sum can be analytically extended to a rational function in $(\widetilde{z}_1, \ldots, z_{\widetilde{n}})$ with poles of orders less than or equal to $\beta(\widetilde{v}_{l',i}, \widetilde{v}_{l'',j})$ allowed $\widetilde{z}_i = \widetilde{z}_j$. Note that local coordinates are considered either on domains of M or domains V_{l+i},

$1 \le i \le m$ which are also coordinates on M. Finally, define the left action of the permutation group S_r on $\overline{W}$ by $\sigma(f)(z_1, \dots, z_r) = f(z_{\sigma(1)}, \dots, z_{\sigma(r)})$, for $f \in \overline{W}$. We require that $\sum_{\sigma \in J_{l;s}^{-1}} (-1)^{|\sigma|} \sigma(X(v_{\sigma(1,1)}, z_{\sigma(1,1)}; \dots; v_{\sigma(l,1)}, \dots, v_{\sigma(1,n)}, \dots, v_{\sigma(l,n)})) = 0$.

In order to work with objects having coordinate invariant formulation, we consider elements of $\overline{W}$ with local coordinates z multiplied by powers of corresponding differentials dz. For all choices of l points of $\mathfrak{P}$ and all choices of vertex operator algebra elements of $\mathfrak{V}$, for $ln \ge 0$ complex variables $(\widetilde{z}_1, \dots, \widetilde{z}_l)$ defined in domains V_k, $1 \le k \le ln$ of the coordinate chart $\mathcal{V}$ on M, let us introduce the following vector $\mathbf{X}(\widetilde{v}_1, \widetilde{z}_1; \dots; \widetilde{v}_{ln}, \widetilde{z}_{ln}) = \left[X \left(\widetilde{v}_1, \widetilde{z}_1 d\widetilde{z}_{i(1)}; \dots; \widetilde{v}_{ln}, \widetilde{z}_{ln} \ d\widetilde{z}_{i(ln)} \right) \right]$. containing $\overline{W}$-rational functions X satisfying conditions stated above in this Section. Here $i(j)$, $j = 1, \dots, ln$, are cycling permutations of $(1, \dots, ln)$ starting with j, and we Denote by $[.]$ the vector with elements given by mappings X. In [4], in the case $n = 1$, they proved that the vertex operator $Y_W(u, z) \ dz^{\mathrm{wt}(u)}$ is an invariant object with respect to changes of the local coordinate. In Section 4 we prove that the vectors $\mathbf{X}$ introduced above as well as vertex operators $\mathcal{Y}(u, z_i) = Y_W(u, z_i) \ dz_i^{\mathrm{wt}(u)}$, $i \ge 0$, for primary $u \in V$, are invariant with respect to changes of coordinates, i.e., to the group of coordinate transformations Aut $\mathcal{O}^{(n)}$ on M $(w_1, \dots, w_r) \mapsto (z_1, \dots, z_r)$, and corresponding differentials.

In [17] the classical approach to cohomology of vector fields of manifolds was initiated. In [10, 23] we find an alternative way to describe cohomology of Lie algebra of vector fields on a manifold in the cosimplicial setup. Taking into account the standard methods of defining canonical (i.e., independent of the choice of covering $\mathcal{U}$ and coordinates) cosimplicial object [10, 23] as well as the Čech-de Rham cohomology construction [8], we formulate here the vertex operator algebra approch to cohomology of a foliation. Let $I_q = (i_k, j_k)$, $1 \le k \le q$, be a subset (with no repetitions) of the set $(z_{i,1}, \dots, z_{i,n})$, $1 \le i \le l$ of local variables corresponding to of l points $(p_1, \dots, p_l)$. Similarly, let $J_r = (i_{k'}, j_{k'})$, $1 \le k' \le r$, be a subset (with no repetitions) of the set $(z_{i,1}, \dots, z_{i,p})$, $l+1 \le i \le l+m$, of local variables corresponding to of m points $(p_{l+1}, \dots, p_{l+m})$. Consider a subspace of vectors $\mathbf{X}(v_{i_1,j_1}, z_{i_1,j_1}; \dots; v_{i_q,j_q}, z_{i_q,j_q})$ associated to I_q and J_r. We associate to any subset J_r a collection of r vertex operators $\mathcal{Y}_W(v_{i'_a,j'_a}, z_{i'_a,j'_a})$, $1 \le a \le r$, with local coordinates $z_{i'_a,j'_a}$ defined in r subdomains $U_{i'_a,b} \subset U_{i'_a,b}$, $1 \le b \ lek$ of k transversal sections $U_{i'_a}$ of a transversal basis $\mathcal{U}$. Form a space $\widehat{W}^q_r = \widehat{W}^q_r(V, W, \mathcal{U}, \mathcal{F})$ as the space $\widehat{W}^q(V, W, U, \mathcal{F})$ of vectors $\mathbf{X}$ with q (v, z)-entries, combined with the set of r vertex operators defined on the intersection $U = \bigcap_{U_{i'_1,1} \xrightarrow{h_{i'_1,1}} \dots \xrightarrow{h_{i'_r,k}} U_{i'_r,k}, 1 \le a \le r, \ 1 \le b \le k,} U_{i'_a,b}$, where the intersection ranges over all r subdomains on k transversal sections $U_{i'_a,b}$ of $\mathcal{U}$, and over all sequences of holonomy embeddings $h_{i'_a,b}$, $1 \le a \le r$, $1 \le b \le k$. In the case $k = 0$ the sequence of holonomy embeddings is empty. The spaces $\widehat{W}^q_r$ are related by the shift operators $\Delta^q_r : \widehat{W}^q_r \to \widehat{W}^{q+1}_{r-1}$, increasing the upper index and decreasing the lower index in elements of $\widehat{W}^q_r$. For $I_{q+1} = (i_k, j_k)$, $1 \le k \le q+1$, and $J_r = (i'_{k'}, j'_{k'})$, $1 \le k' \le r$, and $\mathbf{X} \in \widehat{W}_r^q$ let us define the operator in a standard way [8, 19]

$$\begin{aligned}
&\Delta_r^q \mathbf{X}(v_{i_1,j_1}, z_{i_1,j_1}; \ldots; v_{i_q,j_q}, z_{i_q,j_q}) \\
&= \mathcal{Y}_W(v_{i_1,j_1}, z_{i_1,j_1}) \mathbf{X}\left(v_{i_2,j_2}, z_{i_2,j_2}; \ldots; v_{i_q,j_q}, z_{i_q,j_q}\right) \\
&\quad + \sum_{s=1}^{q} (-1)^s \mathbf{X}\left(v_{i_1,j_1}, z_{i_1,j_1}; \ldots; \mathcal{Y}_W(v_{i_{s-1},j_{s-1}}, z_{i_{s-1},j_{s-1}} - \zeta_s)\right. \\
&\qquad \left. \mathcal{Y}_W(v_{i_{s+1},j_{s+1}}, z_{i_{s+1},j_{s+1}} - \zeta_s) \mathbf{1}_V; \ldots; v_{i_q,j_q}, z_{i_q,j_q}\right) \\
&\qquad + (-1)^{q+1} \mathcal{Y}_W(v_{i_{q+1},j_{q+1}}, z_{i_{q+1},j_{q+1}}) \mathbf{X}\left(v_{i_1,j_1}, z_{i_1,j_1}; \ldots; v_{i_q,j_q}, z_{i_q,j_q}\right).
\end{aligned}$$

The shift operator Δ_r^q is chosen in such a way that characteristics of it would have nice cohomological properties. Note that, after the application of Δ_r^q on an element $\mathbf{X}$ containing local coordinates and corresponding vertex operator algebra elements associated to all n dimensions, of M, the result is of such action is then related to submanifold with less number of local coordinates describing points $(p_1, \ldots, p_l)$. For $q = 2$, there exists a subspace $\widehat{W}_\vartheta^2$ of $\widehat{W}_0^2$ containing $\widehat{W}_r^2$ for all $r \geq 1$ such that Δ_r^2 is defined on this subspace. For the domains $|\widetilde{z}_1 - \zeta| > |\widetilde{z}_2 - \zeta|$, $|\widetilde{z}_2 - \zeta| > 0$, and $|\zeta - \widetilde{z}_3| > |\widetilde{z}_1 - \zeta|, |\widetilde{z}_2 - \zeta| > 0$, respectively, such that for $\widetilde{v}_1$, $\widetilde{v}_2$, $\widetilde{v}_3 \in V$, the characteristic functions of elements $\sum_{s\in\mathbb{C}} \mathcal{Y}_W(\widetilde{v}_1, \widetilde{z}_1)\mathcal{Y}_W(P_s(X(\widetilde{v}_2, \widetilde{z}_2 - \zeta; \widetilde{v}_3, \widetilde{z}_3 - \zeta)), \zeta)$ $+X(\widetilde{v}_1, \widetilde{z}_1; P_s(\mathcal{Y}_V(\widetilde{v}_2, \widetilde{z}_2 - \zeta)\mathcal{Y}_V(\widetilde{v}_3; \widetilde{z}_3 - \zeta)\mathbf{1}_V), \zeta)$, and $\sum_{s\in\mathbb{C}} X(P_s(Y_V(\widetilde{v}_1, \widetilde{z}_1 - \zeta)\mathcal{Y}_V(\widetilde{v}_2; \widetilde{z}_2 - \zeta)\mathbf{1}_V), \zeta)\mathcal{Y}_V(\widetilde{v}_3, \widetilde{z}_3)$ $+\mathcal{Y}_W(\widetilde{v}_3, \widetilde{z}_3)\mathcal{Y}_W(P_s(X(\widetilde{v}_1, \widetilde{z}_1 - \zeta; \widetilde{v}_2, \widetilde{z}_2 - \zeta)), \zeta)$, are absolutely convergent in the regions above and can be analytically extended to rational functions in $\widetilde{z}_1$ and $\widetilde{z}_2$ with poles allowed only at $\widetilde{z}_1$, $\widetilde{z}_2 = 0$, and $\widetilde{z}_1 = \widetilde{z}_2$. For $J_3 = (i_k, j_k)$, $1 \leq k \leq 3$, the operator Δ_ϑ^2 is defined for $X \in \widehat{W}_\vartheta^2$ by

$$\begin{aligned}
&\Delta_\vartheta^2 \mathbf{X}(v_{i_1,j_1}, z_{i_1,j_1}; v_{i_2,j_2}, z_{i_2,j_2}; v_{i_3,j_3}, z_{i_3,j_3}) \\
&\quad = \mathcal{Y}_W(v_{i_1,j_1}, z_{i_1,j_1}) X(v_{i_2,j_2}, z_{i_2,j_2}; v_{i_3,j_3}, z_{i_3,j_3}) \\
&\qquad + \mathbf{X}(v_{i_1,j_1}, z_{i_1,j_1}; \mathcal{Y}_V(v_{i_2,j_2}, z_{i_2,j_2}) \mathcal{Y}_V(v_{i_3,j_3}, z_{i_3,j_3}) \mathbf{1}_V) \\
&\qquad - \mathbf{X}(\mathcal{Y}_V(v_{i_1,j_1}, z_{i_1,j_1}) \mathcal{Y}_V(v_{i_2,j_2}, z_{i_2,j_2}) \mathbf{1}_V; v_{i_3,j_3}, z_{i_3,j_3}) \\
&\qquad + \mathcal{Y}_W(v_{i_3,j_3}, z_{i_3,j_3}) \mathbf{X}(v_{i_2,j_2}, z_{i_2,j_2}; v_{i_3,j_3}, z_{i_3,j_3}).
\end{aligned}$$

Summarizing all above, we introduce the notion of a V-structure for a foliation $\mathcal{F}$. The objects of category $\mathcal{C}$ of V-structures for $\mathcal{F}$ consist of all sets of points $\mathfrak{P}$, all sets of vertex operator algebra states $\mathfrak{V}$, with conditions on corresponding domains of points, and $\mathcal{W}$. Morphisms of $\mathcal{C}$ are provided by maps of points and intertwining operators. The question of V-structure classification will be considered elsewhere.

3.1. **Non-commutative cohomology of $\mathcal{F}$.** By involving the definition of $\widehat{W}_r^q$ it is possible to introduce a non-commutative interpretation of foliation $\mathcal{F}$ cohomology. For all choices of elements from $\mathfrak{P}$, $\mathfrak{V}$, and $\mathcal{W}$, let us consider the spaces $C_r^q = C_r^q(V, W, \mathcal{U}, \mathcal{F})$ containing rational functions given by characteristics $\Omega(X)$ of elements $X \in \widehat{W}$. For any $X \in \widehat{W}_r^q$, the map D_r^q induces the map δ_r^q by $\Omega(X)$. In Section 4 we show that δ_r^q exhibits chain-cochain property. Note that the spaces $\widehat{W}_r^q$ containing non-commutative elements $\mathbf{X}$, as well as their cohomology are described here in terms of their characteristics. Define linear maps $\delta_r^n : C_r^n \to C_{r-1}^{n+1}$, for each pair l, $r \geq 0$, and $\delta^2_\vartheta : C^2_\vartheta \to C^3_0$. Since $C^q_\infty \subset C^q_r$ for any $r \geq 0$, and $C^q_{r_2} \subset C^q_{r_1}$, for r_1, $r_2 \in \geq 0$ with $r_1 \leq r_2$, $\delta^q_r.C^q_\infty$ is independent of r. Let $\delta^q_\infty = \delta^q_r.C^q_\infty : C^q_\infty \to C^{l+1}_\infty$. Thus, we obtain a double complex $((C^q_r, C^2_\vartheta), (\delta^q_r, \delta^2_\vartheta))$, q, $r \geq 0$, in particular, with $r = \infty$,

$$0 \longrightarrow \left\{ \begin{array}{c} \widehat{W}^0_r \\ \Delta_{r,3}\ \widehat{W}^0_r \end{array} \right. \xrightarrow{\Delta^0_r} \left\{ \begin{array}{c} \widehat{W}^1_{r-1} \\ \delta_{r,3}\ \widehat{W}^1_{r-1} \end{array} \right. \xrightarrow{\Delta^1_{r-1}} \left\{ \begin{array}{c} \widehat{W}^2_{r-2} \\ \widehat{W}^2_\vartheta \end{array} \right. \left\{ \begin{array}{c} \ldots \xrightarrow{\Delta^{r-1}_1} \\ \xrightarrow{\Delta^2_\vartheta} \end{array} \right. \left\{ \begin{array}{c} \widehat{W}^r_0 \\ \Delta_{r,3}\ \widehat{W}^r_0 \end{array} \right. ,$$

$$0 \longrightarrow \left\{ \begin{array}{c} C^0_r \\ \delta_{r,3}\ C^0_r \end{array} \right. \xrightarrow{\delta^0_r} \left\{ \begin{array}{c} C^1_{r-1} \\ \delta_{r,3}\ C^1_{r-1} \end{array} \right. \xrightarrow{\delta^1_{m-1}} \left\{ \begin{array}{c} C^2_{r-2} \\ C^2_\vartheta \end{array} \right. \left\{ \begin{array}{c} \ldots \xrightarrow{\delta^{r-1}_1} \\ \xrightarrow{\delta^2_\vartheta} \end{array} \right. \left\{ \begin{array}{c} C^r_0 \\ \delta_{r,3}\ C^r_0 \end{array} \right. \longrightarrow 0,$$

with $\delta^{q+1}_{r-1} \circ \delta^q_r = 0$, $\delta^2_\vartheta \circ \delta^1_2 = 0$. Using the above complex one is able to introduce a cohomology of $\mathcal{F}$. For q, $r \geq 0$ we define the q, r-th vertex operator algebra cohomology $H^q_r = H^q_r(V, W, \mathcal{F})$ of $\mathcal{F}$ with coefficient in $\overline{W}$ depending on mp vertex operators defined on $\mathcal{U}$ to be $H^q_r = \mathrm{Ker} \left\{ \begin{array}{c} \delta^q_r \\ \delta_{q,1}\ \delta^2_\vartheta \end{array} \right. /\mathrm{Im}\ \delta^{q-1}_{r+1}$, including q, $r = \infty$. Note that in general $q \leq l$, $m \leq p$. Thus the cohomology H^q_r describes subdomains of leaves of $\mathcal{F}$ by means of a vertex operator algebra V.

4. Proof of Proposition 1

We first show that vectors $\mathbf{X} \in \widehat{W}^q_r$ are $\mathfrak{G}$-invariant torsors with respect to the product $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ of the group $\mathfrak{G}_1$ of foliation preserving changes of local coordinates $(\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}})$ in the specific domains of points p_i, $1 \leq i \leq l + m$ described in Section 3, and the group $\mathfrak{G}_2$ of changes of transversal basis of $\mathcal{F}$. First, recall definition of a torsor given in Section 2. Consider the vector $\mu = \mathbf{X}(\widetilde{v}_1, \widetilde{z}_1; \ldots; \widetilde{v}_{ln}, \widetilde{z}_{ln}) = \left[X\left(\widetilde{v}_1, \widetilde{z}_1 d\widetilde{z}_{i(1)}; \ldots; \widetilde{v}_{ln}, \widetilde{z}_{ln} d\widetilde{z}_{i(ln)}\right)\right]$. In our case, $\mathfrak{G}_1 = \mathrm{Aut}\ \mathcal{O}^{(\widetilde{n})} = \mathrm{Aut}_{\mathbb{C}}[[\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}]]$, the group of formal automorphisms of $\widetilde{n}$-dimensional formal power series algebra $\mathbb{C}[[\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}]]$. An element of $\mathrm{Aut}\ \mathcal{O}^{(\widetilde{n})}$ can be represented as $(\widetilde{z}'_1, \ldots, \widetilde{z}'_{\widetilde{n}}) \to (\rho_1, \ldots, \rho_{\widetilde{n}})(\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}})$. Here elements of $(\rho_1, \ldots, \rho_{\widetilde{n}})$ are $\rho_i(\widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}) = \sum_{i_1 \geq 0, \ldots, i_{\widetilde{n}} \geq 0, \sum\limits_{j=1}^{\widetilde{n}} i_j \geq 1}$ $a_{i_1} \ldots a_{i_k}\ z^k_{i_1} \ldots z^{i_k}_k$, where $(a_{i_1}, \ldots, a_{i_k}) \in \mathbb{C}^{\otimes k}$, and are the images of ρ_i, $i = 1, \ldots, \widetilde{n}$ in the finite dimensional $\mathbb{C}$-vector space. Note that $d\widetilde{z}'_j = \sum_{i=1}^{\widetilde{n}} d\widetilde{z}_i\ \partial_{\widetilde{z}_i} \rho_j$, $1 \leq j \leq \widetilde{n}$, $\partial_{\widetilde{z}_i} \rho_j = \partial \rho_j / \partial \widetilde{z}_i$. In order to represent the action of the group $\mathrm{Aut}\ \mathcal{O}^{(\widetilde{n})}$ on the variables $(\widetilde{z}_1, \ldots, \widetilde{z}_{ln})$ of $\mathbf{X}$ in terms of an action on elements $(\widetilde{v}_1, \ldots, \widetilde{v}_{ln})$, we have to switch to an exponential form of the transformations $\rho_i(\widetilde{z}_1, \ldots, \widetilde{z}_{ln})$ with corresponding coefficients $\beta^{(j)}_{r_1, \ldots, r_{ln}} \in \mathbb{C}$ which can be found recursively [18] in terms of coefficients $a^{(i)}_{r_1, \ldots, r_{ln}}$. By definition of the action of $\mathrm{Aut}\ \mathcal{O}^{(ln)}$, for $d\widetilde{z}'_i$, we have $\mu' = \widetilde{\mu} \cdot g_1$, $g_1 \in \mathfrak{G}_1$, where $\widetilde{\mu} = \left[X\left(\widetilde{v}_1, \widetilde{z}_1 d\widetilde{z}'_{i(1)}; \ldots; \widetilde{v}_{ln}, \widetilde{z}_{ln} d\widetilde{z}'_{i(ln)}\right)\right]$. But we see that $\widetilde{\mu} = \left[X\left(v_1, z_1 \sum_{j=1}^{ln} \partial_j \rho_{i(1)}\ d\widetilde{z}_j; \ldots; v_{ln}, z_{ln} \sum_{j=1}^{ln} \partial_j \rho_{i(ln)}\ d\widetilde{z}_j\right)\right]$, and $g_1 = \mathrm{R}(\rho_1, \ldots, \rho_{ln})$. We assume that $\overline{W}$ has the action of Virasoro algebra generators $L_V(-r)$ in the form $\left(\widetilde{z}^{r+1}_i \partial_{\widetilde{z}_i}\right)$, $r \in \mathbb{N}$. Its Lie subalgebra $\mathrm{Der}_0\ \mathcal{O}^{(ln)}$ of V is given by the Lie algebra of $\mathrm{Aut}\ \mathcal{O}^{(ln)}$. Since the vector fields $\left(\widetilde{z}^{r+1}_i \partial_{\widetilde{z}_i}\right)$ act on $\overline{W}$ as operators of degree $(-r)$, the action of the Lie subalgebra $\mathrm{Der}_+\ \mathcal{O}^{(ln)}$ is locally nilpotent. The operator $(\widetilde{z}\partial_{\widetilde{z}})$ acts as the grading operator $L_V(-1)$ diagonalizable with integral eigenvalues. The action of Der $\mathcal{O}^{(ln)}$ on $\overline{W}$ can be exponentiated to an action of Aut $\mathcal{O}^{(ln)}$. Therefore, g_1 is given by $\left[\widehat{\partial}_J\rho_{i(I)}\right] = \left[\ \widehat{\partial}_J\rho_{i_1(I)}, \widehat{\partial}_J\rho_{i_2(I)}, \ldots, \widehat{\partial}_J\rho_{i_{ln}(I)}\ \right]$. Here the index operator J takes values of index $\widetilde{z}_j$ of arguments in the vector μ while the index operator I takes the values of index of differentials $d\widetilde{z}_i$ in each entry of μ. The index operator $i(I) = (i_1(I), \ldots, i_{ln}(I))$ is given by consequent cycling permutations of I. Define the operator $\widehat{\partial}_J\rho_a$ given by the exponential of elements $(-\widehat{\partial}_J\rho_a)$, summed over all $r_1, \ldots, r_{ln}$, such that $\sum\limits_{i=1}^{ln} r_i \geq 1$, and $1 \leq J \leq ln$, which contains index operators J as index of a dummy variable ζ_J turning into $\widetilde{z}_j$, $j = 1, \ldots, ln$. $\widehat{\partial}_J\rho_a$ acts on each argument of maps X in μ. Due to properties of $\widehat{W}$ required above, the action of g_1 on $(\widetilde{v}_1, \ldots, \widetilde{v}_{ln}) \in V$ gives a finite sum. By using linearity of the mapping X, we obtain $\mu' = \mu \cdot g_1 = mu$, i.e., it is a $\mathfrak{G}_1$-invariant torsor. According to the definition of the transversal basis given above, for another transversal basis $\mathcal{U}'$ for $\mathcal{F}$, for each transversal section U_i which belong to the original basis $\mathcal{U}$, there exists a holonomy embedding $h'_i : \mu|_{U_i} \hookrightarrow \mu|_{U'_j}$, i.e., it embeds U_i into a section U'_j of a new transversal basis $\mathcal{U}'$. Then one considers the sequnce of holonomy embeddings $h'_{i_{a'}, b'}$ such that $U'_0 \stackrel{h'_{i_1,b_1}}{\longrightarrow} \ldots \stackrel{h'_{i_r,b_r}}{\longrightarrow} U'_k$. For the combination of embeddings $\widetilde{h}_{i_r,b_r}, i \geq 0$, and $U_0 \stackrel{h_{i_1,b_1}}{\longrightarrow} \ldots \stackrel{h_{i_k,b_k}}{\longrightarrow} U_k$, we obtain commutative diagrams. Since the intersection in (**??**) is performed over all set of sequences of homology mappings, then it is independent on the choice of a transversal basis. The insertions of vertex operators $(\mathcal{Y}_W(\widetilde{v}_{i_1}, \widetilde{z}_{i_1}), \ldots, \mathcal{Y}_W(\widetilde{v}_{i_r}, \widetilde{z}_{i_r}))$ described in definition of the spaces $\widehat{W}^q_r$, keep μ invariant with respect to coordinate changes. Therefore elements of $\widehat{W}^q_r$-spaces are invariant under the action of the product of the groups Aut $\mathcal{O}^{(\widetilde{n})}$ and the group of foliation-preserving changes of $\mathcal{U}$. It is natural to put $\widehat{W}^0_m = W$. According to definition of $\widehat{W}^q_r$, it is obvious that $\widehat{W}^q_r \subset \widehat{W}^q_{r-1}$. Definition of $\mathbf{X}(\widetilde{v}_1, \widetilde{z}_1; \ldots; \widetilde{v}_{ln}, \widetilde{z}_{ln}) \in \widehat{W}^q_r$ contains two conditions on $\mathbf{X}$ and $(\mathcal{Y}_W(\widetilde{v}_{l+1}, \widetilde{z}_{l+1}), \ldots, \mathcal{Y}_W(\widetilde{v}_{(l+m)p}, \widetilde{z}_{(l+m)p}))$. The first condition requires existence of positive integers $\beta^n_m(\widetilde{v}_i, \widetilde{v}_j)$ depending on $\widetilde{v}_i$, $\widetilde{v}_j$ only, and the second condition restricts orders of poles of corresponding sums. The operators Δ^q_r include insertions and omitting of (v, z)-arguments and $\mathcal{Y}(v, z)$-dependence in $\mathbf{X}$. According to proposition 2.8 of [19], $\Delta^q_r\mathbf{X}$ satisfies properties described in Section 3 with $m - 1$ vertex operators $(\mathcal{Y}_W(v_{l+1}, z_{l+1}) \ldots \mathcal{Y}_W(v_{l+m}, z_{l+m}))$. As a formal sum, it is absolutely convergent in $(\widetilde{z}_1, \ldots, \widetilde{z}_{(l+1)n})$. Thus the result of insertions of vertex operators remains in $\mathcal{W}^{q+1}_{r-1}$, i.e., $\Delta^q_r\mathbf{X} \in \widehat{W}^{q+1}_{r-1}$ and Δ^q_r is indeed a map whose image is in $\widehat{W}^{q+1}_{r-1}$. X satisfies the definition of Section 3 with mp vertex operators $m \geq 0$. Since exists the lower limit on domain of absolute convergence given in Definition of X, the extension of a sequence of k-homology embeddings by another embedding preserves the conditions applied to the mappings $\mathbf{X}$ which belong to the spaces $\widehat{W}$. Each sequence of $i'_{a,b}$-homology embeddings $h_{i'_{a,b}}$, $1 \leq a \leq r$, $1 \leq b \leq k$, applied to vectors $\mathbf{X}$ preserves the conditions on elements of $\overline{W}$ given in the definition of the spaces $\widehat{W}_r^q$ of Section 3. The conditions for vertex operators $\mathcal{Y}$ on $\mathcal{U}$ are invariant with respect to the change of the set of homology embeddings $h_{i'_{a,b}}$.

Let us finally show that a V-structure induces a chain-cochain complex $((C_r^q, C_\vartheta^2), (\delta_r^q, \delta_\vartheta^2))$ for $\mathcal{F}$. The proof of this makes use of Propositions 3.1 and 4.1 of [19]. Recall that an element $\mathbf{X}$ comply with restrictions on defining domains for mp vertex operators $\mathcal{Y}(\widetilde{v}_i, \widetilde{z}_i)$, $ln_+1 \le i \le \widetilde{n}$ at points on transversal sections. By construction, the summands in $\delta_r^q\,[\Omega(X)]$ satisfy these conditions of Section 3 and belong to C_{r-1}^{q+1}. The construction of the double chain-cochain complex for $n = 1$ case is known. By construction, the action of coboundary operator $\delta_r^q \delta_r^q\,[\Omega(X)]$, in each component of $\mathbf{X}$ contains a one-dimensional case of the action of δ_r^q on X. Thus, $\delta_{r-1}^{q+1} \circ \delta_r^q = 0$. Checking the case of C_ϑ^2, we find $\delta_2^1 C_2^1 \subset C_1^2 \subset C_\vartheta^2$, we derive $\delta_\vartheta^2 \circ C_2^1 = \delta_1^2 \circ \delta_2^1 = 0$.

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