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K-THEORY COHOMOLOGY OF ASSOCIATIVE ALGEBRA TWISTED BUNDLES

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Abstract. We introduce and study a K-theory of twisted bundles for associative algebras $A(\mathfrak{g})$ of formal series with an infinite-Lie algebra coefficients over arbitrary compact topological spaces. Fibers of such bundles are given by elements of algebraic completion of the space of all formal series in complex parameters, sections are provided by rational functions with prescribed analytic properties. In this paper we introduce and study K-groups $K(A(\mathfrak{g}), X)$ of twisted $A(\mathfrak{g})$-bundles as equivalence classes $[\mathcal{E}]$ of $A(\mathfrak{g})$-bundle $\mathcal{E}$. We show that for any twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ there exist another bundle $\widetilde{\mathcal{E}}$ such that an element of $K(A(\mathfrak{g}), X)$ for $\mathcal{E}$ can be represented in the form $[\mathcal{E}]/[\widetilde{\mathcal{E}}]$. The group $K(A(\mathfrak{g}), X)$ homomorphism properties with respect to tensor product, and splitting properties with respect to reductions of X into base points. We determine also cohomology of cells of K-groups for the factor X/Y of two compact spaces X and Y.

1. Data availability statement

The author confirms that:

1.) All data generated or analysed during this study are included in this published article.

2.) Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

2. Introduction

The equivalence classes of bundles [1, 6, 12] associated to various algebraic structures defined on topological spaces allow to use combinations of algebraic and topological properties of non-commutative objects in terms of abelian groups. It is natural to consider bundles using modules of associative algebras. Such bundles are important for computations in cohomology theory on smooth manifolds, [21, 20, 19]. In [23] we introduced and studied fiber twisted bundles related to modules of associative algebras for infinite-dimensional Lie algebras in the form of group of automorphisms torsors originating from local geometry. Our original motivation for this work was to understand continuous cohomology [3, 4, 7, 9, 10, 17, 19] of non-commutatieve structrues over compact topological spaces. In particular [4], one hopes to relate cohomology of infinite-dimensional Lie algebras-valued series considered on complex manifolds with fiber bundles on auxiliary topological spaces [17]. Let $\mathfrak{g}$ be an infinite-dimensional Lie

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algebra [14]. Starting from algebraic completion $G_{(z_1,\ldots,z_n)}$ of the space of $\mathfrak{g}$-valued series in a few formal complex parameters $(z_1, \ldots, z_n)$, we introduce the category $\mathcal{O}_{A(\mathfrak{g})}$ of associative algebra modules for the associative algebra $A(\mathfrak{g})$ originating from $G_{(z_1,\ldots,z_n)}$ by means of factorization with respect to two natural multiplications [22]. Local parts of twisted bundles are constructed as principal bundles of products of $\mathrm{Aut}(\mathfrak{g})$-modules and spaces of all sets of local parameters of a X-covering. As in the untwisted case [6], this result is crucial in defining $A(\mathfrak{g})$ K-groups and studying the cohomology their properties. In [6] they explored the vertex operator algebra approach to K-theory of compact topological spaces. Vertex operator algebra V bundles and associative algebra bundles related to V gave rise to a series of exact sequences ot K-groups for a compact topological space. An alternative approach to vertex operator algebra bundles was given in [2].

In this paper we introduce and determine properties of K-groups $K(A(\mathfrak{g}), X)$ defined for twisted $A(\mathfrak{g})$-bundles for the associative algebras $A(\mathfrak{g})$ for $\mathfrak{g}$ on compact topological space X. In order to formulate the K-theory, we use the axiomatics of prescribed rational functions. We show, in particular, that all elements of $K(A(\mathfrak{g}), X)$ for a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ can be represented in the form $[\mathcal{E}]/[\widetilde{\mathcal{E}}]$ with another bundle $\widetilde{\mathcal{E}}$. The group $K(A(\mathfrak{g}), X)$ exhibits natural homomorphism properties with respect to tensor product of associative algebras, and possesses splitting properties with respect to reductions of X into a point. We study also cohomology of cells of K-groups for the factor X/Y of two compact spaces X and Y. The cells as short exact sequences of K-groups of X/Y, X and Y. The cohomology of K-cells is determined explicitly. K-groups of twisted associative algebra bundles possess non-vanishing cohomology in contrast to K-groups for vertex operator algebras. Studies of K-groups of bundles considered in this paper find their applications in conformal field theory [2, 18], deformation theory [11, 15, 16], vertex algebras [8, 13], and algebraic topology [3, 7].

3. Prescribed rational functions originating from matrix elements

In this Section the space of prescribed rational functions is defined as functions with certain analytical and symmetry properties [13]. They depend on an infinite number of non-commutative parameters. Let $X_{(\alpha)} = \{X_\alpha, \alpha \in \mathbb{Z}_{>0}\}$ be an open covering of a compact topological space X which gives a local trivialization of the $A(\mathfrak{g})$ fiber bundle. Let $\mathfrak{g}$, be an infinite-dimensional Lie algebra. Denote by $G = G_{(z_1,\ldots,z_n)}$ be the graded (with respect to a grading operator K_G) algebraic completion of the space of all formal series in each of complex formal parameters $(z_1, \ldots, z_n)$ individually, with expansion coefficients as elements $g \in G$, and satisfying certain properties described below. We denote by $(x_1, \ldots, x_n) = (g_1, z_1; \ldots; g_n, z_n)$ for $(g_1, \ldots, g_n) \in G^{\otimes n}$. It is assumed that on G there exists a non-degenerate bilinear pairing $(.,.)$. $G' = \coprod_{\lambda\in\mathbb{C}} G^*_\lambda$ denotes the gaded dual for $G = \bigoplus_{\lambda\in\mathbb{C}} G_{(\lambda)}$ with respect to $(.,.)$. For a fixed $\theta \in G^*$, and varying $(x_1, \ldots, x_n)$ consider the matrix elements

$$F(x_1, \ldots, x_n) = (\theta, f(x_1, \ldots, x_n)) , \tag{3.1}$$

where $F(x_1, \ldots, x_n) \in \mathbb{C}(z_1, \ldots, z_n)$ depends implicitly on $(g_1, \ldots, g_n)$. In this paper we consider meromorphic functions of defined on a compact topological space which are extendable to rational functions $R(f(z_1, \ldots, z_n))$ on larger domains of several complex formal parameters $(z_1, \ldots, z_n)$. Denote by $F_n\mathbb{C}$ the configuration space of $n \geqslant 1$ ordered coordinates in $\mathbb{C}^n$, $F_n\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$. In order to work with G-elements $(g_1, \ldots, g_n)$ objects on X, we consider converging rational functions $f(x_1, \ldots, x_n) \in G$ of $(z_1, \ldots, z_n) \in F_n\mathbb{C}$. For an arbitrary fixed $\theta \in G^*$, we call a map linear in $(g_1, \ldots, g_n)$ and $(z_1, \ldots, z_n)$, $F : (x_1, \ldots, x_n) \mapsto R((\theta, f(x_1, \ldots, x_n))$, a rational function in $(z_1, \ldots, z_n)$. The poses are only allowed at at $z_i = z_j$, $i \neq j$. We define left action of the permutation group S_n on $F(z_1, \ldots, z_n)$ by $\sigma(F)(x_1, \ldots, x_n) = F\left(g_1, z_{\sigma(1)}; \ldots; g_n, z_{\sigma(n)}\right)$.

We assume [13] that $G_{(\lambda)} = \{w \in G | K_0 w = \lambda w \ \ \lambda = \mathrm{wt}(w)\}$, such that $G_{(\lambda)} = 0$ when the real part of α is sufficiently negative, Moreover we require that $\dim G_{(\lambda)} < \infty$, i.e., it is finite, and for fixed λ, $G_{(n+\lambda)} = 0$, for all small enough integers n. An admissible G is a $\mathbb{C}$-graded vector space with $G_{(0)} \neq 0$. which satisfies the following conditions. Assume that G is equipped with a map $\omega_g : G[(z_1, \ldots, z_n) \to G[[(z_1, \ldots, z_n), (z_1^{-1}, \ldots, z_n^{-1})]]$, $g \mapsto \omega_g(z_1, \ldots, z_n) \equiv \sum_{l \in \mathbb{C}} g_l \ z^l$. For each element $g \in G$, and $(z_1, \ldots, z_n) \in \mathbb{C}^{\otimes n}$ let us associate a formal series $\omega_g(x) = \omega_g(z_1, \ldots, z_n) = \sum_{(s_1, \ldots, s_n) \in \mathbb{C}^{\otimes n}} g_{(s_1, \ldots, s_n)} \ z_1^{s_1} \ldots z_n^{s_n}$. For $g \in G$, $w \in$, $n \in \mathbb{C}$, $g_n w = 0$, $n \gg 0$, $\omega_{\mathbf{1}}(z_1, \ldots, z_n) = \mathrm{Id}$, For $g \in G$, $\omega_g(z_1, \ldots, z_n) w$ contains only finitely many negative power terms, that is, $\omega_g(z_1, \ldots, z_n) w \in G$. The locality and associativity properties are assumed for matrix elements (3.1), for g_1, $g_2 \in \mathfrak{g}$, $w \in G$, $\theta \in G'$, the series $(\theta, \omega_{g_1}(z_1) \ \omega_{g_2}(z_2) w)$, $(\theta, \omega_{g_2}(z_2) \ \omega_{g_1}(z_1) w)$, $\left(\theta, \omega_{\omega_{g_1}(z_1 - z_2) g_2}(z_2) w\right)$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2. The poles are only allowed at $z_1 = 0 = z_2$, $z_1 = z_2$. If g is homogeneous then $g_m G_{(n)} \subset G_{(\mathrm{wt} u - m - 1 + n)}$. For a subgroup $\mathfrak{G} \subset \mathrm{Aut}\ G$, $\mathfrak{G}$ acts on G as automorphisms if $g \ \omega_h(z_1, \ldots, z_n) \ g^{-1} = \omega_{gh}(z_1, \ldots, z_n)$, for all g, $h \in \mathfrak{G}$. For an admissible G the operator K_G satisfies the derivation property $\omega_{K_G g}(z_1, \ldots, z_n) = \frac{d}{dz} \omega_g(z_1, \ldots, z_n)$. An admissible $\mathbb{C}$-graded $G = \bigoplus_{\lambda \in \mathbb{C}} G_{(\lambda)}$, $G_{(\lambda)} = \{w \in G | K_0 w = \lambda w\}$ is called an ordinary. We require that $\dim G_{(\lambda)}$ is finite and for fixed λ, $G_{(s+\lambda)} = 0$, for all small enough integers s. Let us assume that $G_{(0)} = \mathbb{C}\mathbf{1}$. G is called rational if it is a direct sum of irreducible admissible G^i. From [5, 22] we know that, all of finite number (up to isomorphisms) irreducible admissible G^i of a rational G, are ordinary modules. For each $\lambda \in \mathbb{C}$ denote $D(\lambda) = \{\lambda + n | 0 \leqslant n \in \mathbb{Z}\}$. The category $\mathcal{O}_{\mathfrak{g}}$ of ordinary G is that for there exist finitely many complex weights $(\lambda_1, ..., \lambda_p)$ such that with $P(G) = \{\lambda \in \mathbb{C} | G_\lambda \neq 0\}$, $P(G) \subset \bigcup_{i=1}^p D(\lambda_i)$. Any irreducible module is in $\mathcal{O}_{\mathfrak{g}}$. If G is rational then $\mathcal{O}_{\mathfrak{g}}$ is exactly the category of ordinary modules. For two $\mathfrak{g}$ and $\mathfrak{g}'$ the functor $\mathcal{O}_{\mathfrak{g}} \times \mathcal{O}_{\mathfrak{g}'} \to \mathcal{O}_{\mathfrak{g} \otimes \mathfrak{g}'}$ such that $G^1 \times G^2 \to G^1 \otimes G^2$. Next recall the notion of a contragredient module [8]. We denote the natural bilinear pairing on $G' \times G$ by (w', w) for $w' \in G'$ and $w \in G$. It is called invariant if $(w', \omega_g(z_1, \ldots, z_n) w) = (\omega_{g'} w', w)$, where $g' = e^{z^3 \partial_z} (-z^{-2})^{K_0} z^{-1} g$, for $\omega_{g'} w'_i \in G'$, and $g \in G$. Then [8, 6] G' with $\omega_g G)$ is also a module. It is irreducible if and only if G is irreducible. For any G, $G' \oplus G$ has a natural non-degenerate symmetric invariant bilinear pairing defined by $(u + u', w + w') = (u, w') + (w, u')$ for any u, $w \in G$ and u', $w' \in G'$. In particular, any G can be embedded into a module with a nondegenerate symmetric invariant bilinear pairing.

Let $(z_1, \dots, z_n) \in F_n\mathbb{C}$. Denote by T_G the translation operator [13]. Denote by $(T_G)_i$ the operator acting on the i-th entry. We then define the action of partial derivatives on an element $F(x_1, \dots, x_n)$

$$\begin{aligned} \partial_{z_i} F(x_1, \dots, x_n) &= F((T_G)_i\ (x_1, \dots, x_n)), \\ \sum_{i \geqslant 1} \partial_{z_i} F(x_1, \dots, x_n) &= T_G F(x_1, \dots, x_n), \end{aligned} \tag{3.2}$$

and call it T_G-derivative property. For $z \in \mathbb{C}$, let

$$e^{zT_G} F(x_1, \dots, x_n) = F(g_1, z_1 + z; \dots; g_n, z_n + z). \tag{3.3}$$

Let us denote by $\mathrm{Ins}_i(A)$ the operator of multiplication by $A \in \mathbb{C}$ at the i-th position. Then we assume that both sides of the expression $F\left((g_1, \dots, g_n), \mathrm{Ins}_i(z_1, \dots, z_n)\ (z_1, \dots, z_n)\right) = F\left(\mathrm{Ins}_i(e^{zT_G})\ (x_1, \dots, x_n)\right)$, are absolutely convergent on the open disk $|z| < \min_{i \neq j}\{|z_i - z_j|\}$, and equal as power series expansions in z. A rational function has K_G-property if for $z \in \mathbb{C}^\times$ satisfies $(zz_1, \dots, zz_n) \in F_n\mathbb{C}$,

$$z^{K_G} F(x_1, \dots, x_n) = F\left(z^{K_G}(g_1, zz_1; \dots; g_n, zz_n)\right). \tag{3.4}$$

Now we recall the definition of rational functions with prescribed analytical behavior on a domain of X. We denote by $P_k : G \to G_{(k)}$, $k \in \mathbb{C}$, the projection of G on $G_{(k)}$. Following [13], we formulate the following definition. For i, $j = 1, \dots, (l+k)n$, $k \geqslant 0$, $i \neq j$, $1 \leqslant l', l'' \leqslant n$, let $(l_1, \dots, l_n)$ be a partition of $(l+k)n = \sum\limits_{i \geqslant 1} l_i$, and $k_i = l_1 + \dots + l_{i-1}$. For $\zeta_i \in \mathbb{C}$, define $h_i = F(\omega_{g_{k_1+l_1}}(z_{k_1+l_1} - \zeta_1) \dots \omega_{g_{k_l+l_l}}(z_{k_l+l_l} - \zeta_l))$, for $i = 1, \dots, l$. A rational function F is a rational function with prescribed analytical behavior if it satisfies properties (3.2)–(3.4). In addition to that, it is assumed that the function $\sum\limits_{(r_1, \dots, r_l) \in \mathbb{Z}^l} F(P_{r_1} h_1, \zeta_1; \dots; P_{r_l} h_l, \zeta_l)$, is absolutely convergent to an analytically extension in $(z_1, \dots, z_n)_{l+k}$ in the domains $|z_{k_i+p} - \zeta_i| + |z_{k_j+q} - \zeta_j| < |\zeta_i - \zeta_j|$, for i, $j = 1, \dots, k$, $i \neq j$, and for $p = 1, \dots, l_i$, $q = 1, \dots, l_j$. The convergence and analytic extention do not depend on complex parameters $(\zeta)_l$. On the diagonal of $(z_1, \dots, z_n)_{l+k}$ the order of poles is bounded from above by escribed ppositive numbers $\beta(g_{l',i}, g_{l'',j})$. For $(g_1, \dots, g_{l+k}) \in G^{\otimes(l+k)}$, $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_s| > 0$, for $i = 1, \dots, k$, $s = k+1, \dots, l+k$ the sum $\sum_{q \in \mathbb{C}} F(\omega_{g_1}(x_1) \dots \omega_{g_k}(x_k) P_q(\omega_{g_{1+k}}(z_{1+k}) \dots \omega_{g_{l+k}}(x_{l+k})))$, is absolutely convergent and analytically extendable to a rational function in variables $(z_1, \dots, z_n)_{l+k}$. The order of pole that is allowed at $z_i = z_j$ is bounded from above by the numbers $\beta(g_{l',i}, g_{l'',j})$.

For $m \geqslant 1$ and $1 \leqslant p \leqslant m-1$, let $J_{m;p}$ be the set of elements of the group S_m which preserve the order of the first p numbers and the order of the last $m-p$ numbers, i.e., $J_{m,p} = \{\sigma \in S_m \mid \sigma(1) < \dots < \sigma(p),\ \sigma(p+1) < \dots < \sigma(m)\}$. Denote by $J_{m;p}^{-1} = \{\sigma \mid \sigma \in J_{m;p}\}$. In what follows, we apply the condition

$$\sum_{\sigma \in J_{n;p}^{-1}} (-1)^{|\sigma|} \sigma(F(g_{\sigma(1)}, z_1; \dots; g_{\sigma(n)}, z_n)) = 0. \tag{3.5}$$

on certain rational functions. The space $\Theta\left(n, k, G_{(z_1, \dots, z_n)}, U\right)$ of n formal complex parameters matrix elements $F(x_1, \dots, x_n)$ as the space of restricted rational functions with prescribed analytical behavior on a $F_n\mathbb{C}$-domain $U \subset X$, and satisfying T_G- and K_G-properties (3.2)–(3.4), (3.5).

4. The associative algebra $A(\mathfrak{g})$

In this Section we remind [5, 22] the definition and properties of the associative algebra $A(\mathfrak{g})$ for $\mathfrak{g}$. For any homogeneous vectors h, $\widetilde{h} \in G$, one defines the bilinear extention to $G \times G$ of the multiplications $h *_\kappa \widetilde{h} = \mathrm{Res}_z \left((1+z)^{\mathrm{wt}(h)} \sum_{l \in \mathbb{C}} h_n z^{l-\kappa} \right) .\widetilde{h}$, for $\kappa = 1, 2$. Here Res_z denotes the coefficient in front of z^{-1}. For h, $\widetilde{h} \in G$, define $A(\mathfrak{g}) = G_{(z_1,\ldots,z_n)}/(\mathrm{span}(h *_2 \widetilde{h}))^\theta$, $\theta = 0, 1$. For $\theta = 0$ we get back to $G_{(z_1,\ldots,z_n)}$ with associativity property $G \times G$ and expressed via matrix elements. For $\theta = 1$ we obtain an associative algebra associated with ordinary associativity. In [22, 6] we find the following

Theorem 1. *The bilinear operation $*_1$ turns $A(\mathfrak{g})$ into an associative algebra with the linear map $\phi : g \mapsto \exp\left(-z^2 \partial_z\right) (-1)^{-z\partial_z} g$, inducing an anti-involution ν on $A(\mathfrak{g})$.*

In what follows, let us denote by $W = W_{(z_1,\ldots,z_n)} \subset G_{(z_1,\ldots,z_n)}$ an $A(\mathfrak{g})$-module. The space of lowest weight vectors of G is then defined as $L(W) = \{g \in G, w \in W | g_{\mathrm{wt}(h)+m} w = 0, h \in W, m \geqslant 0\}$. With $W = \bigoplus_{\lambda \in \mathbb{C}} W_{(\lambda)}$, each homogeneous subspace $L(W)_{(\lambda)} = L(W) \cap W_{(\lambda)}$ of the natural grading $L(W) = \bigoplus_{\lambda \in \mathbb{C}} L(W)_{(\lambda)}$ is finite dimensional [6, 22]. It is easy to see the following

Lemma 1. *Let W, $\widetilde{W}$ be two $A(\mathfrak{g})$-modules with an $A(\mathfrak{g})$-module homomorphism $\varphi : W \to \widetilde{W}$. Then $\varphi(L(W)) \subset L(\widetilde{W})$. In particular, if φ is an isomorphism then $\varphi(L(W)) = L(\widetilde{W})$.*

An associative algebra $A(\mathfrak{g})$ is called semisimple if it is a direct sum of full matrix algebras. Let $A(\mathfrak{g})$ and $A(\widetilde{\mathfrak{g}})$ be two associative algebras with anti-involutions $\nu_{A(\mathfrak{g})}$ and $\nu_{A(\widetilde{\mathfrak{g}})}$ respectively. Then one has

Lemma 2. *$A(\mathfrak{g}) \otimes_{\mathbb{C}} A(\widetilde{\mathfrak{g}})$ is an associative algebra with anti-involution $\nu_{A(\mathfrak{g})} \otimes \nu_{A(\widetilde{\mathfrak{g}})}$.*

We denote by W' the dual space to W with respect to the form $(.,.)$. The following lemma is obvious.

Lemma 3. *W' is an $A(\mathfrak{g})$-module such that $(a\, m', m) = (m', \nu(a)\, m)$, for $a \in A(\mathfrak{g})$, $m' \in W'$, $m \in W$, and ν is an anti-involution.*

For $w_i \in W$, $i = 1, 2$, and $a \in A(\mathfrak{g})$, a form $(.,.)$ defined on W is called invariant if $(a\, w_1, w_2) = (w_1, \nu(a)\, w_2)$. The category $\mathcal{O}_{A(\mathfrak{g})}$ consists of $A(\mathfrak{g})$-modules W with $\lambda_i \in \mathbb{C}$, $1 \leqslant i \leqslant p$, such that $W = \bigoplus_{i=1, \, n \geqslant 0}^{p} W_{(\lambda_i)}$, is a direct sum of finite dimensional $A(\mathfrak{g})$-modules, and $\mathrm{Hom}_{A(\mathfrak{g})}\left(W_{(\lambda)}, W_{(\mu)}\right) = 0$, if $\mu \neq \lambda$. From [6] we have

Theorem 2. *For homogeneous $g \in G$ extended linearly to all G and $W_{(0)} \neq 0$, $L(W)$ is a left $A(\mathfrak{g})$-module, and the linear map $g \mapsto g_{(\mathrm{wt}(g)-1)}|L(W)$: $W \to \mathrm{End}(L(W))$, induces a homomorphism from W to $\mathrm{End}(L(W))$. For all $\lambda \in \mathbb{C}$, $L(W)_{(\lambda)}$ is an finite-dimensional $A(\mathfrak{g})$-module.*

5. Twisted $A(\mathfrak{g})$-bundles

In this Section we recall [23] the construction of associative algebra $A(\mathfrak{g})$ twisted bundles. Here we show how to organize elements of the space $\Theta\left(n, k, W_{(z_1,\dots,z_n),\lambda}, X_\alpha\right)$ of prescribed rational functions into sections of a twisted $A(\mathfrak{g})$-bundle on X. Let $\mathcal{H}$ be a subgroup of the group $\mathrm{Aut}_{(z_1,\dots,z_n)}\ \mathcal{O}_X$ of n independent formal parameters $(z_1,\dots,z_n)$ automorphisms on X. A non-empty set $\mathcal{X}$ is called a group $\mathfrak{H}$-torsor [2] if it is endowed with a simply transitive right action of $\mathfrak{H}$. This means that for ξ, $\widetilde{\xi} \in \mathcal{X}$, there exists a unique $h \in \mathfrak{H}$ such that $\xi \cdot h = \widetilde{\xi}$, where for h, $\widetilde{h} \in \mathfrak{H}$ the right action is given by $\xi \cdot (h \cdot \widetilde{h}) = (\xi \cdot h) \cdot \widetilde{h}$. This construction allows us to identify $\mathcal{X}$ with $\mathfrak{H}$ by sending $\xi \cdot h$ to h.

Similar to [2], one sees that certain subspaces $W \subset G_{(z_1,\dots,z_n)}$ are $\mathcal{H}$-modules. For $\mathcal{H}$-torsor X_α of its module W and X_α, let us associate the X_α-twist of W $\mathcal{E}_{X_\alpha} = W \underset{\mathcal{H}}{\times} X_\alpha = W \times X_\alpha/\{(w, a\cdot\xi) \sim (aw, \xi)\}$, for $\xi \in X_\alpha$, $a \in \mathcal{H}$, and $w \in W$. The isomorphisms $i_{(z_1,\dots,z_n),X_\alpha} : W \xrightarrow{\sim} \mathcal{E}_{X_\alpha}$ of W define a representation of $\mathcal{H}$ on W. Then $\mathcal{E}_{X_\alpha}$ is canonically identified with the twist of W by the $\mathcal{H}$-torsor of X_α. Elements of $\Theta\left(n, k, W_{(z_1,\dots,z_n)}, X_\alpha\right)$ form sections $F(x_1,\dots,x_n)$. The principal bundle for the group $\mathcal{H}$ on X provides the construction of local parts of a twisted $A(\mathfrak{g})$-bundle. Let Aut_{X_α} be the space of all sets of local formal parameters on X_α. We define the $\mathcal{H}$-twist of W $\mathcal{E}_{(z_1,\dots,z_n)} = W \underset{\mathcal{H}}{\times} Aut_{X_\alpha}$.

Let us assume that a $\mathbb{C}$-grading on W is induced by a K_0-graidng defined on G Let $\mathfrak{G} \subset \mathrm{Aut}(G)$ be a W-grading preserving subgroup. Denote by $\mathcal{O}_{\mathfrak{G},A(\mathfrak{g})}$ a subcategory of $\mathcal{O}_{A(\mathfrak{g})}$ consisting of $A(\mathfrak{g})$-modules W such that $\mathfrak{G}$ acts on W as automorphisms. By using ideas of [2], let us define the local part $\mathcal{E}(W_{(z_1,\dots,z_n),(\lambda)})$ of a tweisted $A(\mathfrak{g})$-bundle via matrix elements $F(x_1,\dots,x_n)$ that belong to $\Theta\left(n, k, W_{(z_1,\dots,z_n),(\lambda)}\right)$ for all n, $k \geqslant 0$. on a finite part $\{X_\alpha, \alpha \in I_0\}$ of a covering $\{X_\alpha\}$ of X. By using the properties of prescribed rational functions we form a principal $\mathcal{H}$-bundle, which is a fiber bundle $\mathcal{E}(W_{(z_1,\dots,z_n),(\lambda)})$ with the fiber space provided by elements $f(x_1,\dots,x_n) \in W$, and defined by trivializations $i_{(z_1,\dots,z_n)} : F(x_1,\dots,x_n) = (\theta, f(x_1,\dots,x_n)) \to X_\alpha$, with a continuous free and transitive $F(x_1,\dots,x_n)$-preserving right action $F(x_1,\dots,x_n) \times \mathcal{H} \to F(x_1,\dots,x_n)$. The projection $Aut_{X_\alpha} \to X$ is a principal $\mathcal{H}$-bundle similar to [2]. $\mathcal{H}$-torsor Aut_{X_α} is the fiber of this bundle over X_α. For a finite-dimensional $\mathcal{H}$-module $W_{i_{(z_1,\dots,z_n)},\lambda}$, let $\mathcal{E}(W_{(z_1,\dots,z_n),\lambda}) = \bigoplus_{n,k\geqslant 0} W_{i_{(z_1,\dots,z_n)},\lambda} \underset{\mathcal{H}}{\times} Aut_{X_\alpha}$, be the fiber bundle associated to $W_{i_{(z_1,\dots,z_n)},\lambda}$, Aut_{X_α}, with sections provided by elements of $\Theta(n, k, W, X_\alpha)$, for n, $k \geqslant 0$. On X we can choose $\{X_\alpha\}$ such that the bundle $\mathcal{E}(W_{(z_1,\dots,z_n),\lambda})$ over X_α is $X_\alpha \times F(x_1,\dots,x_n)$. The map $\mathcal{E}(W_{(z_1,\dots,z_n),\lambda}) : \mathbb{C}^n \to X$ is the fiber bundle $\mathcal{E}(W_{(z_1,\dots,z_n),\lambda})$ with fiber $f(x_1,\dots,x_n)$, the total space $\mathbb{C}^n$ of $\mathcal{E}(W_{(z_1,\dots,z_n),\lambda})$, and X is its base space. For every X_α of X $i^{-1}_{(z_1,\dots,z_n)}$ is homeomorphic to $X_a \times \mathbb{C}^n$. For $f((x_1,\dots,x_n)) : i^{-1}_{(z_1,\dots,z_n)} \to X_\alpha \times \mathbb{C}^n$, that $\mathcal{P} \circ f((x_1,\dots,x_n)) = i_{(z_1,\dots,z_n)} \circ i^{-1}_{(z_1,\dots,z_n)}(X_\alpha)$, where $\mathcal{P}$ is the projection map on X_α.

A twisted fiber bundle $\mathcal{E}$ over X associated to $A(\mathfrak{g})$-module with the fiber $W \in \mathcal{O}_{\mathfrak{G},A(\mathfrak{g})}$ and $\Theta(n, k, W, X)$, n, $k \geqslant 0$-valued sections is a direct sum of vector bundles $\mathcal{E} = \bigoplus_{\lambda\in\mathbb{C}} \mathcal{E}(W_{(z_1,\dots,z_n),\lambda})$, such that all transition functions are $A(\mathfrak{g})$-module isomorphisms. It includes a family of continuous isomorphisms $H_\alpha = \{H_{\alpha,\lambda}, \lambda \in \mathbb{C}\}$, $H_{\alpha,\lambda} : \mathcal{E}(W_{(\lambda)})|_{X_\alpha} \to W_{i_{(z_1,\dots,z_n)},\lambda} \underset{\mathfrak{G}}{\times} Aut_{X_\alpha}$, of fiber bundles such that for transition functions $g_{\alpha\beta,\lambda} = H_{\alpha,\lambda} *_2 H^{-1}_{\beta,\lambda}$, for all $\lambda \in \mathbb{C}$. Then $g_{\alpha\beta}(x) = (g_{\alpha\beta,\lambda}(\xi)) : W \to W$, are $A(\mathfrak{g})$-module isomorphisms for any $\xi \in X_\alpha \bigcap X_\beta$ where the transition functions $g_{\alpha,\beta}(x)$ are G-valued.

Now we recall properties of twisted $A(\mathfrak{g})$-bundles [23]. For $A(\mathfrak{g}) = \mathbb{C}$, the twisted $A(\mathfrak{g})$-bundle is a classical complex vector bundle over X. We have

Lemma 4. *For $\mathcal{E}_1$, $\mathcal{E}_2$ $A(\mathfrak{g}_1)$- and $A(\mathfrak{g}_2)$-bundles $\mathcal{E}_1 \otimes \mathcal{E}_2$ is $A(\mathfrak{g}_1) \otimes_{\mathbb{C}} A(\mathfrak{g}_2)$-bundle over X.*

For $\mathcal{E}$, let us introduce $\mathcal{E}' = \oplus_{\lambda\in\mathbb{C}} (\mathcal{E}(W_{(\lambda)}))^*$ which is, due to properties of the non-degenerate bilinear pairing $(.,.)$, is also a $A(\mathfrak{g})$-bundle. For two bundles $\mathcal{E}$ and $\mathcal{E}'$ on X, a map $\eta : \mathcal{E} \to \mathcal{E}'$, is called a bundle morphism if there exist a family of continuous morphisms of fiber bundles $\eta_\lambda : \mathcal{E}(W_{(\lambda)}) \to \mathcal{E}'(W_{(\lambda)})$, such that with $\eta = (\eta_\lambda)$, for all $\lambda \in \mathbb{C}$, and $\eta_\lambda : \mathcal{E} \to \mathcal{E}'$, is an $A(\mathfrak{g})$-module morphism. From [6, 23] we find

Lemma 5. *$\mathcal{E} \bigoplus \mathcal{E}'$ is a twisted $A(\mathfrak{g})$-bundle, endowed with non-degenerate symmetric invariant bilinear pairing $\left(g^*_{\alpha\beta}(\xi)\theta, g_{\alpha\beta}(\xi)u\right) = (\theta, u)$, independent of $g_{\alpha\beta}$ for all α, $\beta \in I$, $\xi \in X_\alpha \cap X_\beta$, $\xi \in G$, $\theta \in G'$, and induced by the natural bilinear pairing on $G \oplus G'$.*

A twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ is called trivial if there exists an $A(\mathfrak{g})$-bundle isomorphism $\varphi : \mathcal{E} \to W \times X$, here $W \times X$ is the $A(\mathfrak{g})$-bundle on X with W as fibers. For any $A(\mathfrak{g})$-module $M \in \mathcal{O}_{A(\mathfrak{g})}$ we denote the trivial $A(\mathfrak{g})$-bundle on X by $\mathcal{M}$. Let us now understand how the notion of a trivial bundle is determined by the definition of a twisted $A(\mathfrak{g})$-bundle. Let $W \in \mathcal{O}_{A(\mathfrak{g})}$. A subgroup $\mathcal{H} \subset \mathrm{Aut}_{(z_1,\dots,z_n)}\mathcal{O}_X$ determines a trivial bundle $W \times X$ if $\mathcal{H}$ satisfies the following properties. W is a $\mathcal{H}$-module. The $\mathcal{H}$ twist $\mathcal{E}(X_\alpha)$ preserves W under $\mathrm{Aut}_{(z_1,\dots,z_n)}\mathcal{O}_{X_\alpha}$-actions by isomorphisms $i_{(z_1,\dots,z_n)}$. The local part of the bundle $W \times X$ is given by the principle $\mathcal{H}$-bundle with $\mathcal{H}$-actions preserving sections $F(x_1,\dots,x_n) \to X_\alpha$. The trivial bundle $W \times X$ is the direct sum of trivial bundles with transition functions g given by W-preserving triansition functions of $\{X_\alpha\}$, $\alpha \in I$. From [23] we have

Proposition 1. *For any twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$, there exists a twisted $A(\mathfrak{g})$-bundle $\widetilde{\mathcal{E}}$ such that $\mathcal{E} \bigoplus \widetilde{\mathcal{E}}$ is a trivial twisted $A(\mathfrak{g})$-bundle with a $A(\mathfrak{g})$-module W-preserving action of a subgroup of $\mathrm{Aut}_{(z_1,\dots,z_n)}\,\mathcal{O}_X$.*

Twisted $A(\mathfrak{g})$-bundles possess the following homotopy-stability property:

Proposition 2. *For a homotopy $\tau_t : \widetilde{X} \to X$, $0 \leqslant t \leqslant 1$, of a compact Hausdorff space $\widetilde{X}$, and a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ over X, $\tau_0^*(\mathcal{E}) \simeq \tau_1^*(\mathcal{E})$.*

We then have

Proposition 3. *For a rational $A(\mathfrak{g})$-module W with a decomposition into irreducible modules W^i, any twisted $A(\mathfrak{g})$-bundle over X $\mathcal{E} \simeq \oplus_{i=1}^p \mathcal{V}(\mathcal{E})^i \otimes \mathcal{W}^i$ with trivial twisted bundles $\mathcal{W}^i$ associated to W^i, and vector bundles $\mathcal{V}(\mathcal{E})^i$.*

Proof. Let $\mathcal{E}$ be a twisted $A(\mathfrak{g})$-bundle with fiber $W = \oplus_{i=1}^{p} M_i \otimes W^i$, where M_i is the space of multiplicity of W^i in W. Then each set of transition functions $\{g_{\alpha\beta}\}$ defines a map $h_{\alpha\beta} : X_\alpha \cap X_\beta \to \bigoplus_{i=1}^{p} \mathrm{End}(M_i)$. For each i we define a vector bundle $\mathcal{V}(\mathcal{E})^i$ over X with fiber M_i and transition functions $\{h_{\alpha\beta} | \alpha, \beta \in I\}$. Then we have a $A(\mathfrak{g})$-bundle $\mathcal{V}(\mathcal{E})^i \otimes \mathcal{W}^i$. □

6. $K(A(\mathfrak{g}), X)$-group for twisted $A(\mathfrak{g})$-bundles

In this Seciton, for an associative algebra $A(\mathfrak{g})$, we introduce the K-group $K(A(\mathfrak{g}), X)$ of a twisted $A(\mathfrak{g})$-bundle on a compact topological space X, and study corresponding properties. We follow the set-ups of [1, 6] with certain necessary extensions. According to the definition of a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$, the set of its equivalence classes $[\mathcal{E}]$ is an abelian semigroup with addition given by the direct sum. We denote by $K(A(\mathfrak{g}), X)$ the abelian group generated by the set of equivalence classes $[\mathcal{E}]$ of a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$. For $\mathfrak{g} = \mathbb{C}$, the group $K(A(\mathfrak{g}), X)$ becomes the ordinary group $K(X)$ as in [1]. Let us define by $\Omega_0 = \big(\mathfrak{g}, W \subset G_{(z_1,\ldots,z_n)}, K_G, (.,.), \theta, \beta(g', g''), K_0; \mathcal{H}, i_{(z_1,\ldots,z_n)}, X_\alpha, \alpha \in I\big)$ the extented moduli space for a twisted $A(\mathfrak{g})$-bundle defined on a compact topological space X. Let us now describe the set of parameters the isomorphism classes of twisted $A(\mathfrak{g})$-bundles depend on. Recall that $\mathcal{H} \subset \mathrm{Aut}_{(z_1,\ldots,z_n)}\ \mathcal{O}_X$ of independent formal parameters $(z_1, \ldots, z_n)$ automorphisms on X. The construction involves the category of subsets $W \subset G$ which is a $\mathcal{H}$-module. $\mathcal{H}$-torsors and X_α-twists of W on X_α are determined by $\mathcal{H}$, Aut_{X_α}, and W. The elements of $\Theta\,(n, k, W, X_\alpha)$ give rise to a collection of sections $F(x_1, \ldots, x_n)$ as prescribed rational functions. The space $\Theta\left(n, k, W_{(z_1,\ldots,z_n),\lambda}\right)$ for all n, $k \geqslant 0$, of prescribed rational functions is fixed by assumptions of Lemma. The module $W \subset \mathcal{O}(A(\mathfrak{g}))$ is endowed with a $\mathbb{C}$-grading generated by K_0. Note that the trivializations $i_(z_1, \ldots, z_n) : F(x_1, \ldots, x_n) = (\theta, f(x_1, \ldots, x_n)) \to X_a$ are chosen in such way that they preserve the space $\Theta\left(n, k, W_{(z_1,\ldots,z_n),\lambda}\right)$. The choice of trivializations $i_{(z_1,\ldots,z_n),X_\alpha} : W \xrightarrow{\sim} \mathcal{E}_{X_\alpha}$, is coherent with the choice of $\{X_\alpha\}$. Suppose we take another system of domains $\{X'_{\alpha'}\}$. One shows that the construction of the bundle does not depend on the choice of transition functions on the intersections $X_\alpha \cap X'_{\alpha'}$. Thus, the equivalence classes do not depend on the choice of a covering. Since, by construction, the transition functions $g_{\alpha\beta}(x) = (g_{\alpha\beta,\lambda}(\xi)) : W \to W$, are $A(\mathfrak{g})$-module isomorphisms for any $\xi \in X_\alpha \bigcap X_\beta$, then the construction of the bundle is invariant of the choice of transition functions. The twisted $A(\mathfrak{g})$-bundle $\widetilde{\mathcal{E}}$ in Proposition 1 is constructed as follows [23]. Define an $A(\mathfrak{g})$-bundle injective and bilinear form preserving homomorphism $\psi : \mathcal{E} \to W^{\oplus s} \times X$ for some s, sending $\mathcal{E}$ to a trivial $A(\mathfrak{g})$-bundle. Let us take $\widetilde{\mathcal{E}} = \psi(\mathcal{E})^\dagger$ with respect to the bilinear form $(.,.)$. As the dual to $\mathcal{E}$, $\widetilde{\mathcal{E}}$ is an $A(\mathfrak{g})$-bundle on X. From Proposition 1 we infer

Lemma 6. *For the fixed set of data* ($\mathfrak{g}$, $W \subset G$, K_G*-grading,* $(.,.)$, $\theta \in W'_{(z_1,\ldots,z_n)}$, $\beta(g', g'')$) *of* Ω_0*, the elements of* $K(A(\mathfrak{g}), X)$ *are of the form* $[\mathcal{E}]/[\widetilde{\mathcal{E}}]|_{\mathcal{H}.W}$ *up to* W*-preserving action* $\mathcal{H}.W$ *of a subgroup* $\mathcal{H} \subset \mathrm{Aut}_{(z_1,\ldots,z_n)}\ \mathcal{O}_{\mathcal{H}.X}$*, where* $\mathcal{H}.X$ *denotes the* $\mathcal{H}$ *preserving section and transition functions.*

The statement of following Lemma follows from Lemma 4 and the construction of a twisted $A(\mathfrak{g})$-bundle.

Lemma 7. *For two infinite-dimensional algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$, the tensor product of the twisted $A(\mathfrak{g})$- and $A(\widetilde{\mathfrak{g}})$-bundles induces a natural group homomorphism $K(A(\mathfrak{g}), X) \otimes_{\mathbb{Z}} K(A(\widetilde{\mathfrak{g}}), X) \to K(A(\mathfrak{g}) \otimes A(\widetilde{\mathfrak{g}}), X)$.*

According to the definition of a twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$, the set of isomorphis classes of $\mathcal{E}$ is determined in particular by $\mathcal{H}$-invariant modules W and $\mathcal{H}$-invariant sections $F(z_1, \ldots, z_n)$. The axioms of prescribed rational functions $F(x_1, \ldots, x_n)$ form a functional representation of G with additional analytic behavior properties. Lemma 7 shows that for each s elements of the K-group is the tensor product of $A(\mathfrak{g})$ where each element if represented by commutative elements of $F(z_1, \ldots, z_n)$. Therefore, from Lemma 7 we obtain

Lemma 8. *With $A(\mathfrak{g})^{\otimes 0} \simeq \mathbb{C}$, the group $K = \bigoplus_{s \geqslant 0} K\left(A(\mathfrak{g})^{\otimes s}, X\right)$ form a commutative algebra over $K(A(\mathfrak{g}), X)$.*

Next we state

Lemma 9. *Any element of $K(A(\mathfrak{g}), X)$ has the form $[\mathcal{E}]/[\mathcal{M}]$, where $\mathcal{E}$ is a twisted $A(\mathfrak{g})$-bundle and M is a $A(\mathfrak{g})$-module. For two equivalent classes $[\mathcal{E}]$, $[\mathcal{E}'] \in K(A(\mathfrak{g}), X)$, there exists a $A(\mathfrak{g})$-module M such that $\mathcal{E} \cong \mathcal{E}'$ up to the trivial $A(\mathfrak{g})$-bundle $\mathcal{M}$.*

Proof. As we have shown in Lemma 6, for every $\mathcal{E}$ one constructs a bundle $\mathcal{E}'$ such that every element of $K(A(\mathfrak{g}), X)$ is of the form $[\mathcal{E}]/[\mathcal{E}']$ up to $A(\mathfrak{g})$-module M-preserving action of a subgroup $\mathcal{H} \subset \mathrm{Aut}_{(z_1,\ldots,z_n)}\, \mathcal{O}_X$. By Proposition 1, for any twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$, there exists a twisted $A(\mathfrak{g})$-bundle $\widetilde{\mathcal{E}}$ and a $A(\mathfrak{g})$-module M such that $\mathcal{E} \oplus \widetilde{\mathcal{E}} \cong \mathcal{M}$. Thus we have $[\mathcal{E}']/[\mathcal{E}] = [\mathcal{E}' \oplus \widetilde{\mathcal{E}}]/[\mathcal{E} \oplus \widetilde{\mathcal{E}}] = [\mathcal{E}' \oplus \widetilde{E}]/[\mathcal{M}]$. If $[\mathcal{E}] = [\mathcal{E}']$ then there exists a $A(\mathfrak{g})$-bundle $\mathcal{E}''$ such that $\mathcal{E} \oplus \mathcal{E}'' \cong \widetilde{\mathcal{E}} \oplus \mathcal{E}''$. Let $\mathcal{E}$ be a $A(\mathfrak{g})$-bundle such that $\mathcal{E}'' \oplus \mathcal{E} \cong \mathcal{M}$ for some $A(\mathfrak{g})$-module M. Then we have $E \oplus \mathcal{M} \cong \mathcal{E}' \oplus \mathcal{M}$. □

Here we describe the group $K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$ for $(\xi_1, \ldots, \xi_n) \in X$. According to the construction of a twisted $A(\mathfrak{g})$-bundle, one has a subgroup $\mathcal{H}_0 \subset \mathrm{Aut}_{(z_{\xi_1},\ldots,z_{\xi_n})}\mathcal{O}_{(\xi_1,\ldots,\xi_n)}$ over a set $(\xi_1, \ldots, \xi_n) \in X$, such that an $A(\mathfrak{g})$-module W is $\mathcal{H}_0$-module. The local part of $\mathcal{E}$ is defined by $\mathcal{E}_{(\xi_1,\ldots,\xi_n)}$ $\mathcal{H}_0$-twists of $W_{(z_{\xi_1},\ldots,z_{\xi_n})}$ attached to $(\xi_1, \ldots, \xi_n)$ via mappings $i_{(z_1,\ldots,z_n)} : W_{(z_{\xi_1},\ldots,z_{\xi_n})} \to \mathcal{E}_{(\xi_1,\ldots,\xi_n)}$. An invariant $\mathcal{H}_0$-action on $F(x_1, \ldots, x_n)$ is assumed. Basically, $\mathcal{E}_{(\xi_1,\ldots,\xi_n)} = W_{(z_{\xi_1},\ldots,z_{\xi_n})} \underset{S_n}{\times} Aut_{(\xi_1,\ldots,\xi_n)}$. The equivalence classes $[\mathcal{E}]|_{(\xi_1,\ldots,\xi_n)}$ of twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ over a set of points $(\xi_1, \ldots, \xi_n) \in X$. are given by equivalence classes of $\mathcal{H}_0$-invariant $\mathrm{Aut}_{(z_{\xi_1},\ldots,z_{\xi_n})}\mathcal{O}_{(\xi_1,\ldots,\xi_n)}$ modules $W_{(z_{\xi_1},\ldots,z_{\xi_n})}$. If $A(\mathfrak{g}) = \bigoplus_{i=1}^{p} A(\mathfrak{g})$ is semisimple up to isomorphisms, then $K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$ is isomorphic to the group $\mathbb{Z}^p \times \ldots \times \mathbb{Z}^p$ with generators $[\mathcal{M}^1], \ldots, [\mathcal{M}^p]$ for inequivalent $A(\mathfrak{g})$-modules $M^1, \ldots, M^p$. Generalizing results of [6], we obtain.

Lemma 10. *For a semisimple associative algebra $A(\mathfrak{g})$ $K(A(\mathfrak{g}), X) = K(X) \otimes_{\mathbb{Z}} K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$.*

Proof. For rational G correspoding twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ is determined by certain vector bundles. Assume that its fiber $M = \oplus_{i=1}^{p} E_i \otimes M^i$ where $\{M^1, \ldots, M^p\}$ contains all inequivalent irreducible $A(\mathfrak{g})$-modules, and E_i is the space of multiplicity of M^i in M. Similar to Proposition 3, we know that if $A(\mathfrak{g})$ is semisimple then there exist vector bundles $\mathcal{V}(\mathcal{E})^i$ for $i = 1, \ldots, p$ such that $\mathcal{E} \simeq \bigoplus_{i=1}^{p} \mathcal{V}(\mathcal{E})^i \otimes \mathcal{M}^i$, where trivial bundles $\mathcal{M}^i$ correspond to inequivalent set of $A(\mathfrak{g})$-modules M^i. According our description of a twisted $A(\mathfrak{g})$-bundle over a set $(\xi_1, \ldots, \xi_n) \in X$, one has a map to equivalence classes which gives $K(\bigoplus_{i=1}^{p} A(\mathfrak{g})^i, X) = \bigoplus_{i=1}^{p} K(A(\mathfrak{g})^i, X) = K(\bigoplus_{i=1}^{p} \mathcal{V}(\mathcal{E})^i, X) \otimes_{\mathbb{Z}} K(\mathcal{M}^i) = K(X) \otimes_{\mathbb{Z}} K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$. □

7. Cohomology of K-cells

By using results of [6], we first define K-groups for twisted associative algebra bundles on factor spaces of compact topological spaces. Let $\mathcal{C}_c$ denote the category of compact spaces, $\mathcal{C}_0$ the category of compact spaces with distinguished basepoints. We define a functor $\mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$, by associating two compact topological spaces X, Y to a compact space X/Y with base points $(\xi_1, \ldots, \xi_n) = Y/Y$. In the case $Y \neq \{\varnothing\}$, X/Y is the disjoint union of X with points $(\xi_1, \ldots, \xi_n)$, and for $X \in \mathcal{C}_c$, we denote $X_0 = X/\{\varnothing\}$. If X is in $\mathcal{C}_0$, we define a functor Ker $i_0^*(X)$ to be the kernel of the map $i_0^* : K(A(\mathfrak{g}), X) \to K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$ where $i_0 : (\xi_1, \ldots, \xi_n) \to X$ is the inclusion of the basepoints. If $c : X \to (\xi_1, \ldots, \xi_n)$ is the collapsing map then c^* induces a splitting $K(A(\mathfrak{g}), X) = \text{Ker } i_0^*(X) \oplus K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$. It is clear that $K(A(\mathfrak{g}), X) \simeq \text{Ker } i^*(X_0)$. Now we define $K(A(\mathfrak{g}), X/Y) = \text{Ker } i_0^*(X/Y)$. In particular, $K(A(\mathfrak{g}), X) \simeq K(X_0)$.

Now let us introduce [6] the generalized smash product operator in $\mathcal{C}_0$, for X, $Y \in \mathcal{C}_0$. We put $X \wedge Y = X \times Y / X \vee Y$ for $X \vee Y = X \times \{(\xi_1, \ldots, \xi_n)\} \bigcup \{(\xi_1', \ldots, \xi_n')\} \times Y$, being base-points of X, Y respectively. For any triple of spaces X, Y, $Z \in \mathcal{C}_0$, one has a natural homeomorphism $X \wedge (Y \wedge Z) \simeq (X \wedge Y) \wedge Z$. Let $I(t)$, $0 \leqslant t \leqslant 1$ we denote the boundary $\partial I(t) = \{0, 1\}$. Then $I(t)/\partial I(t) \simeq S^1 \in \mathcal{C}_0$. For $X \in \mathcal{C}_0$ we define the operator $\mathcal{S}$ of the reduced suspension on X as $\mathcal{S}X = S^1 \wedge X \in \mathcal{C}_0$. Then one has the s-th power of iterated suspensions $\mathcal{S}^s X \cong S^s \wedge X$. For $X \in \mathcal{C}_0$ and $s \geqslant 0$ we consider Ker $i_0^*(\mathcal{S}^s X)$. For X, $Y \in \mathcal{C}_c$ let us define $K(A(\mathfrak{g}), X/Y)^{-s} = (\text{Ker } i_0^*)^{-s}$ $(X/Y) = \text{Ker } i_0^*(\mathcal{S}^s(X/Y))$, $K(A(\mathfrak{g}), X)^{-s} = K(A(\mathfrak{g}), X_0)^{-s} = \text{Ker } i_0^*(\mathcal{S}^s(X_0))$. We next define the cone functor $C : \mathcal{C}_c \to \mathcal{C}_0$ on X by $CX = I(t) \times X / \{0\} \times X$, and identify X with the $\{1\} \times X \subset CX$. One calls $CX/X = I(t) \times X / \partial I(t) \times X$ the unreduced suspension of X. In what follows, we assume that X is a finite CW-complex and $Y \subset X$ is a CW sub-complex. Next, we have a modification of a Lemma from [6] in our context of twisted $A(\mathfrak{g})$-bundles:

Lemma 11. *For $\xi \in \text{Ker } i_0^*(\mathcal{S}X)$, $T(t) = 1 - t$, $0 \leqslant t \leqslant 1$ such that $(T \wedge 1) : \mathcal{S}X \to \mathcal{S}X$ and the identity on $X \in \mathcal{C}_0$. Then $(T \wedge 1)^* \xi = -\xi$ for ξ.*

Proof. By construction of a twisted $A(\mathfrak{g})$-bundle and according to Lemma 9, for any $\xi \in \text{Ker } i_0^*(\mathcal{S}X)$, there exist a $A(\mathfrak{g})$-bundle $\mathcal{E}$ and a $A(\mathfrak{g})$-module M such that $\xi = [\mathcal{E}]/[\mathcal{M}]|_{\mathcal{H}.M}$, where $\mathcal{M}$ is a trivial bundle associated to M. For two cones $C_1 X$ and $C_2 X$, on the contructible $C_1 X \cup C_2 X = C_1 X_{C_2} X = \mathcal{S}X$, the restrictions $\mathcal{E}|_{CX}$, $\mathcal{E}|_{C_2 X}$

of $\mathcal{E}$ are trivial. Then we can define maps $\rho_i : \mathcal{E}|_{C_iX} \to M \times C_iX$, $1 \leqslant i \leqslant 2$, such that $\rho = \rho_2 \circ \rho_1^{-1} = \rho^s : X \to \mathrm{Aut}_{G_{(z_1,\ldots,z_n)}}(M)$, $s \in \mathbb{C}$. According to definition of the smash product above, $(T \wedge 1)^*$ acts on $[\mathcal{E}]$ $x \to \rho(x)^{-1} = (\rho^s)^{-1}$ resulting in $\mathcal{E}' \in K(A(\mathfrak{g}), \mathcal{S}X)$. Since ρ_i are isomorphisms of M, from Lemma 5 we obtain that classes of isomorphisms of $\mathcal{E}$ and $\mathcal{E}'$ are the direct sum of classes of $\mathcal{M}$ and $\mathcal{M}'$ in $K(A(\mathfrak{g}), \mathcal{S}X)$. ▫

Now we establish the cohomological properties of $K(A(\mathfrak{g}), X/Y)$.

Lemma 12. *For inclusions $i : Y \to X$, $j : X_0 \to X/Y$, the cohomology of the cell $\mathcal{K}_{X/Y} = K(A(\mathfrak{g}), X/Y) \xrightarrow{j^*} K(A(\mathfrak{g}), X) \xrightarrow{i^*} K(A(\mathfrak{g}), Y)$ is given by $H_{K(A(\mathfrak{g}),X/Y)} = \mathrm{Ker}\ i^*/\mathrm{Im}\ j^* = \mathrm{Aut}_{(z_1,\ldots,z_n)}\mathcal{O}_{X/Y}|_{\mathcal{H}.W}$.*

Proof. Let $Y_0 = Y/\{\varnothing\}$. The composition $i^* \circ j^*$ is induced by the composition $j \circ i : Y_0 \to X/Y$, and $i^* \circ j^* = 0$. Suppose now that $\xi \in \mathrm{Ker}\ i^*$. According to Lemma 9, ξ is represented in the form $([\mathcal{E}]/[\mathcal{M}])|_{\mathcal{H}.W_{(z_1,\ldots,z_n)}}$ where $\mathcal{E}$ is a twisted $A(\mathfrak{g})$-bundle over X and M is a $A(\mathfrak{g})$-module. Since $i^*\xi = 0$, it follows that $[\mathcal{E}]|_Y = [\mathcal{M}]$ in $K(A(\mathfrak{g}), Y)$. According to Lemma 9, this implies that there exists a $A(\mathfrak{g})$-module $M'|_{\mathcal{H}.W}$ such that $(\mathcal{E}\oplus\mathcal{M}')|_{\mathcal{H}.W} = (\mathcal{M}\oplus\mathcal{M}')|_{\mathcal{H}.W}$, and the bundle $\mathcal{M}'$ is trivial. Now as Y is a CW sub-complex of X, there exists an open neighborhood $\widetilde{Y}$ of Y in X such that Y satisfies the following condition with respect to $\widetilde{Y}$. For $0 \leqslant t \leqslant 1$, one can find a map $f(t) : \widetilde{Y} \to \widetilde{Y}$ such that $f(1) = \mathrm{Id}_{\widetilde{Y}}$, $f(0)|_Y = \mathrm{Id}_Y$, and $f(0)(\widetilde{Y}) = Y$. By Lemma 2 the triviality of the blow-up $(\mathcal{E} \oplus \mathcal{M}')|_{\widetilde{Y}}$ of $(\mathcal{E} \oplus \mathcal{M}')|_Y$ is homotopically preserved, on $\widetilde{Y}$. This defines a bundle $\mathcal{E}\oplus\mathcal{M}'/\alpha$ on X/Y, and an element $\tau = [\mathcal{E}\oplus\mathcal{M}'/\gamma]/[\mathcal{M}\oplus \mathcal{M}'] \in \mathrm{Ker}\ i_0^*(X/Y) = K(A(\mathfrak{g}), X/Y)$, where $I^* : K(A(\mathfrak{g}), X) \to K(A(\mathfrak{g}), \xi)$. Since $\mathcal{M}'$ is trivial, $j^*(b\tau) = [\mathcal{E} \oplus \mathcal{M}'/\gamma]/[\mathcal{M} \oplus \mathcal{M}']|_{\mathcal{H}.W} = [\mathcal{E}]/[\mathcal{M}]_{\mathcal{H}.W} = b\xi$, where $b \in \mathrm{Aut}_{(z_1,\ldots,z_n)}\ \mathcal{O}_{X/Y}$. Thus $\mathrm{Ker}\ i^* = \left(\mathrm{Aut}_{(z_1,\ldots,z_n)}\ \mathcal{O}_{X/Y}\right)\ \mathrm{Im}\ j^*$. ▫

Lemma 13. *For $X/Y \in \mathcal{C}_c$ and $Y \in \mathcal{C}_0$, the cohomology of the sequence $K(A(\mathfrak{g}), X/Y) \to \mathrm{Ker}\ i_0^*(X) \to \mathrm{Ker}\ i_0^*(Y)$, is given by $\mathrm{Aut}_{(z_{\xi_1},\ldots,z_{\xi_n})}\ \mathcal{O}_{(\xi_1,\ldots,\xi_n)}|_{\mathcal{H}.W}$ up to the twist $\mathcal{E}_{\mathcal{H}.W}$.*

Proof. The statement follows from Lemma 12 and the decomposition isomorphisms $K(A(\mathfrak{g}), X) \simeq \mathrm{Ker}\ i_0^*(X) \oplus K(A(\mathfrak{g}), (\xi_1, \ldots, \xi_n))$, $K(A(\mathfrak{g}), Y) \simeq \mathrm{Ker}\ i_0^*(Y) \oplus K(A(\mathfrak{g}), (\xi_1', \ldots, \xi_n'))$. Thus, we obtain the statement of the lemma with the twist inserted. ▫

For $s \geqslant 0$ we next define the $(-s)$-th power of the cell of K-groups

$$\mathcal{K}_{X/Y}^{-s} = K(A(\mathfrak{g}), X/Y)^{-s} \xrightarrow{j^*} K(A(\mathfrak{g}), X)^{-s} \xrightarrow{i^*} K(A(\mathfrak{g}), Y)^{-s}.$$

For a map $\nu_1 : Z \to Z/X$, and the isomorphism $\vartheta : K(A(\mathfrak{g}), Z/X) \to K(A(\mathfrak{g}), Y)^{-1}$, let us define the operator $\delta^{(s)} : K(A(\mathfrak{g}), X/Y)^{-s} \to K(A(\mathfrak{g}), X/Y)^{-s-1}$, as the operator $\delta = \nu_1^* \circ \vartheta^{-1}$ acting on $K(A(\mathfrak{g}), Y)^{-s}$. The main proposition of this paper is

Proposition 4. *The s-th cell cohomology $H^s_{\mathcal{K}_{X/Y}} = \mathrm{Ker}\ \delta^{(-s)}/\mathrm{Im}\ \delta^{-(s+1)}$, of the left-infinite sequence*

$$\ldots \xrightarrow{\delta^{(-s-2)}} \mathcal{K}_{X/Y}^{-s-1} \xrightarrow{\delta^{-(s+1)}} \mathcal{K}_{X/Y}^{-s} \xrightarrow{\delta^{(-s)}} \ldots \xrightarrow{\delta^{(-1)}} \mathcal{K}_{X/Y}^{0} \xrightarrow{\delta^{(0)}} 0,$$

for $s \geqslant 0$, is given by $\mathrm{Aut}_{(z_1,\ldots,z_n)}\mathcal{O}_{\mathcal{S}^s(X/Y)}|_{\mathcal{H}.W}$.

Proof. The proof is a modification of the proof of Theorem (5.3) of [6] for our case. Let us set $\mathcal{K}_0^0 = \text{Ker }i_0^*(X/Y) \stackrel{j^*}{\to} \text{Ker }i_0^*(X) \stackrel{i^*}{\to} \text{Ker }i_0^*(Y)$. One can see that the cohomology of the main sequence is equivalent to the cohomology of the sequence

$$\ldots \stackrel{\delta^{(-2)}}{\to} \mathcal{K}_{X/Y}^{-1} \stackrel{\delta^{(-1)}}{\to} \mathcal{K}_0^0 \stackrel{\delta^{(0)}}{\to} 0, \tag{7.1}$$

defined by $H^0_{\mathcal{K}_{X/Y}} = \text{Ker }\delta^{(-1)}/\text{Im }\delta^{(0)}$ is $\mathrm{Aut}_{(z_1,\ldots,z_n)}\ \mathcal{O}_{X/Y}|_{\mathcal{H}.W}$. By replacing X/Y by $\mathcal{S}^s(X/Y)$ for $s \geqslant 1$, we obtain cohomology of an infinite sequence continuing (7.1). Then by replacing X/Y by X_0/Y_0 where X/Y is any pair in $\mathcal{C}_c \times \mathcal{C}_c$, we get the cohomology of the infinite sequence in our Proposition. Recall that Lemma 13 gives the cohomology of mappings inside the cell K_0 of (7.1). To determine cohomology of other parts of the sequence will apply Lemma 13 to $Z = X_C Y = X \bigcup CY$, Z/X and $Z_C X/Z = (Z \bigcup CX)/Z$. Let $\nu_1 : Z \to Z/X$ and $\nu_2 : X \to Z$ bet two inclusions. By considering Z/X we get the cohomology $H_{K_0,Z} = \text{Ker }\nu_2^*/\text{Im }\nu_1^*$ of the sequence $K(A(\mathfrak{g}), Z/X) \stackrel{\nu_1^*}{\to} \text{Ker }i_0^*(A(\mathfrak{g}), Z) \stackrel{\nu_2^*}{\to} \text{Ker }i_0^*(X)$, given by $\mathrm{Aut}_{(z_{\xi_1},\ldots,z_{\xi_n})}\ \mathcal{O}_{(\epsilon_1,\ldots,\epsilon_n)}$ for $(\epsilon_1,\ldots,\epsilon_n)$ beeing base points for Z/X. Let $\widetilde{Y}$ be the neighborhood of Y in X as in Lemma 12. Note that CY is contractible. Then by Lemma 2 any twisted $A(\mathfrak{g})$-bundle $\mathcal{E}$ on Z is trivial on $\widetilde{Y}_C Y$. Therefore $p^* : \text{Ker }i_0^*(X/Y) \to \text{Ker }i_0^*(Z)$ is an isomorphism where $p : Z \to Z/CY = X/Y$ for the collapsing map. The have the composition $j^* = \nu_2^* \circ p^*$ of maps. We then obtain the sequence $K(A(\mathfrak{g}), Y)^{-1} \stackrel{\delta^{(-1)}}{\to} K(A(\mathfrak{g}), X/Y) \stackrel{j^*}{\to} \text{Ker }i_0^*(X)$ given by $\text{Ker }i_0^*(\mathcal{S}Y)^{-1} \stackrel{\delta^{(-1)}}{\to} \text{Ker }i_0^*(X/Y) \stackrel{j^*}{\to} \text{Ker }i_0^*(X)$, which cohomology is $H_{K_0,X/Y} = \text{Ker }j^*/\text{Im }\delta^{(-1)} = \mathrm{Aut}_{(z_{\xi'_1},\ldots,z_{\xi'_n})}\ \mathcal{O}_{(\epsilon'_1,\ldots,\epsilon'_n)}$. We apply Lemma 13 to approximations $(X_{C_1}Y)$ and $(X_{C_1}Y)_{C_2}X/(X_{C_1}Y)$ where we have denoted the cones C_i, $i = 1, 2$. Thus we obtain the sequence

$$K(A(\mathfrak{g}), (X_{C_1}Y)_{C_2}X/(X_{C_1}Y)) \stackrel{\nu_3^*}{\to} \text{Ker }i_0^*((X_{C_1}Y)_{C_2}X) \stackrel{\nu_4^*}{\to} \text{Ker }i_0^*(X_{C_1}Y). \tag{7.2}$$

One shows that this sequence is isomorphic to the sequence in $\mathcal{K}^{-1}_{X/Y}$ of (7.1). According to the definition of δ, it is enough to will show the equivalence of the following two maps

$$\begin{aligned} K(A(\mathfrak{g}), (X_{C_1}Y)_{C_2}X/X_{C_1}Y) &\stackrel{\nu_3^*}{\to} \text{Ker }i_0^*((X_{C_1}Y)_{C_2}X) \stackrel{\nu_5^*}{\to} \text{Ker }i_0^*(C_2X/X) \\ &= K(A(\mathfrak{g}), X)^{-1} \stackrel{i^*}{\to} K(A(\mathfrak{g}), Y)^{-1} = \text{Ker }i_0^*(C_1Y/Y). \end{aligned} \tag{7.3}$$

Using Lemma 11, according to definition of C_i, $i = 1, 2$ we obtain the sequence $\text{Ker }i_0^*(\mathcal{S}Y) \to K(A(\mathfrak{g}), C_1Y/Y) \to K(A(\mathfrak{g}), (C_1Y)_{C_2}Y)$, which is equivalent to $K(A(\mathfrak{g}), C_2Y/Y) \to \text{Ker }i_0^*(\mathcal{S}Y)$. Finally, we get

$$H^0_{\mathcal{K}_{X/Y}} = \text{Ker }\delta^{(0)}/\text{Im }\delta^{(-1)} = H_{\mathcal{K}_0^0} = \mathrm{Aut}_{(z_1,\ldots,z_n)}\ \mathcal{O}_{X/Y}|_{\mathcal{H}.W}.$$

□

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