INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Vladimír Müller
Yuri Tomilov

Preprint No. 38-2022

PRAHA 2022

MATRIX REPRESENTATIONS OF ARBITRARY BOUNDED OPERATORS ON HILBERT SPACES

VLADIMIR MÜLLER AND YURI TOMILOV

Abstract. We show that under natural and quite general assumptions, a large part of a matrix for a bounded linear operator on a Hilbert space can be preassigned. The result is obtained in a more general setting of operator tuples leading to interesting consequences, e.g. when the tuple consists of powers of a single operator. We also prove several variants of this result of independent interest. The paper substantially extends former research on matrix representations in infinite-dimensional spaces dealing mainly with prescribing the main diagonals.

1. Introduction

Let H be a separable infinite-dimensional Hilbert space, and let $B(H)$ stand for the space of bounded linear operators on H. If $T \in B(H)$, then T allows for a variety of matrix representations $A_T := \langle Tu_j, u_n\rangle_{n,j=1}^\infty$ induced by the set of orthonormal bases $(u_n)_{n=1}^\infty$ in H. In other words, for a fixed basis $(u_n)_{n=1}^\infty$ we study the matrix elements $\langle UTU^{-1}u_j, u_n\rangle$ of the unitary orbit $\{UTU^{-1} : U \text{ is unitary}\}$ of T (where then the choice of $(u_n)_{n=1}^\infty$ is essentially irrelevant).

While the study of matrix representations goes back to the birth of operator theory, a number of pertinent facts and insights of their structure were obtained only recently. In particular, starting from the pioneering works [26], [27], [5], and [6] by Kadison and Arveson on the main diagonals of projections (and normal operators with finite spectrum), the research on main diagonals of Hilbert space operators attracted a substantial attention. For sample works in this direction see e.g. [8], [9], [25], [28], [32], [33], and the recent survey [30]. The mainstream of the research on diagonals concentrated around normal operators and their natural subclasses, consisting of unitary and selfadjoint operators, and addressed the problem of characterizing sets of possible main diagonals $\mathcal{D}(T) = \{(\langle Tu_n, u_n\rangle)_{n=1}^\infty\}$ for classes of operators of $T \in B(H)$ when $(u_n)_{n=1}^\infty$ varies through the set all orthonormal bases in H. In [38] we've changed this point of view to a more demanding task of describing the set $\mathcal{D}(T)$, or at least its substantial subsets, for a fixed

1991 *Mathematics Subject Classification.* Primary 47B02, 47A67, 47A08; Secondary 47A12, 47A13.

Key words and phrases. Hilbert space operator, orthonormal bases, matrix representations, numerical range, spectrum.

The first author was supported by grant No. 20-22230L of GA CR and RVO:67985840.

$T \in B(H)$. In addition, in [38], the problem was addressed in a more general framework of operator tuples $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$. We relied on the properties of the so-called essential numerical range $W_e(\mathcal{T})$ and their relations to the essential spectrum $\sigma_e(\mathcal{T})$, especially in the case of power tuples $\mathcal{T} = (T, \dots, T^k)$, $T \in B(H)$. These properties revealed a new structure in $\mathcal{D}(T)$ and led in particular to the so-called non-Blaschke type conditions

$$\sum_{n=1}^{\infty} \operatorname{dist}(\lambda_n, \partial W_e(T)) = \infty$$

on $(\lambda_n)_{n=1}^{\infty}$ in the interior $\operatorname{Int} W_e(T)$ of $W_e(T)$ to be realized by the main diagonal of T. The relevance of $W_e(T)$ in this context was first noted by Stout [47], Fan [17] and Herrero [22], and these works were generalized substantially in [38]. The ideas of [38] appeared to be fruitful and were further developed in [40], where a systematic approach to matrix representations of bounded operators and operator tuples was initiated. Among other things, we found conditions for prescribing three diagonals, for having bands od zeros around the main diagonal, and described several general situations where matrix representations with size restrictions on the set of their matrix elements are possible. In particular, it was shown that if $T \in B(H)$, then $0 \in W_e(\mathcal{T})$ if and only if for each sequence $(a_n)_{n=1}^{\infty} \not\in \ell^1(\mathbb{N})$ there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ in H such that $|\langle Tu_j, u_n\rangle| \le \sqrt{|a_n a_j|}$ for all j and n, see Section 6 for more on this.

However, a more general and natural problem of matching arrays (a_{nj}), $(n, j) \in B$, with $B \subset \mathbb{N} \times \mathbb{N}$ being "large" by a matrix of a given $T \in B(H)$ has been left in [40] widely open. It was not even quite clear whether our methods can handle a band of (a_{nj}) consisting from more than three diagonals. This paper will bridge this gap and show that following the ideas in [40], one can prescribe the arrays of quite a general nature.

We study the following problem:

Problem 1.1. Let $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$, $B \subset \mathbb{N} \times \mathbb{N}$ and $\{a_{nj} : (n, j) \in B\} \subset \mathbb{C}$ be fixed. What are natural assumptions on $\mathcal{T}$, B and (a_{nj}) to ensure the existence of an orthonormal basis $(u_n)_{n=1}^{\infty} \subset H$ such that

$$\langle \mathcal{T} u_j, u_n\rangle = a_{nj}, \qquad (n, j) \in B.$$

In other words, what is the degree of arbitrariness in a matrix representation of $\mathcal{T}$?

Note that a similar problem in the setting of finite-dimensional spaces was studied by a number of authors.

A particular case of Problem 1.1 are so-called sparse matrix representations for $T \in B(H)$ when the arrays (a_{nj}) corresponding to admissible sets B of (n, j) consist solely of zeros. Note that sparse representations appear useful in a number of problems from operator theory. Recall, in particular, that any $T \in B(H)$ admits a universal block three-diagonal form with an exponential control on (finite-dimensional) block sizes. Moreover, such blocks can be further sparsified. To our knowledge, in a full generality, such block three-diagonal forms appeared first in [49] and they appeared to be crucial e.g. in the study of commutators [3], [49], [31] or operator norm estimates [35]. Nice accounts of sparse representations can be found in [42] and [31, Sections 4 and 5]. It is instructive to observe that matrix representations of the opposite kind can be found for any $T \in B(H)$ which is not a scalar multiple of the identity operator: by [43, Theorem 2], one can find a basis $(u_n)_{n=1}^{\infty} \subset H$ such that $\langle \mathcal{T} u_j, u_n\rangle \neq 0$ for all n and j. Note also that the issue of sparse representations arises also in the finite-dimensional setting, where patterns of zeros in a matrix achievable by a unitary transformation are studied. For sample papers in this direction one may consult [24], [23] and [2], though we feel that this setting is rather different from the subject of the present paper.

Let $T \in B(H)$ and $(u_n)_{n=1}^{\infty}$ be an orthonormal basis. It is natural to measure the sparsity of the corresponding matrix A_T by the so-called density given by

$$d(A_T) := \limsup_{N\to\infty} N^{-2}\text{card}\,\{(n,j) \in \mathbb{N}\times\mathbb{N} : n,j \le N, \langle Tu_j, u_n\rangle \neq 0\}.$$

The density is of course bases dependent, and it is of practical interest to have it as small as possible. Among other things, it was proved in [31, Corollary 5.7] that for every operator $T \in B(H)$ there is an orthonormal basis $(u_n)_{n=1}^{\infty}$ in which A_T has density zero. Our technique allows one to show that in fact much stronger statements hold. Our sparse representations are quite different from the ones mentioned above, since, in particular, apart from being sparse in a much stronger sense, the set of their zero elements can have a comparatively general geometry.

To state our results we need to define several notions describing size of subsets in $\mathbb{N}\times\mathbb{N}$. They will be basic for all of our considerations to follow. Denote by Δ the main diagonal of $\mathbb{N}\times\mathbb{N}$, $\Delta = \{(n,n) : n \in \mathbb{N}\}$. A set $B \subset \mathbb{N}\times\mathbb{N}$ is said to be *subdiagonal* if $B \subset \big\{(n,j) \in \mathbb{N}\times\mathbb{N} : n > j\big\}$. We say that a set $B \subset (\mathbb{N}\times\mathbb{N})\setminus\Delta$ is *admissible* if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n > m$, such that $(j,n) \notin B$ and $(n,j) \notin B$ for all $j = 1,\dots,m$.

Clearly a subdiagonal set B is *admissible* if and only if for every $m \in \mathbb{N}$ there exists $n > m$ such that $(n,j) \notin B$ for all $j = 1,\dots,m$.

Note that an admissible set can be quite large. For example, the set

$$\{(n,j) : n > j\} \setminus \{(2^k, j) : k \in \mathbb{N}, j \le k\}$$

is an admissible subdiagonal set. Similarly

$$\{(n,j) : n \neq j\} \setminus \{(2^k, n), (n, 2^k) : k \in \mathbb{N}, n \le k\}$$

is admissible.

First, as a warm-up, we study the sparse representations and prove that any tuple of bounded operators has a very sparse matrix representation.

Theorem 1.2. *Let* $\mathcal{T} = (T_1,\dots,T_k) \in B(H)^k$ *and* $B \subset (\mathbb{N}\times\mathbb{N})\setminus\Delta$ *an admissible set. Then there exists an orthonormal basis* $(u_n)_{n=1}^{\infty} \subset H$ *such* *that*

$$\langle \mathcal{T} u_j, u_n \rangle = 0$$

for all $(n,j) \in B$.

Note that in Theorem 1.2 the orthonormal basis $(u_n)_{n=1}^\infty$ is common for all operators $T_1, \dots, T_k$.

The condition $B \cap \Delta = \emptyset$ is in general necessary even for single operators. Indeed, if $T \in B(H)$ and the numerical range $W(T)$ of T does not contain zero, then the main diagonal of any matrix representation of T consists of non-zero entries.

A better result can be obtained if we assume that $0 \in \operatorname{Int} W_e(\mathcal{T})$. In this situation we can obtain even the zero main diagonal and the matrix representation becomes extremely sparse. In particular, the folowing statement holds.

Theorem 1.3. *Let* $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ *and* $0 \in \operatorname{Int} W_e(\mathcal{T})$. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be any function satisfying* $\lim_{m\to\infty} f(m) = \infty$. *Then there exists an orthonormal basis* $(u_n)_{n=1}^\infty$ *in* H *such that*

$$\operatorname{card}\left\{(n,j) \in \mathbb{N} \times \mathbb{N} : n, j \le m, \langle \mathcal{T} u_j, u_n \rangle \neq 0\right\} \le f(m)$$

for all $m \in \mathbb{N}$.

Let us now consider Problem 1.1 in full generality. Under mild assumptions on $T \in B(H)$ we show that large subdiagonal subsets of a matrix A_T can be preassigned if the size of the corresponding matrix elements is restricted appropriately.

Theorem 1.4. *Let* $T \in B(H)$ *be an operator, which is not of the form* $T = \lambda I + K$ *for some* $\lambda \in \mathbb{C}$ *and a compact operator* $K \in B(H)$. *Then there exists* $\delta > 0$ (*depending only on the diameter of* $W_e(T)$) *with the following property: if* $B \subset \mathbb{N} \times \mathbb{N}$ *is subdiagonal and admissible, and* $\{a_{nj} : (n,j) \in B\} \subset \mathbb{C}$ *satisfy*

$$\sum_{n:(n,j)\in B} |a_{nj}| \le \delta \quad \text{for all } j \qquad \text{and} \qquad \sum_{j:(n,j)\in B} |a_{n,j}| \le \delta \quad \text{for all } n,$$

then there is an orthonormal basis $(u_n)_{n=1}^\infty$ *in* H *such that*

$$\langle T u_j, u_n \rangle = a_{nj}$$

for all $n, j \in \mathbb{N}$ *with* $(n,j) \in B$.

Theorem 1.4 can be formulated also for k-tuples $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$, where none of the operators $T_1, \dots, T_k$ is of the form $\lambda I + K$ with $\lambda \in \mathbb{C}$ and $K \in B(H)$ compact. However, we preferred to prove its simplified version.

A technically more involved framework of operator tuples will be addressed in Theorem 1.5 below. Under assumptions stronger than in Theorem 1.4, we prove that large subsets of the whole of A_T can be preassigned under size restrictions on the matrix elements similar to those in Theorem

1.4. Note however that one requires additional restrictions on the diagonal elements, which reflects a special role of the main diagonal in the matrix representations of T, see e.g. [30] and [38] for more on the topic of main diagonals.

For $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$ write $\|\lambda\|_\infty = \max\{|\lambda_1|, \dots, |\lambda_k|\}$.

Theorem 1.5. *Let $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ be such that $\operatorname{Int} W_e(\mathcal{T}) \neq \emptyset$, and let $\varepsilon > 0$ be fixed. Then there exists $\delta > 0$ with the following property: if $B \subset (\mathbb{N} \times \mathbb{N}) \setminus \Delta$ is admissible, and $\{a_{nj} : (n, j) \in B \cup \Delta\} \subset \mathbb{C}^k$ satisfy :*

(i) *$a_{nn} \in \operatorname{Int} W_e(\mathcal{T})$ and $\operatorname{dist}\{a_{nn}, \partial W_e(\mathcal{T})\} > \varepsilon$ for all $n \in \mathbb{N}$;*
(ii) *$\sum_{j:(n,j)\in B} \|a_{nj}\|_\infty \le \delta$ for all $n \in \mathbb{N}$, and $\sum_{n:(n,j)\in B} \|a_{jn}\|_\infty \le \delta$ for all $j \in \mathbb{N}$,*

then there is an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that

$$\langle \mathcal{T} u_j, u_n \rangle = a_{nj}$$

for all $n, j \in \mathbb{N}$ with $(n, j) \in B \cup \Delta$.

Clearly for any $m \in \mathbb{N}$, the set $\{(n, j) : 1 \le |n - j| \le m\}$ is an admissible set. So in particular we can prescribe any finite number of diagonals in the matrix representation of $\mathcal{T}$ subject to mild restrictions on absolute values of their elements. We formulate this conclusion as a separate statement generalising essentially [40, Theorem 2.4].

Corollary 1.6. *Let $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ be such that $\operatorname{Int} W_e(\mathcal{T}) \neq \emptyset$, and let $\varepsilon > 0$ and $m \in \mathbb{N}$ be fixed. Then there exists $\delta = \delta(k, m, \epsilon) > 0$ such that if $\{a_{nj} : |n - j| \le m\} \subset \mathbb{C}^k$ satisfy :*

(i) *$a_{nn} \in \operatorname{Int} W_e(\mathcal{T})$ and $\operatorname{dist}\{a_{nn}, \partial W_e(\mathcal{T})\} > \varepsilon$ for all $n \in \mathbb{N}$;*
(ii) *$\sup\{\|a_{nj}\|_\infty : 1 \le |n - j| \le m\} \le \delta$;*

then there is an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that

$$\langle \mathcal{T} u_j, u_n \rangle = a_{nj}, \qquad |n - j| \le m.$$

The framework of operator tuples makes it possible to formulate similar results for tuples of powers $\mathcal{T} = (T, T^2, \dots, T^k)$ under the spectral assumptions on $T \in B(H)$ rather than the assumptions on $W_e(\mathcal{T})$ as above, making the obtained results more explicit.

If $T \in B(H)$ is such that $0 \in \operatorname{Int} \hat{\sigma}(T)$, where $\hat{\sigma}(T)$ stands for the polynomial hull of $\sigma(T)$, then $0 \in \operatorname{Int} W_e(T, T^2, \dots, T^k)$ for all $k \in \mathbb{N}$ (see Section 2). So we can prescribe quite large subset of entries simultaneously for any finite number of powers T^j.

Similar results are also proved in the case of an invertible operator $T \in B(H)$ and for tuples consisting of both positive and negative powers of T. In this case we impose a stronger assumption $r\mathbb{T} \cup s\mathbb{T} \subset \sigma_e(T)$, where $0 < r < s$ and $\mathbb{T}$ stands for the unit circle.

2. Preliminaries and notations

2.1. **The relevance of numerical ranges.** First, we recall some standard notation used in the context of operator tuples. For a k-tuple $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ and $x, y \in H$ we write shortly

$$\langle \mathcal{T}x, y\rangle = (\langle T_1x, y\rangle, \dots, \langle T_kx, y\rangle) \in \mathbb{C}^k \quad \text{and} \quad \mathcal{T}x = (T_1x, \dots, T_kx) \in H^k.$$

If $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$ we denote $\mathcal{T} - \lambda = (T_1 - \lambda_1, \dots, T - \lambda_k)$ and

$$\|\lambda\|_\infty = \max\{|\lambda_1|, \dots, |\lambda_k|\}. \tag{2.1}$$

In our studies of matrix representations for a bounded operator T on H and more generally for operator tuples $\mathcal{T} \in B(H)^k$, we will rely on the well-studied notions of the (joint) numerical range $W(\mathcal{T})$, given by

$$W(\mathcal{T}) = \big\{(\langle T_1x, x\rangle, ..., \langle T_kx, x\rangle) : x \in H, \|x\| = 1\big\},$$

and of the essential numerical range $W_e(\mathcal{T})$ of $\mathcal{T}$. Being an approximate version of $W(\mathcal{T})$, the latter notion allows for several equivalent definitions. To fix one of them, for $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ we define the (joint) essential numerical range $W_e(\mathcal{T})$ of $\mathcal{T}$ as the set of all k-tuples $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$ such that there exists an orthonormal sequence $(u_n)_{n=1}^\infty \subset H$ with

$$\lim_{n\to\infty} \langle T_t u_n, u_n\rangle = \lambda_t, \qquad t = 1, \dots, k.$$

Recall that $W_e(\mathcal{T})$ is a nonempty, compact and, in contrast to $W(\mathcal{T})$, *convex* subset of $\overline{W(\mathcal{T})}$, see e.g. [29]. At the same time, $W(\mathcal{T})$ is convex if $k = 1$, and it may be non-convex if $k > 1$. Moreover, even if $k = 1$, then $W(\mathcal{T})$ can be neither closed nor open.

The next properties of $W_e(\mathcal{T})$ and $W(\mathcal{T})$ are crucial for the sequel and will be used frequently.

Proposition 2.1. *Let $\mathcal{T} \in B(H)^k$.*

(a) *One has $\lambda \in W_e(\mathcal{T})$ if and only if for every $\epsilon > 0$ and every subspace $M \subset H$ of finite codimension there is a unit vector $x \in M$ such that*

$$\|\langle \mathcal{T}x, x\rangle - \lambda\|_\infty < \epsilon.$$

(b) *If $\lambda \in \operatorname{Int} W_e(\mathcal{T})$, then for every subspace $M \subset H$ of a finite codimension there is $x \in M$ such that $\|x\| = 1$ and*

$$\langle \mathcal{T}x, x\rangle = \lambda.$$

The proof of the first property is easy and can be found e.g. in [37, Proposition 5.5]. Since $W(\mathcal{T})$ is not in general convex, the second property is more involved, see [36, Corollary 4.5] for its proof and other related statements. The properties (a) and (b) are very useful in inductive constructions of sequences in H. In particular, by absorbing all of the elements constructed after a finite number of induction steps into a finite-dimensional subspace F, one may still use $W_e(\mathcal{T})$ when dealing with vectors from $F^\perp$. The properties will play a similar role in this paper.

To note another usage of numerical ranges, recall that the joint spectrum of a commuting tuple $\mathcal{T} = (T_1, \dots, T_k)$ can be defined as the (Harte) spectrum of the n-tuple $(T_1, \dots, T_k)$ in the algebra $B(H)$. Similarly, the joint essential spectrum $\sigma_e(\mathcal{T})$ is defined as the (Harte) spectrum of the k-tuple $(T_1 + \mathcal{K}(H), \dots, T_k + \mathcal{K}(H))$ in the Calkin algebra $B(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of all compact operators on H. One of the main features of $W(\mathcal{T})$ and $W_e(\mathcal{T})$ is that in view of the inclusions $\sigma(\mathcal{T}) \subset \overline{W(\mathcal{T})}$ and $\sigma_e(\mathcal{T}) \subset W_e(T)$ these numerical ranges help to localize spectrum. Sometimes, when the spectral information is more accessible, one may argue the other way round and to identify big subsets of $W(T)$ and $W_e(T)$ in spectral terms. In particular, this becomes apparent for tuples $\mathcal{T}$ of special form $\mathcal{T} = (T, \dots, T^k), T \in B(H)$. Note that $\sigma(\mathcal{T}) = \{(\lambda, \dots, \lambda^k) : \lambda \in \sigma(T)\}$ and $\sigma_e(\mathcal{T}) = \{(\lambda, \dots, \lambda^k) : \lambda \in \sigma_e(T)\}$ (cf. Section 6) so that the spectral properties of the tuple $(T, \dots, T^k)$ are determined by the spectral properties of T. The relevance of spectrum for the study of numerical ranges can be illustrated by the next "numerical ranges" mapping theorem [36, Theorem 4.6], important for the sequel (see Section 6). To formulate it, recall that if $K \subset \mathbb{C}$ is compact, then the polynomial hull $\widehat{K} := \{\lambda \in \mathbb{C} : |p(\lambda)| \le \sup_{z \in K} |p(z)| \text{ for all polynomials } p\}$ of K can be described as the union of K with all bounded components of the complement $\mathbb{C} \setminus K$. If $\operatorname{conv} K$ stands for the convex hull of K, then clearly $\widehat{K} \subset \operatorname{conv} K$.

Theorem 2.2. *Let $T \in B(H)$. If $\lambda \in \operatorname{Int} \widehat{\sigma}(T)$, then*

$$(\lambda, \lambda^2, \dots, \lambda^k) \in \operatorname{conv} \{(z, z^2, \dots, z^k) : z \in \sigma_e(T)\} \subset \operatorname{Int} W_e(T, T^2, \dots, T^k)$$

for all $k \in \mathbb{N}$.

More information on joint essential numerical range for operator tuples and its relation to spectral theory can be found in [37] and [38], see also [29]. The classical case $k = 1$ is considered in details in [7] and [18].

2.2. **Some notations.** Let $T \in B(H)$ and $u, v \in H$. We write for short $u \perp^{(T)} v$ if $u \perp v, Tv, T^*v$. More generally, if $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ then we write $u \perp^{(\mathcal{T})} v$ if

$$u \perp v, T_1 v, \dots, T_k v, T_1^* v, \dots, T_k^* v.$$

Clearly $u \perp^{(\mathcal{T})} v$ if and only if $v \perp^{(\mathcal{T})} u$. Note that if $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ and $v_1, \dots, v_m \in H$ then the set

$$\{u \in H : u \perp^{(\mathcal{T})} v_1, \dots, v_m\}$$

is a subspace of H of finite codimension.

For a subspace $L \subset H$ denote by P_L the orthogonal projection onto L.

As above, for a compact subset $K \subset \mathbb{C}^n$ we denote by $\operatorname{Int} K$ the interior of K, by ∂K the topological boundary of K, by $\operatorname{conv} K$ the convex hull of K and by $\widehat{K}$ the polynomial hull of K.

3. Sparse representations

First we prove Theorem 1.2 that any tuple of operators has a very sparse matrix representation.

Proof of Theorem 1.2. Let $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ be any k-tuple of operators and $B \subset (\mathbb{N}\times\mathbb{N}) \setminus \Delta$ an admissible set. We show that there exists an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that

$$\langle \mathcal{T} u_j, u_n\rangle = 0$$

for all $(n,j) \in B$.

Let $n_0 = 0$ and construct inductively an increasing sequence of integers $(n_s)_{s=0}^\infty$ such that

$$(j, n_s) \notin B \qquad \text{and} \qquad (n_s, j) \notin B, \qquad j = 1, \dots, n_{s-1}.$$

Let $(y_r)_{r=1}^\infty$ be a sequence of vectors in H such that $\bigvee_{r=1}^\infty y_r = H$.

We construct the vectors u_n inductively.

Let $s \ge 1$ and suppose that orthonormal vectors $u_1, \dots, u_{n_{s-1}} \in H$ satisfy

(i) $\langle T_t u_n, u_j\rangle = 0$ for all $t = 1, \dots, k$ and all $(n,j) \in B$ with $n, j \le n_{s-1}$;

(ii) $y_r \in \bigvee_{n=1}^{n_r} u_n$ for all $r \le s-1$.

For all n, $n_{s-1} < n < n_s$, find inductively unit vectors u_n such that

$$u_n \perp^{(\mathcal{T})} u_1, \dots, u_{n-1}, y_s.$$

Then $\langle \mathcal{T} u_n, u_m\rangle = \langle \mathcal{T} u_m, u_n\rangle = 0$ for all $m \le n-1$, $n_{s-1} < n < n_s$.

In order to construct u_{n_s} we distinguish two cases. If $y_s \in \bigvee_{j=1}^{n_s-1} u_j$ then choose u_{n_s} any unit vector satisfying

$$u_{n_s} \perp^{(\mathcal{T})} u_1, \dots, u_{n_s-1}.$$

If $y_s \notin \bigvee_{j=1}^{n_s-1} u_j$ then set

$$u_{n_s} = \frac{(I - P_{M_{n_s-1}})y_s}{\|(I - P_{M_{n_s-1}})y_s\|},$$

where $P_{M_{n_s-1}}$ is the orthogonal projection onto the subspace $M_{n_s-1} := \bigvee_{j=1}^{n_s-1} u_j$. Clearly $\|u_{n_s}\| = 1$ and $u_{n_s} \perp u_1, \dots, u_{n_s-1}$. Moreover,

$$u_{n_s} \in \bigvee\{y_s, u_1, \dots, u_{n_s-1}\} \perp^{(\mathcal{T})} u_m$$

for all $m, n_{s-1} < m < n_s$ by the construction. So the set $\{u_1, \dots, u_{n_s}\}$ is orthonormal.

If $m < n_s$ and either $(m, n_s) \in B$ or $(n_s, m) \in B$ then $m > n_{s-1}$ and

$$T_t u_{n_s} \in \bigvee\{T_t y_s, T_t u_1, \dots, T_t u_{n_s-1}\} \subset u_m^\perp$$

for all $t = 1, \dots, k$. So $\langle \mathcal{T} u_{n_s}, u_m\rangle = 0$. Similarly, $\langle \mathcal{T} u_m, u_{n_s}\rangle = 0$.

Moreover, we have $y_s \in \bigvee_{j=1}^{n_s} u_j$.

If we construct the vectors $u_n, n \in \mathbb{N}$, in this way then they will form an orthonormal system satisfying

$$\langle \mathcal{T} u_j, u_n \rangle = 0$$

for all $(n,j) \in B$. Moreover, $y_r \in \bigvee_{n=1}^\infty u_n$ for all r, and so $(u_n)_{n=1}^\infty$ form an orthonormal basis.

□

As mentioned in the introduction, the assumption that $B \subset (\mathbb{N} \times \mathbb{N}) \setminus \Delta$ cannot be in general omitted. In general, all entries on the main diagonal may be non-zero for any choice of an orthonormal basis if $0 \notin W(T)$ (e.g. if $\operatorname{Re} T \geq cI, c > 0$).

If we assume that $0 \in \operatorname{Int} W_e(\mathcal{T})$ then it is possible to obtain also the zero main diagonal. The next result is a consequence of a more general Theorem 1.5. However, we give a direct proof because it is much simpler and, at the same time, contains all of the main ideas behind the proof of Theorem 1.5.

Theorem 3.1. *Let $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ satisfy $0 \in \operatorname{Int} W_e(\mathcal{T})$ and let $B \subset (\mathbb{N} \times \mathbb{N}) \setminus \Delta$ be an admissible set. Then there exists an orthonormal basis $(u_n)_{n=1}^\infty \subset H$ such that*

$$\langle \mathcal{T} u_j, u_n \rangle = 0 \tag{3.1}$$

for $(n,j) \in B \cup \Delta$.

Proof. Without loss of generality we may assume that $\|T_t\| \leq 1$ for all $t = 1, \dots, k$.

Let $n_0 = 0$ and construct inductively an increasing sequence of integers $(n_s)_{s=0}^\infty$ such that

$$(j, n_s) \notin B \qquad \text{and} \qquad (n_s, j) \notin B, \qquad j = 1, \dots, n_{s-1}.$$

Fix a number $\eta \in (0,1)$ such that

$$\frac{\eta}{1-\eta} < \operatorname{dist}\{0, \partial W_e(\mathcal{T})\}.$$

Fix a sequence of unit vectors $(y_r)_{r=0}^\infty$ in H such that $\bigvee_{r=0}^\infty y_r = H$.

Each $s \in \mathbb{N}$ can be written as $s = 2^{r(s)}(2l(s) - 1)$ where $r(s) \geq 0$ and $l(s) \geq 1$ are uniquely determined integers.

We construct the vectors u_n inductively.

Let $s \geq 1$ and suppose that orthonormal vectors $u_1, \dots, u_{n_{s-1}} \in H$ satisfying

(i) $\langle \mathcal{T} u_j, u_j \rangle = 0$ for all $j = 1, \dots, n_{s-1}$;
(ii) $\langle T_t u_n, u_j \rangle = 0$ for all $t = 1, \dots, k$ and all $(n,j) \in B$ with $n \neq j$ and $n, j \leq n_{s-1}$;
(iii) $\operatorname{dist}^2\left\{ y_r, \bigvee_{j=1}^{n_{2^r(2l-1)}} u_j \right\} \leq (1-\eta)^l$ for all r, l with $2^r(2l-1) \leq s-1$.

For n, $n_{s-1} < n < n_s$ find inductively unit vectors u_n such that

$$u_n \perp^{(\mathcal{T})} u_1, \dots, u_{n-1}, y_{r(s)}$$

and

$$\langle \mathcal{T} u_n, u_n \rangle = 0.$$

Then $\langle \mathcal{T} u_n, u_m \rangle = \langle \mathcal{T} u_m, u_n \rangle = 0$ for all $m \le n$, $n_{s-1} < n < n_s$.

In order to construct u_{n_s} we distinguish two cases. If $y_{r(s)} \in \bigvee_{j=1}^{n_{s-1}} u_j$ then let u_{n_s} be any unit vector satisfying

$$u_{n_s} \perp^{(\mathcal{T})} u_1, \dots, u_{n_s - 1}$$

and

$$\langle \mathcal{T} u_{n_s}, u_{n_s} \rangle = 0.$$

Then clearly (i)–(iii) are satisfied.

If $y_{r(s)} \notin \bigvee_{j=1}^{n_{s-1}} u_j$ then set

$$b_{n_s} = \frac{(I - P_{M_{n_{s-1}}}) y_{r(s)}}{\|(I - P_{M_{n_{s-1}}}) y_{r(s)}\|},$$

where $P_{M_{n_{s-1}}}$ is the orthogonal projection onto the subspace $M_{n_{s-1}} := \bigvee_{j=1}^{n_{s-1}} u_j$. We have

$$\left| \frac{\eta}{1-\eta} \langle \mathcal{T} b_{n_s}, b_{n_s} \rangle \right| < \operatorname{dist}\{0, \partial W_e(\mathcal{T})\},$$

so there exists a unit vector $v_{n_s} \in H$ such that

$$v_{n_s} \perp^{(\mathcal{T})} u_1, \dots, u_{n-1}, b_{n_s}$$

and

$$\langle \mathcal{T} v_{n_s}, v_{n_s} \rangle = -\frac{\eta}{1-\eta} \langle \mathcal{T} b_{n_s}, b_{n_s} \rangle.$$

Define

$$u_{n_s} = \sqrt{1-\eta}\, v_{n_s} + \sqrt{\eta}\, b_{n_s}.$$

Clearly $\|u_{n_s}\| = 1$ since $v_{n_s} \perp b_{n_s}$.

Clearly $u_{n_s} \perp u_1, \dots, u_{n_{s-1}}$. For $j, n_{s-1} < j < n_s$, we have

$$\langle u_{n_s}, u_j \rangle = \langle \sqrt{\eta} b_{n_s}, u_j \rangle = 0$$

since $b_{n_s} \in \bigvee \{y_{r(s)}, u_1, \dots, u_{n_{s-1}}\} \subset u_j^{\perp}$. So the vectors $u_1, \dots, u_{n_s}$ are orthonormal.

We have

$$\langle \mathcal{T} u_{n_s}, u_{n_s} \rangle = (1-\eta) \langle \mathcal{T} v_{n_s}, v_{n_s} \rangle + \eta \langle \mathcal{T} b_{n_s}, b_{n_s} \rangle = 0.$$

If $j < n_s$ and $(n_s, j) \in B$ then $j > n_{s-1}$ and $\langle \mathcal{T} u_j, u_{n_s} \rangle = \langle \mathcal{T} u_j, \sqrt{\eta} b_{n_s} \rangle = 0$ since

$$b_{n_s} \in \bigvee \{y_{r(s)}, u_1, \dots, u_{n_{s-1}}\} \subset \perp^{(\mathcal{T})} u_j.$$

Similarly, $\langle \mathcal{T} u_{n_s}, u_j \rangle = 0$ if $j < n_s$ and $(j, n_s) \in B$.

Finally,

$$\begin{aligned}
\operatorname{dist}^2\{y_{r(s)}, M_{n_s}\} &= \operatorname{dist}^2\{y_{r(s)}, M_{n_s - 1}\} - |\langle y_{r(s)}, u_{n_s} \rangle|^2 \\
&\le \operatorname{dist}^2\{y_{r(s)}, M_{n_{s-1}}\} - |\langle y_{r(s)}, \sqrt{\eta} b_{n_s} \rangle|^2 \\
&= \operatorname{dist}^2\{y_{r(s)}, M_{n_{s-1}}\}(1-\eta) \le (1-\eta)^{l(s)}
\end{aligned}$$

by the induction assumption.

Suppose that the vectors $u_n, n \in \mathbb{N}$, have been constructed in the way described above. Then the vectors $(u_n)_{n\in\mathbb{N}}$ form an orthonormal system satisfying

$$\langle \mathcal{T} u_j, u_n\rangle = 0$$

for all $(n,j) \in B \cup \Delta$. Moreover, for each $r \geq 0$ we have

$$\operatorname{dist}^2\Bigl\{y_r, \bigvee_{j=1}^{\infty} u_j\Bigr\} = \lim_{l\to\infty} \operatorname{dist}^2\bigl\{y_r, M_{2^r(2l-1)}\bigr\} \leq \lim_{l\to\infty}(1-\eta)^l = 0.$$

So $y_r \in \bigvee_{j=1}^{\infty} u_j$. Since $\bigvee_{r=0}^{\infty} y_r = H$, the vectors $(u_n)_{n=1}^{\infty}$ form an orthonormal basis. □

Theorem 3.1 implies that operators $T \in B(H)$ with $0 \in \operatorname{Int} W_e(T)$ have extremely sparse representations as stated in Theorem 1.3 given in the introduction.

Proof of Theorem 1.3. Without loss of generality we may assume that the function f is nondecreasing (if not, then replace $f(m)$ by $\inf\{f(j) : j \geq m\}$).

Find an increasing sequence $(n_k)_{k=1}^{\infty}$ such that $f(n_k) \geq (k+1)^2$ for each $k \in \mathbb{N}$.

Let

$$B = \mathbb{N} \times \mathbb{N} \setminus \bigl\{(n_k, j), (j, n_k) : k \in \mathbb{N}, j \leq k\bigr\}.$$

For each m, $n_k \leq m < n_{k+1}$, we have

$$\operatorname{card}\bigl\{(n,j) : n, j \leq m, (n,j) \notin B\bigr\} \leq \sum_{r=1}^{k}(2r) \leq (k+1)^2 \leq f(n_k) \leq f(m).$$

Clearly $B \setminus \Delta$ is an admissible set. By Theorem 3.1, there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ such that $\langle \mathcal{T} u_j, u_n\rangle = 0$ for all $(n,j) \in B$. □

Clearly the condition $f(m) \to \infty$ as $m \to \infty$ is in general necessary. It is easy to see that there exists a matrix representation of $T \in B(H)$ with $\operatorname{card}\{(n,j) : \langle Tu_n, u_j\rangle \neq 0\} < \infty$ if an only if T is a finite rank operator.

If we do not assume that $0 \in \operatorname{Int} W_e(\mathcal{T})$ then in general all entries on the main diagonal may be non-zero for all orthonormal bases. So we can state the next version of Theorem 1.3 (having the same proof).

Theorem 3.2. *Let* $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be any function satisfying* $\lim_{m\to\infty} f(m) = \infty$. *Then there exists an orthonormal basis* $(u_n)_{n=1}^{\infty}$ *in* H *such that*

$$\operatorname{card}\bigl\{(n,j) \in \mathbb{N}\times\mathbb{N} : n, j \leq m, \langle \mathcal{T} u_j, u_n\rangle \neq 0\bigr\} \leq m + f(m)$$

for all $m \in \mathbb{N}$.

Theorem 3.1 applies directly to k-tuples of the form $(T, T^2, \dots, T^k)$ where $T \in B(H)$ satisfies $0 \in \operatorname{Int} \hat{\sigma}(T)$. We discuss this in the last section.

4. Prescribing subdiagonal entries

Recall that $T \in B(H)$ is compact if and only if $W_e(T) = \{0\}$. So T is of the form $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and a compact operator $K \in B(H)$ if and only if $W_e(T)$ is a singleton, i.e.,

$$\mathrm{diam} W_e(T) = \max\{|\lambda - \mu| : \lambda, \mu \in W_e(T)\} = 0.$$

The next lemma will be crucial in the inductive construction leading to Theorem 1.4. Similar statements can be found in [10, Section 2,3] and [40, Section 5].

Lemma 4.1. *Let $T \in B(H)$ be an operator, which is not of the form $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and a compact operator $K \in B(H)$, and let $0 < C <$ $\mathrm{diam}\, W_e(T)$. Let $M \subset H$ be a subspace of finite codimension. Then there exist vectors $v, z \in M$ such that*

$$\|v\| \cdot \|z\| \le \frac{2\sqrt{2}}{C}, \quad v \perp z, \quad \textit{and} \quad \langle Tv, z\rangle = 1.$$

Proof. Let $\lambda, \mu \in W_e(T)$ be such that $|\lambda - \mu| = \mathrm{diam}\, W_e(T)$. Choose a positive number ε such that $\varepsilon < (\mathrm{diam}\, W_e - C)/4$. Using Lemma 2.1, (i), let $x \in M \cap T^{-1}M$ be a unit vector such that $|\langle Tx, x\rangle - \lambda| < \varepsilon$.

Similarly, let $y \in M \cap T^{-1}M$, $y \perp^{(T)} x$ be a unit vector such that $|\langle Ty, y\rangle - \mu| < \varepsilon$.

Let $v = \frac{x+y}{\sqrt{2}}$. Then

$$v \in M \cap T^{-1}M, \qquad \|v\| = 1, \qquad \langle Tv, v\rangle = \frac{1}{2}\big(\langle Tx, x\rangle + \langle Ty, y\rangle\big),$$

and so

$$\Big|\langle Tv, v\rangle - \frac{\lambda + \mu}{2}\Big| < \varepsilon.$$

Let

$$w = Tv - \langle Tv, v\rangle v.$$

Then $w \in M$, $w \perp v$ and

$$\begin{aligned}\|w\| \ge |\langle w, x\rangle| = &\big|\langle Tv, x\rangle - \langle Tv, v\rangle\langle v, x\rangle\big| = \frac{1}{\sqrt{2}}\Big|\langle Tx, x\rangle - \langle Tv, v\rangle\Big| \\ \ge &\frac{1}{\sqrt{2}}\Big(\Big|\lambda - \frac{\lambda+\mu}{2}\Big| - 2\varepsilon\Big) = \frac{1}{\sqrt{2}}\frac{|\lambda - \mu|}{2} - \frac{2\varepsilon}{\sqrt{2}} > \frac{C}{2\sqrt{2}}.\end{aligned}$$

Set

$$z = \frac{w}{\|w\|^2}.$$

Then

$$z \in M, \quad z \perp v, \quad \|z\| = \frac{1}{\|w\|} \le \frac{2\sqrt{2}}{C} \quad \text{and} \quad \langle Tv, z\rangle = \langle w, z\rangle = 1,$$

as required. ∎

The proof of Theorem 1.4 is less technically demanding than the proof of its full matrix analogue, Theorem 1.5, and it thus provides a good intuition needed for understanding a more involved argument for Theorem 1.5 in the next section.

Proof of Theorem 1.4. Choose positive numbers C and δ such that $C < \operatorname{diam} W_e(T)$, and

$$\delta < \frac{C}{4\sqrt{2}}.$$

For a given subdiagonal and admissible set $B \subset \mathbb{N}\times\mathbb{N}$ and the corresponding array $\{a_{nj} : (n,j) \in B\}$ subject to the size restrictions

$$\sum_{n:(n,j)\in B} |a_{nj}| \le \delta \quad \text{for all } j \qquad \text{and} \qquad \sum_{j:(n,j)\in B} |a_{n,j}| \le \delta \quad \text{for all } n,$$

we construct an orthonormal basis $(u_n)_{n=1}^{\infty} \subset H$ such that

$$\langle Tu_j, u_n\rangle = a_{nj} \qquad \text{for all } n, j \in \mathbb{N},\ (n,j) \in B. \tag{4.1}$$

To clarify technical details, the proof will be divided into several steps. The construction of $(u_n)_{n=1}^{\infty}$ will be based on several inductive arguments. We start with an appropriate choice of parameters needed for our subsequent considerations.

Fix the next initial settings:

(i) Fix a positive number η such that $\eta < 1 - \frac{4\delta\sqrt{2}}{C}$.

(ii) For $s \in \mathbb{N}$ write $s = 2^{r(s)}(2l(s) - 1)$, where the integers $r(s) \ge 0$ and $l(s) \ge 1$ are defined uniquely.

(iii) Fix (very small) positive numbers ρ_s, $s \in \mathbb{N}$. The precise size of these numbers is not important, we require only that $\sum_{s=1}^{\infty} \rho_s < 1$ and

$$\left((1-\eta/2)^{(l-2)/2} + \rho_s\right)^2 (1-\eta) \le (1-\eta/2)^{l-1} \tag{4.2}$$

for all integers $l \ge 2$.

(iv) Set formally $n_0 = 0$ and define inductively an increasing sequence $(n_s)_{s=1}^{\infty}$ such that $n_1 = 1$ and $(n_s, m) \notin B$ for all $m = 1, \dots, n_{s-1}$.

(v) For $(n,i) \in B$ let

$$\beta_{ni} := |a_{ni}|^{1/2} \arg(a_{ni}) \qquad \text{and} \qquad \gamma_{ni} := |a_{ni}|^{1/2}.$$

The vectors $u_n, n \in \mathbb{N}$, forming an orthonormal basis in H, will be constructed in the form

$$u_n = \alpha_n w_n + \sum_{j:(j,n)\in B} \beta_{jn} v_{jn} + \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni} + b_n,$$

where w_n, v_{jn}, z_{ni} and b_n are suitable elements of H. Each of the pairs v_{jn}, z_{jn} will ensure that $\langle Tu_j, u_n\rangle = a_{nj}$ for $(n,j) \in B$. The vectors w_n are used only to have $\|u_n\| = 1$. The vectors b_n will help to arrange $\bigvee_{n=1}^{\infty} u_n = H$. To this end, we fix in advance an orthonormal basis $(y_r)_{r=0}^{\infty}$ in H, and for every r we construct a sequence $\{y_{r,l} : l \ge 0\}$ such that $\lim_{l\to\infty} y_{r,l} = y_r'$

with $y_r' \in \bigvee_{n=1}^\infty u_n$ and y_r' being close enough to y_r. This will imply that $\bigvee_{r=0}^\infty y_r' = H$ and then $\bigvee_{n=1}^\infty u_n = H$. Set formally $y_{r,0} = y_r$ for all r.

First, by an inductive argument, we construct vectors $b_n, w_n, n \in \mathbb{N}$, and $v_{ni}, z_{ni}, (n,i) \in B$, and numbers $\alpha_n \geq 0$ in the following way:

Let $s \in \mathbb{N}$ and suppose that the vectors

$$b_n,\ w_n,\ v_{ni},\ z_{ni}, \quad n \leq n_{s-1}, (n,i) \in B$$

$$y_{r,l}, \quad 2^r(2l-1) \leq s-1,$$

and numbers $\alpha_n, \quad n \leq n_{s-1}$, have already been constructed in such a way that if

$$u_{n,s-1} := \alpha_n w_n + \sum_{\substack{j:(j,n)\in B \\ j \leq n_{s-1}}} \beta_{jn} v_{jn} + \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni} + b_n, \qquad n = 1, \dots, n_{s-1},$$

then the vectors $u_{1,s-1}, \dots, u_{n_{s-1},s-1}$ are mutually orthogonal,

$$\|u_{n,s-1}\|^2 = 1 - \sum_{j > n_{s-1}, (j,n)\in B} |a_{jn}| \frac{2\sqrt{2}}{C}, \qquad n = 1, \dots, n_{s-1},$$

and

$$\|y_{r,l} - y_{r,l-1}\| \leq \rho_{2^r(2l-1)} \quad \text{for all} \quad r, l \quad \text{with} \quad 2^r(2l-1) \leq s-1.$$

(A) Define first vectors $b_n, n_{s-1} < n \leq n_s$.

If $n_{s_1} < n < n_s$, then set $b_n = 0$.

Write $s = 2^{r(s)}(2l(s)-1)$ and define

$$L_{s-1} = \bigvee_{n=1}^{n_{s-1}} u_{n,s-1}.$$

If $y_{r(s),l(s)-1} \notin L_{s-1}$ then set $y_{r(s),l(s)} = y_{r(s),l(s)-1}$. If otherwise $y_{r(s),l(s)-1} \in L_{s-1}$, then choose $y_{r(s),l(s)} \notin L_{s-1}$ such that

$$\|y_{r(s),l(s)}\| \leq 1 \qquad \text{and} \qquad \left\|y_{r(s),l(s)} - y_{r(s),l(s)-1}\right\| < \rho_s.$$

In both cases $y_{r(s),l(s)} \notin L_{s-1}$, so that we can set

$$b_{n_s} = \frac{(I - P_{L_{s-1}}) y_{r(s),l(s)}}{\|(I - P_{L_{s-1}}) y_{r(s),l(s)}\|} \cdot \sqrt{\eta},$$

where $P_{L_{s-1}}$ is the orthogonal projection onto L_{s-1}.

(B) For $n_{s-1} < n \le n_s$ and i such that $(n,i) \in B$, using Lemma 4.1, define inductively vectors $v_{ni}, z_{ni} \in H$ such that

$$
\begin{aligned}
&\|v_{ni}\|^2 = \frac{2\sqrt{2}}{C}, \qquad \|z_{ni}\|^2 \le \frac{2\sqrt{2}}{C},\\
&v_{ni}, z_{ni} \perp^{(T)} v_{m,i'}, z_{m,i'}, \quad m \le n_s, (m,i') \in B, (m,i') \ne (n,i),\\
&v_{ni}, z_{ni} \perp^{(T)} w_m, \qquad m \le n_{s-1},\\
&v_{ni}, z_{ni} \perp^{(T)} b_m, \qquad m \le n_s,\\
&v_{ni}, z_{ni} \perp y_{r,l}, \qquad 2^r(2l-1) \le s,\\
&\quad z_{ni} \perp v_{ni},\\
&\langle T v_{ni}, z_{ni}\rangle = 1.
\end{aligned}
\tag{4.3}
$$

(C) For $n_{s-1} < n \le n_s$ find inductively vectors $w_n \in H$ such that

$$
\begin{aligned}
&\|w_n\| = 1,\\
&\quad w_n \perp^{(T)} v_{m,i}, z_{m,i}, \qquad m \le n_s, (m,i) \in B,\\
&\quad w_n \perp^{(T)} w_m, \qquad m \le n_s, m \ne n,\\
&\quad w_n \perp^{(T)} b_m, \qquad m \le n_s,\\
&\quad w_n \perp y_{r,l}, \qquad 2^r(2l-1) \le s.
\end{aligned}
\tag{4.4}
$$

(D) If n is such that $n_{s-1} < n \le n_s$, then

$$
\begin{aligned}
&\sum_{j:(j,n)\in B} |\beta_{jn}|^2 \cdot \frac{2\sqrt{2}}{C} + \Big\| \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni} + b_n \Big\|^2\\
&\le \frac{2\delta\sqrt{2}}{C} + \sum_{i:(n,i)\in B} |\gamma_{ni}|^2 \|z_{ni}\|^2 + \|b_n\|^2\\
&\le \frac{4\delta\sqrt{2}}{C} + \eta \le 1.
\end{aligned}
$$

Thus we can set

$$
\alpha_n = \Big(1 - \sum_{j:(j,n)\in B} |\beta_{jn}|^2 \cdot \frac{2\sqrt{2}}{C} - \Big\| \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni} + b_n \Big\|^2\Big)^{1/2}.
$$

For $n \le n_s$ set

$$
u_{n,s} = \alpha_n w_n + \sum_{\substack{j:(j,n)\in B\\ j \le n_s}} \beta_{jn} v_{jn} + \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni} + b_n.
$$

Observe that

$$\|u_{n,s}\|^2 = \alpha_n^2 + \sum_{\substack{j\le n_s\\(j,n)\in B}} |\beta_{jn}|^2\cdot\frac{2\sqrt{2}}{C} + \Big\|\sum_{i:(n,i)\in B}\gamma_{ni}z_{ni}+b_n\Big\|^2$$

$$= 1 - \sum_{\substack{j> n_s\\(j,n)\in B}} |\beta_{jn}|^2\cdot\frac{2\sqrt{2}}{C} = 1 - \sum_{\substack{j> n_s\\(j,n)\in B}} |a_{jn}|\cdot\frac{2\sqrt{2}}{C}.$$

Moreover, in view of (4.3) and (4.4), the vectors $u_{1,s},\dots,u_{n_s,s}$ are mutually orthogonal.

This finishes our inductive construction.

(E) Suppose now that the vectors $b_n, w_n, v_{ni}, z_{ni}, n\in\mathbb{N}, (n,i)\in B$, and $y_{r,l}, r,l\ge 0$ have been constructed in the way described above. For $n\in\mathbb{N}$ set

$$u_n = \alpha_n w_n + \sum_{j:(j,n)\in B}\beta_{jn}v_{jn} + \sum_{i:(n,i)\in B}\gamma_{ni}z_{ni} + b_n,$$

and note that

$$u_n = \lim_{s\to\infty} u_{n,s}.$$

So

$$\|u_n\|^2 = \lim_{s\to\infty}\|u_{n,s}\|^2 = \lim_{s\to\infty}\Big(1 - \sum_{\substack{j> n_s\\(j,n)\in B}} |a_{jn}|\frac{2\sqrt{2}}{C}\Big) = 1$$

for every $n\in\mathbb{N}$. Moreover, for any $m\in\mathbb{N}, m\ne n$, we have

$$\langle u_m, u_n\rangle = \lim_{s\to\infty}\langle u_{m,s}, u_{n,s}\rangle = 0.$$

Hence $(u_n)_{n=1}^\infty$ is an orthonormal system in H.

(F) Next we show that $\langle Tu_m, u_n\rangle = a_{nm}, (n,m)\in B$, so that T will have the required matrix with respect to $(u_n)_{n=1}^\infty$ after we prove that $(u_n)_{n=1}^\infty$ is a basis.

Fix $(n,m)\in B$, and note that $m<n$. To evaluate $\langle Tu_m, u_n\rangle$, decompose it as follows:

$$\langle Tu_m, u_n\rangle = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7,$$

where

$$\begin{aligned}
A_1 =&\Big\langle \alpha_m T w_m + \sum_{j':(j',m)\in B} \beta_{j',m} T v_{j',m}, \alpha_n w_n + \sum_{j:(j,n)\in B} \beta_{jn} v_{jn}\Big\rangle,\\
A_2 =&\Big\langle \alpha_m T w_m + \sum_{j':(j',m)\in B} \beta_{j',m} T v_{j',m}, \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni}\Big\rangle,\\
A_3 =&\Big\langle \sum_{i:(m,i)\in B} \gamma_{m,i} T z_{m,i}, \alpha_n w_n + \sum_{j:(j,n)\in B} \beta_{jn} v_{jn}\Big\rangle,\\
A_4 =&\Big\langle \sum_{i':(m,i')\in B} \gamma_{m,i'} T z_{m,i'}, \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni}\Big\rangle,\\
A_5 =&\Big\langle T b_m, \alpha_n w_n + \sum_{j:(j,n)\in B} \beta_{jn} v_{jn}\Big\rangle,\\
A_6 =&\Big\langle T b_m, \sum_{i:(n,i)\in B} \gamma_{ni} z_{ni}\Big\rangle,\\
A_7 =&\langle T u_m, b_n\rangle.
\end{aligned}$$

Using the properties given in (4.3) and (4.4), it is direct to verify that

$$A_1 = A_3 = A_4 = A_5 = A_6 = 0.$$

To evaluate A_7, note that clearly $A_7 = 0$ if $n \notin \{n_1, n_2, \dots\}$. If otherwise $n = n_s$, then $(n,m) \in B$ implies that $m > n_{s-1}$. In particular, $b_m = 0$, so

$$T u_m \in \bigvee\{T w_m, T v_{jm}, T z_{mi} : (j,m) \in B, (m,i) \in B\} \subset \{b_n\}^\perp$$

by our construction. Hence $A_7 = 0$ for all $n \in \mathbb{N}$. Finally, using (4.3), we infer that

$$A_2 = \big\langle \beta_{nm} T v_{nm}, \gamma_{nm} z_{nm}\big\rangle = a_{nm}.$$

and $\langle T u_m, u_n\rangle = a_{nm}$ as required.

(G) It remains to prove that $(u_n)_{n=1}^\infty$ is a basis in H, i.e., that $(u_n)_{n=1}^\infty$ is complete. For $n \in \mathbb{N}$ write $M_n = \bigvee\{u_1, \dots, u_n\}$, and for every $r \geq 0$ let

$$y'_r = \lim_{l\to\infty} y_{r,l},$$

where $y_{r,l}$ are defined in Step A and the limit exists by construction and the initial setting (iii).

We show by induction on l that

$$\operatorname{dist}^2\big\{y_{r,l}, M_{n_{2^r(2l-1)}}\big\} \leq (1-\eta/2)^{l-1} \tag{4.5}$$

for all $l \in \mathbb{N}$. This is clear if $l = 1$. For $l \geq 2$ let $n = n_{2^r(2l-1)}$ and suppose (4.5) is true with l replaced by $l-1$. Then using (4.2), we have

$$\begin{aligned}
\operatorname{dist}^2\{y_{r,l}, M_n\} =&\operatorname{dist}^2\{y_{r,l}, M_{n-1}\} - |\langle y_{r,l}, u_n\rangle|^2\\
=&\operatorname{dist}^2\{y_{r,l}, M_{n-1}\} - |\langle y_{r,l}, b_n\rangle|^2,
\end{aligned}$$

by the construction of u_n. Moreover,

$$|\langle y_{r,l}, b_n\rangle|^2 = \eta\|(I - P_{L_{s-1}})y_{r,l}\|^2 = \eta\|(I - P_{M_{n-1}})y_{r,l}\|^2,$$

where $s = 2^r(2l-1)$. Thus

$$\begin{aligned}\operatorname{dist}^2\{y_{r,l}, M_n\} =&\|(I - P_{M_{n-1}})y_{r,l}\|^2(1-\eta)\\ \le&\big(\|(I - P_{M_{n-1}}y_{r,l-1}\| + \|y_{r,l} - y_{r,l-1}\|\big)^2(1-\eta)\\ \le&\big((1-\eta/2)^{(l-2)/2} + \rho_s\big)^2(1-\eta) \le (1-\eta/2)^{l-1}.\end{aligned}$$

So, for every $r \ge 0$,

$$\operatorname{dist}^2\Big\{y'_r, \bigvee_{n=1}^{\infty} u_n\Big\} = \lim_{l\to\infty} \operatorname{dist}^2\big\{y_{r,l}, M_{2^r(2l-1)}\big\} \le \lim_{l\to\infty}(1-\eta/2)^{l-1} = 0.$$

Hence $y'_r \in \bigvee_{n=1}^{\infty} u_n$ for all $r \ge 0$. Now using (5.4) observe that

$$\sum_{r=0}^{\infty}\|y'_r - y_r\| \le \sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\|y_{r,l+1} - y_{rl}\| < \sum_{s=1}^{\infty}\rho_s < 1.$$

Then by a standard perturbation result for bases, see e.g. [1, Theorem 1.3.9], $(y'_m)_{m=1}^{\infty}$ is a (not necessarily orthonormal) basis in H. Therefore, $\bigvee_{m=0}^{\infty} y'_m = H$, and then $\bigvee_{n=1}^{\infty} u_n = H$ as well. Thus, the vectors $(u_n)_{n=1}^{\infty}$ form an orthonormal basis of H. □

As noted in the introduction, Theorem 1.4 can be formulated also for k-tuples $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$, where none of the operators $T_1, \dots, T_k$ is of the form $\lambda I + K$ with $\lambda \in \mathbb{C}$ and $K \in B(H)$ compact. The arguments given above can be easily adapted to the multi-operator setting. So the proof remains unchanged and we omit it.

5. Prescribing matrix entries: general case

In this section we address a more general setting of operator tuples $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ and prescribing elements within the whole of matrix representation for $\mathcal{T}$ rather than its sub (or upper)-diagonal. We start with proving counterparts of Lemma 4.1. Necessarily they are a bit more involved though based on the same idea.

Lemma 5.1. *Let $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$, $k \in \mathbb{N}$, let $0 \in \operatorname{Int} W_e(T)$, and assume that*

$$\operatorname{dist}\{0, \partial W_e(\mathcal{T})\} > \varepsilon > 0,$$

for some $\epsilon > 0$. Then for any subspace $M \subset H$ of finite codimension, there exist vectors $v, z \in M$ satisfying the following conditions:

(a) $\|v\| = 1$;
(b) $\|z\| \le \frac{2}{\varepsilon}$;
(b) $\langle \mathcal{T}v, v\rangle = 0$;
(c) $z \in \bigvee_{j=1}^{k}\{T_j v, T_j^* v\}$;
(d) $z \perp v$;

(e) $\langle T_1 v, z\rangle = 1$, $\langle T_1^* v, z\rangle = 0$, *and*

$$\langle T_j v, z\rangle = \langle T_j^* v, z\rangle = 0 \qquad \textit{for all } j = 2, \dots, k.$$

Proof. Denote for short $\mathcal{T}^{-1}M := \{x \in H : T_j x \in M, 1 \le j \le k\}$. Since $(\pm\varepsilon, 0, \dots, 0)$ and $(\pm i\varepsilon, 0, \dots, 0) \in \operatorname{Int} W_e(\mathcal{T})$, by Lemma 2.1, (ii) there exists a unit vector $x_1 \in M \cap \mathcal{T}^{-1}M \cap \mathcal{T}^{*-1}M$ such that

$$\langle \mathcal{T} x_1, x_1\rangle = (\varepsilon, 0, \dots, 0).$$

Similarly, there exists a unit vector

$$x_2 \in M \cap \mathcal{T}^{-1}M \cap (\mathcal{T}^*)^{-1}M \cap \{u \in H : u \perp^{(\mathcal{T})} x_1\}$$

such that

$$\langle \mathcal{T} x_2, x_2\rangle = (i\varepsilon, 0, \dots, 0),$$

and there exist unit vectors

$$x_3 \in M \cap \mathcal{T}^{-1}M \cap (\mathcal{T}^*)^{-1}M \cap \{u \in H : u \perp^{(\mathcal{T})} x_1, x_2\}$$

and

$$x_4 \in M \cap \mathcal{T}^{-1}M \cap (\mathcal{T}^*)^{-1}M \cap \{u \in H : u \perp^{(\mathcal{T})} x_1, x_2, x_3\}$$

with

$$\langle \mathcal{T} x_3, x_3\rangle = (-\varepsilon, 0, \dots)$$

and

$$\langle \mathcal{T} x_4, x_4\rangle = (-i\varepsilon, 0, \dots, 0).$$

Let

$$v = \frac{1}{2}(x_1 + x_2 + x_3 + x_4).$$

Then $v \in M \cap \mathcal{T}^{-1}M \cap (\mathcal{T}^*)^{-1}M$, $\|v\| = 1$ and

$$\langle \mathcal{T} v, v\rangle = \frac{1}{4}\Big(\langle \mathcal{T} x_1, x_1\rangle + \langle \mathcal{T} x_2, x_2\rangle + \langle \mathcal{T} x_3, x_3\rangle + \langle \mathcal{T} x_4, x_4\rangle\Big) = 0.$$

Define

$$L := \bigvee_{j=1}^{k} \{T_j v, T_j^* v\}$$

and

$$L' := \bigvee \{T_2 v, \dots, T_k v, T_1^* v, \dots, T_k^* v\}.$$

If we let

$$u := \alpha T_1^* v + \sum_{j=2}^{k} (\beta_j T_j v + \gamma_j T_j^* v),$$

then $u \in L'$, and we have

$$\|T_1 v + u\| \ge |\langle T_1 v + u, x_1\rangle| = \Big|\frac{\langle T_1 x_1, x_1\rangle}{2} + \alpha\langle T_1^* x_1, x_1\rangle\Big| = \Big|\frac{\varepsilon}{2} + \alpha\varepsilon\Big|.$$

Similarly,

$$\|T_1 v + u\| \ge |\langle T_1 v + u, x_2\rangle| = \Big|\frac{\langle T_1 x_2, x_2\rangle}{2} + \alpha\langle T_1^* x_2, x_2\rangle\Big| = \Big|\frac{i\varepsilon}{2} - i\varepsilon\alpha\Big|.$$

So

$$\|T_1v+u\| \geq \varepsilon \max\Big\{\Big|\frac{1}{2}+\alpha\Big|,\Big|\frac{1}{2}-\alpha\Big|\Big\} \geq \frac{\varepsilon}{2}$$

and dist $\{T_1v, L'\} \geq \frac{\varepsilon}{2}$.

Denoting by $P_{L'}$ the orthogonal projection from L onto L', set finally

$$z=\frac{(I-P_{L'})T_1v}{\|(I-P_{L'})T_1v\|^2}.$$

We have $\|(I-P_{L'})T_1v\| = \operatorname{dist}\{T_1v, L'\} \geq \frac{\varepsilon}{2}$ and $\|z\| \leq \frac{2}{\varepsilon}$.

Moreover, $\langle T_1v, z\rangle = 1$ and $z \perp L'$. Thus, v and z satisfy all of the conditions (a)–(e). □ □

Next we generalise Lemma 5.1 by "symmerising" it over all of the elements $T_1, \dots, T_k$ of the tuple $\mathcal{T}$.

Lemma 5.2. *Let $\mathcal{T}=(T_1,\dots,T_k) \in B(H)^k$, $k \in \mathbb{N}$, and let $\lambda \in \operatorname{Int} W_e(\mathcal{T})$. For any $0<\varepsilon<\operatorname{dist}\{\lambda, \partial W_e(\mathcal{T})\}$ and subspace $M \subset H$ of finite codimension, there exist vectors $v_1,\dots,v_k,\tilde v_1,\dots,\tilde v_k, z_1,\dots,z_k,\tilde z_1,\dots,\tilde z_k \in M$ satisfying the following conditions for all $j=1,\dots,k$:*

(a) $\|v_j\|\cdot\|z_j\| \leq \frac{2}{\varepsilon}$ *and* $\|\tilde v_j\|\cdot\|\tilde z_j\| \leq \frac{2}{\varepsilon}$;

(b) $\langle \mathcal{T}v_j, v_j\rangle = \lambda\|v_j\|^2$ *and* $\langle \mathcal{T}\tilde v_j, \tilde v_j\rangle = \lambda\|\tilde v_j\|^2$;

(c) *if* $i \neq j, 1 \leq i,j \leq k$, *then*

$$v_j, \tilde v_j, z_j, \tilde z_j \perp^{(\mathcal{T})} v_i, \tilde v_i, z_i, \tilde z_i,$$

$$\tilde v_j, \tilde z_j \perp^{(\mathcal{T})} v_j, z_j,$$

$$z_r \in \bigvee_{j=1}^{k}\{T_jv_r, T_j^*v_r\},$$

$$\tilde z_r \in \bigvee_{j=1}^{k}\{T_j\tilde v_r, T_j^*\tilde v_r\},$$

$$z_j \perp v_j, T_j^*v_j, T_rv_j, T_r^*v_j, \quad r \neq j,$$

$$\tilde z_j \perp \tilde v_j, T_j\tilde v_j, T_r\tilde v_j, T_r^*\tilde v_j \quad r \neq j,$$

(d) $\langle T_jv_j, z_j\rangle = \langle T_j^*\tilde v_j, \tilde z_j\rangle$=*1.*

Proof. Without loss of generality we may assume that $\lambda=0$. Otherwise, replace $\mathcal{T}$ by $\mathcal{T}-\lambda I$ and construct the vectors $\{v_i, \tilde v_i, z_i, \tilde z_i : 1 \leq i \leq k\}$ for the k-tuple $\mathcal{T}-\lambda I$. Since the vectors $v_1,\dots,v_k,\tilde v_1,\dots,\tilde v_k, z_1,\dots,z_k,\tilde z_1,\dots,\tilde z_k$ are mutually orthogonal, these vectors satisfy all the conditions required for the k-tuple $\mathcal{T}$ as well.

By Lemma 5.1, construct vectors $v_1, z_1 \in M$ satisfying conditions (a)–(d).

Consider the k-tuple $(T_1^*, T_2, \dots, T_k)$. By Lemma 5.1, find vectors

$$\tilde v_1, \tilde z_1 \in M \cap \{u \in H : u \perp^{(\mathcal{T})} v_1, z_1\}$$

satisfying the conditions (a)–(d).

Consider now the k-tuple $(T_2, T_3, \dots, T_k, T_1)$ and using Lemma 5.1 again construct vectors

$$v_2, z_2 \in M \cap \{u \in H : u \perp^{(\mathcal{T})} v_1, z_1, \tilde{v}_1, \tilde{z}_1\}$$

satisfying the conditions (a)–(d).

Continuing this procedure for tuples

$$(T_2^*, T_3, \dots, T_k, T_1), (T_3, \dots, T_k, T_1, T_2), \dots, (T_k^*, T_1, \dots, T_{k-1})$$

we construct a family $\{v_i, \tilde{v}_i, z_i, \tilde{z}_i : 1 \le i \le k\} \subset H$ with the required properties. □

Now we are ready to prove Theorem 1.5, which is one of the main results of this paper. Recall that for $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$, we denote $\|\lambda\|_\infty = \max\{|\lambda_1|, \dots, |\lambda_k|\}$.

Proof of Theorem 1.5. Given $\epsilon > 0$ and an admissible set $B \subset \mathbb{N} \times \mathbb{N}$, let δ be such that

$$0 < \delta < \frac{\varepsilon\sqrt{\varepsilon}}{18k}.$$

For a given array $\{a_{nj} : (n, j) \in B \cup \Delta\} \subset \mathbb{C}^k$ satisfying the size conditions

$$a_{nn} \in \operatorname{Int} W_e(\mathcal{T}), \quad \operatorname{dist}\{a_{nn}, \partial W_e(\mathcal{T})\} > \varepsilon \quad \text{for all } n \in \mathbb{N}$$

and

$$\sum_{j:(n,j)\in B} \|a_{nj}\|_\infty \le \delta \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \sum_{n:(n,j)\in B} \|a_{jn}\|_\infty \le \delta \quad \text{for all } j \in \mathbb{N},$$

we construct an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that

$$\langle \mathcal{T} u_j, u_n \rangle = a_{nj} \qquad \text{for all } n, j \in \mathbb{N}, (n, j) \in B \cup \Delta. \tag{5.1}$$

Similarly to the proof of Theorem 1.4, we will use inductive arguments, and for clarification purposes, we will divide them into several steps.

For $(j, n) \in B \cup \Delta$ let $a_{jn} = (a_{jn}^{(1)}, \dots, a_{jn}^{(k)})$.

Fix the following initial settings:

(i) Without loss of generality we can assume that $\|T_t\| \le 1$ for all $t = 1, \dots, k$.
(ii) We can also assume that B is symmetric, i.e., $(j, n) \in B \iff (n, j) \in B$. Let $B_\Delta := \{(j, n) \in B, j > n\}$.
(iii) Fix η such that $0 < \eta \le \delta$.
(iv) For $s \in \mathbb{N}$ write $s = 2^{r(s)}(2l(s) - 1)$, where the integers $r(s) \ge 0$ and $l(s) \ge 1$ are defined uniquely.
(v) Fix positive numbers $\rho_s, s \in \mathbb{N}$, such that $\sum_{s=1}^\infty \rho_s < 1$ and (4.2) holds.
(vi) Set formally $n_0 = 0$. Fix an increasing sequence $(n_s)_{s=1}^\infty$ such that $n_1 = 1$ and $(n_s, 1), (n_s, 2), \dots, (n_s, n_{s-1}) \notin B_\Delta$ for all $s \ge 2$.

(vii) For $(n,i)\in B_\Delta$ and $t\in\{1,\dots,k\}$ let

$$\begin{aligned}\beta_{ni}^{(t)} &=|a_{ni}^{(t)}|^{1/2}\arg(a_{ni}^{(t)}),\\ \gamma_{ni}^{(t)} &=|a_{ni}^{(t)}|^{1/2},\\ \tilde\beta_{ni}^{(t)} &=|a_{in}^{(t)}|^{1/2}\overline{\arg(a_{in}^{(t)})},\\ \tilde\gamma_{ni}^{(t)} &=|a_{in}^{(t)}|^{1/2}.\end{aligned}$$

The construction is similar to the one in the subdiagonal case, though technical details deviate at many steps. We look for vectors $u_n, n\in\mathbb{N}$, forming the required orthonormal basis by writing them in the form

$$\begin{aligned}u_n = \alpha_n w_n + &\sum_{t=1}^k \sum_{j:(j,n)\in B_\Delta} \big(\beta_{jn}^{(t)} v_{jn}^{(t)} + \tilde\beta_{jn}^{(t)}\tilde v_{jn}^{(t)}\big)\\ &+\sum_{t=1}^k \sum_{i:(n,i)\in B_\Delta} \big(\gamma_{ni}^{(t)} z_{ni}^{(t)} + \tilde\gamma_{ni}^{(t)}\tilde z_{ni}^{(t)}\big) + b_n,\end{aligned}$$

where $w_n, b_n, v_{jn}^{(t)}, \tilde v_{jn}^{(t)}, z_{ni}^{(t)}, \tilde z_{ni}^{(t)} \in H$ with $(j,n),(n,i)\in B_\Delta, t=1,\dots,k$. Similarly to the subdiagonal case, the pairs $v_{j,n}, z_{j,n}$ for $(j,n)\in B_\Delta$ will be constructed in order to have $\langle T_t u_n, u_j\rangle = a_{jn}^{(t)}$ for all $1\le t\le k$, and the pairs $\tilde v_{jn}^{(t)}, \tilde z_{jn}^{(t)}$ will ensure that $\langle T_t u_j, u_n\rangle = a_{nj}^{(t)}, 1\le t\le k$.

As in the subdiagonal case, we fix in advance an orthonormal basis $(y_r)_{r=0}^\infty$ in H. Then for each $r\ge 0$, setting $y_{r,0}=y_r$, we construct a sequence $\{y_{r,l} : l\ge 0\}$ such that $\lim_{l\to\infty} y_{r,l} = y_r'$, $y_r'\in\bigvee_{n=1}^\infty u_n$ and y_r' is close to y_r. From here we derive that $\bigvee_{r=0}^\infty y_r' = H$, hence $\bigvee_{n=1}^\infty u_n = H$ as well.

Arguing inductively we construct vectors

$$b_n, w_n\in H,\quad n\in\mathbb{N}, \text{and}\quad v_{ni}^{(t)}, \tilde v_{ni}^{(t)}, z_{ni}^{(t)}, \tilde z_{ni}^{(t)}\in H,\quad (n,i)\in B_\Delta, t\in\{1,\dots,k\},$$

numbers $\alpha_n\ge 0$, and k-tuples $\lambda_n\in \operatorname{Int} W_e(\mathcal{T})$ as follows:

Let $s\in\mathbb{N}$ and suppose that the vectors

$$w_n, b_n, v_{ni}^{(t)}, \tilde v_{ni}^{(t)}, z_{ni}^{(t)}, \tilde z_{ni}^{(t)}\in H,\qquad n\le n_{s-1}, (n,i)\in B_\Delta, t\in\{1,\dots,k\},$$

vectors $y_{r,l}$, $\;2^r(2l-1)\le s-1$, numbers $\alpha_n\ge 0$ and k-tuples $\lambda_n\in \operatorname{Int} W_e(\mathcal{T})$, $n\le n_{s-1}$, have been already constructed in such a way that if for $1\le n\le n_{s-1}$ one sets

$$\begin{aligned}u_{n,s-1} :=& \alpha_n w_m + \sum_{t=1}^k \sum_{\substack{j\le n_{s-1}\\ (j,n)\in B_\Delta}} \big(\beta_{jn}^{(t)} v_{jn}^{(t)} + \tilde\beta_{jn}^{(t)}\tilde v_{jn}^{(t)}\big)\\ &+\sum_{t=1}^k \sum_{i:(n,i)\in B_\Delta} \big(\gamma_{ni}^{(t)} z_{ni}^{(t)} + \tilde\gamma_{ni}^{(t)}\tilde z_{ni}^{(t)}\big) + b_n,\end{aligned}$$

then the elements

$$u_{1,s-1},\dots,u_{n_{s-1},s-1} \quad \text{are mutually orthogonal} \tag{5.2}$$

and

$$\|u_{n,s-1}\|^2 = 1-\sum_{t=1}^{k}\sum_{\substack{j>n_{s-1}\\(j,n)\in B_\Delta}}(|a_{jn}|+|a_{nj}|)\cdot\frac{4}{\varepsilon^{3/2}}. \tag{5.3}$$

Moreover, we assume that $\|y_{r,l}-y_{r,l-1}\|\le \eta_{2^r(2l-1)}$ for all $r\ge 0, l\ge 1$ with $2^r(2l-1)\le s-1$.

(A) We first define $b_n, n_{s-1}<n\le n_s$:
For $n_{s-1}<n<n_s$ set $b_n=0$. Let $s=2^{r(s)}(2l(s)-1)$, and define

$$L_{s-1} := \bigvee_{n=1}^{n_{s-1}} u_{n,s-1}.$$

Find $y_{r(s),l(s)}\in H\setminus L_{s-1}$ such that $\|y_{r(s),l(s)}\|\le 1$ and

$$\left\|y_{r(s),l(s)}-y_{r(s),l(s)-1}\right\| < \rho_s. \tag{5.4}$$

Then set

$$b_{n_s} := \frac{(I-P_{L_{s-1}})y_{r(s),l(s)}}{\left\|(I-P_{L_{s-1}})y_{r(s),l(s)}\right\|}\cdot\sqrt{\eta}. \tag{5.5}$$

(B) Next, arguing inductively for $n=n_{s-1}+1,\dots,n_s$, we construct vectors $v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)}\in H$ and k-tuples $\lambda_n\in \operatorname{Int} W_e(\mathcal{T})$.

Let $n_{s-1}<n<n_s$ and suppose that the vectors $v_{mi}^{(t)},\tilde v_{mi}^{(t)},z_{mi}^{(t)},\tilde z_{mi}^{(t)}\in H$ and $\lambda_m\in\operatorname{Int} W_e(\mathcal{T})$ have been constructed for all $m<n, (m,i)\in B_\Delta$ and $t\in\{1,\dots,k\}$.

Using Lemma 5.2 repeatedly, find for all $i,(n,i)\in B_\Delta$ and $t=1,\dots,k$ vectors $v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)}\in H$ such that for all $m\le n,(n,i),(m,j)\in$

$B_\Delta, 1\le t,t'\le k, (n,i,t)\ne(m,j,t')$,

$$\|v_{ni}^{(t)}\|=\|\tilde v_{ni}^{(t)}\|=\frac{2}{\varepsilon^{3/4}}, \tag{5.6}$$

$$\|z_{ni}^{(t)}\|,\|\tilde z_{ni}^{(t)}\|\le \varepsilon^{-1/4},$$

$$v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)}\perp^{(\mathcal T)} v_{mj}^{(t')},\tilde v_{mj}^{(t')},z_{mj}^{(t')},\tilde z_{mj}^{(t')}$$

$$\tilde v_{ni}^{(t)},\tilde z_{ni}^{(t)}\perp^{(\mathcal T)} v_{ni}^{(t)},z_{ni}^{(t)},$$

$$v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)}\perp^{(\mathcal T)} b_m,\quad m\le n_s,$$

$$v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)}\perp^{(\mathcal T)} w_m,\quad m\le n_{s-1},$$

$$v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)}\perp^{(\mathcal T)} y_{r,l},\quad 2^r(2l-1)\le s, \tag{5.7}$$

$$z_{ni}^{(t)}\perp v_{ni}^{(t)},\tilde z_{ni}^{(t)}\perp \tilde v_{ni}^{(t)},$$

$$\langle \mathcal T^* v_{ni}^{(t)},z_{ni}^{(t)}\rangle=\langle \mathcal T\tilde v_{ni}^{(t)},\tilde z_{ni}^{(t)}\rangle=0,$$

$$\langle T_t v_{ni}^{(t')},z_{ni}^{t'}\rangle=\langle \mathcal T_t^*\tilde v_{ni}^{(t')},\tilde z_{ni}^{(t')}\rangle=\delta_{t,t'},$$

where $\delta_{t,t'}$ is the Kronecker symbol and

$$\langle \mathcal T v_{ni}^{(t)},v_{ni}^{(t)}\rangle=\langle \mathcal T\tilde v_{ni}^{(t)},\tilde v_{ni}^{(t)}\rangle=\frac{4\lambda_i}{\varepsilon^{3/2}}$$

(note that $i<n$, and so $\lambda_i\in \operatorname{Int} W_e(\mathcal T)$ was already constructed).

Write for short

$$x_n=\sum_{t=1}^k\sum_{i:(n,i)\in B}\big(\gamma_{ni}^{(t)}z_{ni}^{(t)}+\tilde\gamma_{ni}^{(t)}\tilde z_{ni}^{(t)}\big)+b_n.$$

We have

$$\|x_n\|^2=\sum_{t=1}^k\sum_{i:(n,i)\in B_\Delta}(|\gamma_{ni}^{(t)}|^2+|\tilde\gamma_{ni}^{(t)}|^2)+\|b_n\|^2\le 2k\delta+\eta\le(2k+1)\delta\le\frac{\varepsilon}{6}\le\frac16.$$

Thus if we set

$$\lambda_n=\frac{a_{nn}-\langle \mathcal T x_n,x_n\rangle}{1-\|x_n\|^2},$$

then

$$\|\lambda_n-a_{nn}\|_\infty\le\Big\|a_{nn}-\frac{a_{nn}}{1-\|x_n\|^2}\Big\|_\infty+\Big|\frac{\|x_n\|^2}{1-\|x_n\|^2}\Big|\le\frac{2\|x_n\|^2}{1-\|x_n\|^2}\le\frac{\varepsilon/3}{1-\frac16}\le\frac{\varepsilon}{2}.$$

So $\lambda_n\in \operatorname{Int} W_e(T)$ and $\operatorname{dist}\{\lambda_n,\partial W_e(T)\}>\varepsilon/2$.

(C) Suppose that the vectors

$$v_{ni}^{(t)},\tilde v_{ni}^{(t)},z_{ni}^{(t)},\tilde z_{ni}^{(t)},\qquad n\le n_s,(n,i)\in B_\Delta, t=1,\dots,k,$$

and the k-tuples $\lambda_n\in \operatorname{Int} W_e(\mathcal T)$ have been constructed.

Choose inductively vectors $w_n, n_{s-1} < n \le n_s$, satisfying

$$
\begin{aligned}
&\|w_n\| = 1,\\
&\langle \mathcal{T} w_n, w_n\rangle = \lambda_n,\\
&w_n \perp^{(\mathcal{T})} v_{mi}^{(t)}, \tilde{v}_{mi}^{(t)}, z_{mi}^{(t)}, \tilde{z}_{mi}^{(t)}, \quad m \le n_s, (m,i) \in B_\Delta, t = 1, \dots, k,\\
&w_n \perp^{(\mathcal{T})} b_m, \quad m \le n_s,\\
&w_n \perp^{(\mathcal{T})} w_m, \quad m \ne n,\\
&w_n \perp y_{r,l} \quad 2^r(2l-1) \le s.
\end{aligned}
\tag{5.8}
$$

(D) For every n such that $n_{s-1} + 1 \le n \le n_s$ we have

$$\sum_{t=1}^{k} \sum_{j:(j,n)\in B} (|\beta_{jn}^{(t)}|^2 + |\tilde{\beta}_{jn}^{(t)}|^2)\frac{4}{\varepsilon^{3/2}} + \|x_n\|^2 \le \frac{8k\delta}{\varepsilon^{3/2}} + \varepsilon/6 \le 1.$$

Define now

$$\alpha_n = \Big(1 - \sum_{t=1}^{k} \sum_{j:(j,n)\in B} \big(|\beta_{jn}^{(t)}|^2 + |\tilde{\beta}_{jn}^{(t)}|^2\big)\frac{4}{\varepsilon^{3/2}} + \|x_n\|^2\Big)^{1/2}.$$

For $n \le n_s$ set

$$
\begin{aligned}
u_{n,s} = \alpha_n w_n + &\sum_{t=1}^{k} \sum_{\substack{j \le n_s\\ (j,n)\in B_\Delta}} \big(\beta_{jn}^{(t)} v_{jn}^{(t)} + \tilde{\beta}_{jn}^{(t)} \tilde{v}_{jn}^{(t)}\big)\\
+ &\sum_{t=1}^{k} \sum_{i:(n,i)\in B_\Delta} \big(\gamma_{ni}^{(t)} z_{ni}^{(t)} + \tilde{\gamma}_{ni}^{(t)} \tilde{z}_{ni}^{(t)}\big) + b_n.
\end{aligned}
$$

Then the vectors $u_{1,s}, \dots, u_{n_s,s}$ are mutually orthogonal and

$$\|u_{n,s}\|^2 = 1 - \sum_{t=1}^{k} \sum_{\substack{j > n_s\\ (j,n)\in B_\Delta}} (|a_{jn}| + |a_{nj}|)\frac{4}{\varepsilon^{3/2}}.$$

In other words, (5.2) and (5.3) hold with $s-1$ replaced by s.

(E) Suppose that the vectors $w_n, b_n, n \in \mathbb{N}$, and $v_{ni}^{(t)}, \tilde{v}_{ni}^{(t)}, z_{ni}^{(t)}, \tilde{z}_{ni}^{(t)}, (n,i) \in B_\Delta, t = 1, \dots, k$, are constructed.

Set

$$
\begin{aligned}
u_n := \alpha_n w_n + &\sum_{t=1}^{k} \sum_{j:(j,n)\in B_\Delta} \big(\beta_{jn}^{(t)} v_{jn}^{(t)} + \tilde{\beta}_{jn}^{(t)} \tilde{v}_{jn}^{(t)}\big)\\
+ &\sum_{t=1}^{k} \sum_{i:(n,i)\in B_\Delta} \big(\gamma_{ni}^{(t)} z_{ni}^{(t)} + \tilde{\gamma}_{ni}^{(t)} \tilde{z}_{ni}^{(t)}\big) + b_n.
\end{aligned}
\tag{5.9}
$$

Since

$$u_n = \lim_{s\to\infty} u_{n,s}$$

for all $n \in \mathbb{N}$, we have

$$\|u_n\| = \lim_{s\to\infty} \|u_{n,s}\| = 1 \qquad \text{for all} \quad n \in \mathbb{N}.$$

Moreover, for all $m, n \in \mathbb{N}$, $m \neq n$,

$$\langle u_m, u_n\rangle = \lim_{s\to\infty} \langle u_{m,s}, u_{n,s}\rangle = 0.$$

Hence the vectors $(u_n)_{n=1}^\infty$ form an orthonormal system in H.

(F) Let $(n,m) \in B_\Delta$ and $t \in \{1,\dots,k\}$. To evaluate the inner product $\langle T_t u_m, u_n\rangle$ we use definition (5.9) of $(u_n)_{n=1}^\infty$ and decompose $\langle T_t u_m, u_n\rangle$ as

$$\langle T_t u_m, u_n\rangle = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7,$$

where

$$\begin{aligned}
A_1 =& \Big\langle \alpha_m T_t w_m + \sum_{t'=1}^k \sum_{j:(j,m)\in B_\Delta} \big(\beta_{jm}^{(t')} T_t v_{jm}^{(t')} + \tilde\beta_{jm} T_t \tilde v_{jm}^{(t')}\big), \\
&\alpha_n w_n + \sum_{t''=1}^k \sum_{j':(j'n)\in B_\Delta} \big(\beta_{j'n}^{(t'')} v_{j'n}^{(t'')} + \tilde\beta_{j'n}^{(t'')} \tilde z_{j'n}^{(t'')}\big)\Big\rangle, \\
A_2 =& \Big\langle \alpha_m T_t w_m + \sum_{t'=1}^k \sum_{j:(j,m)\in B_\Delta} \big(\beta_{jm}^{(t')} T_t v_{jm}^{(t')} + \tilde\beta_{jm} T_t \tilde v_{jm}^{(t')}\big), \\
&\sum_{t''=1}^k \sum_{i:(n,i)\in B} \big(\gamma_{ni}^{(t'')} z_{ni}^{(t'')} + \tilde\gamma_{ni}^{(t'')} \tilde z_{ni}^{(t'')}\big)\Big\rangle, \\
A_3 =& \Big\langle \sum_{t'=1}^k \sum_{i:(m,i)\in B_\Delta} \big(\gamma_{mi}^{(t')} T_t z_{mi}^{(t')} + \tilde\gamma_{mi}^{(t')} T_t \tilde z_{mi}^{(t')}\big), \\
&\alpha_n w_n + \sum_{t''=1}^k \sum_{j':(j'n)\in B_\Delta} \big(\beta_{j'n}^{(t'')} v_{j'n}^{(t'')} + \tilde\beta_{j'n}^{(t'')} \tilde z_{j'n}^{(t'')}\big)\Big\rangle,
\end{aligned}$$

$$
\begin{aligned}
A_4 =& \Big\langle \sum_{t'=1}^{k} \sum_{i:(m,i)\in B_\Delta} \big(\gamma_{mi}^{(t')} T_t z_{mi}^{(t')} + \tilde{\gamma}_{mi}^{(t')} T_t \tilde{z}_{mi}^{(t')}\big), \\
& \sum_{t''=1}^{k} \sum_{i:(n,i)\in B} \big(\gamma_{ni}^{(t'')} z_{ni}^{(t'')} + \tilde{\gamma}_{ni}^{(t'')} \tilde{z}_{ni}^{(t'')}\big) \Big\rangle, \\
A_5 =& \Big\langle T_t b_m, \alpha_n w_n + \sum_{t''=1}^{k} \sum_{j':(j'n)\in B_\Delta} \big(\beta_{j'n}^{(t'')} v_{j'n}^{(t'')} + \tilde{\beta}_{j'n}^{(t'')} \tilde{z}_{j'n}^{(t'')}\big) \Big\rangle, \\
A_6 =& \Big\langle T_t b_m, \sum_{t''=1}^{k} \sum_{i:(n,i)\in B} \big(\gamma_{ni}^{(t'')} z_{ni}^{(t'')} + \tilde{\gamma}_{ni}^{(t'')} \tilde{z}_{ni}^{(t'')}\big) \Big\rangle, \\
A_7 =& \langle T_t u_m, b_n \rangle.
\end{aligned}
$$

By construction, using the properties of w_m, v_{ni}, and z_{ni} listed in (5.6) and (5.8), we have

$$A_1 = A_3 = A_4 = A_5 = A_6 = 0.$$

Similarly,

$$A_2 = \big\langle \beta_{nm}^{(t)} T_t v_{nm}^{(t)}, \gamma_{nm}^{(t)} z_{nm}^{(t)} \big\rangle = a_{nm}^{(t)}.$$

It remains to consider A_7. Clearly $A_7 = 0$ if $n \notin \{n_1, n_2, \dots\}$. Suppose that $n = n_s$ for some $s \in \mathbb{N}$. Since $(n, m) \in B_\Delta$, we have $m > n_{s-1}$. So $b_m = 0$ and

$$
\begin{aligned}
T_t u_m \in \bigvee \Big\{ & T_t w_m, T_t v_{jm}^{(t')}, T_t \tilde{v}_{jm}^{(t')}, T_t z_{mi}^{(t')}, T_t \tilde{z}_{mi}^{(t')} : \\
& (m,i) \in B_\Delta, (j,m) \in B_\Delta, t' = 1, \dots, k \Big\} \subset \{b_n\}^\perp.
\end{aligned}
$$

So $A_7 = 0$. Thus, summarising the above,

$$\big\langle T_t u_m, u_n \big\rangle = a_{nm}^{(t)}.$$

Similarly,

$$\langle T_t u_n, u_m \rangle = \overline{\langle T_t^* u_m, u_n \rangle} = \overline{\tilde{\beta}_{nm}^{(t)} T_t \tilde{v}_{nm}^{(t)}, \tilde{\gamma}_{nm}^{(t)} \tilde{z}_{nm}^{(t)} \rangle} = a_{m,n}^{(t)}.$$

(G) Finally, for each $n \in \mathbb{N}$, we compute the diagonal elements $\langle \mathcal{T} u_n, u_n \rangle$. We have

$$\begin{aligned}\langle \mathcal{T} u_n, u_n\rangle =& \alpha_n^2 \langle \mathcal{T} w_n, w_n\rangle + \sum_{t=1}^{k} \sum_{j:(j,n)\in B_\Delta} \Big(\langle \mathcal{T} v_{jn}^{(t)}, v_{jn}^{(t)}\rangle \|v_{jn}^{(t)}\|^2 \\ &+ \langle \mathcal{T} v_{jn}^{(t)}, v_{jn}^{(t)}\rangle \Big) \|v_{jn}^{(t)}\|^2 + \langle \mathcal{T} x_n, x_n\rangle \\ =& \lambda_n \Big(\alpha_n^2 + \sum_{t=1}^{k} \sum_{j:(j,n)\in B_\Delta} (|a_{jn}| + |a_{nj}|) \frac{4}{\varepsilon^{3/2}} \Big) + \langle \mathcal{T} x_n, x_n\rangle \\ =& \lambda_n (1 - \|x_n\|^2) + \langle \mathcal{T} x_n, x_n\rangle \\ =& a_{nn}.\end{aligned}$$

Thus, $(u_n)_{n=1}^\infty$ satisfies (5.1).

(H) It remains to show that $(u_n)_{n=1}^\infty$ is a basis in H. The argument in this step is analogous to the corresponding argument in the proof of Theorem 1.4, step G, and its sketch below is given for completness.

Fix $r \ge 0$ and let

$$y_r' = \lim_{l\to\infty} y_{r,l},$$

where the sequence $\{y_{r,l} : l \ge 0\}$ is defined in Step A, and the limit exists in view of the initial setting (v). Setting $M_n = \bigvee\{u_1, \dots, u_n\}, n \in \mathbb{N}$, we show by induction on l that

$$\operatorname{dist}^2\big\{y_{r,l}, M_{n_{2^r(2l-1)}}\big\} \le (1 - \eta/2)^{l-1} \tag{5.10}$$

for all $l \in \mathbb{N}$. Since the inequality is obvious if $l = 1$, we let $l \ge 2$ and $n = n_{2^r(2l-1)}$. Assuming that (5.10) is true with l replaced by $l - 1$, as in the step G from the proof of Theorem 1.4, we infer that

$$\begin{aligned}\operatorname{dist}^2\{y_{r,l}, M_n\} =& \operatorname{dist}^2\{y_{r,l}, M_{n-1}\}(1-\eta) \\ \le& \big((1-\eta/2)^{(l-2)/2} + \rho_s\big)^2 (1-\eta) \\ \le& (1-\eta/2)^{l-1}.\end{aligned}$$

From here it follows that

$$\operatorname{dist}^2\Big\{y_r', \bigvee_{n=1}^{\infty} u_n\Big\} = \lim_{l\to\infty} \operatorname{dist}^2\big\{y_{r,l}, M_{2^r(2l-1)}\big\} \le \lim_{l\to\infty} (1-\eta/2)^{l-1} = 0,$$

hence $y_r' \in \bigvee_{n=1}^\infty u_n$. Since, taking into account (5.4),

$$\sum_{r=0}^{\infty} \|y_r' - y_r\| \le \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \|y_{r,l+1} - y_r\| < \sum_{s=1}^{\infty} \rho_s < 1,$$

we conclude, similarly to the proof of Theorem 1.4, step G, that $(y_r')_{r=0}^\infty$ is a basis by [1, Theorem 1.3.9]. Hence $\bigvee_{n=1}^\infty u_n = H$, so that the vectors $(u_n)_{n=1}^\infty$ form an orthonormal basis in H. This finishes the proof. $\square$

6. Tuples consisting of powers of a single operator

Theorem 1.5 can be directly applied to any k-tuple $(T, T^2, \dots, T^k)$, where $T \in B(H)$ satisfies $\operatorname{Int} \hat{\sigma}(T) \neq \emptyset$. By Theorem 2.2,

$$(\lambda, \lambda^2, \dots, \lambda^k) \in \operatorname{Int} W_e(T, \dots, T^k)$$

for all $\lambda \in \operatorname{Int} \hat{\sigma}(T)$. In particular we have the following corollary.

Corollary 6.1. *Let $T \in B(H)$ satisfy $0 \in \operatorname{Int} \hat{\sigma}(T)$, and $k \in \mathbb{N}$. Then there exists $\delta > 0$ with the following property: if $B \subset (\mathbb{N} \times \mathbb{N}) \setminus \Delta$ is admissible, and $\{a_{nj} = (a_{nj}^{(1)}, \dots, a_{nj}^{(k)}) : (n,j) \in B \cup \Delta\} \subset \mathbb{C}^k$ is such that $\sum_{j:(n,j) \in B \cup \Delta} \|a_{nj}\|_\infty \leq \delta$ for all $n \in \mathbb{N}$, and $\sum_{n:(n,j) \in B \cup \Delta} \|a_{jn}\|_\infty \leq \delta$ for all $j \in \mathbb{N}$, then there is an orthonormal basis $(u_n)_{n=1}^\infty$ in H satisfying*

$$\langle T^t u_j, u_n \rangle = a_{nj}^{(t)}$$

for all $t = 1, \dots, k$ and $n, j \in \mathbb{N}$ with $(n,j) \in B \cup \Delta$.

In particular, there exists an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that

$$\langle T^t u_j, u_n \rangle = 0$$

for all $t = 1, \dots, k$ and $n, j \in \mathbb{N}$ with $(n,j) \in B \cup \Delta$.

It is a standard intuition behind many properties in operator theory that if $T \in B(H)$ is invertible then the operators T and T^{-1} cannot be "small" at the same time. However, as far as matrix representations are concerned our technique allows one to get several statements which seem to oppose this general principle. One statement of this kind concerns the size of matrix elements for T and T^{-1}.

Recall that as it was proved in [40, Theorems 2.5 and 6.2], for $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ one has $0 \in W_e(\mathcal{T})$ if and only if for any $(a_n)_{n=1}^\infty \subset (0, \infty)$ satisfying $(a_n)_{n=1}^\infty \notin \ell^1(\mathbb{N})$ there exists an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that

$$|\langle T_t u_n, u_j \rangle| \leq \sqrt{a_n a_j}$$

for all $n, j \in \mathbb{N}$ and $t = 1, \dots, k$.

Then, given an invertible $T \in B(H)$ and considering the tuple $\mathcal{T} = (T^{-1}, T)$ the next statement was obtained ([40, Theorem 6.3]).

Theorem 6.2. *Let $T \in B(H)$ be an invertible operator such that there exists a nonzero $\lambda \in \mathbb{C}$ satisfying $\{\lambda, -\lambda\} \subset \sigma_e(T)$. Then for any $(a_n)_{n=1}^\infty \subset (0, \infty)$ satisfying $(a_n)_{n=1}^\infty \notin \ell^1(\mathbb{N})$ there exists an orthonormal basis $(u_n)_{n=1}^\infty \subset H$ such that*

$$|\langle T u_n, u_j \rangle| \leq \sqrt{a_n a_j} \qquad \text{and} \qquad |\langle T^{-1} u_n, u_j \rangle| \leq \sqrt{a_n a_j}$$

for all $n, j \in \mathbb{N}$.

Developing this interesting effect a bit further, we will consider more general $(2k)$-tuples $\mathcal{T}$ of the form

$$\mathcal{T} = (T^{-k}, \dots, T^{-1}, T, \dots, T^k), \quad k \in \mathbb{N}, \tag{6.1}$$

where $T \in B(H)$ is an invertible operator. We will be interested in providing sparse matrix representations for the elements of $\mathcal{T}$ with respect to the same basis and for the same admissible set $B \subset \mathbb{N} \times \mathbb{N}$. It is instructive to recall that the set B can be quite large.

To deduce them from Theorem 1.5, note first that

$$\sigma_e(\mathcal{T}) = \{(\lambda^{-k}, \dots, \lambda^{-1}, \lambda, \dots, \lambda^k) : \lambda \in \sigma_e(T)\}$$

by the spectral mapping theorem for the essential spectrum, see [34, Corollary 30.11]. Since the spectral mapping theorem in this generality is quite involved we offer a simple argument for our particular situation. To this aim and to make a link to the studies in [40], recall that $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k \in \sigma_e(\mathcal{T})$ if and only if there exists a sequence of unit vectors $(x_n)_{n=1}^\infty \in H$ converging weakly to zero such that either $\|T_t x_n - \lambda_t x_n\| \to 0$, $n \to \infty$, for every $1 \le t \le k$, or $\|T_t^* x_n - \bar{\lambda}_t x_n\| \to 0$, $n \to \infty$, $1 \le t \le k$, see e.g. [11, p. 122-123]. Rather than $\|x_n\| = 1, n \in \mathbb{N}$, it suffices to require $\inf_n \|x_n\| > 0$. On the other hand, the sequence $(x_n)_{n=1}^\infty$ can be chosen orthonormal, see e.g. [11, p. 123]. The next simple lemma uses the case $k = 1$ of the above, and addresses just one implication needed in the sequel.

Lemma 6.3. *Let $T \in B(H)$ be an invertible operator, $k \in \mathbb{N}$, and let $\mathcal{T}$ be given by* (6.1)*. If $\lambda \in \sigma_e(T)$, then*

$$(\lambda^{-k}, \dots, \lambda^{-1}, \lambda, \dots, \lambda^k) \in \sigma_e(\mathcal{T}).$$

Proof. Let $(x_n)_{n=1}^\infty \subset H$ be a sequence converging weakly to 0 such that $\inf_{n \ge 1} \|x_n\| > 0$ and either $\|(T - \lambda)x_n\| \to 0$ or $\|(T^* - \bar{\lambda})x_n\| \to 0$ as $n \to \infty$.

In the first case let $y_n = \lambda^k T^k x_n, n \ge 1$. Then $\inf_n \|y_n\| > 0$ since T is invertible, $y_n \to 0, n \to \infty$, weakly and

$$(T^j - \lambda^j)y_n = \lambda^k T^k (T^{j-1} + \lambda T^{j-2} + \dots + \lambda^{j-1})(T - \lambda)x_n \to 0$$

for all $j = 1, \dots, k$. Furthermore, $(T^{-j} - \lambda^{-j})y_n = \lambda^{k-j} T^{k-j} (\lambda^j - T^j)x_n \to 0$ for all $j = 1, \dots, k$. So

$$(\lambda^{-k}, \dots, \lambda^{-1}, \lambda, \dots, \lambda^k) \in \sigma_e(\mathcal{T}).$$

In the second case set $z_n = \bar{\lambda}^k T^{*k} x_n$. We have similarly $(T^{*j} - \bar{\lambda}^j)z_n \to 0, n \to \infty$, for all $j = \pm 1, \dots, \pm k$, hence again $(\lambda^{-k}, \dots, \lambda^{-1}, \lambda, \dots, \lambda^k) \in \sigma_e(\mathcal{T})$.

□

To be able to apply Theorem 1.5 to $(2k)$-tuples $\mathcal{T}$ of the form (6.1), we should ensure that $0 \in \operatorname{Int} W_e(\mathcal{T})$, and the statement below provides $\mathcal{T}$ with this property under natural geometric spectral assumptions on T.

Proposition 6.4. *Let $T \in B(H)$ be an invertible operator, and let $s > r > 0$ be fixed. Suppose that $\{re^{i\varphi}, se^{i\varphi} : 0 \le \varphi < 2\pi\} \subset \sigma_e(T)$. Let $k \in \mathbb{N}$ and $\mathcal{T} = (T^{-k}, \dots, T^{-1}, T, \dots, T^k) \in B(H)^k$. Then $0 \in \operatorname{Int} W_e(\mathcal{T})$.*

Proof. By Theorem 2.2, there exists $a > 0$ such that if $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ satisfies $\|z\|_\infty < a$, then $z \in \operatorname{conv}\{(\lambda, \dots, \lambda^k) : |\lambda| = r\}$. So for all $z = (z_1, \dots, z_k)$ with $\|z\|_\infty \le a$, using Lemma 6.3, one has

$$\Big(\frac{\bar{z}_k^k}{r^{2k}}, \dots, \frac{\bar{z}_1}{r^2}, z_1, \dots, z_k\Big) \in \operatorname{conv}\sigma_e(\mathcal{T}).$$

Similarly, there exists $a' > 0$ such that all $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ with $\|z\|_\infty \le a'$ satisfy $z \in \operatorname{conv}\{\lambda, \dots, \lambda^k) : |\lambda| = s\}$, hence

$$\Big(\frac{-\bar{z}_k}{s^{2k}}, \dots, \frac{-\bar{z}_1}{s^2}, -z_1, \dots, -z_k\Big) \in \operatorname{conv}\sigma_e(\mathcal{T}).$$

Therefore, for all $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ with $\|z\|_\infty \le \min\{a, a'\}$ we have

$$\Big(\frac{\bar{z}_k}{2r^{2k}} - \frac{\bar{z}_k}{2s^{2k}}, \dots, \frac{\bar{z}_1}{2r^2} - \frac{\bar{z}_1}{2s^2}, 0, \dots, 0\Big) \in \operatorname{conv}\sigma_e(\mathcal{T}) \subset W_e(\mathcal{T}).$$

since $W_e(\mathcal{T})$ is a convex set. So $(z_{-k}, \dots, z_{-1}, 0, \dots, 0) \in W_e(\mathcal{T})$ for all $z_{-k}, \dots, z_{-1} \in \mathbb{C}$ with sufficiently small $\max\{|z_{-j}| : 1 \le j \le k\}$.

Similarly, $(0, \dots, 0, z_1, \dots, z_k) \in W_e(\mathcal{T})$ for all $z_1, \dots, z_k \in \mathbb{C}$ with $\max\{|z_j| : 1 \le j \le k\}$ small enough. Combining these two inclusions and using the convexity of $W_e(\mathcal{T})$ again, we obtain that $(0, \dots, 0) \in \operatorname{Int} W_e(\mathcal{T})$. □

This together with Theorem 1.5 proves the following corollary.

Corollary 6.5. *Let $T \in B(H)$ be an invertible operator, and $B \subset (\mathbb{N}\times\mathbb{N})\setminus\Delta$ be an admissible set. Assume that there exist $s > r > 0$ such that $s\mathbb{T} \cup r\mathbb{T} \subset \sigma_e(T)$, where $\mathbb{T}$ stands for the unit circle. Then for every $k \in \mathbb{N}$ there exists an orthonormal basis $(u_n)_{n\in\mathbb{N}}$ in H such that*

$$\langle T^t u_j, u_n\rangle = 0$$

for all $t = \pm1, \dots, \pm k$ and $(n, j) \in B \cup \Delta$.

Simple examples enjoying spectral assumptions in Corollary 6.5 can be found already within the class of normal operators. In particular, one may consider a multiplication operator $Mf(z) = zf(z)$ on $L^2(S, d\mu)$, where $S \subset \mathbb{C}$ is a Borel set containing $r\mathbb{T}$ and $s\mathbb{T}, s > r > 0$, and μ is a Borel measure on S whose essential support contains $r\mathbb{T}$ and $s\mathbb{T}$ as well. Of course, more general examples of this kind can be provided by replacing z with a function $g \in L^\infty(S, d\mu)$. Even in this case the existence of sparse representations provided by Corollary 6.5 is far from being obvious.

Another class of examples of operators fitting Corollary 6.5 is provided by invertible composition operators C considered e.g. in [45]. If the spectral radius $r(C)$ of C equals s and $r(C^{-1}) = r^{-1}$, then by rotation invariance ([45, Theorem B]) we have $\partial\sigma(C) \supset s\mathbb{T}$ and $\partial\sigma(C^{-1}) \supset r^{-1}\mathbb{T}$ whenever r and s are different from 1. So $s\mathbb{T} \subset \sigma_e(T)$ and $r\mathbb{T} \subset \sigma_e(T)$ since $s\mathbb{T}$ and $r\mathbb{T}$ belong to $\partial\sigma(C)$. A concrete example of such a composition operator on $L^2(0, 1)$ is considered e.g. in [45, Example (2)]. A similar and quite general example now addressing composition operators on the Hardy space $H^2(\mathbb{D})$

is analysed in [41, Theorem 6]. In the framework of composition operators, the availability of sparse representations seem to be also highly nontrivial

Remark finally that similar applications of our results can be provided in the setting of subdiagonal arrays B, but we omit easy formulations of the corresponding corollaries in this setting.

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Institute of Mathematics, Czech Academy of Sciences, ul. Žitna 25, Prague, Czech Republic
Email address: muller@math.cas.cz

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich str.8, 00-656 Warsaw, Poland
Email address: ytomilov@impan.pl