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Abstract

In numerical simulations a smooth domain occupied by a fluid has to be approximated by a computational domain that typically does not coincide with a physical domain. Consequently, in order to study convergence and error estimates of a numerical method domain-related discretization errors, the so-called variational crimes, need to be taken into account.

In this paper we present an elegant alternative to a direct, but rather technical, analysis of variational crimes by means of the penalty approach. We embed the physical domain into a large enough cubed domain and study the convergence of a finite volume method for the corresponding domain-penalized problem. We show that numerical solutions of the penalized problem converge to a generalized, the so-called dissipative weak, solution of the original problem. If a strong solution exists, the dissipative weak solution emanating from the same initial data coincides with the strong solution. In this case, we apply a novel tool of the relative energy and derive the error estimates between the numerical solution and the strong solution. Extensive numerical experiments that confirm theoretical results are presented.

Keywords: compressible Navier–Stokes system, convergence, error estimates, finite volume method, penalization method, dissipative weak solution, relative energy

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1 Introduction

In this paper, we study compressible fluid flow in a smooth physical domain Ω^f modeled by Navier–Stokes equations

Navier–Stokes problem on a smooth domain Ω^f

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}), \tag{1.1b}$$

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \frac{2}{d} \mathrm{div}_x \boldsymbol{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \boldsymbol{u} \mathbb{I}, \ \mu > 0, \ \lambda \geq 0.$$

Dirichlet boundary condition

$$\boldsymbol{u}|_{\partial \Omega^f} = 0, \quad \Omega^f \subset \mathbb{R}^d, \ d = 2, 3. \tag{1.1c}$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0, \ \varrho \boldsymbol{u}(0, \cdot) = \boldsymbol{m}_0. \tag{1.1d}$$

Here, ϱ and $\boldsymbol{u}$ are the fluid density and velocity, respectively. Moreover, the constants μ and λ are the viscosity coefficients. For the sake of simplicity, we consider the *isentropic* state equation:

$$p(\varrho) = a \varrho^\gamma, \ a > 0, \ \gamma > 1 \text{ with the associated pressure potential } \mathcal{P}(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma. \tag{1.2}$$

Throughout the paper we always assume that the initial data satisfy

$$\varrho_0 \geq \underline{\varrho} > 0, \quad \varrho_0 \in L^\infty(\Omega^f), \quad \boldsymbol{m}_0 \in L^\infty(\Omega^f; \mathbb{R}^d). \tag{1.3}$$

On the one hand, mathematical analysis of the Navier–Stokes system (1.1) is currently available either for periodic boundary conditions or for no-slip boundary conditions applied in a smooth physical domain occupied by a fluid. On the other hand, numerical methods, e.g., finite volume or finite element methods, typically require a polygonal computational domain.

Hence, a smooth physical domain has to be approximated by a polygonal computational domain and additional approximation errors arise. Let us note that in the case of complicated geometry the generation of a suitable polygonal approximation may be computationally very costly.

To overcome these difficulties we apply a penalty method originally used in the context of incompressible Navier–Stokes equations by Angot et al. [2]. Thus, the physical domain Ω^f is embedded into a large cube on which the periodic boundary conditions are imposed, see Figure 1. The original boundary conditions are enforced through a penalty term, represented by a singular friction term in the momentum equation. The resulting penalized problem (1.4a)–(1.4c) is solved on a flat torus $\mathbb{T}^d$ by an upwind-type finite volume (FV) method.

Penalized Navier–Stokes system on $\mathbb{T}^d$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0, \tag{1.4a}$$

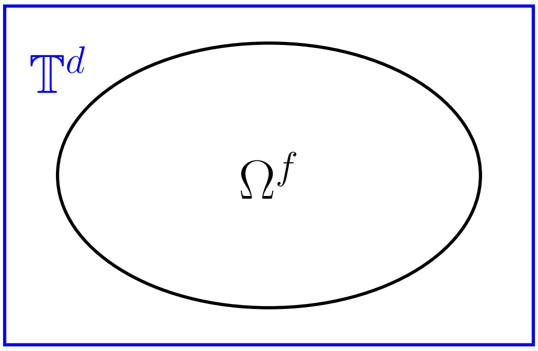


Figure 1: A fluid domain Ω^f embed into a torus $\mathbb{T}^d$.

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}) - \frac{\mathbb{1}_{\Omega^s}}{\epsilon_s} \boldsymbol{u}, \tag{1.4b}$$

where

$$\mathbb{1}_{\Omega^s}(x) = \begin{cases} 1, & \text{if } x \in \Omega^s := \mathbb{T}^d \setminus \Omega^f, \\ 0, & \text{if } x \in \Omega^f. \end{cases}$$

Boundary conditions: periodic boundary condition
Initial conditions

$$\widetilde{\varrho}(0,\cdot) := \widetilde{\varrho}_0 = \begin{cases} \text{some } \varrho_0^s, & \text{if } x \in \Omega^s, \\ \varrho_0, & \text{if } x \in \Omega^f, \end{cases} \quad \widetilde{\boldsymbol{m}}(0,\cdot) := \widetilde{\boldsymbol{m}}_0 = \begin{cases} 0, & \text{if } x \in \Omega^s, \\ \varrho_0 \boldsymbol{u}_0, & \text{if } x \in \Omega^f. \end{cases}$$

$$\widetilde{\varrho}_0 > 0 \quad \text{satisfying the periodic boundary condition,} \tag{1.4c}$$

The idea to penalize a complicated physical domain and solve numerically the corresponding problem on a simple domain is quite often used in the literature. We refer a reader to [18, 19] for the immersed boundary method and to [11, 12, 15] for the fictitious domain method developed in the context of incompressible Navier–Stokes equations. In [4] a penalization method has been applied to approximate a moving domain in the fluid-structure interaction problem. Further, in [13, 14] penalization of boundary conditions for the compressible Navier–Stokes-Fourier system was applied for the spectral method. Error estimates between exact and penalized numerical solutions were presented in [2] for the incompressible Navier–Stokes equations and in [17, 20, 21, 22] for elliptic boundary problems. We mention also Basarić et al. [3] and Feireisl et al. [10], where the penalization method was used to prove the existence of weak solutions. In [10] the penalization method has been used to show the existence of a weak solution to the compressible Navier–Stokes equations on a moving domain and in [3] the existence of a weak solution to the Navier–Stokes–Fourier system with the Dirichlet conditions on a rough (Lispchitz) domain was proved.

The present paper is organized in the following way. We introduce the concept of generalized, the so-called *dissipative weak* solution for the Dirichlet boundary problem (1.1) and the corresponding penalized problem (1.4) in Section 2. Numerical method, *the finite volume method* (3.4) for the approximation of the penalized problem (1.4) is introduced in Section 3. Section 4 and Section 5 are devoted to the main results of the paper: convergence analysis of the finite volume method as well as error estimates between the finite volume solutions of the penalized problem and the exact strong solution of the Navier–Stokes system with the Dirichlet boundary conditions. Here we consider the errors with respect to the mesh (discretization) parameter as well as the penalization parameter. The paper is closed with Section 6, where several numerical experiments illustrate our theoretical results.

2 Dissipative weak solution

Following [8] we introduce the *dissipative weak (DW) solutions* to the Navier–Stokes system. We consider both, the penalized problem on $\mathbb{T}^d$ (1.4) and the original Dirichlet problem on Ω^f (1.1). We will show in Section 4 that the DW solutions arise as a natural limit of numerical approximations and build a suitable tool for the convergence analysis.

Definition 2.1 (DW solution of the penalized problem)**.** *We say that $(\varrho, \boldsymbol{u})$ is a DW solution of the Navier–Stokes system (1.4) if the following hold:*

- **Integrability.**

$$\begin{aligned} &\varrho \geq 0, \ \ \varrho \in L^\infty(0,T;L^\gamma(\mathbb{T}^d)), \quad \boldsymbol{u} \in L^2(0,T;W^{1,2}(\mathbb{T}^d;\mathbb{R}^d)), \\ &\varrho\boldsymbol{u} \in L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;\mathbb{R}^d)), \quad \mathbb{S} \in L^2((0,T)\times\mathbb{T}^d;\mathbb{R}^{d\times d}_{sys}). \end{aligned} \tag{2.1}$$

- **Energy inequality.**

$$\begin{aligned} &\left[\int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \mathcal{P}(\varrho)\right) \mathrm{d}x\right](\tau,\cdot) + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} \,\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{\epsilon_s}\int_0^\tau \int_{\Omega^s} |\boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} d\mathfrak{E}(\tau) + \int_0^\tau \int_{\mathbb{T}^d} d\mathfrak{D}(\tau)\mathrm{d}t \leq \int_{\mathbb{T}^d} \left(\frac{|\widetilde{\boldsymbol{m}}_0|^2}{2\widetilde{\varrho}_0} + \mathcal{P}(\widetilde{\varrho}_0)\right) \mathrm{d}x \end{aligned} \tag{2.2}$$

  for any $\tau \in [0,T]$ and the energy defect measures

$$\mathfrak{E} \in L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^d)), \quad \mathfrak{D} \in \mathcal{M}^+((0,T)\times\mathbb{T}^d).$$

- **Equation of continuity.**

$$-\int_{\mathbb{T}^d} \widetilde{\varrho}_0 \phi(0,\cdot)\,\mathrm{d}x = \int_0^T \int_{\mathbb{T}^d} (\varrho\partial_t\phi + \varrho\boldsymbol{u}\cdot\nabla_x\phi)\,\mathrm{d}x\mathrm{d}t \tag{2.3}$$

  for any test function $\phi \in C^1([0,T)\times\mathbb{T}^d)$.

- **Momentum equation.**

$$\begin{aligned} -\int_{\mathbb{T}^d} \widetilde{\boldsymbol{m}}_0 \cdot \boldsymbol{\phi}(0,\cdot)\,\mathrm{d}x &= \int_0^T \int_{\mathbb{T}^d} (\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{\phi} + p(\varrho)\mathrm{div}_x\boldsymbol{\phi})\,\mathrm{d}x\mathrm{d}t \\ &\quad - \frac{1}{\epsilon_s}\int_0^T \int_{\Omega^s} \boldsymbol{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x\boldsymbol{u}) : \nabla_x\boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \nabla_x\boldsymbol{\phi} : d\mathfrak{R}(t)\,\mathrm{d}x \end{aligned} \tag{2.4}$$

for any test function $\boldsymbol{\phi} \in C^1([0,T) \times \mathbb{T}^d; \mathbb{R}^d)$ *with Reynolds defect*

$$\mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\mathbb{T}^d; \mathbb{R}^{d\times d}_{sym}))$$

satisfying

$$\underline{d}\mathfrak{E} \le tr[\mathfrak{R}] \le \overline{d}\mathfrak{E} \quad \text{for some constants } 0 < \underline{d} \le \overline{d}. \tag{2.5}$$

Definition 2.2 (DW solution of the Dirichlet problem)**.** *We say that* $(\varrho, \boldsymbol{u})$ *is a DW solution of the Navier–Stokes system* (1.1) *if the following hold:*

- **Integrability.**

$$\begin{aligned} &\varrho \ge 0, \ \ \varrho \in L^\infty(0,T; L^\gamma(\Omega^f)), \quad \boldsymbol{u} \in L^2(0,T; W^{1,2}(\Omega^f; \mathbb{R}^d)), \\ &\varrho\boldsymbol{u} \in L^\infty([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega^f; \mathbb{R}^d)), \quad \mathbb{S} \in L^2((0,T)\times\Omega^f; \mathbb{R}^{d\times d}_{sys}). \end{aligned} \tag{2.6}$$

- **Energy inequality.**

$$\begin{aligned} &\left[\int_{\Omega^f} \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \mathcal{P}(\varrho)\right) \mathrm{d}x\right](\tau,\cdot) + \int_0^\tau \int_{\Omega^f} \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} \,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{\overline{\Omega^f}} d\mathfrak{E}(\tau) + \int_0^\tau \int_{\overline{\Omega^f}} d\mathfrak{D}(\tau)\mathrm{d}t \le \int_{\Omega^f} \left(\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_0} + \mathcal{P}(\varrho_0)\right) \mathrm{d}x \end{aligned} \tag{2.7}$$

for any $\tau \in [0,T]$ *with the energy defect measure*

$$\mathfrak{E} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega^f})), \quad \mathfrak{D} \in \mathcal{M}^+((0,T)\times\overline{\Omega^f}).$$

- **Equation of continuity.**

$$-\int_{\Omega^f} \varrho_0 \cdot \phi(0,\cdot)\,\mathrm{d}x = \int_0^T \int_{\Omega^f} \left(\varrho\partial_t\phi + \varrho\boldsymbol{u}\cdot\nabla_x\phi\right) \mathrm{d}x\mathrm{d}t \tag{2.8}$$

for any test function $\phi \in C^1_c([0,T) \times \overline{\Omega^f})$.

- **Momentum equation.**

$$\begin{aligned} -\int_{\Omega^f} \boldsymbol{m}_0 \cdot \boldsymbol{\phi}(0,\cdot)\,\mathrm{d}x &= \int_0^T \int_{\Omega^f} \left(\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{\phi} + p(\varrho)\mathrm{div}_x\boldsymbol{\phi}\right) \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^T \int_{\Omega^f} \mathbb{S}(\nabla_x\boldsymbol{u}) : \nabla_x\boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\overline{\Omega^f}} \nabla_x\boldsymbol{\phi} : d\mathfrak{R}(t)\mathrm{d}t \end{aligned} \tag{2.9}$$

for any test function $\boldsymbol{\phi} \in C^1_c([0,T) \times \overline{\Omega^f}; \mathbb{R}^d)$ *with Reynolds defect*

$$\mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega^f}; \mathbb{R}^{d\times d}_{sym}))$$

satisfies

$$\underline{d}\mathfrak{E} \le tr[\mathfrak{R}] \le \overline{d}\mathfrak{E} \quad \text{for some constants } 0 < \underline{d} \le \overline{d}. \tag{2.10}$$

3 Numerical scheme

In this section we generalize the finite volume method proposed in [7] to approximate the penalized problem (1.4).

3.1 Space discretization

Mesh. Let $\mathcal{T}_h$ be a uniform structured (square for $d = 2$ or cuboid for $d = 3$) mesh of $\mathbb{T}^d$ with h being the mesh parameter. We denote by $\mathcal{E}$ the set of all faces of $\mathcal{T}_h$ and by $\mathcal{E}_i$, $i = 1, \ldots, d$, the set of all faces that are orthogonal to $\boldsymbol{e}_i$ – the basis vector of the canonical system. Moreover, we denote by $\mathcal{E}(K)$ the set of all faces of a generic element $K \in \mathcal{T}_h$. Then, we write $\sigma = K|L$ if $\sigma \in \mathcal{E}$ is the common face of neighbouring elements K and L. Further, we denote by x_K and $|K| = h^d$ (resp. x_σ and $|\sigma| = h^{d-1}$) the center and the Lebesgue measure of an element $K \in \mathcal{T}_h$ (resp. a face $\sigma \in \mathcal{E}$), respectively.

With the above notations, we further define Ω_h^f as the set of all elements inside the physical domain Ω^f, i.e.

$$\Omega_h^f = \left\{ K \mid K \subset \Omega^f \right\} \quad \text{and} \quad \Omega_h^s = \mathcal{T}_h \setminus \Omega_h^f,$$

cf. Figure 3. It yields $\Omega_h^f \subset \Omega^f$ and $\Omega^s \subset \Omega_h^s$. Here and hereafter, we assume that

$$\operatorname{dist}(\partial\Omega^f, \Omega_h^f) \sim h, \tag{3.1}$$

which gives $|\Omega^f \setminus \Omega_h^f| \sim h$ and $|\Omega_h^s \setminus \Omega^s| \sim h$.

Dual mesh. For any $\sigma = K|L \in \mathcal{E}_i$, we define a dual cell $D_\sigma := D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) is defined as

$$D_{\sigma,K} = \{x \in K \mid x_i \in \text{co}\{(x_K)_i, (x_\sigma)_i\}\} \text{ for any } \sigma \in \mathcal{E}_i,\ i = 1, \ldots, d,$$

with $\text{co}\{A, B\} \equiv [\min\{A, B\}, \max\{A, B\}]$. We refer to Figure 2 for a two-dimensional illustration of a dual cell.

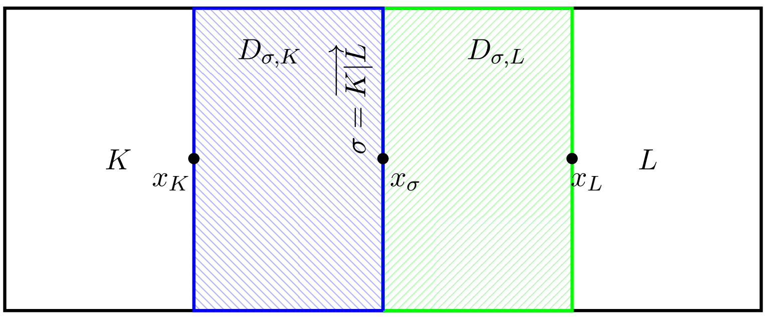


Figure 2: Dual mesh $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$

Function space. The symbol Q_h stands for the set of piecewise constant functions on the grid $\mathcal{T}_h$. Note that hereafter $\boldsymbol{v}_h \in (Q_h)^d$ means that every component of a vector–valued function $\boldsymbol{v}_h$ belongs to the set Q_h. The projection of any $\phi \in L^1(\mathbb{T}^d)$ onto the space Q_h is given by

$$\Pi_{\mathcal{T}}\phi(x) = \sum_{K\in\mathcal{T}_h} \frac{\mathbb{1}_K(x)}{|K|} \int_K \phi \, \mathrm{d}x.$$

Further, we use the following notations for the average and jump operators for $v \in Q_h$

$$\{\!\{v\}\!\}(x) = \frac{v^{\mathrm{in}}(x) + v^{\mathrm{out}}(x)}{2}, \qquad [\![v]\!](x) = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x)$$

with

$$v^{\mathrm{out}}(x) = \lim_{\delta\to 0+} v(x + \delta\boldsymbol{n}), \qquad v^{\mathrm{in}}(x) = \lim_{\delta\to 0+} v(x - \delta\boldsymbol{n}),$$

whenever $x \in \sigma \in \mathcal{E}$ and $\boldsymbol{n}$ is the outer normal vector to σ. In addition, if $\sigma \in \mathcal{E}_i$, we write $\{\!\{v\}\!\}$ and $[\![v]\!]$ as $\{\!\{v\}\!\}^{(i)}$ and $[\![v]\!]^{(i)}$, respectively. Moreover, we define an upwind quantity of $v \in Q_h$ associated to the velocity field $\boldsymbol{u} \in (Q_h)^d$ at a generic face σ

$$v^{\mathrm{up}} = \begin{cases} v^{\mathrm{in}}, & \text{if } \{\!\{\boldsymbol{u}\}\!\} \cdot \boldsymbol{n} \geq 0, \\ v^{\mathrm{out}}, & \text{if } \{\!\{\boldsymbol{u}\}\!\} \cdot \boldsymbol{n} < 0. \end{cases}$$

Discrete operators. For piecewise constant functions $r_h \in Q_h$, $\boldsymbol{v}_h \in (Q_h)^d$ we define the following discrete gradient, divergence and Laplace operators as

$$\begin{aligned}
\nabla_{\mathcal{E}} r_h(x) &= \sum_{\sigma\in\mathcal{E}} (\nabla_{\mathcal{E}} r_h)_{D_\sigma} 1_{D_\sigma}(x), & (\nabla_{\mathcal{E}} r_h)_{D_\sigma} &= \frac{1}{h} [\![r_h]\!] \boldsymbol{n}, \quad \nabla_{\mathcal{E}} \boldsymbol{v}_h = (\nabla_{\mathcal{E}} v_{1,h}, \ldots, \nabla_{\mathcal{E}} v_{d,h})^T, \\
\mathrm{div}_h \boldsymbol{v}_h(x) &= \sum_{K\in\mathcal{T}_h} (\mathrm{div}_h \boldsymbol{v}_h)_K 1_K(x), & (\mathrm{div}_h \boldsymbol{v}_h)_K &= \frac{1}{h} \sum_{\sigma\in\mathcal{E}(K)} \{\!\{\boldsymbol{v}_h\}\!\} \cdot \boldsymbol{n}, \\
\Delta_h r_h(x) &= \sum_{K\in\mathcal{T}_h} (\Delta_h r_h)_K 1_K(x), & (\Delta_h r_h)_K &= \frac{1}{h^2} \sum_{\sigma\in\mathcal{E}(K)} [\![r_h]\!].
\end{aligned}$$

It is easy to verify the interpolation errors

$$|[\![\Pi_{\mathcal{T}}\phi]\!]| \lesssim h \|\nabla_x \phi\|_{L^\infty(\Omega)}, \ \|\Pi_{\mathcal{T}}\phi - \phi\|_{L^\infty(\Omega)} \lesssim h \|\nabla_x \phi\|_{L^\infty(\Omega)}, \ \|\nabla_{\mathcal{E}} \Pi_{\mathcal{T}}\phi\|_{L^\infty(\Omega)} \lesssim \|\nabla_x \phi\|_{L^\infty(\Omega)} \tag{3.2}$$

for any $\phi \in W^{1,\infty}(\Omega)$, and the following discrete integration-by-parts formula

$$\int_{\mathbb{T}^d} f_h \Delta_h r_h \, \mathrm{d}x = -\frac{1}{h} \int_{\mathcal{E}} [\![r_h]\!] [\![f_h]\!] \, \mathrm{d}S_x \tag{3.3}$$

for any $r_h, f_h \in Q_h$.

Time discretization. Given a time step $\Delta t > 0$ we divide the time interval $[0, T]$ into $N_T = T/\Delta t$ uniform parts, and denote $t^k = k\Delta t$. Then $v_h^k \approx v_h(t^k)$ is the approximation of a function v_h at time t^k, $k = 1, \dots, N_T$. By $v_h(t) \in L_{\Delta t}(0, T; Q_h)$ we denote a piecewise constant in time function of discrete values $v_h^k \in Q_h$

$$v_h(t, \cdot) = v_h^0 \ \text{ for } \ t < \Delta t, \quad v_h(t, \cdot) = v_h^k \ \text{ for } \ t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, \cdots, N_T.$$

Further, we define the discrete time derivative by a backward difference formula

$$D_t v_h(t) = \frac{v_h(t) - v_h^{\triangleleft}}{\Delta t} \quad \text{with} \quad v_h^{\triangleleft} = v_h(t - \Delta t).$$

Hereafter, we work with the couple $(\varrho_h(t), \boldsymbol{u}_h(t))$ that represents the (piecewise constant in space and time) discrete density and velocity, respectively. Moreover, we set $\boldsymbol{m}_h = \varrho_h \boldsymbol{u}_h$ and $p_h = p(\varrho_h)$.

3.2 Finite volume method for the Navier–Stokes system on $\mathbb{T}^d$

We are now ready to propose a finite volume method presented in the weak form.

Definition 3.1 (Finite volume method)**.** *Let the initial data* (1.1d) *be extended by* $\widetilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0$ *as in* (1.4c) *and let* $(\varrho_h^0, \boldsymbol{m}_h^0) := (\Pi_{\mathcal{T}} \widetilde{\varrho}_0, \Pi_{\mathcal{T}} \widetilde{\boldsymbol{m}}_0)$. *We say that* $(\varrho_h^{\epsilon_s}, \boldsymbol{u}_h^{\epsilon_s}) \in L_{\Delta t}(0, T; Q_h \times (Q_h)^d)$ *is a finite volume approximation of the penalized problem* (1.4) *if the following system of algebraic equations hold*

$$\int_{\mathbb{T}^d} D_t \varrho_h^{\epsilon_s} \phi_h \,\mathrm{d}x - \int_{\mathcal{E}} F_h^{\varepsilon}(\varrho_h^{\epsilon_s}, \boldsymbol{u}_h^{\epsilon_s}) \, [\![\phi_h]\!] \,\mathrm{d}S_x = 0, \qquad \textit{for all } \phi_h \in Q_h, \tag{3.4a}$$

$$\begin{aligned} &\int_{\mathbb{T}^d} D_t(\varrho_h^{\epsilon_s} \boldsymbol{u}_h^{\epsilon_s}) \cdot \boldsymbol{\phi}_h \,\mathrm{d}x - \int_{\mathcal{E}} \mathbf{F}_h^{\varepsilon}(\varrho_h^{\epsilon_s} \boldsymbol{u}_h^{\epsilon_s}, \boldsymbol{u}_h^{\epsilon_s}) \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S_x - \int_{\mathbb{T}^d} p_h^{\epsilon_s} \mathrm{div}_h \boldsymbol{\phi}_h \,\mathrm{d}x + \frac{1}{\epsilon_s} \int_{\Omega_h^s} \boldsymbol{u}_h^{\epsilon_s} \cdot \boldsymbol{\phi}_h \,\mathrm{d}x \\ &+ \mu \int_{\mathbb{T}^d} \nabla_{\mathcal{E}} \boldsymbol{u}_h^{\epsilon_s} : \nabla_{\mathcal{E}} \boldsymbol{\phi}_h \,\mathrm{d}x + \nu \int_{\mathbb{T}^d} \mathrm{div}_h \boldsymbol{u}_h^{\epsilon_s} \, \mathrm{div}_h \boldsymbol{\phi}_h \,\mathrm{d}x = 0, \qquad \textit{for all } \boldsymbol{\phi}_h \in (Q_h)^d. \end{aligned} \tag{3.4b}$$

where $\nu = \lambda + \frac{d-2}{d}\mu$ *and the flux* $F_h^{\varepsilon}(r_h, \boldsymbol{u}_h)$ *reads*

$$F_h^{\varepsilon}(r_h, \boldsymbol{u}_h) = Up[r_h, \boldsymbol{u}_h] - h^{\varepsilon} [\![r_h]\!], \qquad \textit{with} \quad Up[r_h, \boldsymbol{u}_h] = r_h^{\mathrm{up}} \{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n} \ \textit{and} \ \varepsilon > -1. \tag{3.5}$$

Remark 3.2. *We shall write* $(\varrho_h^{\epsilon_s}, \boldsymbol{u}_h^{\epsilon_s})$ *as* $(\varrho_h, \boldsymbol{u}_h)$ *for simplicity if there is no confussion.*

4 Convergence

In this section we study the convergence of the finite volume method (3.4). To this goal we first discuss the stability and consistency of the finite volume method (3.4).

4.1 Stability

We begin with the following lemma reported in Feireisl et al. [8, Lemmas 11.2 and 11.3].

Lemma 4.1 (Properties of scheme (3.4))**.** *Let* $\widetilde{\varrho}_0 > 0$. *Then there exists at least one solution to the FV method* (3.4). *Moreover, any solution* $(\varrho_h, \boldsymbol{u}_h)$ *to* (3.4) *satisfies for all* $t \in (0, T)$ *that*

- *Positivity of density.*
$$\varrho_h(t) > 0;$$
- *Conservation of mass.*
$$\int_{\mathbb{T}^d} \varrho_h(t)\,\mathrm{d}x = \int_{\mathbb{T}^d} \widetilde{\varrho}_0\,\mathrm{d}x;$$
- *Internal energy balance.*
$$\begin{aligned}&\int_{\mathbb{T}^d} D_t\mathcal{P}(\varrho_h)\,\mathrm{d}x + \int_{\mathbb{T}^d} p(\varrho_h)\mathrm{div}_h\boldsymbol{u}_h\,\mathrm{d}x\\ &\quad= -\int_{\mathbb{T}^d}\frac{\Delta t}{2}\mathcal{P}''(\varrho_h^\star)|D_t\varrho_h|^2\,\mathrm{d}x - \int_{\mathcal{E}}\left(h^\varepsilon + \frac{1}{2}|\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\right)\mathcal{P}''(\varrho_{h,\dagger})\,[\![\varrho_h]\!]^2\,\mathrm{d}S_x, \end{aligned}\tag{4.1}$$

where $\varrho_h^\star \in \mathrm{co}\{\varrho_h^\triangleleft, \varrho_h\}$ *and* $\varrho_{h,\dagger} \in \mathrm{co}\{\varrho_h^{\mathrm{in}}, \varrho_h^{\mathrm{out}}\}$ *for any* $\sigma \in \mathcal{E}$*.*

It is easy to check the energy stability of the FV method.

Lemma 4.2 (Energy stability)**.** *Let* $(\varrho_h, \boldsymbol{u}_h)$ *be a numerical solution of the FV method* (3.4)*. Then it holds*

$$\begin{aligned}&D_t\int_{\mathbb{T}^d}\left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2 + \mathcal{P}(\varrho_h)\right)\mathrm{d}x + \int_{\mathbb{T}^d}\left(\mu|\nabla_{\mathcal{E}}\boldsymbol{u}_h|^2 + \nu|\mathrm{div}_h\boldsymbol{u}_h|^2\right)\mathrm{d}x\\ &\quad= -\frac{1}{\epsilon_s}\int_{\Omega_h^s}|\boldsymbol{u}_h|^2\,\mathrm{d}x - D_{num} = -\frac{1}{\epsilon_s}\int_{\Omega^s}|\boldsymbol{u}_h|^2\,\mathrm{d}x - D_{num}^{new},\end{aligned}\tag{4.2}$$

where $D_{num}^{new} \geq D_{num} \geq 0$ *represent the numerical dissipations, which read*

$$\begin{aligned}D_{num}^{new} &= D_{num} + \frac{1}{\epsilon_s}\int_{\Omega_h^s\setminus\Omega^s}|\boldsymbol{u}_h|^2\,\mathrm{d}x,\\ D_{num} &= h^\varepsilon\int_{\mathcal{E}}\{\!\{\varrho_h\}\!\}\,|[\![\boldsymbol{u}_h]\!]|^2\,\mathrm{d}S_x + \frac{\Delta t}{2}\int_{\mathbb{T}^d}\varrho_h^\triangleleft|D_t\boldsymbol{u}_h|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{E}}\varrho_h^{\mathrm{up}}|\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}||[\![\boldsymbol{u}_h]\!]|^2\,\mathrm{d}S_x\\ &\quad+\int_{\mathbb{T}^d}\frac{\Delta t}{2}\mathcal{P}''(\varrho_h^\star)|D_t\varrho_h|^2\,\mathrm{d}x + \int_{\mathcal{E}}\left(h^\varepsilon + \frac{1}{2}|\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\right)\mathcal{P}''(\varrho_{h,\dagger})\,[\![\varrho_h]\!]^2\,\mathrm{d}S_x.\end{aligned}$$

Proof. We start by recalling the kinetic energy balance, cf. [7, equation (3.4)],

$$\begin{aligned}&D_t\int_{\mathbb{T}^d}\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2\,\mathrm{d}x + \mu\int_{\mathbb{T}^d}|\nabla_{\mathcal{E}}\boldsymbol{u}_h|^2\,\mathrm{d}x + \nu\int_{\mathbb{T}^d}|\mathrm{div}_h\boldsymbol{u}_h|^2\,\mathrm{d}x + \frac{1}{\epsilon_s}\int_{\Omega_h^s}|\boldsymbol{u}_h|^2\,\mathrm{d}x - \int_{\mathbb{T}^d}p_h\mathrm{div}_h\boldsymbol{u}_h\,\mathrm{d}x\\ &\quad+h^\varepsilon\int_{\mathcal{E}}\{\!\{\varrho_h\}\!\}\,|[\![\boldsymbol{u}_h]\!]|^2\,\mathrm{d}S_x + \frac{\Delta t}{2}\int_{\mathbb{T}^d}\varrho_h^\triangleleft|D_t\boldsymbol{u}_h|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{E}}\varrho_h^{\mathrm{up}}|\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}||[\![\boldsymbol{u}_h]\!]|^2\,\mathrm{d}S_x = 0.\end{aligned}$$

Recalling the internal energy balance (4.1) and combining it with the above kinetic energy balance we finish the proof. □

Next, using the above energy balance (4.2) and recalling the Sobolev-Poincaré inequality [8, Theorem 16] we obtain the following a priori bounds for the numerical solution of our FV method (3.4).

Lemma 4.3 (Uniform bounds)**.** *Let $(\varrho_h, \boldsymbol{u}_h)$ be a numerical solution of the FV method (3.4). Then the following hold*

$$\begin{aligned}&\|\varrho_h\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^d))} + \|\varrho_h \boldsymbol{u}_h\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d))} + \|p_h\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \\ &\quad + \|\boldsymbol{u}_h\|_{L^2(0,T;L^p(\mathbb{T}^d))} + \|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2((0,T)\times\mathbb{T}^d)} + \|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2((0,T)\times\mathbb{T}^d)} \leq C,\end{aligned} \tag{4.3a}$$

$$h^\varepsilon \int_0^T \int_{\mathcal{E}} \{\!\{\varrho_h\}\!\} \left|[\![\boldsymbol{u}_h]\!]\right|^2 \mathrm{d}S_x \mathrm{d}t + \int_0^T \int_{\mathcal{E}} \left(h^\varepsilon + \frac{1}{2}|\,\{\!\{\boldsymbol{u}_h\}\!\}\cdot \boldsymbol{n}|\right) \mathcal{P}''(\varrho_{h,\dagger})\, [\![\varrho_h]\!]^2 \,\mathrm{d}S_x \mathrm{d}t \leq C, \tag{4.3b}$$

$$\frac{1}{\epsilon_s} \|\boldsymbol{u}_h\|^2_{L^2((0,T)\times\Omega^s)} \leq \frac{1}{\epsilon_s} \|\boldsymbol{u}_h\|^2_{L^2((0,T)\times\Omega^s_h)} \leq C, \tag{4.3c}$$

where $\varrho_{h,\dagger} \in \mathrm{co}\{\varrho_h^{\mathrm{in}}, \varrho_h^{\mathrm{out}}\}$. The parameter $p \in [1,\infty)$ for $d=2$ and $p=6$ for $d=3$. The constant C depends on the mass $M = \int_{\mathbb{T}^d} \widetilde{\varrho}_0 \,\mathrm{d}x > 0$ and the initial energy $E_0 = \int_{\mathbb{T}^d} \left(\frac{1}{2}\widetilde{\varrho}_0 |\widetilde{\boldsymbol{u}}_0|^2 + \mathcal{P}(\widetilde{\varrho}_0)\right) \mathrm{d}x > 0$, but it is independent of the computational parameters $(h, \Delta t)$ as well as the penalization parameter ϵ_s.

4.2 Consistency formulation

Having shown the stability of our numerical method, we need to show its consistency. As the consistency proof is quite technical and moreover the idea and structure are analogous to [9, Section 2.7] and [8, Section 11.3], we postpone it into Appendix B. Note that our result is more general than in [8] since we need here less regularity of the test function in the continuity equation (4.4). This will be required later for the error estimates in Section 5.

Lemma 4.4 (Consistency formulation)**.** *Let $(\varrho_h, \boldsymbol{u}_h)$ be a solution of the FV scheme (3.4) with $\Delta t, h \in (0,1)$, $\gamma > 1$ and $\varepsilon > -1$. Then we have for any $\tau \in [0,T]$*

- *For all $\phi \in W^{1,\infty}((0,T)\times\mathbb{T}^d)$, $\partial_t^2 \phi \in L^\infty((0,T)\times\mathbb{T}^d)$ it holds that*

$$\left[\int_{\mathbb{T}^d} \varrho_h \phi \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t \phi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \phi\right) \mathrm{d}x\mathrm{d}t \; + e_\varrho(\tau, \Delta t, h, \phi) \tag{4.4a}$$

  with consistency error bounded by

$$|e_\varrho(\tau, \Delta t, h, \phi)| \leq C_\varrho\big(\Delta t + h^{(1+\varepsilon)/2} + h^{(1+\beta_R)/2} + h^{1+\beta_D}\big); \tag{4.4b}$$

- *For all $\boldsymbol{\phi} \in W^{1,\infty}((0,T)\times\mathbb{T}^d;\mathbb{R}^d) \cap L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d)), \partial_t^2 \phi \in L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d)$ it holds that*

$$\begin{aligned}\left[\int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \boldsymbol{\phi} \,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\phi} + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\phi} + p_h \mathrm{div}_x \boldsymbol{\phi}\right) \mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau \int_{\mathbb{T}^d} \left(\mu \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \boldsymbol{\phi} + \nu \mathrm{div}_h \boldsymbol{u}_h \,\mathrm{div}_x \boldsymbol{\phi}\right) \mathrm{d}x\mathrm{d}t - \frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega^s} \boldsymbol{u}_h \cdot \boldsymbol{\phi} \,\mathrm{d}x\mathrm{d}t \; + e_{\boldsymbol{m}}(\tau, \Delta t, h, \boldsymbol{\phi})\end{aligned} \tag{4.5a}$$

  with consistency errors bounded by

$$|e_{\boldsymbol{m}}(\tau, \Delta t, h, \boldsymbol{\phi})| \leq C_{\boldsymbol{m}}\big(\sqrt{\Delta t} + h + h^{1+\varepsilon} + h^{1+\beta_M} + (h/\epsilon_s)^{1/2} + (\Delta t/\epsilon_s)^{1/2}\big). \tag{4.5b}$$

Here the constant C_ϱ depends on

$$E_0, \quad T, \quad \|\phi\|_{W^{1,\infty}((0,T)\times\mathbb{T}^d)}, \quad \left\|\partial_t^2\phi\right\|_{L^\infty([0,T]\times\mathbb{T}^d)},$$

and the constant $C_{\boldsymbol{m}}$ depends on

$$E_0, \quad T, \quad \|\phi\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d))}, \quad \|\phi\|_{W^{1,\infty}((0,T)\times\mathbb{T}^d)}, \quad \left\|\partial_t^2\phi\right\|_{L^\infty([0,T]\times\mathbb{T}^d)}$$

and $\beta_D, \beta_R, \beta_M$ are defined by

$$\beta_D = \begin{cases} \min\limits_{p\in[1,\infty)} \left\{\frac{p(\varepsilon+1)+4}{2p}, 1\right\} \cdot \frac{\gamma-2}{\gamma}, & \text{if } d=2, \gamma\in(1,2), \\ \min\left\{\frac{\varepsilon+2}{3}, 1\right\} \cdot \frac{3(\gamma-2)}{2\gamma}, & \text{if } d=3, \gamma\in(1,2), \\ 0, & \text{if } \gamma\geq 2, \end{cases}$$

$$\beta_R = \begin{cases} 0, & \text{if } d=2, \\ \min\left\{\frac{1+\varepsilon}{2}, 1\right\} \cdot \frac{5\gamma-6}{2\gamma}, & \text{if } d=3, \gamma\in\left(1,\frac{6}{5}\right), \\ 0, & \text{if } d=3, \gamma\geq\frac{6}{5}, \end{cases}$$

$$\beta_M = \begin{cases} \max\limits_{p\in\left[\frac{2\gamma}{\gamma-1},\infty\right)} \left\{-\frac{p(\varepsilon+1)+4}{2p\gamma}, \frac{p(\gamma-2)-2\gamma}{p\gamma}\right\}, & \text{if } d=2, \gamma\leq 2, \\ 0, & \text{if } d=2, \gamma>2, \\ \max\left\{-\frac{\varepsilon+2}{2\gamma}, \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma}\right\}, & \text{if } d=3, \gamma\leq 2, \\ \frac{\gamma-3}{\gamma}, & \text{if } d=3, \gamma\in(2,3), \\ 0, & \text{if } d=3, \gamma\geq 3. \end{cases}$$

Remark 4.5 (Observations on the parameters $\beta_R, \beta_D, \beta_M$ and ε)**.**

- *It is easy to verify that*

$$0 \geq \beta_R \geq \beta_D \geq \beta_M \text{ and } \beta_D > -1.$$

  Moreover, $\beta_M > -1$ if one of the following conditions holds

  - $d = 2$,
  - $d = 3$ *and* $\gamma > \frac{3}{2}$,
  - $d = 3$ *and* $\gamma \leq \frac{3}{2}$ *with* $\varepsilon < 2(\gamma - 1)$.

- *We point out that the parameters $\beta_D, \beta_R, \beta_M$ are independent of ε if $\varepsilon \geq 1$. Indeed, for $\varepsilon \geq 1$ we have simpler forms of $\beta_D, \beta_R, \beta_M$, i.e.*

$$\beta_D = \begin{cases} \frac{d(\gamma-2)}{2\gamma}, & \text{if } \gamma<2, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta_R = \begin{cases} \frac{(5\gamma-6)}{2\gamma}, & \text{if } d=3, \gamma<\frac{6}{5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta_M = \begin{cases} \max\limits_{p\in\left[\frac{2\gamma}{\gamma-1},\infty\right)} \frac{p(\gamma-2)-2\gamma}{p\gamma}, & \text{if } d=2, \gamma\leq 2, \\ 0, & \text{if } d=2, \gamma>2, \\ \max\left\{\frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma}\right\}, & \text{if } d=3, \gamma<3, \\ 0, & \text{if } d=3, \gamma\geq 3. \end{cases}$$

- *Our consistency errors (involving the terms $\beta_D, \beta_R, \beta_M$) are better than the results in [8] due to sharper interpolation inequalities proved in Appendix B.1.*

4.3 Convergence of the FV method (3.4) to the DW solution on $\mathbb{T}^d$

We take ϵ_s fixed and pass to the limit with $h \to 0$. First, we deduce from a priori estimate (4.3) that up to a subsequence

$$
\begin{aligned}
\varrho_h &\to \varrho_{\epsilon_s} \text{ weakly-}(*) \text{ in } L^\infty(0,T;L^\gamma(\mathbb{T}^d)),\ \varrho \geq 0,\\
\boldsymbol{u}_h &\to \boldsymbol{u}_{\epsilon_s} \text{ weakly in } L^2(0,T;L^6(\mathbb{T}^d;\mathbb{R}^d)),\\
\nabla_{\mathcal{E}}\boldsymbol{u}_h &\to \nabla_x \boldsymbol{u}_{\epsilon_s} \text{ weakly in } L^2((0,T)\times\mathbb{T}^d;\mathbb{R}^{d\times d}), \quad \text{where } \boldsymbol{u}_{\epsilon_s} \in L^2(0,T;W^{1,2}(\mathbb{T}^d;\mathbb{R}^d))
\end{aligned}
$$

and

$$
\varrho_h \boldsymbol{u}_h \to \boldsymbol{m}_{\epsilon_s} \quad \text{weakly-}(*) \text{ in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;\mathbb{R}^d)).
$$

Realizing that $(\varrho_h, \boldsymbol{u}_h)$ satisfies the consistency formulation (4.4) for the mass conservation equation, applying [1, Lemma 3.7] (see similar result in [16, Lemma 7.1]) we obtain

$$
\boldsymbol{m}_{\epsilon_s} = \varrho_{\epsilon_s}\boldsymbol{u}_{\epsilon_s}.
$$

Further, due to the fact that the total energy $E = \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \mathcal{P}(\varrho)$ is a convex function of $(\varrho, \boldsymbol{m})$ and $|\nabla_x \boldsymbol{u}|^2 + |\mathrm{div}_x \boldsymbol{u}|^2 = |\nabla_x \boldsymbol{u}|^2 + |\mathrm{tr}(\nabla_x \boldsymbol{u})|^2$ is a convex function of $\nabla_x \boldsymbol{u}$, we deduce from [5, Lemma 2.7] that

$$
\begin{aligned}
\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2 + \mathcal{P}(\varrho_h) &\to \overline{\frac{|\boldsymbol{m}|^2}{2\varrho} + \mathcal{P}(\varrho)} \text{ weakly-}(*) \text{ in } L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^d)),\\
\varrho_h\boldsymbol{u}_h\otimes\boldsymbol{u}_h + p(\varrho_h)\mathbb{I} &\to \overline{\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho} + p(\varrho)\mathbb{I}} \text{ weakly-}(*) \text{ in } L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^d;\mathbb{R}^{d\times d}_{\mathrm{sym}})),\\
\mu|\nabla_{\mathcal{E}}\boldsymbol{u}_h|^2 + \nu|\mathrm{div}_h\boldsymbol{u}_h|^2 &\to \overline{\mu|\nabla_x\boldsymbol{u}|^2 + \nu|\mathrm{div}_x\boldsymbol{u}|^2} \text{ weakly-}(*) \text{ in } \mathcal{M}^+((0,T)\times\mathbb{T}^d).
\end{aligned}
$$

Note that the defects

$$
\begin{aligned}
\mathfrak{E} &= \overline{\frac{|\boldsymbol{m}|^2}{2\varrho} + \mathcal{P}(\varrho)} - \left(\frac{1}{2}\varrho_{\epsilon_s}|\boldsymbol{u}_{\epsilon_s}|^2 + \mathcal{P}(\varrho_{\epsilon_s})\right) \geq 0,\\
\mathfrak{R} &= \overline{\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho} + p(\varrho)\mathbb{I}} - (\varrho_{\epsilon_s}\boldsymbol{u}_{\epsilon_s}\otimes\boldsymbol{u}_{\epsilon_s} + p(\varrho_{\epsilon_s})\mathbb{I}) \geq 0,\\
\mathfrak{D} &= \overline{\mu|\nabla_x\boldsymbol{u}|^2 + \nu|\mathrm{div}_x\boldsymbol{u}|^2} - \left(\mu|\nabla_x\boldsymbol{u}_{\epsilon_s}|^2 + \nu|\mathrm{div}_x\boldsymbol{u}_{\epsilon_s}|^2\right) \geq 0
\end{aligned}
$$

satisfy

$$
\underline{d}\mathfrak{E} \leq \mathrm{tr}[\mathfrak{R}] \leq \overline{d}\mathfrak{E}, \quad \underline{d} = \min\left(2, d(\gamma-1)\right), \ \overline{d} = \max\left(2, d(\gamma-1)\right).
$$

Together with

$$
\int_{\mathbb{T}^d} E(\varrho_h^0, \boldsymbol{m}_h^0)\,\mathrm{d}x \to \int_{\mathbb{T}^d} E(\widetilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0)\,\mathrm{d}x,
$$

the consistency formulations (4.4) and (4.5), and the energy balance (4.2), the limit $(\varrho_{\epsilon_s}, \boldsymbol{u}_{\epsilon_s})$ is a DW solution of the penalized problem (1.4) in the sense of Definition 2.1. We summarize the obtained result on the weak convergence of FV solutions in the following theorem.

Theorem 4.6. (**Weak convergence for the penalized problem** (1.4)) Let p satisfy (1.2) with $\gamma > 1$ and $\epsilon_s > 0$ is a fixed penalization parameter. Let $\{\varrho_h, \boldsymbol{u}_h\}_{h\downarrow 0}$ be a family of numerical solutions obtained by the FV method (3.4) with initial data satisfying (1.3). Let $\varepsilon > -1$. If $d = 3$ and $\gamma \leq \frac{3}{2}$ we assume $\varepsilon < 2(\gamma - 1)$.

Then, up to a subsequence, the FV solution $\{\varrho_h, \boldsymbol{u}_h\}$ converges for $\Delta t, h \to 0$ in the following sense

$$\begin{aligned}
\varrho_h &\longrightarrow \varrho_{\epsilon_s} \text{ weakly-(*) in } L^\infty((0,T); L^\gamma(\mathbb{T}^d)),\\
\boldsymbol{u}_h &\longrightarrow \boldsymbol{u}_{\epsilon_s} \text{ weakly in } L^2((0,T)\times\mathbb{T}^d;\mathbb{R}^d),\\
\nabla_{\mathcal{E}}\boldsymbol{u}_h &\longrightarrow \nabla_x \boldsymbol{u}_{\epsilon_s} \text{ weakly in } L^2((0,T)\times\mathbb{T}^d;\mathbb{R}^{d\times d}),\\
\mathrm{div}_h \boldsymbol{u}_h &\longrightarrow \mathrm{div}_x \boldsymbol{u}_{\epsilon_s} \text{ weakly in } L^2((0,T)\times\mathbb{T}^d),
\end{aligned} \tag{4.6}$$

where $(\varrho_{\epsilon_s}, \boldsymbol{u}_{\epsilon_s})$ is a DW solution of the penalized problem (1.4) in the sense of Definition 2.1.

4.4 Convergence of the FV method (3.4) to the DW solution on Ω^f

Our aim now is to consider the limit process for $\epsilon_s \to 0$ and $h \to 0$. According to a priori bound (4.3c) we obtain that up to a subsequences,

$$\boldsymbol{u}_h \to 0 \;\text{ weakly in } L^2((0,T)\times\Omega^s;\mathbb{R}^d)$$

yielding

$$\boldsymbol{u} \in \; L^2(0,T; W_0^{1,2}(\Omega^f;\mathbb{R}^d)).$$

Moreover, together with the fact that ϱ satisfies the equation of continuity (2.3), see Theorem 4.6, we have

$$\partial_t \varrho = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega^s),$$

which means

$$\varrho = \widetilde{\varrho}_0 \text{ in } \Omega^s \quad \text{for all } t \in (0,T). \tag{4.7}$$

Consequently, we have

$$\int_{\mathbb{T}^d} \left(\frac{1}{2}\frac{|\boldsymbol{m}_h^0|^2}{\varrho_h^0} + \mathcal{P}(\varrho_h^0)\right) \mathrm{d}x \to \int_{\Omega^f} \left(\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_0} + \mathcal{P}(\varrho_0)\right) \mathrm{d}x + \int_{\Omega^s} \mathcal{P}(\widetilde{\varrho}_0)\,\mathrm{d}x \tag{4.8}$$

and

$$\int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2 + \mathcal{P}(\varrho_h)\right) \mathrm{d}x \to \int_{\overline{\Omega^f}} d\mathfrak{E}(\tau) + \int_{\Omega^f} \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \mathcal{P}(\varrho)\right) \mathrm{d}x + \int_{\Omega^s} \mathcal{P}(\widetilde{\varrho}_0)\,\mathrm{d}x, \tag{4.9}$$

which yields the energy inequality (2.7) as the last terms in (4.8) and (4.9) cancel with each other after passing to the limit in (4.2).

Together with the consistency formulations (4.4), (4.5) and the energy balance (4.2), the limit $(\varrho, \boldsymbol{u})$ is a DW solution of the Navier–Stokes system (1.1) with the Dirichlet boundary conditions in the sense of Definition 2.2.

Theorem 4.7. (**Weak convergence for the Dirichlet problem** (1.1)) In addition to the assumption of Theorem 4.6, Then, up to a subsequence, the FV solution $\{\varrho_h, \boldsymbol{u}_h\}$ converges for $\Delta t, h, \epsilon_s \to 0$ together with $h^2\Delta t/\epsilon_s \to 0, h^3/\epsilon_s \to 0$ in the following sense

$$
\begin{aligned}
\varrho_h &\longrightarrow \varrho \text{ weakly-(*) in } L^\infty((0,T);L^\gamma(\Omega^f)),\\
\boldsymbol{u}_h &\longrightarrow \boldsymbol{u} \text{ weakly in } L^2(0,T;L_0^2(\Omega^f;\mathbb{R}^d)),\\
\nabla_{\mathcal{E}}\boldsymbol{u}_h &\longrightarrow \nabla_x \boldsymbol{u} \text{ weakly in } L^2((0,T)\times\Omega^f;\mathbb{R}^{d\times d}),\\
\mathrm{div}_h \boldsymbol{u}_h &\longrightarrow \mathrm{div}_x \boldsymbol{u} \text{ weakly in } L^2((0,T)\times\Omega^f),
\end{aligned}
\tag{4.10}
$$

where $(\varrho, \boldsymbol{u})$ is a DW solution of the Dirichlet problem of Navier–Stokes system (1.1) in the sense of Definition 2.2.

Proof. With the test function $\phi \in C_c^1([0,T]\times\overline{\Omega^f})$, we can improve the consistency error E_{ϵ_s} in (B.17) by $h(\Delta t)^{1/2}\epsilon_s^{-1/2} + h^{3/2}\epsilon_s^{-1/2}$, resulting $e_{\boldsymbol{m}} \to 0$. This finishes the proof. □

4.5 Convergence of the FV method (3.4) to the strong solution

As a consequence of the dissipative weak–strong uniqueness result stated in [1], we have the following convergence result. For the local existence of the strong solution of Navier–Stokes system (1.1), we refer the reader to Feireisl et al. [8, Theorem 3.1].

Definition 4.8. (Strong solution) *Let $\Omega^f \subset \mathbb{R}^d, d = 2,3,$ be a bounded domain with a smooth boundary $\partial\Omega^f$. We say that $(\varrho, \boldsymbol{u})$ is the strong solution of the Navier–Stokes problem* (1.1) *if*

$$
\begin{aligned}
&\varrho \in C^1([0,T]\times\overline{\Omega^f}) \cap C(0,T;W^{k,2}(\Omega^f)),\\
&\boldsymbol{u} \in C^1([0,T]\times\overline{\Omega^f};\mathbb{R}^d) \cap C(0,T;W^{k,2}(\Omega^f;\mathbb{R}^d)),\ k \geq 4
\end{aligned}
\tag{4.11}
$$

and equations (1.1) *are satisfied pointwise.*

Theorem 4.9. (**Strong convergence**) Let $\gamma > 1$. Suppose that the Navier–Stokes system (1.1) admits a classical solution $(\varrho, \boldsymbol{u})$ belonging to the class (4.11) with the initial data $(\varrho_0, \boldsymbol{u}_0)$ satisfying $\varrho_0 > 0$. Let $\{\varrho_h, \boldsymbol{u}_h\}_{h\downarrow 0}$ be a family of numerical solutions obtained by the FV method (3.4) with a smoothly extended initial data $(\widetilde{\varrho}_0, \widetilde{\boldsymbol{u}}_0) \in W^{1,\infty}(\mathbb{T}^d)$. Further, we assume that the parameters $\varepsilon, \Delta t, h, \epsilon_s$ satisfy the same conditions as in Theorem 4.7.

Then the FV solutions $(\varrho_h, \boldsymbol{u}_h)$ converge strongly to the strong solution $(\varrho, \boldsymbol{u})$ in the following sense

$$
\begin{aligned}
\varrho_h &\to \varrho \text{ (strongly) in } L^\gamma((0,T)\times\Omega^f),\\
\boldsymbol{u}_h &\to \boldsymbol{u} \text{ (strongly) in } L^2((0,T)\times\Omega^f;\mathbb{R}^d).
\end{aligned}
$$

5 Error estimates

In this section our goal is to study the error between the FV approximation of penalized problem (1.4) and the exact strong solution to the Navier–Stokes equations (1.1) with Dirichlet boundary conditions, cf. Definition 4.8.

For simplicity of the presentation of main ideas, here and hereafter we consider a semi-discrete version of the FV method. In other words, we only study the error with respect to the spatial discretization.

In order to measure a distance between FV solutions of the penalized problem (1.4) and the strong solution to the Navier–Stokes problem (1.1) we introduce the relative energy functional

$$R_E(\varrho_h, \boldsymbol{u}_h|\widetilde{\varrho}, \widetilde{\boldsymbol{u}}) = \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 + \mathbb{E}(\varrho_h|\widetilde{\varrho}) \right) \mathrm{d}x, \ \mathbb{E}(\varrho_h|\widetilde{\varrho}) = \mathcal{P}(\varrho_h) - \mathcal{P}'(\widetilde{\varrho})(\varrho_h - \widetilde{\varrho}) - \mathcal{P}(\widetilde{\varrho}). \quad (5.1)$$

Here, $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}})$ represents a suitable extension of the a strong solution $(\varrho, \boldsymbol{u})$ from Ω^f to $\mathbb{T}^d$. Thanks to (4.7), in what follows we work with the following extension.

Definition 5.1 (Extension of the strong solution). *Let $(\varrho, \boldsymbol{u})$ be the strong solution in the sense of Definition 4.8. We say that $\widetilde{\varrho}, \widetilde{\boldsymbol{u}}$ is the extension of the strong solution $(\varrho, \boldsymbol{u})$ if*

$$(\widetilde{\varrho}, \widetilde{\boldsymbol{u}})(x) = \begin{cases} (\varrho_0^s, 0), & \text{if } x \in \Omega^s, \\ (\varrho, \boldsymbol{u}), & \text{if } x \in \Omega^f, \end{cases} \quad \text{for any } t \in [0, T]. \quad (5.2)$$

Here ϱ_0^s is defined in the initial extension (1.4c) which satisfies $\widetilde{\varrho}_0 \in W^{1,\infty}(\mathbb{T}^d)$.

Remark 5.2. *In view of the Lipschitz continuity of $\nabla_x \widetilde{\boldsymbol{u}}$ we obtain the regularity of $\widetilde{\varrho}, \widetilde{\boldsymbol{u}}$*

$$\widetilde{\varrho} \in W^{1,\infty}(\mathbb{T}^d), \ \ \widetilde{\boldsymbol{u}} \in W^{1,\infty}(\mathbb{T}^d; \mathbb{R}^d) \quad (5.3)$$

and

$$\|\widetilde{\varrho}\|_{W^{1,\infty}(\mathbb{T}^d)} \lesssim \|\varrho\|_{C^1(\overline{\Omega^f})} + \|\varrho_0^s\|_{W^{1,\infty}(\mathbb{T}^d \setminus \Omega^f)}, \quad \|\widetilde{\boldsymbol{u}}\|_{W^{1,\infty}(\mathbb{T}^d)} \lesssim \|\boldsymbol{u}\|_{C^1(\overline{\Omega^f})}. \quad (5.4)$$

Definition 5.3 (Splitting of the mesh). *We split the mesh $\mathcal{T}_h$ into three pieces, see Figure 3. Ω_h^C denotes the area containing the neighbourhood of the fluid boundary $\partial\Omega^f$*

$$\Omega_h^C = \left\{ K \mid \cup_{L \cap K \neq \emptyset} L \cap \partial\Omega^f \neq \emptyset \right\}.$$

The inner domain Ω_h^I and the outer domain Ω_h^O are given as

$$\Omega_h^I \colon = \Omega^f \setminus \Omega_h^C \quad \text{and} \quad \Omega_h^O \colon = \Omega^s \setminus \Omega_h^C.$$

Remark 5.4. *With the above definitions we know that for any cell $K \in \Omega_h^C$, either K or one of its neighbours intersects with the fluid boundary $\partial\Omega^f$. Moreover, we have*

$$\Omega_h^I \subset \Omega_h^f \subset \Omega^f, \quad \Omega_h^O \subset \Omega^s \subset \Omega_h^s, \quad (5.5a)$$

$$\left|\Omega_h^C\right| \lesssim h, \quad |\widetilde{\boldsymbol{u}}| \lesssim \|\widetilde{\boldsymbol{u}}\|_{W^{1,\infty}(\mathbb{T}^d)} h, \ \text{if } x \in \Omega_h^C, \quad (5.5b)$$

$$|\nabla_h(\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})| + |\mathrm{div}_h(\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})| + |\nabla_{\mathcal{E}}(\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})| \lesssim \begin{cases} \|\widetilde{\boldsymbol{u}}\|_{W^{1,\infty}(\mathbb{T}^d)}, & \text{if } x \in \Omega_h^I \cup \Omega_h^C, \\ 0, & \text{if } x \in \Omega_h^O, \end{cases} \quad (5.5c)$$

$$|\Delta_h \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}| \lesssim \begin{cases} \|\widetilde{\boldsymbol{u}}\|_{W^{2,\infty}(\mathbb{T}^d)}, & \text{if } x \in \Omega_h^I, \\ \|\widetilde{\boldsymbol{u}}\|_{W^{1,\infty}(\mathbb{T}^d)} h^{-1}, & \text{if } x \in \Omega_h^C, \\ 0, & \text{if } x \in \Omega_h^O. \end{cases} \quad (5.5d)$$

In the following we analyze the error of the penalized FV solution via the relative energy. It shall be done in three steps.

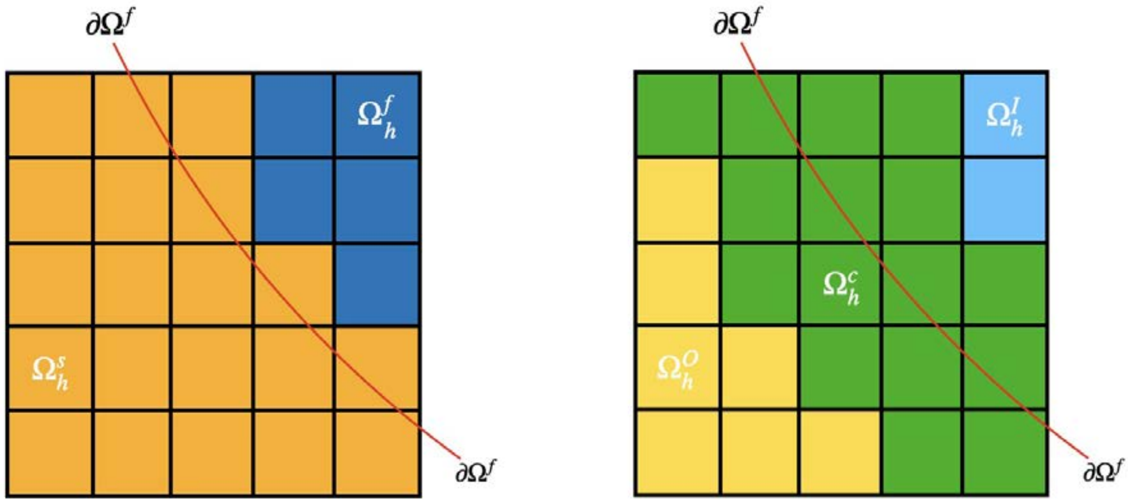


Figure 3: Zoom-in of domain splitting.

Step 1. In this step we derive the equation satisfied by the relative energy. Analogously as in [9], by collecting the energy estimate (4.2), the consistency formulation (4.4a) with the test function $\phi = \frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - \mathcal{P}'(\widetilde{\varrho})$, and the momentum consistency formulation (4.5a) with the test function $\boldsymbol{\phi} = -\widetilde{\boldsymbol{u}}$, we obtain

$$
\begin{aligned}
\left[R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\right]_{t=0}^{t=\tau} &+ \int_0^\tau \int_{\mathbb{T}^d} \left(\mu |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 + \nu |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2 \right) \, \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega_h^s} |\boldsymbol{u}_h|^2 \, \mathrm{d}x\mathrm{d}t = -\int_0^\tau D_{num} \, \mathrm{d}t + e_S + \sum_{i=1}^7 R_i
\end{aligned} \tag{5.6}
$$

with

$$
\begin{aligned}
& e_S = e_\varrho \left(\tau, h, |\widetilde{\boldsymbol{u}}|^2/2 - \mathcal{P}'(\widetilde{\varrho}) \right) + e_{\boldsymbol{m}}(\tau, h, -\widetilde{\boldsymbol{u}}), \\
& e_\varrho = -\sum_{i=1}^4 E_i(\varrho_h), \quad e_{\boldsymbol{m}} = -\sum_{i=1}^4 E_i(\varrho_h \boldsymbol{u}_h) + E_{\nabla_x \boldsymbol{u}} - E_p + E_{\epsilon_s},
\end{aligned} \tag{5.7}
$$

where $E_i, i = 1, \dots, 4, E_{\nabla_x \boldsymbol{u}}, E_p, E_{\epsilon_s}$ are given in (B.11), (B.12), (B.14) and (B.16), respectively. Further, the R_i terms read

$$
\begin{aligned}
R_1 &= \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h - \widetilde{\varrho})(\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot \left(\partial_t \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}} + \frac{\nabla_x p(\widetilde{\varrho})}{\widetilde{\varrho}} \right) \, \mathrm{d}x\mathrm{d}t, \\
R_2 &= -\int_0^\tau \int_{\mathbb{T}^d} \varrho_h (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) \otimes (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) : \nabla_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x\mathrm{d}t, \\
R_3 &= -\mu \int_0^\tau \int_{\mathbb{T}^d} \left(\nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \widetilde{\boldsymbol{u}} + \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \right) \, \mathrm{d}x\mathrm{d}t, \\
R_4 &= -\nu \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \widetilde{\boldsymbol{u}} + \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \right) \, \mathrm{d}x\mathrm{d}t, \\
R_5 &= -\int_0^\tau \int_{\mathbb{T}^d} \Big(p_h - p'(\widetilde{\varrho})(\varrho_h - \widetilde{\varrho}) - p(\widetilde{\varrho}) \Big) \mathrm{div}_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x\mathrm{d}t, \qquad (5.8) \\
R_6 &= \int_0^\tau \int_{\mathbb{T}^d} (\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot \Big(\widetilde{\varrho}(\partial_t \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}}) + \nabla_x p(\widetilde{\varrho}) - \mu \Delta_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} - \nu \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \Big) \, \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\mathbb{T}^d} \frac{(\varrho_h - \widetilde{\varrho})}{\widetilde{\varrho}} \cdot p'(\widetilde{\varrho}) \Big(\partial_t \widetilde{\varrho} + \mathrm{div}_x (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) \Big) \, \mathrm{d}x\mathrm{d}t \\
&= -\int_0^\tau \int_{\Omega^s} \boldsymbol{u}_h \cdot \nabla_x p(\widetilde{\varrho}) \, \mathrm{d}x\mathrm{d}t, \\
R_7 &= \frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega^s} \boldsymbol{u}_h \cdot \widetilde{\boldsymbol{u}} \, \mathrm{d}x\mathrm{d}t = 0,
\end{aligned}
$$

where we have used the fact that

$$\partial_t \widetilde{\varrho} = 0 \quad \text{and} \quad \widetilde{\boldsymbol{u}} = \boldsymbol{0} \quad \text{on } \Omega^s$$

and the following identity as $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}) = (\varrho, \boldsymbol{u})$ is the classical solution on Ω^f

$$\partial_t \widetilde{\varrho} + \mathrm{div}_x (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) = 0 \text{ and } \widetilde{\varrho}(\partial_t \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}}) + \nabla_x p(\widetilde{\varrho}) - \mu \Delta_x \widetilde{\boldsymbol{u}} - \nu \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} = 0 \text{ on } \Omega^f.$$

We should point out that in (5.6) we only consider the consistency errors arising due to spatial discretization since we have a semi-discrete problem. Consequently, we have $t_{n+1} = \tau$ in the consistency estimates derived in Appendix B.3.

Step 2. In this step, we estimate the terms on the right hand side of (5.6) to obtain a relative energy inequality. We start with the estimate of R_i terms. Firstly, by using (C.4) we have

$$|R_1| \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t + \delta\mu \int_0^\tau \int_{\mathbb{T}^d} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t + h.$$

Next, it is obvious that

$$|R_2 + R_5| \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t.$$

Further, using (C.5) and (C.6) we get

$$|R_3 + R_4| \lesssim h + \frac{\epsilon_s}{h} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \delta\mu \left\|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\right\|^2_{L^2((0,\tau)\times\mathbb{T}^d)}$$
$$+ \delta\nu \left\|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}\right\|^2_{L^2((0,\tau)\times\mathbb{T}^d)} .$$

For the estimate of R_6 we apply (C.1h) and obtain

$$|R_6| \lesssim \int_0^\tau \int_{\Omega^s} |\boldsymbol{u}_h| \;\mathrm{d}x\mathrm{d}t \lesssim \frac{\epsilon_s}{\delta} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s}. \tag{5.9}$$

Next, we re-estimate the consistency error e_S. In view of (4.4) the consistency error of the continuity equation is controlled by

$$\left| e_\varrho \left(\tau, h, |\widetilde{\boldsymbol{u}}|^2/2 - \mathcal{P}'(\widetilde{\varrho})\right)\right| \lesssim h^{(1+\varepsilon)/2} + h^{(1+\beta_R)/2} + h^{1+\beta_D}.$$

Now we are left with $e_{\boldsymbol{m}}(\tau, h, -\widetilde{\boldsymbol{u}})$. In what follows we analyze $e_{\boldsymbol{m}}(\tau, h, -\widetilde{\boldsymbol{u}})$ term by term.

- Penalty term E_{ϵ_s}: With (5.5b) we have

$$|E_{\epsilon_s}| = \left|\frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega_h^s \setminus \Omega^s} \boldsymbol{u}_h \cdot \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t\right|$$
$$\lesssim \left|\frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega_h^s \setminus \Omega^s} \left(\delta \boldsymbol{u}_h^2 + \frac{1}{\delta}\widetilde{\boldsymbol{u}}^2\right) \mathrm{d}x\mathrm{d}t\right| \lesssim \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2(0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \epsilon_s^{-1} h^3. \tag{5.10}$$

- Viscosity terms $E_{\nabla_x \boldsymbol{u}}$: Recalling (C.2e) and (C.2f) we have

$$|E_{\nabla_x \boldsymbol{u}}| \lesssim \frac{h}{\delta} + h + \delta\mu \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \;\mathrm{d}x\mathrm{d}t + \delta\nu \int_0^\tau \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2 \;\mathrm{d}x\mathrm{d}t$$
$$\lesssim h + \delta\mu \left\|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\right\|^2_{L^2((0,\tau)\times\mathbb{T}^d)} + \delta\nu \left\|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}\right\|^2_{L^2((0,\tau)\times\mathbb{T}^d)} .$$

- Pressure term E_p: Recalling (C.2d) and (C.1b) we have

$$|E_p| \lesssim h + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\mathrm{d}t. \tag{5.11}$$

- $E_4(\varrho_h \boldsymbol{u}_h)$: Recalling (C.2c) and (C.1d) we have

$$|E_4(\varrho_h \boldsymbol{u}_h)| \lesssim h^2 + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\mathrm{d}t.$$

- $E_3(\varrho_h \boldsymbol{u}_h) = h^\varepsilon \int_0^\tau \int_{\mathcal{E}} [\![\varrho_h \boldsymbol{u}_h]\!] \, [\![\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}]\!] \;\mathrm{d}S_x\mathrm{d}t$: Recalling (3.3), (5.5d), and (C.1c)

$$|E_3(\varrho_h \boldsymbol{u}_h)| = h^{\varepsilon+1} \left| \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \Delta_h \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \;\mathrm{d}x\mathrm{d}t \,\mathrm{d}x\mathrm{d}t \right|$$

$$
\begin{aligned}
&\leq h^{\varepsilon+1}\left|\int_0^\tau \int_{\Omega_h^I} \varrho_h \boldsymbol{u}_h \Delta_h \Pi_{\mathcal{T}} \boldsymbol{u}\, \mathrm{d}x\mathrm{d}t\right| + h^{\varepsilon+1}\left|\int_0^\tau \int_{\Omega_h^C} \varrho_h \boldsymbol{u}_h \Delta_h \Pi_{\mathcal{T}} \boldsymbol{u}\, \mathrm{d}x\mathrm{d}t\right| \\
&\lesssim h^{\varepsilon+1} \left\|\varrho_h \boldsymbol{u}_h\right\|_{L^1((0,T)\times\mathbb{T}^d)} \|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,\infty}(\Omega^f;\mathbb{R}^d))} + h^\varepsilon \int_0^\tau \int_{\Omega_h^C} |\varrho_h \boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \\
&\lesssim h^\varepsilon \left(h + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\mathrm{d}t \right).
\end{aligned}
$$

- $E_2(\varrho_h \boldsymbol{u}_h)$: This term can be analyzed exactly as in [8], i.e.

$$
\begin{aligned}
&|E_2(\varrho_h \boldsymbol{u}_h)| \lesssim h \left\|\widetilde{\boldsymbol{u}}\right\|_{C_1(\mathbb{T}^d)} \left(\int_0^\tau \int_{\mathcal{E}} |[\![\boldsymbol{u}_h]\!]|^2 \,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h \boldsymbol{u}_h]\!]|^2 \,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \\
&\lesssim h \cdot h^{1/2} \cdot \left(\int_0^\tau \int_{\mathcal{E}} \left\{\!\!\left\{ |\varrho_h \boldsymbol{u}_h|^2 \right\}\!\!\right\} \ \mathrm{d}S_x\mathrm{d}t \right)^{1/2} \lesssim h^{1+\beta_M}.
\end{aligned}
$$

- $E_1(\varrho_h \boldsymbol{u}_h)$: Recalling the integration by parts formula (3.3), the estimates (C.1j) and (C.1k) we get

$$
\begin{aligned}
|E_1(\varrho_h \boldsymbol{u}_h)| &= \left| \int_0^\tau \int_{\mathcal{E}} [\![\varrho_h \boldsymbol{u}_h]\!] \, [\![|\{\!\{ \boldsymbol{u}_h \}\!\} \cdot \boldsymbol{n}| \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}]\!] \ \mathrm{d}S_x\mathrm{d}t \right| \\
&= \left| h \sum_{i=1}^d \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \eth_{\mathcal{T}}^{(i)} \left(|\{\!\{ u_{i,h} \}\!\}| \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \right) \ \mathrm{d}x\mathrm{d}t \right| \\
&= \left| h \sum_{i=1}^d \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \left(\Delta_h^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \Pi_{\mathcal{T}} |\{\!\{ u_{i,h} \}\!\}| + (\Pi_{\mathcal{T}} \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \eth_{\mathcal{T}}^{(i)} |\{\!\{ u_{i,h} \}\!\}| \right) \ \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim h^{1+\beta_M} + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\mathrm{d}t.
\end{aligned}
$$

In summary, choosing any $\delta \in (0,1)$ and collecting all terms we obtain the relative energy inequality:

$$
\begin{aligned}
&[R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})]_{t=0}^{t=\tau} + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 + \nu |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2 \right) \ \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega_h^s} |\boldsymbol{u}_h|^2 \ \mathrm{d}x\mathrm{d}t \lesssim (1 + h^\varepsilon) \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\mathrm{d}t + h^{\beta_{RE}} + \frac{h^3}{\epsilon_s} + \epsilon_s + \frac{\epsilon_s}{h}
\end{aligned}
\tag{5.12}
$$

with

$$
\beta_{RE} = \min((1+\varepsilon)/2,\ (1+\beta_R)/2,\ 1+\beta_D,\ 1+\beta_M). \tag{5.13}
$$

Let us focus on the case: $\varepsilon \geq 0$. From the study of convergence, cf. Theorem 4.6, we always take $\varepsilon \in [0, 2(\gamma-1))$ for the case of $d = 3, \gamma \leq \frac{3}{2}$. Under this condition, after straightforward case-by-case calculations we obtain

$$
\beta_{RE} = \min((1+\beta_R)/2,\ 1+\beta_M) = \begin{cases} 1+\beta_M, & \text{if } d=2, \gamma \leq \min\left(\frac{4}{3}, 1+\varepsilon\right), \\ \frac{1}{2}, & \text{if } d=2, \gamma > \min\left(\frac{4}{3}, 1+\varepsilon\right), \\ 1+\beta_M, & \text{if } d=3, \gamma < 2, \\ \frac{1}{2}, & \text{if } d=3, \gamma \geq 2. \end{cases} \tag{5.14}
$$

Step 3. Applying Gronwall's lemma, together with the continuity of the initial data, we obtain

$$R_E(\varrho_h, \boldsymbol{u}_h|\widetilde{\varrho}, \widetilde{\boldsymbol{u}}) + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 + \nu|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2\right) \mathrm{d}x\mathrm{d}t + \frac{1}{\epsilon_s}\int_0^\tau \int_{\Omega_h^s} |\boldsymbol{u}_h|^2 \, \mathrm{d}x\mathrm{d}t$$
$$\lesssim \frac{h^3}{\epsilon_s} + \frac{\epsilon_s}{h} + h^{\beta_{RE}} + R_E(\varrho_h^0, \boldsymbol{u}_h^0|\Pi_{\mathcal{T}}\widetilde{\varrho}_0, \Pi_{\mathcal{T}}\widetilde{\boldsymbol{u}}_0) \lesssim \frac{h^3}{\epsilon_s} + \frac{\epsilon_s}{h} + h^{\beta_{RE}}.$$

Summarizing the above calculations we have obtained the following error estimates.

Theorem 5.5. (**Error estimates**) *Let $\gamma > 1$. Suppose that the Navier–Stokes system (1.1) admits a classical solution $(\varrho, \boldsymbol{u})$ belonging to the class (4.11) with initial data $(\varrho_0, \boldsymbol{u}_0)$ satisfying $\varrho_0 > 0$. Let $\{\varrho_h, \boldsymbol{u}_h\}_{h\downarrow 0}$ be a family of numerical solutions obtained by the FV method (3.4) with a smoothly extended initial data $(\widetilde{\varrho}_0, \widetilde{\boldsymbol{u}}_0) \in W^{1,\infty}(\mathbb{T}^d)$.*

Let $\varepsilon \geq 0, h \in (0,1)$. If $d = 3$ and $\gamma \leq \frac{3}{2}$ we additionally assume $\varepsilon < 2(\gamma - 1)$. Then the following error estimate holds

$$R_E(\varrho_h, \boldsymbol{u}_h|\widetilde{\varrho}, \widetilde{\boldsymbol{u}}) + \int_0^\tau \int_{\mathbb{T}^d} \left(\mu|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 + \nu|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2\right) \mathrm{d}x\mathrm{d}t + \frac{1}{\epsilon_s}\int_0^\tau \int_{\Omega^s} |\boldsymbol{u}_h|^2 \, \mathrm{d}x\mathrm{d}t$$
$$\lesssim h^{\beta_{RE}} + \frac{h^3}{\epsilon_s} + \frac{\epsilon_s}{h}, \tag{5.15}$$

where β_{RE} is defined in (5.14).

Remark 5.6. *The error estimates in Theorem 5.5 confirm convergence of the penalty method (3.4). By a closer inspection we see that the first error term $h^{\beta_{RE}}$ in (5.15) is less than $\sqrt{h}$ and choosing, e.g., $\epsilon_s = h^2$, the second and third error terms are $\mathcal{O}(h)$.*

6 Numerical experiments

In this section, we aim to validate our theoretical convergence results. To this end, we compute the following errors with respect to the reference solution with a fixed ϵ_s and a small parameter h_{ref}:

$$E_\varrho^{\epsilon_s} = \|\varrho_h^{\epsilon_s} - \varrho_{h_{ref}}^{\epsilon_s}\|_{L^\gamma(\mathbb{T}^2)}, \quad E_{\boldsymbol{u}}^{\epsilon_s} = \|\boldsymbol{u}_h^{\epsilon_s} - \boldsymbol{u}_{h_{ref}}^{\epsilon_s}\|_{L^2(\mathbb{T}^2)},$$
$$E_{\nabla \boldsymbol{u}}^{\epsilon_s} = \|\nabla_{\mathcal{E}} \boldsymbol{u}_h^{\epsilon_s} - \nabla_{\mathcal{E}} \boldsymbol{u}_{h_{ref}}^{\epsilon_s}\|_{L^2(\mathbb{T}^2)}, \quad R_E^{\epsilon_s} = R_E\left(\varrho_h^{\epsilon_s}, \boldsymbol{u}_h^{\epsilon_s} \mid \varrho_{h_{ref}}^{\epsilon_s}, \boldsymbol{u}_{h_{ref}}^{\epsilon_s}\right)$$

and with a reference parameter pair $(h_{ref}, \epsilon_{s,ref})$:

$$E_\varrho = \|\varrho_h^{\epsilon_s} - \varrho_{h_{ref}}^{\epsilon_{s,ref}}\|_{L^\gamma(\mathbb{T}^2)}, \quad E_{\boldsymbol{u}} = \|\boldsymbol{u}_h^{\epsilon_s} - \boldsymbol{u}_{h_{ref}}^{\epsilon_{s,ref}}\|_{L^2(\mathbb{T}^2)},$$
$$E_{\nabla \boldsymbol{u}} = \|\nabla_{\mathcal{E}} \boldsymbol{u}_h^{\epsilon_s} - \nabla_{\mathcal{E}} \boldsymbol{u}_{h_{ref}}^{\epsilon_{s,ref}}\|_{L^2(\mathbb{T}^2)}, \quad R_E = R_E\left(\varrho_h^{\epsilon_s}, \boldsymbol{u}_h^{\epsilon_s} \mid \varrho_{h_{ref}}^{\epsilon_{s,ref}}, \boldsymbol{u}_{h_{ref}}^{\epsilon_{s,ref}}\right).$$

In the simulation we take the following parameters

$$\varepsilon = 0.6, \; T = 0.1, \; \mu = 0.1, \; \nu = 0, \; a = 1, \; \gamma = 1.4.$$

We point out that $E_\varrho^{\epsilon_s}, E_{\boldsymbol{u}}^{\epsilon_s}, E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}, R_E^{\epsilon_s}$ are used to verify the convergence rate only with respect to mesh parameter h, cf. Theorem 4.6. Errors $E_\varrho, E_{\boldsymbol{u}}, E_{\nabla_x \boldsymbol{u}}, R_E$ (with respect to the parameter pair $(h, \epsilon_s(h))$) are used to illustrate our convergence results in Theorem 4.7, Theorem 4.9 and Theorem 5.5.

6.1 Experiment 1: Ring domain - continuous extension

In this experiment we take the physical fluid domain to be a ring, i.e. $\Omega^f \equiv B_{0.7} \setminus B_{0.2}$, where $B_r = \{x \mid |x| < r\}$. The initial data (including the smooth extension) are given by

$$(\varrho, \boldsymbol{u})(0, x) = \begin{cases} (1,\, 0,\, 0), & x \in B_{0.2}, \\ \left(1,\, \frac{\sin(4\pi(|x|-0.2))x_2}{|x|},\, -\frac{\sin(4\pi(|x|-0.2))x_1}{|x|}\right), & x \in \Omega^f \equiv B_{0.7} \setminus B_{0.2}, \\ (1,\, 0,\, 0), & x \in \mathbb{T}^2 \setminus B_{0.7}. \end{cases}$$

Figure 4 shows the numerical solutions ϱ_h and $\boldsymbol{u}_h$ at time $T = 0.1$ with fixed mesh size $h = 0.2 \cdot 2^{-4}$ but different penalization parameter $\epsilon_s = 4^{-3}, \ldots, 4^{-6}$. We can observe that the velocity vanishes in the penalized region with decreasing ϵ_s. Further, in Figure 5 we present the errors $E_{\varrho}^{\epsilon_s}, E_{\boldsymbol{u}}^{\epsilon_s}, E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}, R_E^{\epsilon_s}$ with respect to $h = 0.2 \cdot 2^{-m}, m = 0, \ldots, 4$ for fixed $\epsilon_s \in \{4^{-2}, 4^{-3}, 4^{-4}, 4^{-5}, 4^{-6}\}$. Figure 6 depicts the errors $E_{\varrho}, E_{\boldsymbol{u}}, E_{\nabla_x \boldsymbol{u}}, R_E$ with respect to the parameter pair $(h, \epsilon_s(h))$ with $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^{1/2}))$, $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^2))$ and $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^4))$.

Our numerical results indicate first order convergence rate for $\varrho, \boldsymbol{u}, \nabla_x \boldsymbol{u}$ and second order convergence rate for R_E. We would like to point out that these experimental convergence rates are better than our theoretical result. This implies suboptimality of theoretical error estimates. Deviation of optimal theoretical error estimates is a challenging task for future study.

6.2 Experiment 2: Ring domain - discontinuous extension

In the second experiment we consider the same physical fluid domain, but different initial extension of density, i.e.

$$(\varrho, \boldsymbol{u})(0, x) = \begin{cases} (0.01,\, 0,\, 0), & x \in B_{0.2}, \\ \left(1,\, \frac{\sin(4\pi(|x|-0.2))x_2}{|x|},\, -\frac{\sin(4\pi(|x|-0.2))x_1}{|x|}\right), & x \in \Omega^f \equiv B_{0.7} \setminus B_{0.2}, \\ (2,\, 0,\, 0), & x \in \mathbb{T}^2 \setminus B_{0.7}. \end{cases}$$

The effect of different penalization parameters $\epsilon_s = 4^{-3}, \ldots, 4^{-6}$ is present in Figure 7. The errors $E_{\varrho}^{\epsilon_s}, E_{\boldsymbol{u}}^{\epsilon_s}, E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}, R_E^{\epsilon_s}$ with respect to h for fixed penalization parameters are shown in Figure 8, and Figure 9 gives the errors $E_{\varrho}, E_{\boldsymbol{u}}, E_{\nabla_x \boldsymbol{u}}, R_E$ with respect to the pair $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^{1/2}))$, $(h, \mathcal{O}(h^2))$ and $(h, \mathcal{O}(h^4))$. Figures 8 and 9 indicate a similar convergence behaviour as in Experiment 1.

6.3 Experiment 3: Complex domain - discontinuous extension

In the last experiment, we consider a more complicated geometry of the fluid domain, i.e.

$$\Omega^f = \hat{B}_{0.7} \setminus B_{0.2}, \quad \hat{B}_{0.7} := \left\{ x \,\middle|\, |x| < (0.7 + \delta) + \delta \cos(8\phi),\ \tan(\phi) = \frac{x}{y} \right\}.$$

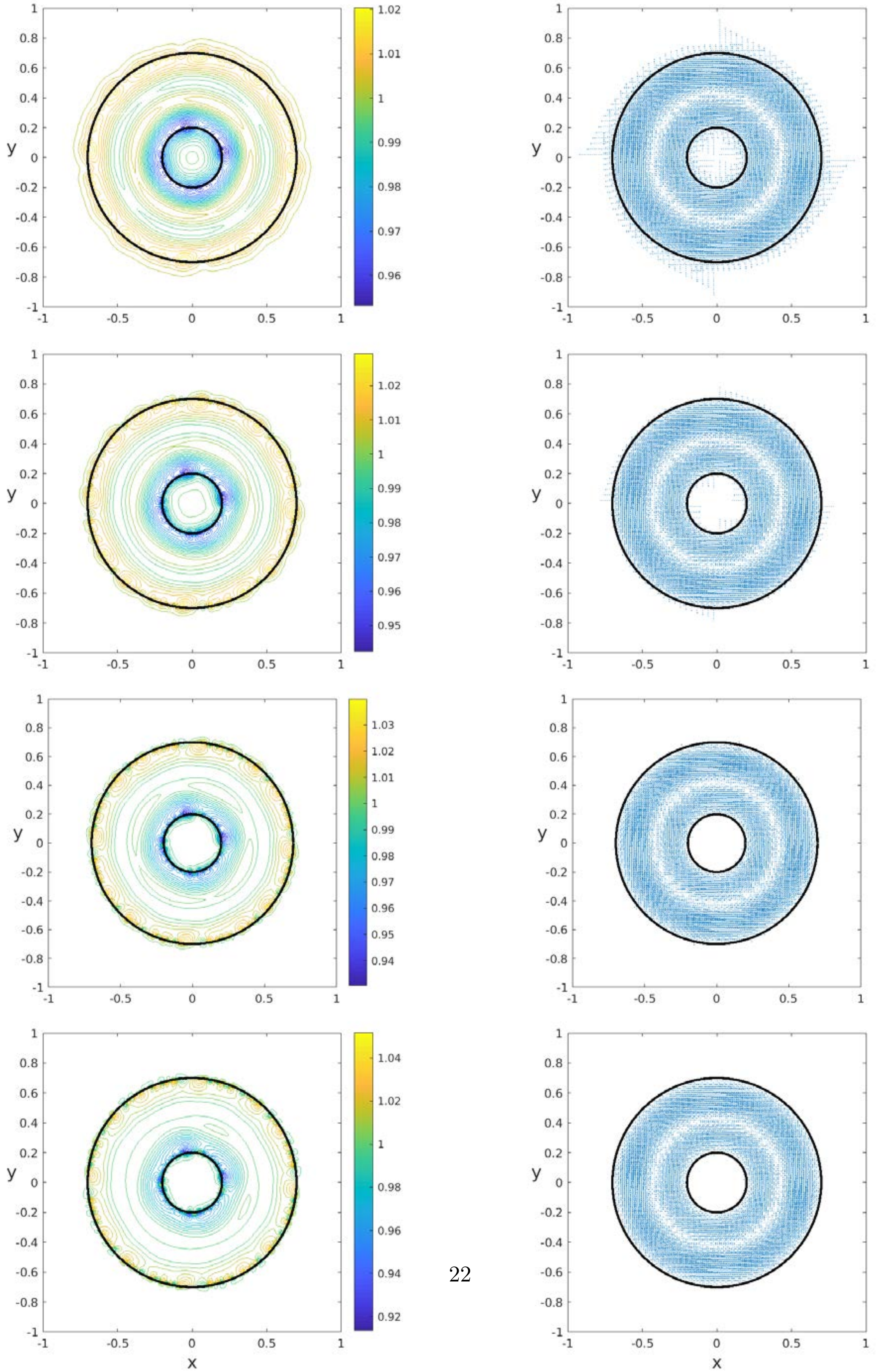
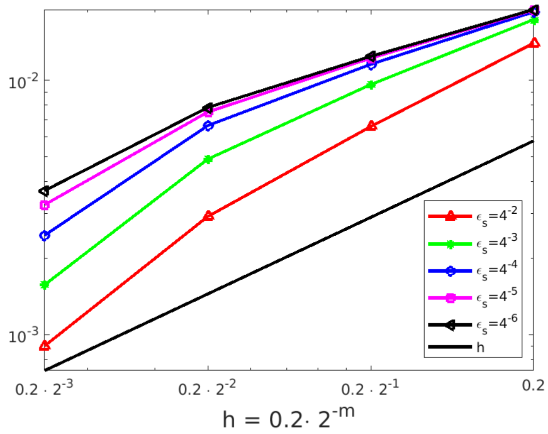
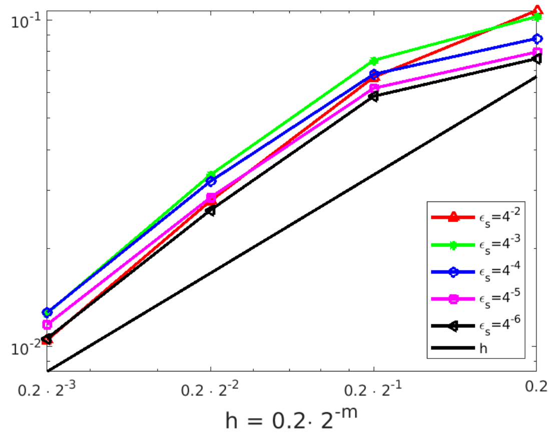


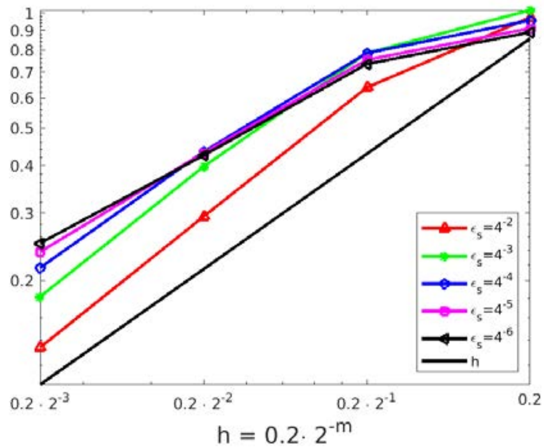
Figure 4: Experiment 1: Numerical solutions ϱ_h (left) and $\boldsymbol{u}_h$ (right) obtained with $h = 0.2 \cdot 2^{-4}$ for different $\epsilon_s = 4^{-m-2}, m = 1, \ldots, 4$ from top to bottom.



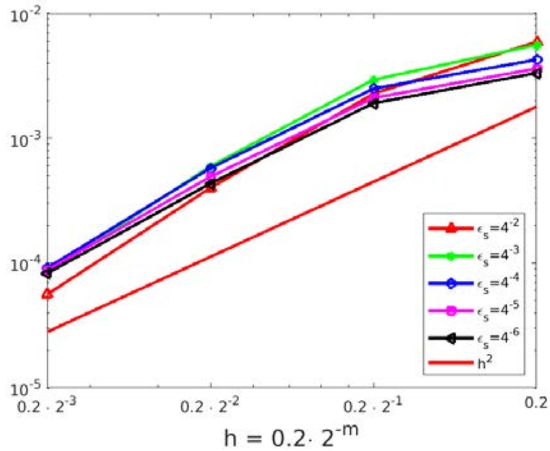
(a) $E_{\varrho}^{\epsilon_s}$



(b) $E_{\boldsymbol{u}}^{\epsilon_s}$



(c) $E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}$



(d) $R_E^{\epsilon_s}$

Figure 5: Experiment 1: The errors $E_{\varrho}^{\epsilon_s}, E_{\boldsymbol{u}}^{\epsilon_s}, E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}, R_E^{\epsilon_s}$ with respect to h for different but fixed ϵ_s. The black and red solid lines without any marker denote the reference slope of h and h^2, respectively.

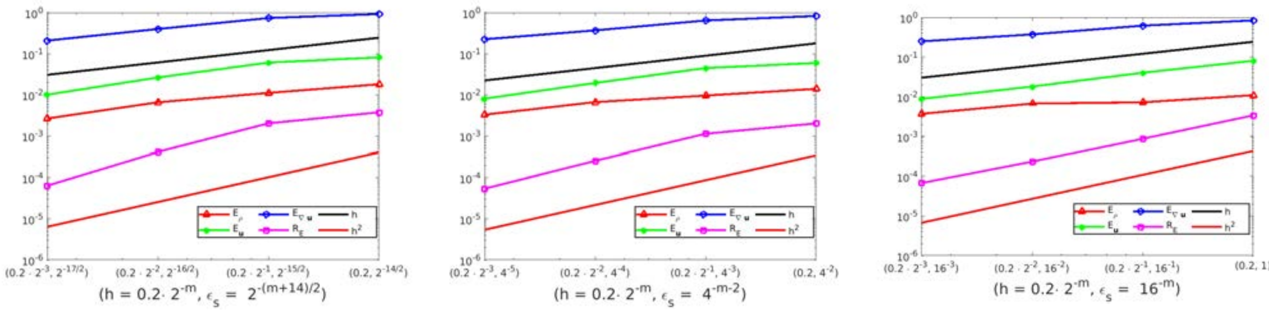


Figure 6: Experiment 1: Errors $E_{\varrho}, E_{\boldsymbol{u}}, E_{\nabla_x \boldsymbol{u}}$ and relative energy R_E with respect to the pair $(h, \epsilon_s(h))$ for $h \in h_s$, $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^{1/2}))$ (left), $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^2))$ (middle) and $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^4))$ (right). The black and red solid lines without any marker denote the reference slope of h and h^2, respectively.

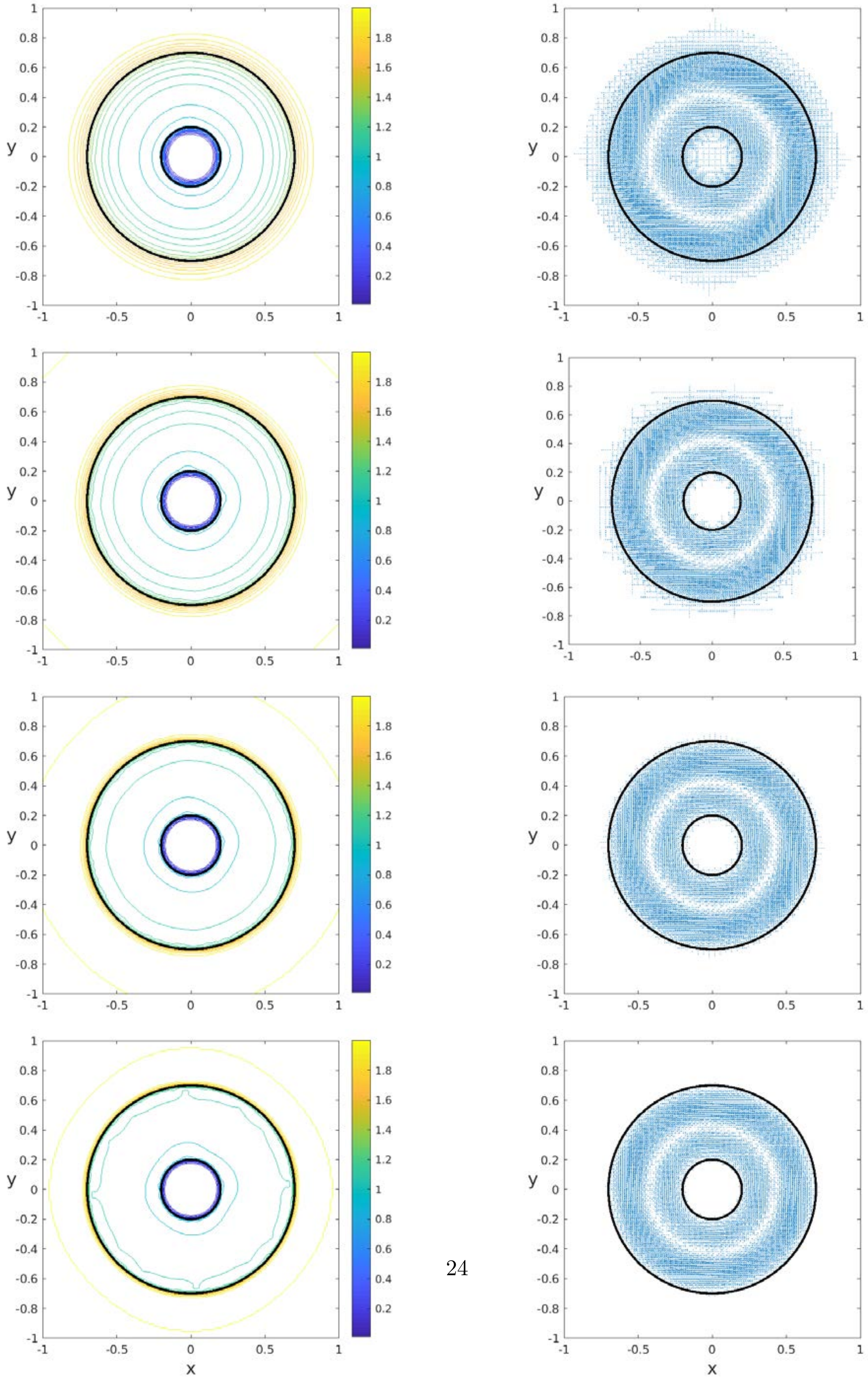
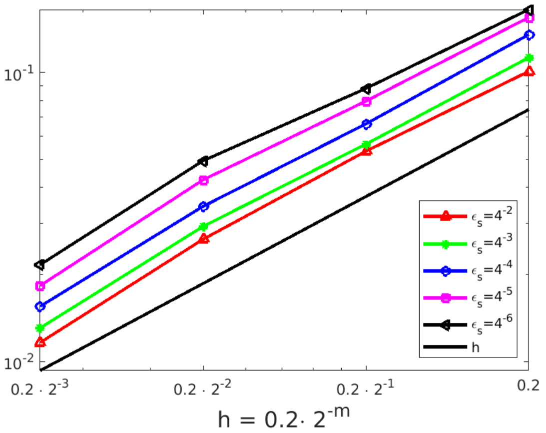
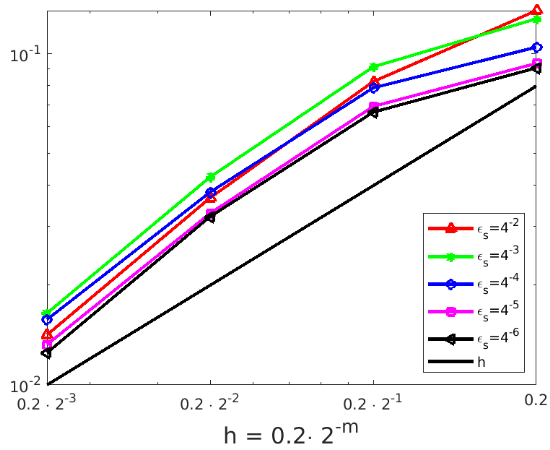


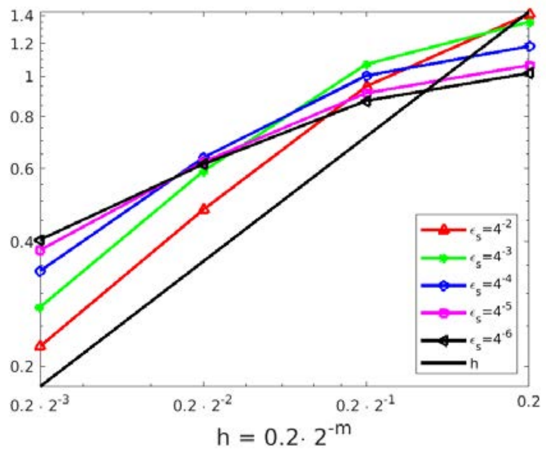
Figure 7: Experiment 2: Numerical solutions ϱ_h (left) and $\boldsymbol{u}_h$ (right) obtained with $h = 0.2 \cdot 2^{-4}$ for different $\epsilon_s = 4^{-m-2}, m = 1, \ldots, 4$ from top to bottom.



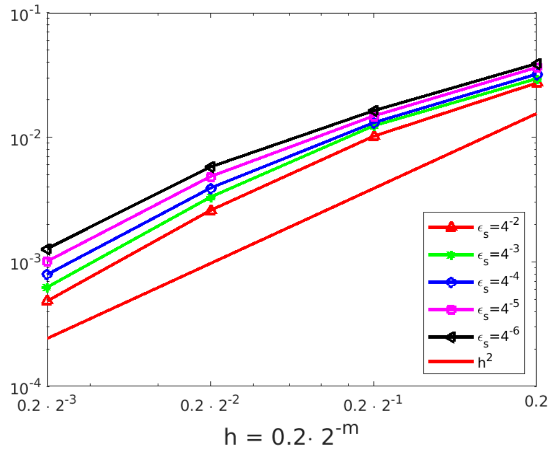
(a) $E_{\varrho}^{\epsilon_s}$



(b) $E_{\boldsymbol{u}}^{\epsilon_s}$



(c) $E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}$



(d) $R_E^{\epsilon_s}$

Figure 8: Experiment 2: The errors $E_{\varrho}^{\epsilon_s}, E_{\boldsymbol{u}}^{\epsilon_s}, E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}, R_E^{\epsilon_s}$ with respect to h for different but fixed ϵ_s. The black and red solid lines without any marker denote the reference slope of h and h^2, respectively.

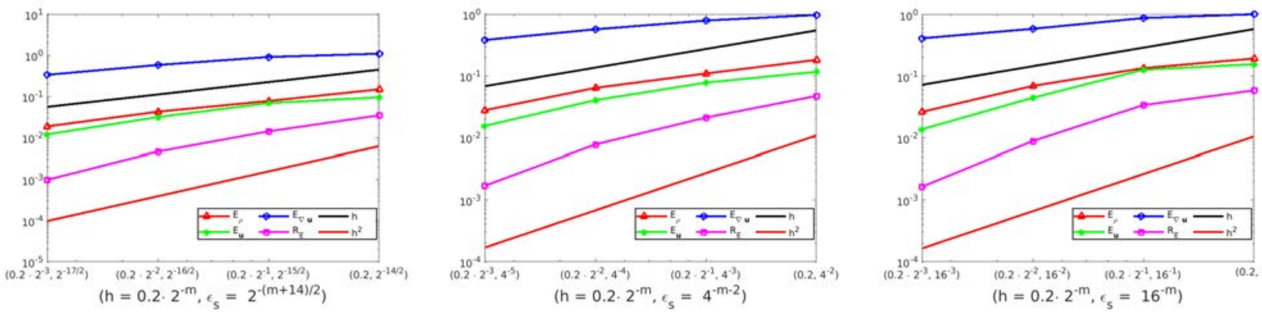


Figure 9: Experiment 2: Errors $E_{\varrho}, E_{\boldsymbol{u}}, E_{\nabla_x \boldsymbol{u}}$ and relative energy R_E with respect to the pair $(h, \epsilon_s(h))$ for $h \in h_s$, $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^{1/2}))$ (left), $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^2))$ (middle) and $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^4))$ (right). The black and red solid lines without any marker denote the reference slope of h and h^2, respectively.

The initial data (including a discontinuous extension) are given by

$$(\varrho, \boldsymbol{u})(0,x) = \begin{cases} (0.01,\ 0,\ 0), & x \in B_{0.2}, \\ \left(1,\ \frac{\left[1-\cos(8\pi(|x|-0.2))\right]x_2}{|x|},\ -\frac{\left[1-\cos(8\pi(|x|-0.2))\right]x_1}{|x|}\right), & x \in B_{0.45} \setminus B_{0.2}, \\ \left(1,\ \frac{\left[-1+\cos(8\pi(|x|-0.2))\right]x_2}{|x|},\ -\frac{\left[-1+\cos(8\pi(|x|-0.2))\right]x_1}{|x|}\right), & x \in B_{0.7} \setminus B_{0.45}, \\ (1,\ 0,\ 0), & x \in \hat{B}_{0.7} \setminus B_{0.7}, \\ (0.01,\ 0,\ 0), & x \in \mathbb{T}^2 \setminus \hat{B}_{0.7}. \end{cases}$$

In the simulation we take $\delta = 0.05$ and compute till $T = 0.1$. The numerical solutions ϱ_h and $\boldsymbol{u}_h$ at time $T = 0.1$ with fixed mesh size $h = 0.2 \cdot 2^{-4}$ but different penalization parameter $\epsilon_s = 4^{-m-2}, m = 1, \ldots, 4$ are presented in Figure 10. Figure 11 shows the errors with respect to h for fixed ϵ_s. The errors with respect to the pair $(h, \epsilon_s(h))$ are displayed in Figure 12. Analogously as above, the numerical results indicate a first order convergence rate for the numerical solutions $\varrho, \boldsymbol{u}, \nabla_x \boldsymbol{u}$ and a second order convergence rate for the relative energy R_E.

Let us point out that the initial data (including the extension) in Experiments 2 and 3 only belong to the class $L^\infty(\mathbb{T}^d)$, which is weaker than the assumption in Theorem 5.5. Nevertheless, the numerical results still indicate the strong convergence with first order convergence rate for numerical solutions.

In addition, our analysis of error estimates, cf. Theorem 5.5, holds for $\epsilon_s \in (\mathcal{O}(h^3), \mathcal{O}(h))$. Again, the numerical results indicate that the convergence rates with $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^{1/2})), (h, \mathcal{O}(h^4))$ are similar as those with $(h, \mathcal{O}(h^2))$, cf. Figures 6, 9, 12. Extension of our theoretical error estimates to a more general setting as indicated in Experiments 2 and 3 is an interesting task for future work.

A Preliminaries

First, we recall a generalized Sobolev-Poincaré inequality, see [8, Theorem 17].

Lemma A.1 ([8]). *Let $\gamma > 1$ and $\varrho_h \geq 0$ satisfy*

$$0 < c_M \leq \int_{\mathbb{T}^d} \varrho_h \,\mathrm{d}x \ \text{ and } \int_{\mathbb{T}^d} \varrho_h^\gamma \,\mathrm{d}x \leq c_E,$$

where $\gamma > 1$, c_M and c_E are positive constants. Then there exists $c = c(c_M, c_E, \gamma)$ independent of h such that

$$\|f_h\|^2_{L^q(\mathbb{T}^d)} \leq c \left(\|\nabla_{\mathcal{E}} f_h\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |f_h|^2 \,\mathrm{d}x \right),$$

where $q = 6$ if $d = 3$, and $q \in [1, \infty)$ if $d = 2$.

Next, we recall the following essential-residual splitting from [8, Lemma 14.3].

Lemma A.2 ([8]). *Let $\gamma > 1$, $\varrho \geq 0$, $\underline{r} = \frac{1}{2} \min_{(t,x)\in Q_T} \widetilde{\varrho} > 0$ and $\overline{r} = 2 \max_{(t,x)\in Q_T} \widetilde{\varrho}$. Then there exists $C = C(\underline{r}, \overline{r}) > 0$ such that*

$$(\varrho - \widetilde{\varrho})^2 \mathbb{1}_{\text{ess}}(\varrho) + \varrho \mathbb{1}_{\text{res}}(\varrho) \stackrel{<}{\sim} (\varrho - \widetilde{\varrho})^2 \mathbb{1}_{\text{ess}}(\varrho) + (1 + \varrho^\gamma) \mathbb{1}_{\text{res}}(\varrho) \leq C \mathbb{E}(\varrho \,|\, \widetilde{\varrho}), \tag{A.1}$$

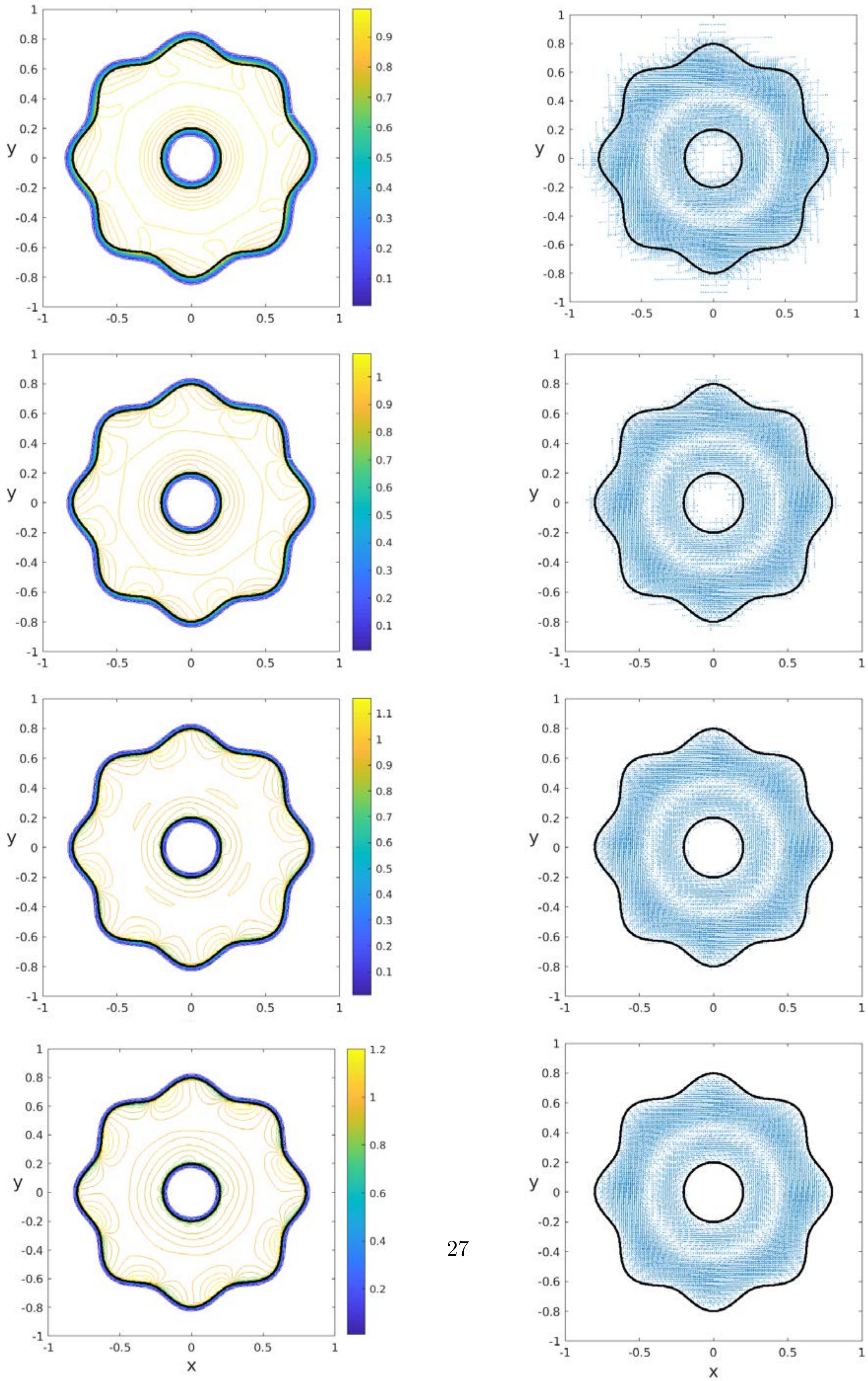
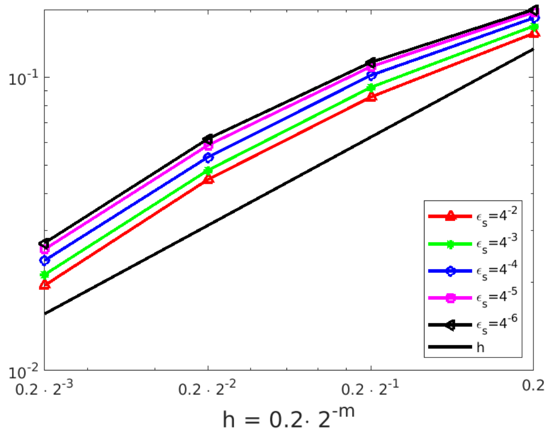
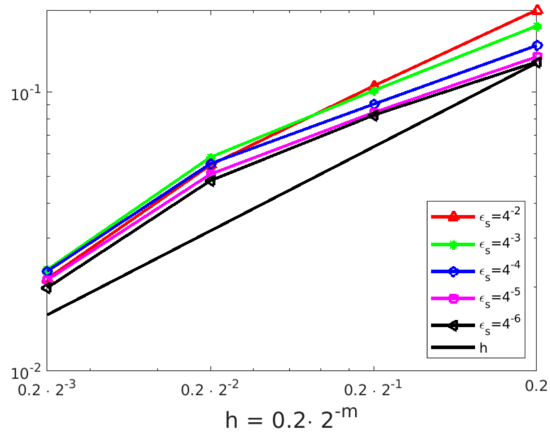


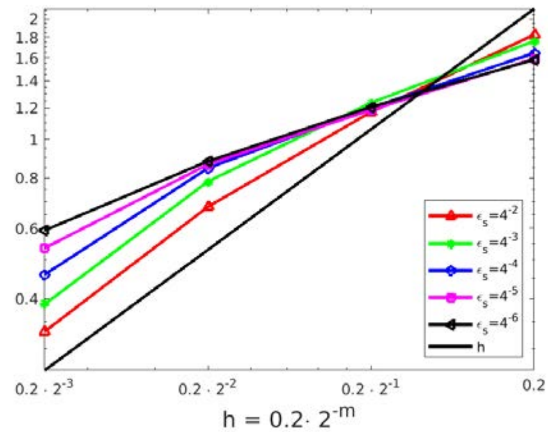
Figure 10: Experiment 3: Numerical solutions ϱ_h (left) and $\boldsymbol{u}_h$ (right) obtained with $h = 0.2 \cdot 2^{-4}$ for different $\epsilon_s = 4^{-m-2}, m = 1, \dots, 4$ from top to bottom.



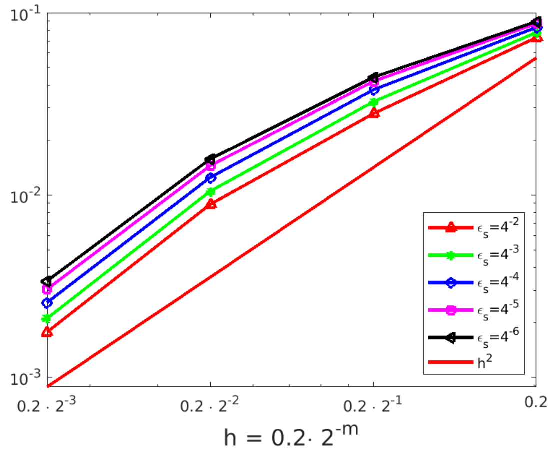
(a) $E_{\varrho}^{\epsilon_s}$



(b) $E_{\boldsymbol{u}}^{\epsilon_s}$



(c) $E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}$



(d) $R_E^{\epsilon_s}$

Figure 11: Experiment 3: The errors $E_{\varrho}^{\epsilon_s}, E_{\boldsymbol{u}}^{\epsilon_s}, E_{\nabla_x \boldsymbol{u}}^{\epsilon_s}, R_E^{\epsilon_s}$ with respect to h for different but fixed ϵ_s. The black and red solid lines without any marker denote the reference slope of h and h^2, respectively.

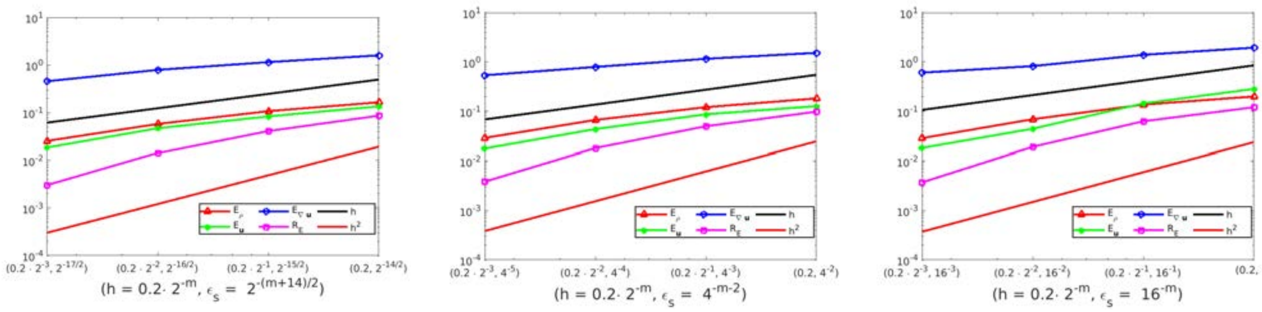


Figure 12: Experiment 3: Errors $E_{\varrho}, E_{\boldsymbol{u}}, E_{\nabla_x \boldsymbol{u}}$ and relative energy R_E with respect to the pair $(h, \epsilon_s(h))$ for $h \in h_s$, $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^{1/2}))$ (left), $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^2))$ (middle) and $(h, \epsilon_s(h)) = (h, \mathcal{O}(h^4))$ (right). The black and red solid lines without any marker denote the reference slope of h and h^2, respectively.

where $\mathbb{E}(\varrho\,|\,\widetilde{\varrho})$ *is defined in* (5.1) *and*

$$(\mathbb{1}_{\rm ess}(\varrho), \mathbb{1}_{\rm res}(\varrho)) :\,= (\mathbb{1}_{[\underline{r},\overline{r}]}(\varrho), \mathbb{1}_{\mathbb{R}^+\setminus[\underline{r},\overline{r}]}(\varrho)) = \begin{cases} (1,0) & \text{if } \varrho \in [\underline{r},\overline{r}], \\ (0,1) & \text{if } \varrho \in \mathbb{R}^+ \setminus [\underline{r},\overline{r}]. \end{cases} \tag{A.2}$$

Further, we recall [9] for the following estimates that are useful in our error analysis.

Lemma A.3. *Let* $\gamma > 1$*,* $\varrho_h > 0$ *and*

$$\widetilde{\varrho} \in [\underline{\varrho}, \overline{\varrho}], \quad |\widetilde{\boldsymbol{u}}| \leq \overline{u}, \quad \underline{\varrho}, \overline{u} > 0.$$

Then the following estimates hold

$$\|\widetilde{\varrho} - \varrho_h\|^2_{L^\gamma(\mathbb{T}^d)} + \|\varrho_h \boldsymbol{u}_h - \widetilde{\varrho}\widetilde{\boldsymbol{u}}\|^2_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d)} \lesssim R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}), \quad \text{if } \gamma \leq 2, \tag{A.3}$$

$$\|\widetilde{\varrho} - \varrho_h\|^\gamma_{L^\gamma(\mathbb{T}^d)} + \|\widetilde{\varrho} - \varrho_h\|^2_{L^2(\mathbb{T}^d)} \lesssim R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}), \quad \text{if } \gamma > 2, \tag{A.4}$$

$$\|\varrho_h \boldsymbol{u}_h - \widetilde{\varrho}\widetilde{\boldsymbol{u}}\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d)} \lesssim \left(R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\right)^{1/2} + \left(R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}})\right)^{1/\gamma}, \quad \text{if } \gamma > 2. \tag{A.5}$$

B Proof of consistency formulation

In this section we are devoted to showing the consistency formulation stated in Lemma 4.4. For simplicity, hereafter we frequently write $L^p(0,T;L^q(\mathbb{T}^d))$ as L^pL^q.

B.1 Negative estimates of density and momentum

To begin, we introduce two negative density estimates.

Lemma B.1. *Let* $\varepsilon > -1$*,* $h \in (0,1)$*,* $(\varrho_h, \boldsymbol{u}_h)$ *be a solution of the FV scheme* (3.4)*,* $s \in (0,\gamma]$*. If* $d = 2$ *then* $p > 1$*, if* $d = 3$ *then* $p = 6$*. Then for* $\gamma \in (1,2]$ *it holds that*

$$\|\varrho_h\|^\gamma_{L^\gamma L^{p\gamma/2}} = \left\|\varrho_h^{\gamma/2}\right\|^2_{L^2L^p} = \int_0^T \left\|\varrho_h^{\gamma/2}\right\|^2_{L^p} \mathrm{d}t \lesssim h^{-\varepsilon-1}, \tag{B.1}$$

$$\|\varrho_h\|^s_{L^sL^\infty} = \int_0^T \|\varrho_h\|^s_{L^\infty} \mathrm{d}t \lesssim h^{-s(2d+p(\varepsilon+1))/(p\gamma)}. \tag{B.2}$$

Proof. First, it it easy to check the equivalence of the norms in (B.1), i.e.,

$$\|\varrho_h\|^\gamma_{L^\gamma L^{p\gamma/2}} = \int_0^T \left(\int_{\mathbb{T}^d} \varrho_h^{p\gamma/2}\,\mathrm{d}x\right)^{\frac{2}{p\gamma}\times\gamma} \mathrm{d}t = \int_0^T \left(\int_{\mathbb{T}^d} (\varrho_h^{\gamma/2})^p\,\mathrm{d}x\right)^{\frac{1}{p}\times 2} \mathrm{d}t = \int_0^T \left\|\varrho_h^{\gamma/2}\right\|^2_{L^p} \mathrm{d}t = \left\|\varrho_h^{\gamma/2}\right\|^2_{L^2L^p}.$$

Further, recalling the proof of [8, Lemma 11.4], we deduce from the estimate (4.3b) that

$$\left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|^2_{L^2L^2} \lesssim h^{-1}\int_0^T \int_{\mathcal{E}} \mathcal{P}''(\varrho_{h,\dagger})\,[\![\varrho_h]\!]^2\,\mathrm{d}S_x\mathrm{d}t \lesssim h^{-\varepsilon-1} \quad \text{for } \gamma \in (1,2].$$

Next, applying the Sobolev-Poincaré inequality, cf. Lemma A.1, and the density estimate in (4.3a) to get

$$\int_0^T \left\|\varrho_h^{\gamma/2}\right\|_{L^p}^2 \,\mathrm{d}t \lesssim \int_0^T \left(\left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2}^2 + \left\|\varrho_h^{\gamma/2}\right\|_{L^2}^2\right) \mathrm{d}t = \left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2L^2}^2 + \|\varrho_h\|_{L^\gamma L^\gamma}^\gamma \lesssim h^{-\varepsilon-1},$$

where $p > 1$ in the case of $d = 2$ and $p = 6$ for the case of $d = 3$. Together with the inverse estimates we obtain

$$\begin{aligned}
&\int_0^T \|\varrho_h\|_{L^\infty}^s \,\mathrm{d}t = \int_0^T \left\|\varrho_h^{\gamma/2}\right\|_{L^\infty}^{2s/\gamma} \mathrm{d}t \lesssim \int_0^T \left(h^{-d/p}\left\|\varrho_h^{\gamma/2}\right\|_{L^p}\right)^{2s/\gamma} \mathrm{d}t = h^{-2sd/(p\gamma)} \int_0^T \left(\left\|\varrho_h^{\gamma/2}\right\|_{L^p}^2\right)^{s/\gamma} \mathrm{d}t \\
&\lesssim h^{-2sd/(p\gamma)} \int_0^T \left(\left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2}^{2s/\gamma} + \left\|\varrho_h^{\gamma/2}\right\|_{L^2}^{2s/\gamma}\right) \mathrm{d}t = h^{-2sd/(p\gamma)} \left(\left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^{2s/\gamma}L^2}^{2s/\gamma} + \|\varrho_h\|_{L^sL^\gamma}^s\right) \\
&\lesssim h^{-2sd/(p\gamma)} \left(\left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2L^2}^{2s/\gamma} + \|\varrho_h\|_{L^\infty L^\gamma}^s\right) \lesssim h^{-2sd/(p\gamma)} \left((h^{-\varepsilon-1})^{s/\gamma} + 1\right) \\
&\lesssim h^{-s(2d+p(\varepsilon+1))/(p\gamma)},
\end{aligned}$$

which completes the proof. □

Further, we obtain the following negative estimates of density and momentum.

Lemma B.2 (Negative estimates of density and momentum)**.** *Let $(\varrho_h, \boldsymbol{u}_h)$ be a solution of the FV scheme* (3.4) *with $h \in (0,1), \varepsilon > -1$ and $\gamma > 1$. Then the following hold:*

$$\|\varrho_h\|_{L^2((0,T)\times\mathbb{T}^d)} \lesssim h^{\beta_D}, \qquad \beta_D = \begin{cases} \min\limits_{p\in[1,\infty)} \left\{\frac{p(\varepsilon+1)+4}{2p}, 1\right\} \cdot \frac{\gamma-2}{\gamma}, & \text{if } d=2, \gamma\in(1,2), \\ \min\left\{\frac{\varepsilon+2}{3}, 1\right\} \cdot \frac{3(\gamma-2)}{2\gamma}, & \text{if } d=3, \gamma\in(1,2), \\ 0, & \text{if } \gamma \geq 2, \end{cases} \tag{B.3}$$

$$\|\varrho_h\|_{L^2(0,T;L^{6/5}(\mathbb{T}^d))} \lesssim h^{\widetilde{\beta_R}}, \qquad \widetilde{\beta_R} = \begin{cases} \min\limits_{p\in\left[\frac{12}{5\gamma},\infty\right)} \left\{\frac{(1+\varepsilon)p}{2(p-2)}, 1\right\} \cdot \frac{5\gamma-6}{3\gamma}, & \text{if } d=2, \gamma\in\left(1,\frac{6}{5}\right), \\ \min\left\{\frac{1+\varepsilon}{2}, 1\right\} \cdot \frac{5\gamma-6}{2\gamma}, & \text{if } d=3, \gamma\in\left(1,\frac{6}{5}\right), \\ 0, & \text{if } \gamma \geq \frac{6}{5}, \end{cases} \tag{B.4}$$

$$\|\varrho_h\boldsymbol{u}_h\|_{L^2((0,T)\times\mathbb{T}^d)} \lesssim h^{\beta_M}, \quad \beta_M = \begin{cases} \max\limits_{p\in\left[\frac{2\gamma}{\gamma-1},\infty\right)} \left\{-\frac{p(\varepsilon+1)+4}{2p\gamma}, \frac{p(\gamma-2)-2\gamma}{p\gamma}\right\}, & \text{if } d=2, \gamma\leq 2, \\ 0, & \text{if } d=2, \gamma>2, \\ \max\left\{-\frac{\varepsilon+2}{2\gamma}, \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma}\right\}, & \text{if } d=3, \gamma\leq 2, \\ \frac{\gamma-3}{\gamma}, & \text{if } d=3, \gamma\in(2,3), \\ 0, & \text{if } d=3, \gamma\geq 3. \end{cases} \tag{B.5}$$

Proof. We start with estimating the density in the L^2L^2-norm, i.e. (B.3). For $\gamma \geq 2$ we easily check

$$\|\varrho_h\|_{L^2L^2} \lesssim \|\varrho_h\|_{L^\infty L^\gamma} \lesssim 1, \quad \text{meaning } \beta_D = 0.$$

Now, let us focus on the case $\gamma < 2$. On the one hand, thanks to the inverse estimate we have

$$\|\varrho_h\|_{L^2L^2} \lesssim h^{\frac{\gamma-2}{2\gamma}d} \|\varrho_h\|_{L^\infty L^\gamma} \lesssim h^{\frac{\gamma-2}{2\gamma}d}.$$

On the other hand, in view of Lemma B.1 we have

$$\|\varrho_h\|_{L^{2-\gamma}L^\infty}^{2-\gamma} \lesssim h^{-(2-\gamma)(2d+p(\varepsilon+1))/(p\gamma)},$$

which yields

$$\begin{aligned}\|\varrho_h\|_{L^2L^2} &= \left(\int_0^T \int_{\mathbb{T}^d} \varrho_h^\gamma \cdot \varrho_h^{2-\gamma} \,\mathrm{d}x\mathrm{d}t\right)^{1/2} \leq \left(\int_0^T \|\varrho_h\|_{L^\infty(\mathbb{T}^d)}^{2-\gamma} \int_{\mathbb{T}^d} \varrho_h^\gamma \,\mathrm{d}x\mathrm{d}t\right)^{1/2} \\ &= \left(\int_0^T \|\varrho_h\|_{L^\infty(\mathbb{T}^d)}^{2-\gamma} \|\varrho_h\|_{L^\gamma(\mathbb{T}^d)}^\gamma \,\mathrm{d}t\right)^{1/2} \leq \|\varrho_h\|_{L^\infty L^\gamma}^{\gamma/2} \|\varrho_h\|_{L^{2-\gamma}L^\infty}^{(2-\gamma)/2} \lesssim h^{\frac{p(\varepsilon+1)+2d}{p}\cdot\frac{\gamma-2}{2\gamma}}.\end{aligned}$$

This completes the proof of (B.3).

Next, we prove (B.4) for the estimates of the $L^2L^{6/5}$-norm of the density. Considering $\gamma \geq 6/5$ it is obvious that

$$\|\varrho_h\|_{L^2L^{6/5}} \lesssim \|\varrho_h\|_{L^\infty L^\gamma} \lesssim 1.$$

For $\gamma < \frac{6}{5}$ the proof can be done in the following two ways. In the first approach we apply the inverse estimates to get

$$\|\varrho_h\|_{L^2L^{6/5}} \lesssim h^{\frac{5\gamma-6}{6\gamma}d} \|\varrho_h\|_{L^\infty L^\gamma} \lesssim h^{\frac{5\gamma-6}{6\gamma}d}.$$

In the second approach, recalling estimate (B.1) and applying the interpolation inequality we obtain

$$\|\varrho_h\|_{L^2L^{6/5}} \lesssim \|\varrho_h\|_{L^\infty L^\gamma}^\alpha \|\varrho_h\|_{L^\gamma L^{p\gamma/2}}^{1-\alpha} \lesssim h^{-\frac{1+\varepsilon}{\gamma}\times(1-\alpha)} \quad \text{for } p > \frac{12}{5\gamma} > 2.$$

Here α satisfies

$$\frac{1}{2} \geq \frac{\alpha}{\infty} + \frac{1-\alpha}{\gamma} \text{ and } \frac{6}{5} \geq \frac{\alpha}{\gamma} + \frac{1-\alpha}{\gamma p/2} \Longleftrightarrow \frac{2-\gamma}{2} \leq \alpha \leq \frac{5\gamma p - 12}{6p-12}.$$

Hence, the optimal bound is achieved by choosing $\alpha = \frac{5\gamma p-12}{6(p-2)}$, i.e.

$$\|\varrho_h\|_{L^2L^{6/5}} \lesssim h^{-\frac{1+\varepsilon}{\gamma}\times(1-\alpha)} = h^{-\frac{1+\varepsilon}{\gamma}\cdot\frac{(6-5\gamma)p}{6(p-2)}}.$$

Collecting the above estimates we obtain (B.4).

Finally, we are left with the estimate of $\|\varrho_h \boldsymbol{u}_h\|_{L^2L^2}$. It is easy to check

$$\begin{aligned}\|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} &= \left\|\varrho_h^2 \boldsymbol{u}_h^2\right\|_{L^1L^1}^{1/2} \lesssim \left(\left\|\varrho_h \boldsymbol{u}_h^2\right\|_{L^\infty L^{p\gamma/(p\gamma-2)}} \|\varrho_h\|_{L^\gamma L^{p\gamma/2}}\right)^{1/2} \\ &\lesssim \left(h^{-\frac{2d}{p\gamma}} h^{-\frac{\varepsilon+1}{\gamma}}\right)^{1/2} = h^{-\frac{p(\varepsilon+1)+2d}{2p\gamma}}, \quad \text{for } \gamma \in (1,2]\end{aligned} \tag{B.6}$$

and

$$\|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} \lesssim h^{-\frac{d}{2\gamma}} \|\varrho_h \boldsymbol{u}_h\|_{L^\infty L^{2\gamma/(\gamma+1)}} \lesssim h^{-\frac{d}{2\gamma}}. \tag{B.7}$$

Moreover, by Hölder's inequality we have

$$\|\varrho_h \boldsymbol{u}_h\|_{L^2L^{\gamma p/(\gamma+p)}} \lesssim \|\varrho_h\|_{L^\infty L^\gamma} \|\boldsymbol{u}_h\|_{L^2L^p} \quad \text{for any } p > 1,$$

from which we obtain

$$\begin{aligned}
\text{for } d=2: &\text{ if } \gamma > 2,\ \|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} \lesssim \|\varrho_h \boldsymbol{u}_h\|_{L^2L^{p\gamma/(\gamma+p)}} \lesssim 1 \quad \text{with } p \geq 2 + \frac{4}{\gamma-2},\\
&\text{ if } \gamma \leq 2,\ \|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} \lesssim h^{\left(\frac{1}{2}-\frac{(\gamma+p)}{p\gamma}\right)d} \|\varrho_h \boldsymbol{u}_h\|_{L^2L^{p\gamma/(\gamma+p)}} \lesssim h^{\frac{p(\gamma-2)-2\gamma}{p\gamma}} \quad \text{for any } p > 1,\\
\text{for } d=3: &\text{ if } \gamma \geq 3,\ \|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} \lesssim \|\varrho_h \boldsymbol{u}_h\|_{L^2L^{6\gamma/(\gamma+6)}} \lesssim 1,\\
&\text{ if } \gamma < 3,\ \|\varrho_h \boldsymbol{u}_h\|_{L^2L^2} \lesssim h^{\left(\frac{1}{2}-\frac{(\gamma+6)}{6\gamma}\right)d} \|\varrho_h \boldsymbol{u}_h\|_{L^2L^{6\gamma/(\gamma+6)}} \lesssim h^{\frac{(\gamma-3)}{3\gamma}d} = h^{\frac{(\gamma-3)}{\gamma}}.
\end{aligned}$$

Consequently, collecting (B.6), (B.7) and the above estimates, we obtain

$$\begin{aligned}
\text{for } d=2: &\text{ if } \gamma > 2,\ \beta_M = 0,\\
&\text{ if } \gamma \leq 2,\ \beta_M = \max_{p\in[1,\infty)} \left\{ -\frac{p(\varepsilon+1)+4}{2p\gamma}, \frac{p(\gamma-2)-2\gamma}{p\gamma}, -\frac{1}{\gamma} \right\}\\
&\qquad\qquad = \max_{p\in\left[\frac{2\gamma}{\gamma-1},\infty\right)} \left\{ -\frac{p(\varepsilon+1)+4}{2p\gamma}, \frac{p(\gamma-2)-2\gamma}{p\gamma} \right\},\\
\text{for } d=3: &\text{ if } \gamma \geq 3,\ \beta_M = 0,\\
&\text{ if } \gamma \in (2,3),\ \beta_M = \max\left\{ \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma} \right\} = \frac{\gamma-3}{\gamma},\\
&\text{ if } \gamma \leq 2,\ \beta_M = \max\left\{ -\frac{\varepsilon+2}{2\gamma}, \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma} \right\},
\end{aligned}$$

which concludes the proof. □

B.2 Negative variational estimates

Having the negative estimates of density and momentum we can present some useful negative variational estimates that shall be used later for the consistency formulation. These proofs are analogous to Lemma 11.5 and Lemma 11.6 of [8].

Lemma B.3. *Let $(\varrho_h, \boldsymbol{u}_h)$ be a solution of the FV method (3.4) with $h \in (0,1)$ and $\gamma > 1$. Then the following hold:*

$$\int_0^\tau \int_{\mathcal{E}} \frac{[\![\varrho_h]\!]^2}{\max\{\rho_h^{\text{in}}, \rho_h^{\text{out}}\}} \cdot (h^\varepsilon + |\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}|)\, \mathrm{d}S_x \mathrm{d}t \lesssim h^{\beta_D}, \tag{B.8a}$$

$$\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]|\, \mathrm{d}S_x \mathrm{d}t \lesssim h^{-(\varepsilon+1)/2}, \tag{B.8b}$$

$$\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]| \cdot |\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}| \,\mathrm{d}S_x \mathrm{d}t \lesssim h^{(\beta_R-1)/2}, \tag{B.8c}$$

$$\int_0^\tau \int_{\mathcal{E}} (|[\![\varrho_h]\!]| + \{\!\{\varrho_h\}\!\}) |[\![\boldsymbol{u}_h]\!] \cdot \boldsymbol{n}| \,\mathrm{d}S_x \mathrm{d}t \lesssim h^{\beta_D}, \tag{B.8d}$$

$$\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h \boldsymbol{u}_h]\!] \cdot \boldsymbol{n}| \,\mathrm{d}S_x \mathrm{d}t \lesssim h^{(\beta_R-1)/2} + h^{\beta_D}, \tag{B.8e}$$

where β_D is given in (B.3) and

$$\beta_R = \begin{cases} 0, & \text{if } d = 2, \\ \widetilde{\beta_R}, & \text{if } d = 3. \end{cases} \tag{B.8f}$$

Proof. We start by showing (B.8a). For any $\varrho > 0$, taking $B(\varrho)$ and ϕ_h in the renormalized continuity equation [8, Lemma 8.2] as $\varrho \ln \varrho - \varrho$ and 1, respectively, we obtain

$$\begin{aligned} & B'(\varrho) = \ln(\varrho), \quad B''(\varrho) = \frac{1}{\varrho} > 0, \\ & \int_{\mathbb{T}^d} D_t(\varrho_h \ln \varrho_h) \,\mathrm{d}x + \int_{\mathbb{T}^d} \varrho_h \mathrm{div}_h \boldsymbol{u}_h \,\mathrm{d}x \\ & \leq - \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(h^\varepsilon - (\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n})^-\right) \Big([\![B(\varrho_h)]\!] - B'(\varrho_h) [\![\varrho_h]\!] \Big). \end{aligned} \tag{B.9}$$

Due to the convexity of $B(\varrho)$, i.e.

$$[\![B(\varrho_h)]\!] - B'(\varrho_h) [\![\varrho_h]\!] = \frac{1}{2} B''(\xi) [\![\varrho_h]\!]^2 \geq \frac{[\![\varrho_h]\!]^2}{2 \max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}}, \quad \xi \in \mathrm{co}\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\},$$

we obtain

$$h^\varepsilon \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \Big([\![B(\varrho_h)]\!] - B'(\varrho_h) [\![\varrho_h]\!] \Big) \geq h^\varepsilon \int_{\mathcal{E}} \frac{[\![\varrho_h]\!]^2}{\max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}} \,\mathrm{d}S_x.$$

Moreover, we have

$$- \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} (\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n})^- \ \Big([\![B(\varrho_h)]\!] - B'(\varrho_h) [\![\varrho_h]\!] \Big) = \sum_{\sigma \in \mathcal{E}} |\sigma| \frac{1}{2} \left| \{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n} \right| B''(\xi) [\![\varrho_h]\!]^2$$

with $\xi \in \mathrm{co}\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}$, which implies

$$- \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left((\{\!\{u\}\!\} \cdot \boldsymbol{n})^-\right) \Big([\![B(\varrho_h)]\!] - B'(\varrho_h) [\![\varrho_h]\!] \Big) \geq \int_{\mathcal{E}} |\{\!\{u\}\!\} \cdot \boldsymbol{n}| \frac{[\![\varrho_h]\!]^2}{2 \max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}} \,\mathrm{d}S_x.$$

Collecting the above estimates we derive from (B.9) that

$$\int_0^\tau \int_{\mathcal{E}} \frac{[\![\varrho_h]\!]^2}{\max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}} \left(h^\varepsilon + \frac{|\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}|}{2} \right) \mathrm{d}S_x \mathrm{d}t$$

$$
\begin{aligned}
&\leq \int_0^\tau \sum_{K\in\mathcal{T}_h} |K| \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(h^\varepsilon - (\{\!\{u\}\!\}\cdot\boldsymbol{n})^-\right) \Big([\![B(\varrho_h)]\!] - B'(\varrho_h)\,[\![\varrho_h]\!] \Big)\mathrm{d}t \\
&\leq \int_{\mathbb{T}^d} \varrho_h^0 \ln(\varrho_h^0)\,\mathrm{d}x - \int_{\mathbb{T}^d} \varrho_h \ln(\varrho_h)\,\mathrm{d}x - \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h \mathrm{div}_h \boldsymbol{u}_h)\,\mathrm{d}x \\
&\lesssim 1 + \left| \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h \mathrm{div}_h \boldsymbol{u}_h)\,\mathrm{d}x \right| \leq 1 + \|\varrho_h\|_{L^2L^2} \|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2L^2}\,.
\end{aligned}
$$

Note that we have used here the inequality $|\varrho\ln(\varrho)| \lesssim 1+\varrho^\gamma$. Consequently, applying Lemma B.2 concludes the proof of (B.8a).

Secondly, thanks to Hölder's inequality and trace inequality, together with the density dissipation (B.8a) and uniform bounds (4.3) we obtain (B.8b) in the following way:

$$
\begin{aligned}
\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]|\,\mathrm{d}S_x\mathrm{d}t &\lesssim \left(\int_0^\tau \int_{\mathcal{E}} \frac{[\![\varrho_h]\!]^2}{\max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}}\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_{\mathcal{E}} \max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \\
&\lesssim h^{-\varepsilon/2} h^{-1/2} = h^{-(1+\varepsilon)/2} \quad \text{for } \gamma \geq 2, \\
\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]|\,\mathrm{d}S_x\mathrm{d}t &\lesssim \int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]| \sqrt{\mathcal{P}''(\varrho_{h,\dagger})\,(\varrho_{h,\dagger}+1)}\,\mathrm{d}S_x\mathrm{d}t \\
&\lesssim \left(\int_0^\tau \int_{\mathcal{E}} [\![\varrho_h]\!]^2\,\mathcal{P}''(\varrho_{h,\dagger})\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_{\mathcal{E}} (\varrho_{h,\dagger}+1)\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \\
&\lesssim h^{-\varepsilon/2} h^{-1/2} = h^{-(1+\varepsilon)/2} \quad \text{for } \gamma \in (1,2),
\end{aligned}
$$

where we have used the inequality

$$
\mathbb{1}_{(0,\infty)}(\varrho)\,\mathcal{P}''(\varrho)(\varrho+1) \gtrsim 1 \text{ for } \gamma \in (1,2)
$$

and $\varrho_{h,\dagger}$ is given in (4.3b).

Thirdly, we can derive (B.8c) in an analogous way. On the one hand, we have that for $\gamma \geq 2$

$$
\begin{aligned}
&\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]\,\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \\
&\lesssim \left(\int_0^\tau \int_{\mathcal{E}} \frac{[\![\varrho_h]\!]^2}{\max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\}} |\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_{\mathcal{E}} \max\{\rho_h^{\mathrm{in}}, \rho_h^{\mathrm{out}}\} |\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \\
&\lesssim h^{-1/2} \|\varrho_h\|_{L^2L^{p'}}^{1/2} \|\boldsymbol{u}_h\|_{L^2L^p}^{1/2} \lesssim h^{-1/2} \|\varrho_h\|_{L^2L^{p'}}^{1/2}\,,
\end{aligned}
$$

where $p' = \frac{p}{p-1}$, for any $p>1$ in the case of $d=2$ and $p=6$ for the case of $d=3$. On the other hand, for $\gamma < 2$ we have

$$
\begin{aligned}
&\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]| \cdot |\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \lesssim \int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]| \cdot |\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}| \cdot \sqrt{\mathcal{P}''(\varrho_{h,\dagger})\cdot(\varrho_{h,\dagger}+1)}\,\mathrm{d}S_x\mathrm{d}t \\
&\lesssim \left(\int_0^\tau \int_{\mathcal{E}} [\![\varrho_h]\!]^2\,\mathcal{P}''(\varrho_{h,\dagger}) |\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_{\mathcal{E}} (\varrho_{h,\dagger}+1) |\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \right)^{1/2}
\end{aligned}
$$

$$\lesssim h^{-1/2}(\|\varrho_h\|_{L^2L^{p'}}^{1/2} + 1)\,\|\boldsymbol{u}_h\|_{L^2L^p}^{1/2} \lesssim h^{-1/2}\,\|\varrho_h\|_{L^2L^{p'}}^{1/2}.$$

In view of the above two estimates, the proof of (B.8c) reduces to show $\|\varrho_h\|_{L^2L^{p'}} \lesssim h^{\beta_R}$. If $d=2$, we take $p' \in (1,\gamma]$ (i.e. $p \geq \frac{\gamma}{\gamma-1}$) and obtain $\|\varrho_h\|_{L^2L^{p'}} \lesssim 1$. If $d=3$, we choose $p=6$ and $p'=\frac{6}{5}$. Then we apply (B.4) to get

$$\|\varrho_h\|_{L^2L^{p'}} = \|\varrho_h\|_{L^2L^{6/5}} \lesssim h^{\widetilde{\beta_R}},$$

which completes the proof of (B.8c).

The fourth estimate (B.8d) is straightforward:

$$\int_0^\tau \int_{\mathcal{E}} (|[\![\varrho_h]\!]| + \{\!\{\varrho_h\}\!\})|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \lesssim \left(\int_0^\tau \int_{\mathcal{E}} h\,\{\!\{\varrho_h^2\}\!\}\,\mathrm{d}S_x\mathrm{d}t\right)^{1/2} \left(\int_0^\tau \int_{\mathcal{E}} \frac{|[\![\boldsymbol{u}_h]\!]|^2}{h}\,\mathrm{d}S_x\mathrm{d}t\right)^{1/2}$$
$$\lesssim \|\varrho_h\|_{L^2L^2}\,\|\nabla_{\mathcal{E}}\boldsymbol{u}_h\|_{L^2L^2} \lesssim h^{\beta_D}.$$

Finally, recalling the product rule for the equality

$$[\![\varrho_h\boldsymbol{u}_h]\!] = [\![\varrho_h]\!]\,\{\!\{\boldsymbol{u}_h\}\!\} + \{\!\{\varrho_h\}\!\}\,[\![\boldsymbol{u}_h]\!]$$

we may employ (B.8b) and (B.8d) to derive (B.8e)

$$\int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t \leq \int_0^\tau \int_{\mathcal{E}} |[\![\varrho_h]\!]\,\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t + \int_0^\tau \int_{\mathcal{E}} \{\!\{\varrho_h\}\!\}\,|[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}|\,\mathrm{d}S_x\mathrm{d}t$$
$$\lesssim h^{(\beta_R-1)/2} + h^{\beta_D},$$

which completes the proof. □

B.3 Consistency proof

Equipped with Lemma B.2 and B.3, we are now ready to prove the consistency formulation, which is quite analogous to [8, Section 11.3], [7, Theorem 4.1] and [9, Section 2.7]. Hence, in the following we only give the idea and framework of the proof.

Proof of Lemma 4.4. Let $\tau \in [t_n, t_{n+1})$.

Step 1 – time derivative terms: Let r_h stand for ϱ_h or $\varrho_h\boldsymbol{u}_h$. Recalling [9, equation(2.17)] we know that

$$\left|\left[\int_{\mathbb{T}^d} r_h\phi\,\mathrm{d}x\right]_{t=0}^{\tau} - \int_0^{t^{n+1}}\int_{\mathbb{T}^d} \left(D_t r_h(t)\Pi_{\mathcal{T}}\phi(t) + r_h(t)\partial_t\phi(t)\right)\mathrm{d}x\mathrm{d}t\right|$$
$$\lesssim \left(\left\|\partial_t^2\phi\right\|_{L^\infty L^\infty} + \|\partial_t\phi\|_{L^\infty L^\infty}\right)\Delta t \lesssim \Delta t \tag{B.10}$$

Step 2 – convective terms: To deal with the convective terms, it is convenient to recall [7, Lemma 2.5]

$$\int_0^{t^{n+1}} \int_{\mathbb{T}^d} r_h \cdot \nabla_x \phi \, \mathrm{d}x\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathcal{E}} F_h^{up}[r_h, \boldsymbol{u}_h] \, [\![\Pi_{\mathcal{T}}\phi]\!] \, \mathrm{d}S_x\mathrm{d}t = \sum_{i=1}^4 E_i(r_h),$$

where

$$\begin{aligned}
E_1(r_h) &= \frac{1}{2} \int_0^{t^{n+1}} \int_{\mathcal{E}} |\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}| \, [\![r_h]\!] \, [\![\Pi_{\mathcal{T}}\phi]\!] \, \mathrm{d}S_x\mathrm{d}t, \\
E_2(r_h) &= \frac{1}{4} \int_0^{t^{n+1}} \int_{\mathcal{E}} [\![\boldsymbol{u}_h]\!] \cdot \boldsymbol{n} \, [\![r_h]\!] \, [\![\Pi_{\mathcal{T}}\phi]\!] \, \mathrm{d}S_x\mathrm{d}t, \\
E_3(r_h) &= h^{\varepsilon} \int_0^{t^{n+1}} \int_{\mathcal{E}} [\![r_h]\!] \, [\![\Pi_{\mathcal{T}}\phi]\!] \, \mathrm{d}S_x\mathrm{d}t, \\
E_4(r_h) &= \int_0^{t^{n+1}} \int_{\mathbb{T}^d} r_h \boldsymbol{u}_h \cdot (\nabla_x \phi - \nabla_h(\Pi_{\mathcal{T}}\phi)) \, \mathrm{d}x\mathrm{d}t = \int_0^{t^{n+1}} \int_{\mathcal{E}} [\![r_h \boldsymbol{u}_h]\!] \cdot \boldsymbol{n}(\phi - \{\!\{\Pi_{\mathcal{T}}\phi\}\!\}) \, \mathrm{d}S_x\mathrm{d}t.
\end{aligned}$$

Applying Hölder's inequality with (3.2) and (B.8) we obtain for $r_h = \varrho_h$ that

$$\left| - \sum_{i=1}^4 E_i(\varrho_h) \right| \lesssim h^{(1+\beta_R)/2} + h^{(1+\varepsilon)/2} + h^{1+\beta_D} \qquad \text{for } \phi \in L^\infty(0,T; W^{1,\infty}(\mathbb{T}^d;\mathbb{R})).$$

Directly following [7, Theorem 4.1] we obtain for $r_h = \varrho_h \boldsymbol{u}_h$ that

$$\left| - \sum_{i=1}^4 E_i(\varrho_h \boldsymbol{u}_h) \right| \lesssim h + h^{1+\varepsilon} + h^{1+\beta_M} \qquad \text{for } \phi \in L^\infty(0,T; W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d)).$$

Moreover, it is obvious that

$$\Big| \int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} r_h \cdot \nabla_x \phi \, \mathrm{d}x\mathrm{d}t \Big| \lesssim \Delta t \|\nabla_x \phi\|_{L^\infty L^\infty} \|r_h\|_{L^\infty L^1} \lesssim \Delta t, \quad r_h = \varrho_h \text{ or } \varrho_h \boldsymbol{u}_h.$$

Consequently, we obtain

$$\Big| \int_0^\tau \int_{\mathbb{T}^d} r_h \cdot \nabla_x \phi \, \mathrm{d}x\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathcal{E}} F_h^{up}[r_h, \boldsymbol{u}_h] \, [\![\Pi_{\mathcal{T}}\phi]\!] \, \mathrm{d}S_x\mathrm{d}t \Big| \lesssim \Big| \sum_{i=1}^4 E_i(r_h) \Big| + \Delta t. \tag{B.11}$$

Step 3 – viscosity terms: Recalling [9, Section 2.7] we have the consistency errors contributed by viscosity terms in the momentum equation

$$\begin{aligned}
E_{\nabla_x \boldsymbol{u}} = & \int_0^\tau \int_{\mathbb{T}^d} (\mu \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \widetilde{\boldsymbol{u}} + \nu \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \widetilde{\boldsymbol{u}}) \, \mathrm{d}x\mathrm{d}t \\
& - \int_0^{t^{n+1}} \int_{\mathbb{T}^d} (\mu \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_{\mathcal{E}}(\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) + \nu \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_h(\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})) \, \mathrm{d}x,
\end{aligned} \tag{B.12}$$

which can be controlled by

$$
\begin{aligned}
|E_{\nabla_x \boldsymbol{u}}| \lesssim & h \left(\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2L^2} + \|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2L^2} \right) \left\| \nabla_x^2 \boldsymbol{\phi} \right\|_{L^\infty L^\infty} \\
& + (\Delta t)^{1/2} \left(\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2L^2} + \|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2L^2} \right) \|\nabla_x \phi\|_{L^\infty L^\infty}.
\end{aligned} \tag{B.13}
$$

Step 4 – pressure term: Recalling [9, Section 2.7] we have the consistency errors contributed by the pressure term in the momentum equation

$$
E_p = \int_0^\tau \int_{\mathbb{T}^d} p_h \mathrm{div}_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathbb{T}^d} p_h \mathrm{div}_h (\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \, \mathrm{d}x\mathrm{d}t, \tag{B.14}
$$

which can be controlled by

$$
|E_p| \lesssim h \|p_h\|_{L^\infty L^1} \left\| \nabla_x^2 \boldsymbol{\phi} \right\|_{L^\infty L^\infty} + \Delta t \|p_h\|_{L^\infty L^1} \|\nabla_x \phi\|_{L^\infty L^\infty}. \tag{B.15}
$$

Step 5 – penalization term: The consistency error contributed by the penalty term can be written as

$$
\begin{aligned}
E_{\epsilon_s} :=& \frac{1}{\epsilon_s} \int_0^\tau \int_{\Omega^s} \boldsymbol{u}_h \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \frac{1}{\epsilon_s} \int_0^{t^{n+1}} \int_{\Omega_h^s} \boldsymbol{u}_h \cdot \Pi_{\mathcal{T}} \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \\
=& - \frac{1}{\epsilon_s} \int_\tau^{t_{n+1}} \int_{\Omega^s} \boldsymbol{u}_h \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t + \frac{1}{\epsilon_s} \int_0^{t^{n+1}} \int_{\Omega_h^s \setminus \Omega^s} \boldsymbol{u}_h \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{B.16}
$$

where we have used

$$
\int_{\Omega_h^s} \boldsymbol{u}_h \cdot \boldsymbol{\phi} \, \mathrm{d}x - \int_{\Omega_h^s} \boldsymbol{u}_h \cdot \Pi_{\mathcal{T}} \boldsymbol{\phi} \, \mathrm{d}x = \sum_{K \in \Omega_h^s} \boldsymbol{u}_K \cdot \int_K (\boldsymbol{\phi} - \Pi_{\mathcal{T}} \boldsymbol{\phi}) \, \mathrm{d}x = 0.
$$

With

$$
\Delta t \|\boldsymbol{u}_h(t_n)\|_{L^2(\Omega^s)}^2 \leq \Delta t \sum_{i=0}^{N_T - 1} \|\boldsymbol{u}_h(t_i)\|_{L^2(\Omega^s)}^2 = \|\boldsymbol{u}_h\|_{L^2((0,T)\times\Omega^s)}^2 \lesssim \epsilon_s
$$

the first term in (B.16) can be controlled by

$$
\left| \frac{1}{\epsilon_s} \int_\tau^{t_{n+1}} \int_{\Omega^s} \boldsymbol{u}_h \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \right| \lesssim \frac{\|\boldsymbol{\phi}\|_{L^\infty((\tau,t_{n+1})\times\Omega^s)}}{\epsilon_s} \Delta t \|\boldsymbol{u}_h(t_n)\|_{L^1(\Omega^s)} \lesssim \frac{\Delta t}{\epsilon_s} \|\boldsymbol{u}_h(t_n)\|_{L^2(\Omega^s)} \lesssim (\Delta t/\epsilon_s)^{1/2},
$$

resulting the following estimate of E_{ϵ_s}

$$
|E_{\epsilon_s}| \lesssim (\Delta t/\epsilon_s)^{1/2} + \frac{\|\boldsymbol{u}_h\|_{L^2((0,T)\times\Omega_h^s)}}{\epsilon_s} \|\boldsymbol{\phi}\|_{L^\infty L^\infty} (|\Omega_h^s \setminus \Omega^s|)^{1/2} \lesssim (\Delta t/\epsilon_s)^{1/2} + (h/\epsilon_s)^{1/2}. \tag{B.17}
$$

In summary, combining (B.10), (B.11), (B.13), (B.15), and (B.17) we have

$$
\begin{aligned}
|e_\varrho| &\lesssim \Delta t + h^{(1+\varepsilon)/2} + h^{(1+\beta_R)/2} + h^{1+\beta_D}, \\
|e_{\boldsymbol{m}}| &\lesssim (\Delta t)^{1/2} + h + h^{1+\varepsilon} + h^{1+\beta_M} + (\Delta t/\epsilon_s)^{1/2} + (h/\epsilon_s)^{1/2},
\end{aligned}
$$

which concludes the proof.

□

C Some useful estimates for the error analysis

Lemma C.1. *Let $\delta > 0$ be an arbitrary constant, $h \in (0,1)$, pressure p satisfy (1.2) with $\gamma > 1$, $(\varrho_h, \boldsymbol{u}_h)$ be a solution of the FV method (3.4), $(\varrho, \boldsymbol{u})$ be the strong solution in the sense of Definition 4.8, and $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}})$ be given by Definition 5.1. Then*

$$\int_0^\tau \int_{\Omega_h^C} \varrho_h \ \mathrm{d}x\mathrm{d}t \lesssim h + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t, \tag{C.1a}$$

$$\int_0^\tau \int_{\Omega_h^C} p_h \ \mathrm{d}x\mathrm{d}t \lesssim h + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t, \tag{C.1b}$$

$$\int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \lesssim h + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t, \tag{C.1c}$$

$$\int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h|^2 \ \mathrm{d}x\mathrm{d}t \lesssim h^2 + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t, \tag{C.1d}$$

$$\int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \lesssim h + \delta \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \ \mathrm{d}x\mathrm{d}t. \tag{C.1e}$$

$$\int_0^\tau \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \lesssim h + \delta \int_0^\tau \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2 \ \mathrm{d}x\mathrm{d}t. \tag{C.1f}$$

$$\int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h| \lesssim \delta h \frac{\|\boldsymbol{u}_h\|_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \frac{\epsilon_s}{\delta}. \tag{C.1g}$$

$$\int_0^\tau \int_{\Omega^s} |\boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \leq \int_0^\tau \int_{\Omega_h^s} |\boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \lesssim \frac{\epsilon_s}{\delta} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s}. \tag{C.1h}$$

$$\int_0^\tau \int_{\Omega^s} \varrho_h |\boldsymbol{u}_h| \ \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t + \frac{\epsilon_s}{\delta} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s}. \tag{C.1i}$$

$$\left| h \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \left(\Delta_h^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \right) \Pi_{\mathcal{T}} \left| \{\!\{ u_{i,h} \}\!\}^{(i)} \right| \ \mathrm{d}x\mathrm{d}t\mathrm{d}t \right| \lesssim h^{1+\beta_M} + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t. \tag{C.1j}$$

$$\left| h \sum_{i=1}^d \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot (\Pi_{\mathcal{T}} \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \eth_{\mathcal{T}}^{(i)} |\{\!\{ u_{i,h} \}\!\}| \ \mathrm{d}x\mathrm{d}t \right| \lesssim h^{1+\beta_M}. \tag{C.1k}$$

Proof. First, we use Lemma A.2 for the essential–residual splitting and the inequality $\left|\Omega_h^C\right| \lesssim h$ to get (C.1a)

$$\begin{aligned} \int_0^\tau \int_{\Omega_h^C} \varrho_h \ \mathrm{d}x\mathrm{d}t &= \int_0^\tau \int_{\Omega_h^C} \mathbb{1}_{\mathrm{res}}(\varrho_h) \varrho_h \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_h^C} \mathbb{1}_{\mathrm{ess}}(\varrho_h) \varrho_h \ \mathrm{d}x\mathrm{d}t \\ &\lesssim \int_0^\tau \int_{\Omega_h^C} \mathbb{E}(\varrho_h | \widetilde{\varrho}) \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_h^C} 1 \ \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t + h. \end{aligned}$$

Second, the same process applies to p_h and gives (C.1b). Third, by triangular inequality, Young's inequality and (5.5b) we get (C.1c)

$$\int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t \leq \int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}| \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_h^C} \varrho_h |\widetilde{\boldsymbol{u}}| \, \mathrm{d}x\mathrm{d}t$$
$$\lesssim \int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t + (1+h) \int_0^\tau \int_{\Omega_h^C} \varrho_h \, \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t + h.$$

Fourth, we apply an analogous analysis to get (C.1d)

$$\int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h|^2 \, \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_h^C} \varrho_h |\widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t$$
$$\lesssim \int_0^\tau \int_{\Omega_h^C} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t + h^2 \int_0^\tau \int_{\Omega_h^C} \varrho_h \, \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t + h^2.$$

Fifth, we use triangular inequality, Young's inequality, and (5.5b) to get (C.1e)

$$\int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t \leq \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}| \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_h^C} |\nabla_x \widetilde{\boldsymbol{u}}| \, \mathrm{d}x\mathrm{d}t$$
$$\lesssim \int_0^\tau \int_{\Omega_h^C} \left(\delta |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 + \frac{1}{\delta} \right) \, \mathrm{d}x\mathrm{d}t + \|\nabla_x \widetilde{\boldsymbol{u}}\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})} h \lesssim h + \delta \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t.$$

The same process applies to $\mathrm{div}_h \boldsymbol{u}_h$ and yields (C.1f). Sixth, by Young's inequality we get (C.1g)

$$\int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h| \mathrm{d}t \leq \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \left(\frac{\delta h}{2\epsilon_s} |\boldsymbol{u}_h|^2 + \frac{\epsilon_s}{2\delta h} \right) \mathrm{d}t \lesssim \delta h \frac{\|\boldsymbol{u}_h\|_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \frac{\epsilon_s}{\delta}.$$

and (C.1h)

$$\int_0^\tau \int_{\Omega^s} |\boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t \leq \int_0^\tau \int_{\Omega_h^s} |\boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t \lesssim \epsilon_s^{1/2} \frac{\|\boldsymbol{u}_h\|_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s^{1/2}} \lesssim \frac{\epsilon_s}{\delta} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s}.$$

Seventh, the proof of (C.1i) relies on Lemma A.2 for the essential–residual splitting. On the one hand, for the essential part it is obvious that

$$\int_0^\tau \int_{\Omega^s} \mathbb{1}_{\mathrm{ess}}(\varrho_h) \varrho_h |\boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau \int_{\Omega^s} |\boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t \lesssim \frac{\epsilon_s}{\delta} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s}.$$

On the other hand, for the residual part we have

$$\int_0^\tau \int_{\Omega^s} \mathbb{1}_{\mathrm{res}}(\varrho_h) \varrho_h |\boldsymbol{u}_h| \, \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\Omega^s} \mathbb{1}_{\mathrm{res}}(\varrho_h) \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}| \, \mathrm{d}x\mathrm{d}t$$
$$\lesssim \left(\int_0^\tau \int_{\Omega^s} \mathbb{1}_{\mathrm{res}}(\varrho_h) \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \, \mathrm{d}x\mathrm{d}t \right)^{1/2} \left(\int_0^\tau \int_{\Omega^s} \mathbb{1}_{\mathrm{res}}(\varrho_h) \varrho_h \, \mathrm{d}x\mathrm{d}t \right)^{1/2} \lesssim \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t,$$

which completes the proof of (C.1i). Eighth, we show the proof via mesh splitting. By using the triangular inequality together with the estimates (4.3a), (B.5), (5.5b), (5.5d), and (C.1d) on Ω_h^C we derive

$$\begin{aligned}
&\left| h \int_0^\tau \int_{\Omega_h^C} \varrho_h \boldsymbol{u}_h \left(\Delta_h^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \right) \Pi_{\mathcal{T}} |\{\!\{ u_{i,h} \}\!\}| \, \mathrm{d}x\mathrm{d}t \right| \lesssim \int_0^\tau \int_{\Omega_h^C} |\varrho_h \boldsymbol{u}_h| \Pi_{\mathcal{T}} \left(|\{\!\{ u_{i,h} \}\!\} - u_{i,h}| + |u_{i,h}| \right) \, \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2} \int_0^\tau \int_{\Omega_h^C} |\varrho_h \boldsymbol{u}_h| \, \Pi_{\mathcal{T}} |[\![u_{i,h}]\!]| \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_h^C} |\varrho_h \boldsymbol{u}_h| \, |u_{i,h}| \, \mathrm{d}x\mathrm{d}t \\
&\lesssim h \left\| \varrho_h \boldsymbol{u}_h \right\|_{L^2 L^2} \left\| \nabla_{\mathcal{E}} \boldsymbol{u}_h \right\|_{L^2 L^2} + h^2 + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t \\
&\lesssim h^{1+\beta_M} + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t.
\end{aligned}$$

Combining the triangular inequality with the estimates (4.3a), (B.5), and (5.5d) on Ω_h^I, we derive

$$\left| h \int_0^\tau \int_{\Omega_h^I} \varrho_h \boldsymbol{u}_h \left(\Delta_h^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \right) \Pi_{\mathcal{T}} |\{\!\{ u_{i,h} \}\!\}| \, \mathrm{d}x\mathrm{d}t \right| \lesssim h \left\| \varrho_h \boldsymbol{u}_h \right\|_{L^2 L^2} \left\| \boldsymbol{u}_h \right\|_{L^2 L^2} \lesssim h^{1+\beta_M}.$$

Consequently, combing the above two estimates and recalling that $\widetilde{\boldsymbol{u}} = 0$ on Ω_h^O we get (C.1j)

$$\left| h \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \left(\Delta_h^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \right) \Pi_{\mathcal{T}} |\{\!\{ u_{i,h} \}\!\}| \, \mathrm{d}x\mathrm{d}t \right| \lesssim h^{1+\beta_M} + h^2 + \int_0^\tau R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \mathrm{d}t.$$

Finally, recalling Hölder's inequality, the estimates (4.3a), (B.5)we derive the last estimate (C.1k)

$$\begin{aligned}
&\left| h \sum_{i=1}^d \int_0^\tau \int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h (\Pi_{\mathcal{T}} \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \eth_{\mathcal{T}}^{(i)} |\{\!\{ u_{i,h} \}\!\}| \, \mathrm{d}x\mathrm{d}t \right| \lesssim h \left\| \varrho_h \boldsymbol{u}_h \right\|_{L^2 L^2} \left\| \widetilde{\boldsymbol{u}} \right\|_{W^{1,\infty}} \left\| \eth_{\mathcal{T}}^{(i)} |\{\!\{ u_{i,h} \}\!\}| \right\|_{L^2 L^2} \\
&\leq h \left\| \varrho_h \boldsymbol{u}_h \right\|_{L^2 L^2} \left\| \widetilde{\boldsymbol{u}} \right\|_{W^{1,\infty}} \left\| \nabla_h \boldsymbol{u}_h \right\|_{L^2 L^2} \lesssim h \left\| \varrho_h \boldsymbol{u}_h \right\|_{L^2 L^2} \left\| \widetilde{\boldsymbol{u}} \right\|_{W^{1,\infty}} \left\| \nabla_{\mathcal{E}} \boldsymbol{u}_h \right\|_{L^2 L^2} \lesssim h^{1+\beta_M}.
\end{aligned}$$

□

Lemma C.2. *Let $h \in (0,1)$, $v \in L^1(\mathbb{T}^d)$, $\boldsymbol{u} \in W^{2,\infty}(\Omega^f; \mathbb{R}^d)$, $\widetilde{\boldsymbol{u}}$ be given by Definition 5.1. Let $(\varrho_h, \boldsymbol{u}_h)$ be a solution of the FV method (3.4). Then*

$$\left\| \nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} \right\|_{L^2(\mathbb{T}^d)}^2 + \left\| \nabla_x \widetilde{\boldsymbol{u}} - \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \right\|_{L^2(\mathbb{T}^d)}^2 + \left\| \mathrm{div}_x \widetilde{\boldsymbol{u}} - \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}} \right\|_{L^2(\mathbb{T}^d; \mathbb{R}^d)}^2 \lesssim h, \tag{C.2a}$$

$$\left| \int_{\mathbb{T}^d} v (\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \, \mathrm{d}x \right| \lesssim h + \int_{\Omega_h^C} |v| \, \mathrm{d}x, \tag{C.2b}$$

$$\left| \int_{\mathbb{T}^d} v (\nabla_x \widetilde{\boldsymbol{u}} - \nabla_h \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \, \mathrm{d}x \right| \lesssim h + \int_{\Omega_h^C} |v| \, \mathrm{d}x, \tag{C.2c}$$

$$\left| \int_{\mathbb{T}^d} v (\mathrm{div}_x \widetilde{\boldsymbol{u}} - \mathrm{div}_h \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}) \, \mathrm{d}x \right| \lesssim h + \int_{\Omega_h^C} |v| \, \mathrm{d}x, \tag{C.2d}$$

$$\left|\int_{\mathbb{T}^d} \nabla_{\mathcal{E}} \boldsymbol{u}_h : (\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| \lesssim h + \frac{h}{\delta} + \delta \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2\,\mathrm{d}x, \tag{C.2e}$$

$$\left|\int_{\mathbb{T}^d} \mathrm{div}_h \boldsymbol{u}_h\ (\mathrm{div}_x \widetilde{\boldsymbol{u}} - \mathrm{div}_h \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| \lesssim h + \frac{h}{\delta} + \delta \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2\,\mathrm{d}x \tag{C.2f}$$

for any constant $\delta > 0$, where

$$\Pi_{\mathcal{E}} \boldsymbol{\phi} = \left(\Pi_{\mathcal{E}}^{(1)} \phi_1, \cdots, \Pi_{\mathcal{E}}^{(d)} \phi_d\right) \ \textit{and}\ \Pi_{\mathcal{E}}^{(i)} \phi = \sum_{\sigma \in \mathcal{E}_i} \frac{1_{D_\sigma}}{|\sigma|} \int_\sigma \phi\,\mathrm{d}S_x.$$

Proof. Recalling Definition 5.1 we consider separately three regions $\Omega_h^I, \Omega_h^C, \Omega_h^O$ and get

$$\begin{aligned}&\|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|_{L^2(\mathbb{T}^d)}^2 = \|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|_{L^2(\Omega_h^I)}^2 + \|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|_{L^2(\Omega_h^C)}^2 + \|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|_{L^2(\Omega_h^O)}^2 \\ &\lesssim h^2 \|\boldsymbol{u}\|_{W^{2,\infty}(\Omega^f)}^2 + \int_{\Omega_h^C} 1\,\mathrm{d}x\, \|\nabla_x \widetilde{\boldsymbol{u}}\|_{L^\infty(\mathbb{T}^d)}^2 + 0 \lesssim h.\end{aligned}$$

Analogously,we have

$$\|\nabla_x \widetilde{\boldsymbol{u}} - \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}}\|_{L^2(\mathbb{T}^d;\mathbb{R}^d)}^2 \lesssim h, \quad \|\mathrm{div}_x \widetilde{\boldsymbol{u}} - \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}}\|_{L^2(\mathbb{T}^d;\mathbb{R}^d)}^2 \lesssim h,$$

which proves (C.2a)

Similarly, we get (C.2b)

$$\begin{aligned}&\left|\int_{\mathbb{T}^d} v(\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| \\ &\le \left|\int_{\Omega_h^I} v(\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| + \left|\int_{\Omega_h^C} v(\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| + \left|\int_{\Omega_h^O} v(\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| \\ &\lesssim \|v\|_{L^1(\Omega_h^I)} \|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|_{L^\infty(\Omega_h^I;\mathbb{R}^{d\times d})} + \|v\|_{L^1(\Omega_h^C)} \|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|_{L^\infty(\Omega_h^C;\mathbb{R}^{d\times d})} + 0 \\ &\lesssim h \|\boldsymbol{u}\|_{W^{2,\infty}(\Omega^f;\mathbb{R}^d)} \|v\|_{L^1(\Omega_h^I)} + \|\nabla_x \widetilde{\boldsymbol{u}}\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})} \|v\|_{L^1(\Omega_h^C)} \lesssim h + \|v\|_{L^1(\Omega_h^C)}.\end{aligned}$$

The proofs of (C.2c) and (C.2d) are omitted as they can be done exactly in the same way.

Next, as a consequence of (C.2b) and (C.1e), we derive (C.2e)

$$\left|\int_{\mathbb{T}^d} \nabla_{\mathcal{E}} \boldsymbol{u}_h : (\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| \lesssim h + \|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^1(\Omega_h^C)} \lesssim h + \delta \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2\,\mathrm{d}x$$

Thanks to (C.2d) and (C.1f), we have

$$\left|\int_{\mathbb{T}^d} \mathrm{div}_h \boldsymbol{u}_h\ (\mathrm{div}_x \widetilde{\boldsymbol{u}} - \mathrm{div}_h \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\,\mathrm{d}x\right| \lesssim h + \|\mathrm{div}_h \boldsymbol{u}_h\|_{L^1(\Omega_h^C)} \lesssim h + \delta \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2\,\mathrm{d}x.$$

□

Next, we report the following lemma, see similar result in [9, Lemmas B.2 and B.4].

Lemma C.3. *Let $\gamma > 1$, $h \in (0,1)$ and $(\varrho_h, \boldsymbol{u}_h)$ be a solution obtained by the FV method (3.4). Further, let $\boldsymbol{u} \in W^{2,\infty}(\Omega^f;\mathbb{R}^d)$ and $\widetilde{\boldsymbol{u}}$ be given by Definition 5.1. Then there exist $C_0 = C_0(\underline{r}, \overline{r}, M, E_0, \gamma) > 0$, $C_1 = C_1(M, E_0, \gamma) > 0$ and $C_2 = C_2(M, E_0, \gamma, \|\boldsymbol{u}\|_{W^{2,\infty}(\Omega^f;\mathbb{R}^d)}) > 0$ such that*

$$\|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \lesssim C_1 \left(\|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right) + C_2 h, \tag{C.3}$$

$$\int_{\mathbb{T}^d} |(\varrho_h - \widetilde{\varrho})(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}})| \,\mathrm{d}x \lesssim \left(\frac{C_0}{\delta} + C_1 \delta \right) R_E(\varrho_h, \boldsymbol{u}_h | \widetilde{\varrho}, \widetilde{\boldsymbol{u}}) + C_1 \delta \|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d)} + C_2 \delta h, \tag{C.4}$$

$$\begin{aligned} &\left| \int_0^\tau \int_{\mathbb{T}^d} \left(\nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \widetilde{\boldsymbol{u}} + \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \right) \,\mathrm{d}x\mathrm{d}t \right| \\ &\qquad \lesssim h + \frac{\epsilon_s}{h} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \delta \|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2((0,\tau)\times\mathbb{T}^d)}, \end{aligned} \tag{C.5}$$

$$\begin{aligned} &\left| \int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \widetilde{\boldsymbol{u}} + \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \right) \,\mathrm{d}x\mathrm{d}t \right| \\ &\lesssim h + \frac{\epsilon_s}{h} + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \delta \|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2((0,\tau)\times\mathbb{T}^d)} + \delta \|\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}\|^2_{L^2((0,\tau)\times\mathbb{T}^d)}, \end{aligned} \tag{C.6}$$

where M and E_0 are the total fluid mass and initial energy, respectively. The constants $\underline{r}, \overline{r}$ are given in Lemma A.2.

Proof. Firstly, by setting $f_h = \boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}$ in Lemma A.1 we know that

$$\|\boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \leq C_1 \left(\|\nabla_{\mathcal{E}} (\boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right),$$

where the constant C_1 depends on the total mass M, initial energy E_0 and γ.

Then by the triangular inequality and projection error we derive

$$\begin{aligned} &\|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \leq \|\boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} + \|\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \\ &\leq C_1 \left(\|\nabla_{\mathcal{E}} (\boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}})\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right) + \left(h \|\nabla_x \widetilde{\boldsymbol{u}}\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})} \right)^2 \\ &\leq C_1 \left(\|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^{d\times d})} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right) \\ &\quad + C_1 \left(\|\nabla_x \widetilde{\boldsymbol{u}} - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^{d\times d})} + \int_{\mathbb{T}^d} \varrho_h |\Pi_{\mathcal{T}} \widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right) + h^2 \|\nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})} \\ &\leq C_1 \left(\|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^{d\times d})} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right) \\ &\quad + C_1 \left(h^2 \|\boldsymbol{u}\|^2_{W^{2,\infty}(\Omega^f;\mathbb{R}^d)} + h \|\widetilde{\boldsymbol{u}}\|_{W^{1,\infty}(\Omega^f;\mathbb{R}^d)} + h^2 \|\nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})} \int_{\mathbb{T}^d} \varrho_h \,\mathrm{d}x \right) + h^2 \|\nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^\infty(\mathbb{T}^d;\mathbb{R}^{d\times d})} \end{aligned}$$

$$= C_1 \left(\|\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d;\mathbb{R}^{d\times d})} + \int_{\mathbb{T}^d} \varrho_h |\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x \right) + C_2(h^2 + h),$$

where C_2 depends on $C_1, \|\boldsymbol{u}\|_{W^{2,2}(\Omega^f;\mathbb{R}^d)}, \|\widetilde{\boldsymbol{u}}\|_{W^{1,\infty}(\Omega^f;\mathbb{R}^d)}$ and the total mass, which proves (C.3). We omit the proof of (C.4) as it can be done exactly in the same way as [9, Lemma B.4]. Next, by Gauss theorem we know that

$$\int_K \mathrm{div}_x \boldsymbol{U} \,\mathrm{d}x = \int_K \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \boldsymbol{U} \text{ for any } K \in \mathcal{T}_h.$$

Thus we have

$$\int_{\Omega_h^f} \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x = \int_{\Omega_h^f} \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x.$$

Using this identity we observe

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \,\mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\Omega_h^f} \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t \\
&= - \int_0^\tau \int_{\mathbb{T}^d} \nabla_{\mathcal{E}} \boldsymbol{u}_h : \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t + I_0
\end{aligned}$$

where

$$\begin{aligned}
I_0 &= - \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t \\
&= - \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \mathrm{div}_{\mathcal{T}} \Pi_{\mathcal{E}} \nabla_x \boldsymbol{u} \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \Delta_x \boldsymbol{u} \,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

can be controlled via Hölder's inequality and (C.1g)

$$|I_0| \lesssim (1 + h^{-1}) \|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,\infty}(\Omega^f \setminus \Omega_h^f;\mathbb{R}^d))} \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h| \,\mathrm{d}x\mathrm{d}t \lesssim \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \frac{\epsilon_s}{h}.$$

Further, recalling the estimate (C.1e) we obtain

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{T}^d} \left(\nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_x \widetilde{\boldsymbol{u}} + \boldsymbol{u}_h \cdot \Delta_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \right) \,\mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\mathbb{T}^d} \nabla_{\mathcal{E}} \boldsymbol{u}_h : \left(\nabla_x \widetilde{\boldsymbol{u}} - \Pi_{\mathcal{E}} \nabla_x \widetilde{\boldsymbol{u}} \right) \,\mathrm{d}x\mathrm{d}t + I_0 \\
&\lesssim h + \int_0^\tau \int_{\Omega_h^C} |\nabla_h \boldsymbol{u}_h| \,\mathrm{d}x\mathrm{d}t + I_0 \lesssim h + \delta \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \,\mathrm{d}x\mathrm{d}t + \epsilon_s + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s} + \frac{\epsilon_s}{h},
\end{aligned}$$

which proves (C.6). Again we use the Gauss theorem to get

$$\int_{D_\sigma} \partial_i \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x = \int_{D_\sigma} \eth_{\mathcal{E}}^{(i)} \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x \quad \forall \sigma \in \mathcal{E}_i,$$

where $\Pi_\epsilon \phi|_K = \frac{1}{|\epsilon|}\int_\epsilon \phi$ for ϵ being the face of D_σ that crosses the center of element $K \in \mathcal{T}_h$. Using the above identity we have

$$
\begin{aligned}
&\int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} 1_{\Omega^f} \,\mathrm{d}x = \int_{\Omega_h^f} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \\
&= \int_{\Omega_h^f} \{\{\boldsymbol{u}_h\}\} \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + \int_{\Omega_h^f} (\boldsymbol{u}_h - \{\{\boldsymbol{u}_h\}\}) \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x \\
&= \sum_{i=1}^d \int_{\mathbb{T}^d} \{\{u_{i,h}\}\}^{(i)} \eth_{\mathcal{E}}^{(i)} \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + I_1 = -\sum_{i=1}^d \int_{\mathbb{T}^d} \mathrm{div}_h \boldsymbol{u}_h \eth_{\mathcal{E}}^{(i)} \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + I_1
\end{aligned}
$$

where

$$
I_1 = -\sum_{i=1}^d \int_{\Omega^f \setminus \Omega_h^f} \{\{u_{i,h}\}\}^{(i)} \eth_{\mathcal{E}}^{(i)} \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + \int_{\Omega_h^f} (\boldsymbol{u}_h - \{\{\boldsymbol{u}_h\}\}) \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x + \int_{\Omega^f \setminus \Omega_h^f} \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x
$$

satisfies the following estimate

$$
\begin{aligned}
&\left|\int_0^\tau I_1 \mathrm{d}t\right| \lesssim \left|\sum_{i=1}^d \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \{\{u_{i,h}\}\}^{(i)} \eth_{\mathcal{E}}^{(i)} \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x\mathrm{d}t\right| + \left(\|\boldsymbol{u}_h - \{\{\boldsymbol{u}_h\}\}\|_{L^2L^2} + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h|\right) \|\boldsymbol{u}\|_{L^\infty W^{2,\infty}} \\
&\lesssim h^{-1} \sum_{i=1}^d \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} \left|\{\{u_{i,h}\}\}^{(i)} \pm u_{i,h}\right| \mathrm{d}x\mathrm{d}t + h \left\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\right\|_{L^2L^2} + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h| \mathrm{d}t \\
&\lesssim \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\nabla_{\mathcal{E}} \boldsymbol{u}_h| \mathrm{d}t + h^{-1} \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h| \mathrm{d}t + h + \int_0^\tau \int_{\Omega^f \setminus \Omega_h^f} |\boldsymbol{u}_h| \mathrm{d}t \\
&\lesssim \frac{\epsilon_s}{h} + \delta \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \;\mathrm{d}x\mathrm{d}t + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s},
\end{aligned}
$$

where we have used the estimates (C.1g) and (C.1e). Finally, with (C.1f) we finish the proof of (C.6)

$$
\begin{aligned}
&\left|\int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_x \widetilde{\boldsymbol{u}} + \boldsymbol{u}_h \cdot \nabla_x \mathrm{div}_x \widetilde{\boldsymbol{u}} 1_{\Omega^f}\right) \;\mathrm{d}x\mathrm{d}t\right| = \left|\int_0^\tau \int_{\mathbb{T}^d} \left(\mathrm{div}_h \boldsymbol{u}_h (\mathrm{div}_x \widetilde{\boldsymbol{u}} - \Pi_\epsilon \mathrm{div}_x \widetilde{\boldsymbol{u}}\right) \;\mathrm{d}x\mathrm{d}t + \int_0^\tau I_1 \mathrm{d}t\right| \\
&\lesssim h + \int_0^\tau \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h| \;\mathrm{d}x\mathrm{d}t + \left|\int_0^\tau I_1 \mathrm{d}t\right| \\
&\lesssim h + \delta \int_0^\tau \int_{\Omega_h^C} |\mathrm{div}_h \boldsymbol{u}_h - \mathrm{div}_x \widetilde{\boldsymbol{u}}|^2 \;\mathrm{d}x\mathrm{d}t + \frac{\epsilon_s}{h} + \delta \int_0^\tau \int_{\Omega_h^C} |\nabla_{\mathcal{E}} \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}|^2 \;\mathrm{d}x\mathrm{d}t + \delta \frac{\|\boldsymbol{u}_h\|^2_{L^2((0,\tau)\times\Omega_h^s)}}{\epsilon_s}.
\end{aligned}
$$

□

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