



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**Embeddings between generalized  
weighted Lorentz spaces**

*Amiran Gogatishvili*

*Zdeněk Mihula*

*Luboš Pick*

*Hana Turčinová*

*Tuğçe Ünver*

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# EMBEDDINGS BETWEEN GENERALIZED WEIGHTED LORENTZ SPACES

AMIRAN GOGATISHVILI, ZDENĚK MIHULA, LUBOŠ PICK, HANA TURČINOVÁ AND TUĞÇE ÜNVER

ABSTRACT. We give a new characterization of a continuous embedding between two function spaces of type  $GF$ . Such spaces are governed by functionals of type

$$\|f\|_{GF(r,q;w,\delta)} := \left( \int_0^L \left( \frac{1}{\Delta(t)} \int_0^t f^*(s)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} < \infty,$$

in which  $f^*$  is the nonincreasing rearrangement of  $f$ ,  $r, q \in (0, \infty)$ ,  $w, \delta$  are weights and  $\Delta$  is the primitive of  $\delta$ . To characterize the embedding of such a space, say  $GF(r_1, q_1; w_1, \delta_1)$ , into another,  $GF(r_2, q_2; w_2, \delta_2)$ , means to find a balance condition on the four positive real parameters and the four weights in order that an appropriate inequality holds for every admissible function. We develop a new discretization technique which will enable us to get rid of restrictions on parameters imposed in earlier work such as the non-degeneracy conditions or certain relations between the  $r$ 's and  $q$ 's. Such restrictions were caused mainly by the use of duality techniques, which we avoid in this paper. On the other hand we consider here only the case when  $q_1 \leq q_2$  to keep the paper in a reasonable length.

## 1. INTRODUCTION

Discretizing and antidiscretizing techniques have been successfully applied to solving several rather difficult problems in the function space theory that had looked almost impossible before. The method itself is technical and not very attractive, but it yields the desired results. Numerous dismal attempts to avoid it and to get equally strong results using different approaches have been tried heavily, all markedly unsuccessful. In this paper, we have a different mission. Our aim is not to circumvent the discretization technique, but rather to enhance it, and to suggest a lateral point of view allowing one to overcome certain restrictions that have been littering it thus far. Roughly speaking, we are going to cleanse the discretization method from several assumptions on weights involved that have been appearing regularly in earlier work, and which we now confute as unnecessary, a pivotal instance of these being the

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non-degeneracy conditions. As a result, we obtain a considerably stronger characterization of embeddings between  $G\Gamma$ -spaces, but the impact of the improvement is wider as it extends to natural applications of the embeddings obtained. The earlier work made it clear that one of the main sources of the necessity for taking various restrictions was the use of duality as a crucial step in existing techniques. Our main achievement here is that the duality techniques are replaced by different ones based on exploiting the subtle interplay between discrete Hardy inequalities and the localization brought in by the discretization method, allowing us to obtain results of required generality and versatility.

Now is the time to be more precise. Let  $(\mathcal{R}, \mu)$  be a  $\sigma$ -finite nonatomic measure space such that  $\mu(\mathcal{R}) = L \in (0, \infty]$ , and let  $\mathfrak{M}(\mathcal{R}, \mu)$  be the set of all  $\mu$ -measurable functions on  $\mathcal{R}$  whose values belong to  $[-\infty, \infty]$  and  $\mathfrak{M}^+(\mathcal{R}, \mu) = \{f \in \mathfrak{M}(\mathcal{R}, \mu) : f \geq 0 \text{ } \mu\text{-a.e.}\}$ . By  $f^*$  we denote the nonincreasing rearrangement of  $f \in \mathfrak{M}(\mathcal{R}, \mu)$  defined by

$$f^*(t) = \inf\{\lambda \in [0, \infty) : \mu(\{x \in \mathcal{R} : |f(x)| > \lambda\}) \leq t\}, \quad t \in (0, \infty).$$

If  $X$  and  $Y$  are (quasi-)Banach spaces of measurable functions on the same measure space and the identity operator  $\text{Id}$  is bounded from  $X$  to  $Y$  in the sense that there exists a positive constant  $C$  such that  $\|f\|_Y \leq C\|f\|_X$  for all  $f \in X$ , then we say that  $X$  is *embedded* into  $Y$ , a fact which we denote by  $X \hookrightarrow Y$ . The least such a constant  $C$  in the embedding  $X \hookrightarrow Y$  is equal to  $\|\text{Id}\|_{X \rightarrow Y}$ .

Let  $r, q \in (0, \infty)$  and  $w, \delta$  be *weights* on  $(0, L)$ , that is, measurable functions on  $(0, L)$  that are positive a.e. on  $(0, L)$  and integrable near 0. By integrable near 0 we mean that  $\int_0^t w(s)ds < \infty$  for every  $t \in (0, L)$ , and the same goes for  $\delta$ . The *generalized Gamma space*  $G\Gamma(r, q; w, \delta)$  is the collection of all functions  $f \in \mathfrak{M}(\mathcal{R}, \mu)$  such that

$$\|f\|_{G\Gamma(r, q; w, \delta)} := \left( \int_0^L \left( \frac{1}{\Delta(t)} \int_0^t f^*(s)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} < \infty,$$

where we used the notation

$$\Delta(t) = \int_0^t \delta(s) ds \quad \text{for } t \in (0, L).$$

We will use this convention throughout, for example  $\Delta_1(t)$  will denote  $\int_0^t \delta_1(s) ds$ ,  $U(t)$  will denote  $\int_0^t u(s) ds$ , and so on.

The roots of generalized Gamma spaces reach to the pivotal paper [24] by Sawyer, in which the spaces of type Gamma were first introduced in connection with duality questions for the so-called classical Lorentz spaces of type Lambda, which were introduced and studied by Lorentz in [21], and with boundedness of classical operators on these spaces. Sawyer's results unleashed a tsunami of papers, and it is impossible to cite the whole lot of them. Let us just recall that important weak versions of the Gamma-type spaces were studied in the early 1990's, see e.g. [6, 7]. In [10], a simpler form of spaces  $G\Gamma(r, q; w, \delta)$  (involving the outer weight but not the inner) was introduced, see also [11]. It was pointed out that these spaces play a key role for boundedness of Sobolev-type functions (in this connection see also [14]), and they moreover constitute a natural environment for seeking solutions to certain variational inequalities. In [9] it was observed that their special cases are equivalent to the so-called *small Lebesgue spaces*. These are associated to the well-known *grand Lebesgue spaces*, introduced in [17] in connection with pointwise behavior of Jacobians and with classical discoveries of Müller [22] and Ball [2]. Direct applications of the  $G\Gamma$ -spaces to the study of the existence,

uniqueness and regularity of the so-called ‘very weak solutions’ to Dirichlet and Neumann problems for the equation  $-\Delta u = f$  in nonstandard function spaces can be found for example in [23]. The spaces  $G\Gamma$  play an interesting role in the interpolation theory, as pointed out in [1]. In [14], Köthe duals of simplified  $G\Gamma$ -spaces were studied and the question of when they are Banach algebras was rounded off there. Some more connections, applications, and historical notes can be found in the last cited paper, too.

Our aim here is to investigate embeddings between pairs of  $G\Gamma$ -spaces, that is,

$$G\Gamma(r_1, q_1; w_1, \delta_1) \hookrightarrow G\Gamma(r_2, q_2; w_2, \delta_2).$$

This amounts to finding a balance condition that would characterize all parameters  $r_i, q_i$  and weights  $w_i, \delta_i$ ,  $i = 1, 2$ , for which there exists a positive constant  $C$ , depending possibly only on these parameters and weights, such that the inequality

$$\begin{aligned} & \left( \int_0^L \left( \frac{1}{\Delta_2(t)} \int_0^t f^*(s)^{r_2} \delta_2(s) ds \right)^{\frac{q_2}{r_2}} w_2(t) dt \right)^{\frac{1}{q_2}} \\ & \leq C \left( \int_0^L \left( \frac{1}{\Delta_1(t)} \int_0^t f^*(s)^{r_1} \delta_1(s) ds \right)^{\frac{q_1}{r_1}} w_1(t) dt \right)^{\frac{1}{q_1}} \end{aligned} \quad (1.1)$$

holds for every  $\mu$ -measurable function  $f$ . We begin the analysis by ridding of one of the parameters and expressing the inequality in an equivalent but slightly simpler form. By a standard rescaling argument based on replacing  $(f^*)^{r_1}$  with  $f^*$ , and then denoting  $r = r_2/r_1$ ,  $q = q_2/r_1$ ,  $p = q_1/r_1$ ,  $u = \delta_1$ ,  $\delta = \delta_2$ ,  $v = w_1$  and  $w = w_2$ , we easily observe that (1.1) is equivalent to

$$\begin{aligned} & \left( \int_0^L \left( \frac{1}{\Delta(t)} \int_0^t f^*(s)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ & \leq C \left( \int_0^L \left( \frac{1}{U(t)} \int_0^t f^*(s) u(s) ds \right)^p v(t) dt \right)^{\frac{1}{p}}, \end{aligned} \quad (1.2)$$

again with  $C$  universal for any  $f$ .

Let us concentrate on inequality the (1.2). The first step relaxed a little the number of dangers to worry about, but we still face a more serious problem which consists in the fact that the inequality is formulated for symmetrized versions of functions. Put another way, it constitutes a weighted inequality restricted to nonincreasing functions. Since such a restriction makes inequalities notoriously hard to manage, our next step will be a reduction of (1.2) to an unrestricted equivalent inequality. However, we will pay for the reduction by appearance of one more integral operator.

Assume that  $0 < p, q, r < \infty$  and let  $v, w, u, \delta$  be weights on  $(0, L)$ . Then the inequality (1.2) holds if and only if there exists a positive constant  $C$  such that the inequality

$$\begin{aligned} & \left( \int_0^L \left( \frac{1}{\Delta(t)} \int_0^t \left( \int_s^L h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ & \leq C \left( \int_0^L \left( \frac{1}{U(t)} \int_0^t \left( \int_s^L h \right) u(s) ds \right)^p v(t) dt \right)^{\frac{1}{p}} \end{aligned} \quad (1.3)$$

holds for all  $h \in \mathfrak{M}^+(0, L)$ . This type of equivalence is quite standard. Indeed, the fact that (1.2) implies (1.3) amounts to finding, to a given nonnegative function  $h: (0, L) \rightarrow [0, \infty)$ , a function  $f: \mathcal{R} \rightarrow [0, \infty)$  such that  $f^*(s) = \int_s^L h$  for almost every  $s \in (0, L)$ , which is possible owing to the classical Sierpiński theorem (see [3]\*Chapter 2, Corollary 7.8). Conversely, assume that (1.3) holds. For every  $f \in \mathfrak{M}(\mathcal{R}, \mu)$ , there exists a sequence  $\{g_n\}_{n=1}^\infty$  of nonnegative measurable functions whose support is bounded and such that the sequence  $\{\int_t^\infty g_n(s) ds\}_{n=1}^\infty$  is nondecreasing in  $n$  for every fixed  $t > 0$ , and  $\lim_{n \rightarrow \infty} \int_t^\infty g_n(s) ds = f^*(t)$  for almost all  $t > 0$  ([15]\*Proposition 2.1). Then, using the monotone convergence theorem, we get (1.2).

So now it is (1.3) to worry about. From now on, we shall denote by  $C$  the optimal (smallest) constant in (1.3). This can be formally written as

$$C = \sup_{h \in \mathfrak{M}^+(0, L)} \frac{\left( \int_0^L \left( \frac{1}{\Delta(t)} \int_0^t \left( \int_s^L h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}}}{\left( \int_0^L \left( \frac{1}{U(t)} \int_0^t \left( \int_s^L h \right) u(s) ds \right)^p v(t) dt \right)^{\frac{1}{p}}}. \quad (1.4)$$

And, the ultimate task is to prove two-sided estimates of  $C$  in terms of quantities defined in easily computable way and dependent solely on parameters and weights. In fact it is quite remarkable that something like that is possible at all. As always in the theory of weighted inequalities, the form of the characterizing expressions will heavily depend on the comparison of the parameters  $p, q, r$ , inevitably forcing us to split the result into several cases. In this paper we handle the convex variant of the inequality, that is, the case  $p \leq q$  (which corresponds to the relation  $q_1 \leq q_2$  in (1.1), mentioned in the abstract).

Before stating the main result, we need to fix some more notation. We shall write  $A \lesssim B$  in cases when the quantity  $A$  is bounded above with a constant multiple of  $B$ , in which the multiplicative constant does not depend on any vital parameters in  $A$  or  $B$ . We also write  $A \gtrsim B$  with the obvious meaning. The relation  $A \approx B$  means that  $A \lesssim B$  and  $B \lesssim A$  hold together (with possibly different multiplicative constants). We shall specify throughout the parameters on which constants are allowed to depend on.

We finally introduce auxiliary functions  $\varphi$  and  $\sigma$  by setting

$$\varphi(t) = \int_0^L \min\{U(t)^p, U(s)^p\} \frac{v(s)}{U(s)^p} ds, \quad t \in (0, L), \quad (1.5)$$

and

$$\sigma(t) = \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p}(s) v(s) ds \right) U(t)^{p-1} u(t), \quad t \in (0, L), \quad (1.6)$$

in order to simplify the statement of our main result.

**Theorem 1.1.** *Let  $0 < p \leq q < \infty$ ,  $0 < r < \infty$  and  $u, \delta, v, w$  be weights on  $(0, L)$ . Assume that there is  $t_0 \in (0, L)$  such that  $0 < \varphi(t_0) < \infty$ . Then  $C$  defined by (1.4) satisfies the following relations.*

(i) If  $p \leq q$ ,  $p \leq r$ ,  $1 \leq q$ ,  $1 \leq r$ , then

$$C \approx B_1 + B_2,$$

where

$$B_1 := \sup_{t \in (0, L)} W(t)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}},$$

$$B_2 := \sup_{t \in (0, L)} \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}}.$$

(ii) If  $p \leq r < 1 \leq q$ , then

$$C \approx B_1 + B_2 + B_3,$$

where

$$B_3 := \sup_{t \in (0, L)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}}.$$

(iii) If  $1 \leq r < p \leq q$ , then

$$C \approx B_1 + B_2 + B_4,$$

where

$$B_4 := \sup_{t \in (0, L)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s, t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}}.$$

(iv) If  $r < p \leq q$ ,  $r < 1 \leq q$ , then

$$C \approx B_1 + B_2 + B_3 + B_5,$$

where

$$B_5 := \sup_{t \in (0, L)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^t \sigma(s) \left( \int_0^t \Delta(\tau)^{\frac{1}{1-r}} \delta(\tau) U(\tau)^{-\frac{r}{1-r}} \right. \right. \\ \left. \left. \times \min \left\{ U(s)^{\frac{r}{1-r}}, U(\tau)^{\frac{r}{1-r}} \right\} d\tau \right)^{\frac{p(1-r)}{p-r}} ds \right)^{\frac{p-r}{pr}}.$$

(v) If  $p \leq q < 1 \leq r$ , then

$$C \approx B_1 + B_2 + B_6 + B_7,$$

where

$$B_6 := \sup_{t \in (0, L)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\ \left. \times \sup_{\tau \in (t, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} ds \right)^{\frac{1-q}{q}}, \\ B_7 := \sup_{t \in (0, L)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^L W^{\frac{q}{1-q}} w U^{-\frac{q}{1-q}} \right)^{\frac{1-q}{q}}.$$

(vi) If  $p \leq q < 1$ ,  $p \leq r < 1$ , then

$$C \approx B_1 + B_2 + B_3 + B_7 + B_8,$$

where

$$B_8 := \sup_{t \in (0, L)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right)$$

$$\times \left( \int_t^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \Big)^{\frac{1-q}{q}}.$$

(vii) If  $r < p \leq q < 1$ , then

$$C \approx B_1 + B_2 + B_3 + B_5 + B_7 + B_8.$$

Moreover, the multiplicative constants in all the equivalences above depend only on  $p, q, r$ .

The first particular result in this direction was obtained in [12] under the restriction  $q_2 \geq r_2$  in (1.1), which translates to  $q \geq r$  in (1.4). It is also stated there that the solution in the converse case is left as an open problem. In this paper, we solve this problem, at least for the convex variant of the inequality.

Let us summarize the content of the following sections. Elements of the discretization technique are collected in Section 2. Fine analysis of indispensable discrete inequalities is carried out in Section 3. The converse process of antidiscretization is the content of Section 4. Finally, in the last Section 5, we prove Theorem 1.1.

## 2. PRELIMINARIES

In this section, we shall fix notation and recall preliminary results. It is essentially borrowed from [18]\*Section 2, which draws from [8], and we include it to make this paper as self-contained as possible.

Throughout the entire paper,  $L \in (0, \infty]$  is fixed. We say that a positive function defined on  $(0, L)$  is *admissible* if it is increasing and continuous. In this section, we shall assume that  $\varrho$  is an admissible function. A function  $h: (0, L) \rightarrow [0, \infty)$  is said to be  $\varrho$ -*quasiconcave* if  $h$  is nondecreasing on  $(0, L)$  and the function  $\frac{h}{\varrho}$  is nonincreasing on  $(0, L)$ . If this is the case, we write  $h \in Q_\varrho(0, L)$ . Let  $h$  denote a function from  $Q_\varrho(0, L)$  in the rest of this section. Thanks the monotonicity properties of  $\varrho$ -quasiconcave functions,  $h$  does not vanish identically on  $(0, L)$  if and only if  $h(t) \neq 0$  for every  $t \in (0, L)$ . Note that  $h^p$  is a  $\varrho^p$ -quasiconcave function for every  $p > 0$ , and so is  $\frac{\varrho}{h}$  provided that  $h \neq 0$ . A nonnegative linear combination of  $\varrho$ -quasiconcave functions is a  $\varrho$ -quasiconcave function. Furthermore, if  $k \in \mathbb{N}$  and  $h_j \in Q_{\varrho_j}(0, L)$  for  $j = 1, 2, \dots, k$ , where each  $\varrho_j$  is admissible, then the product  $h_1 h_2 \cdots h_k$  is a  $(\varrho_1 \varrho_2 \cdots \varrho_k)$ -quasiconcave function.

Let  $h \neq 0$  and  $a \in (1, \infty)$ . There are (possibly infinite) numbers  $M, N \in \mathbb{Z} \cup \{-\infty, \infty\}$  such that  $-\infty \leq N \leq 0 \leq M \leq \infty$ , and an increasing sequence  $\{x_k\}_{k=N}^M \subseteq [0, L]$  having the following six properties.

- $M = \infty$  if and only if

$$\lim_{t \rightarrow L^-} h(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow L^-} \frac{\varrho(t)}{h(t)} = \infty.$$

If  $M = \infty$ , then  $\lim_{k \rightarrow \infty} x_k = L$ . Otherwise,  $x_M = L$ .

- $N = -\infty$  if and only if

$$\lim_{t \rightarrow 0^+} h(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \frac{\varrho(t)}{h(t)} = 0.$$

If  $N = -\infty$ , then  $\lim_{k \rightarrow -\infty} x_k = 0$ . Otherwise,  $x_N = 0$ .



- For every  $k \in \mathbb{Z}$  such that  $N + 2 \leq k \leq M - 1$ , one has

$$ah(x_{k-1}) \leq h(x_k) \quad \text{and} \quad a \frac{\varrho(x_{k-1})}{h(x_{k-1})} \leq \frac{\varrho(x_k)}{h(x_k)}.$$

- For every  $k \in \mathbb{Z}$  such that  $N + 2 \leq k \leq M - 1$ , one has

$$\frac{1}{a}h(x_k) \leq h(t) \leq h(x_k) \quad \text{for each } t \in [x_{k-1}, x_k]$$

or

$$\frac{1}{a} \frac{\varrho(x_k)}{h(x_k)} \leq \frac{\varrho(t)}{h(t)} \leq \frac{\varrho(x_k)}{h(x_k)} \quad \text{for each } t \in [x_{k-1}, x_k].$$

- If  $M < \infty$ , then

$$h(x_{M-1}) \leq h(t) \leq ah(x_{M-1}) \quad \text{for each } t \in [x_{M-1}, L)$$

or

$$\frac{\varrho(x_{M-1})}{h(x_{M-1})} \leq \frac{\varrho(t)}{h(t)} \leq a \frac{\varrho(x_{M-1})}{h(x_{M-1})} \quad \text{for each } t \in [x_{M-1}, L).$$

- If  $N > -\infty$ , then

$$\frac{1}{a}h(x_{N+1}) \leq h(t) \leq h(x_{N+1}) \quad \text{for each } t \in (0, x_{N+1}]$$

or

$$\frac{1}{a} \frac{\varrho(x_{N+1})}{h(x_{N+1})} \leq \frac{\varrho(t)}{h(t)} \leq \frac{\varrho(x_{N+1})}{h(x_{N+1})} \quad \text{for each } t \in (0, x_{N+1}].$$

Such a sequence is called a *covering sequence for  $h$ ,  $\varrho$  and  $a$* . We denote the set of all covering sequences for  $h$ ,  $\varrho$  and  $a$  by  $CS(h, \varrho, a)$ . Note that all sequences in  $CS(h, \varrho, a)$  share the same values of  $N$  and  $M$ . Moreover, it is independent of the parameter  $a$  whether  $N$  and  $M$  are finite or infinite. If  $\{x_k\}_{k=N}^M \in CS(h, \varrho, a)$ , then  $\{x_k\}_{k=N}^M \in CS(\frac{\varrho}{h}, \varrho, a)$ , and  $\{x_k\}_{k=N}^M \in CS(h^p, \varrho^p, a^p)$  for every  $p \in (0, \infty)$ . Furthermore, it follows from the properties of covering sequences that

$$(0, L) \subseteq \bigcup_{k=N+1}^M (x_{k-1}, x_k] \subseteq (0, L];$$

moreover, the first inclusion is strict if and only if  $M \neq \infty$ .

Given a covering sequence  $\{x_k\}_{k=N}^M \in CS(h, \varrho, a)$ , the index set  $\mathcal{K}^+ = \{k \in \mathbb{Z}: N + 1 \leq k \leq M\}$  can be decomposed into  $\mathcal{K}^+ = \mathcal{Z}_1 \cup \mathcal{Z}_2$ , where  $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$ , in such a way that

$$h(t) \approx h(x_k) \quad \text{for all } t \in [x_{k-1}, x_k] \text{ and every } k \in \mathcal{Z}_1,$$

and

$$\frac{\varrho(t)}{h(t)} \approx \frac{\varrho(x_k)}{h(x_k)} \quad \text{for all } t \in [x_{k-1}, x_k] \text{ and every } k \in \mathcal{Z}_2,$$

in which the equivalence constants depend only on the parameter  $a$ . The interested reader can find the construction of covering sequences and proofs of their properties in [8]\*Chapter 3.

We shall conclude this section by recalling a result that, in a way, bridges the divide between the discrete world and the continuous one. Let  $p > 0$  and  $\tilde{w} \in \mathfrak{M}^+(0, L)$ . Set

$$\tilde{\varphi}(t) = \int_0^L \min\{\varrho(t), \varrho(s)\} \tilde{w}(s) ds, \quad t \in (0, L),$$

and assume that there is  $t_0 \in (0, L)$  such that  $0 < \tilde{\varphi}(t_0) < \infty$ . It is easy to see that  $\tilde{\varphi} \in Q_\varrho(0, L)$ . Let  $\{x_k\}_{k=N}^M \in CS(\tilde{\varphi}, \varrho, a)$  with  $a$  being large enough—namely  $a > 108$ . It follows from [18]\*Lemma 3.4 with  $\alpha = \beta = 0$ , see also (4.3) that

$$\begin{aligned} \int_0^L \left( \int_0^L \frac{\varrho(t)^{\frac{1}{p}} g(s)}{\varrho(t)^{\frac{1}{p}} + \varrho(s)^{\frac{1}{p}}} ds \right)^p \tilde{w}(t) dt &\approx \sum_{k=N}^M \tilde{\varphi}(x_k) \left( \int_0^L \frac{g(t)}{\varrho(x_k)^{\frac{1}{p}} + \varrho(t)^{\frac{1}{p}}} dt \right)^p \\ &\approx \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \frac{\tilde{\varphi}(t)^{\frac{1}{p}}}{\varrho(t)^{\frac{1}{p}}} g(t) dt \right)^p \end{aligned} \quad (2.1)$$

for every  $g \in \mathfrak{M}^+(0, L)$ , in which the multiplicative constants depend only on  $a$  and  $p$ . The assumption that  $a > 108$ , which is dictated by the assumptions of [18]\*Lemma 3.4, is merely technical and not restrictive at all.

Let  $N, M \in \mathbb{Z} \cup \{-\infty, \infty\}$ ,  $N < M$ , and  $\{\varrho_k\}_{k=N}^M$  be a sequence of positive numbers. We say that  $\{\varrho_k\}_{k=N}^M$  is *strongly increasing* or *strongly decreasing* if

$$\inf \left\{ \frac{\varrho_{k+1}}{\varrho_k} : N \leq k < M \right\} > 1 \quad (2.2)$$

or

$$\sup \left\{ \frac{\varrho_{k+1}}{\varrho_k} : N \leq k < M \right\} < 1, \quad (2.3)$$

respectively. We shall frequently use the following equivalences involving strongly monotone sequences. Let  $\{a_k\}_{k=N}^M$  be a sequence of nonnegative numbers and  $p > 0$ . If  $\{\varrho_k\}_{k=N}^M$  is strongly increasing, then

$$\sum_{k=N}^M \varrho_k \left( \sum_{i=k}^M a_i \right)^p \approx \sum_{k=N}^M \varrho_k a_k^p, \quad (2.4)$$

$$\sum_{k=N}^M \varrho_k \left( \sup_{k \leq i \leq M} a_i \right)^p \approx \sum_{k=N}^M \varrho_k a_k^p \quad (2.5)$$

and

$$\sup_{N \leq k \leq M} \varrho_k \left( \sum_{i=k}^M a_i \right)^p \approx \sup_{N \leq k \leq M} \varrho_k a_k^p. \quad (2.6)$$

If  $\{\varrho_k\}_{k=N}^M$  is strongly decreasing, then

$$\sum_{k=N}^M \varrho_k \left( \sum_{i=N}^k a_i \right)^p \approx \sum_{k=N}^M \varrho_k a_k^p \quad (2.7)$$

and

$$\sup_{N \leq k \leq M} \varrho_k \left( \sum_{i=N}^k a_i \right)^p \approx \sup_{N \leq k \leq M} \varrho_k a_k^p. \quad (2.8)$$

Moreover, all the equivalence constants depend only on the value of (2.2) or (2.3) and  $p$ . Such inequalities involving strongly monotone sequences are classical; e.g., see [16]\*Proposition 2.1 (cf. [19, 20]).

Finally, we shall also make use of the following equivalent expression for optimal constants in discrete Hardy inequalities with weights. Let  $0 < p, q, r < \infty$ ,  $\{a_k\}_{k=N}^M$  and  $\{b_k\}_{k=N}^M$  be sequences of nonnegative numbers,  $N, M \in \mathbb{Z} \cup \{-\infty, \infty\}$ ,  $N < M$ . Set

$$D = \sup_{\{x_k\}_{k=N}^M} \frac{\left( \sum_{k=N}^M \left( \sum_{i=N}^k x_i^r b_i \right)^{\frac{q}{r}} a_k \right)^{\frac{1}{q}}}{\left( \sum_{k=N}^M x_k^p \right)^{\frac{1}{p}}},$$

where the supremum extends over all sequences  $\{x_k\}_{k=N}^M$  of nonnegative numbers. Owing to [4], we have

$$D \approx \begin{cases} \sup_{N \leq k \leq M} \left( \sum_{i=k}^M a_i \right)^{\frac{1}{q}} b_k^{\frac{1}{r}} & \text{if } p \leq \min\{r, q\}, \\ \sup_{N \leq k \leq M} \left( \sum_{i=k}^M a_i \right)^{\frac{1}{q}} \left( \sum_{i=N}^k b_i^{\frac{p}{p-r}} \right)^{\frac{p-r}{pr}} & \text{if } r < p \leq q, \end{cases} \quad (2.9)$$

in which the equivalence constants depend only on  $p, q$  and  $r$ .

### 3. EQUIVALENT DISCRETE INEQUALITIES

We start with an auxiliary lemma.

**Lemma 3.1.** *Let  $0 < p, q, r < \infty$  and  $u, \delta, v, w$  be weights on  $(0, L)$ . Let  $\varphi$  be the function from (1.5). Assume that there is  $t_0 \in (0, L)$  such that  $0 < \varphi(t_0) < \infty$ . Let  $\{x_k\}_{k=N}^M \in CS(\varphi, U^p, a)$  with  $a > 108$ . Denote by  $\widetilde{C}_i$ ,  $i = 1, 2, 3, 4$ , the optimal constants in the inequalities:*

$$\begin{aligned} & \left( \sum_{k=N+1}^M \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ & \leq \widetilde{C}_1 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}} h \right)^p \right)^{\frac{1}{p}}, \end{aligned} \quad (3.1)$$

$$\begin{aligned} & \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ & \leq \widetilde{C}_2 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}} h \right)^p \right)^{\frac{1}{p}}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} & \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k \int_{x_j}^{x_{j+1}} h \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ & \leq \widetilde{C}_3 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}} h \right)^p \right)^{\frac{1}{p}}, \end{aligned} \quad (3.3)$$

$$\left( \sum_{k=N+1}^{M-1} \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} h \right)^q \left( \int_{x_{k-1}}^{x_k} w \right) \right)^{\frac{1}{q}} \leq \widetilde{C}_4 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}} h \right)^p \right)^{\frac{1}{p}} \quad (3.4)$$

for every  $h \in \mathfrak{M}^+(0, L)$ . Then the  $C$  defined by (1.4) satisfies

$$C \approx \widetilde{C}_1 + \widetilde{C}_2 + \widetilde{C}_3 + \widetilde{C}_4,$$

in which the equivalence constants depend only on the parameters  $p, q, r$  and  $a$ .

**Proof.** First, since

$$\frac{1}{U(s) + U(t)} \approx \min \left\{ \frac{1}{U(t)}, \frac{1}{U(s)} \right\} \quad \text{for every } s, t \in (0, L), \quad (3.5)$$

we have

$$\int_0^t \left( \int_s^L h \right) u(s) ds \approx \int_0^L \frac{U(s)U(t)}{U(s) + U(t)} h(s) ds \quad \text{for every } t \in (0, L).$$

Then

$$\text{RHS (1.3)} \approx C \left( \int_0^L \left( \int_0^L \frac{U(s)U(t)}{U(s) + U(t)} h(s) ds \right)^p \frac{v(t)}{U(t)^p} dt \right)^{\frac{1}{p}}$$

and, applying (2.1) to  $\varrho = U^p$ ,  $g = hU$ ,  $\tilde{w} = \frac{v}{U^p}$ , and  $\tilde{\varphi} = \varphi$ , we obtain

$$\text{RHS (1.3)} \approx C \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}} h \right)^p \right)^{\frac{1}{p}}. \quad (3.6)$$

Next,

$$\begin{aligned} \text{LHS (1.3)} &= \left( \sum_{k=N+1}^M \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_0^t \left( \int_s^{x_k} h + \int_{x_k}^L h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ &\approx \left( \sum_{k=N+1}^M \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_0^t \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ &\quad + \left( \sum_{k=N+1}^{M-1} \left( \int_{x_k}^L h \right)^q \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_0^t \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ &=: \text{I} + \text{II}. \end{aligned} \quad (3.7)$$

We shall first deal with I. Decomposing the integral  $\int_0^t$  into the sum  $\int_0^{x_{k-1}} + \int_{x_{k-1}}^t$  and using the fact that  $x_{k-1} = 0$  if  $k = N + 1$ , which is possible if and only if  $N > -\infty$ , we obtain

$$\begin{aligned} \text{I} &\approx \left( \sum_{k=N+2}^M \left( \int_0^{x_{k-1}} \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\quad + \left( \sum_{k=N+1}^M \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ &=: \text{I}_1 + \text{I}_2. \end{aligned} \tag{3.8}$$

Note that  $\text{I}_1$  can be written as

$$\begin{aligned} \text{I}_1 &= \left( \sum_{k=N+2}^M \left( \sum_{i=N+1}^{k-1} \int_{x_{i-1}}^{x_i} \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\approx \left( \sum_{k=N+2}^M \left( \sum_{i=N+1}^{k-1} \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\quad + \left( \sum_{k=N+2}^M \left( \sum_{i=N+1}^{k-1} \left( \int_{x_i}^{x_k} h \right)^r \int_{x_{i-1}}^{x_i} \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &= \left( \sum_{k=N+2}^M \left( \sum_{i=N+1}^{k-1} \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\quad + \left( \sum_{k=N+2}^M \left( \sum_{i=N+1}^{k-1} \left( \sum_{j=i}^{k-1} \int_{x_j}^{x_{j+1}} h \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &=: \text{I}_{1,1} + \text{I}_{1,2}. \end{aligned} \tag{3.9}$$

Now we shall deal with II. The very definition of  $\Delta$  yields

$$\text{II} = \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} h \right)^q \left( \int_{x_{k-1}}^{x_k} w \right) \right)^{\frac{1}{q}}.$$

Altogether,

$$\text{LHS (1.3)} \approx \text{I}_{1,1} + \text{I}_{1,2} + \text{I}_2 + \text{II}.$$

Consequently, reindexing  $(k-1) \mapsto k$  in  $\text{I}_{1,1}$  and  $\text{I}_{1,2}$ , we get from (3.7), (3.8) and (3.9)

$$\text{LHS (1.3)} \approx \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}}$$

$$\begin{aligned}
& + \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k \int_{x_j}^{x_{j+1}} h \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\
& + \left( \sum_{k=N+1}^M \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\
& + \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} h \right)^q \left( \int_{x_{k-1}}^{x_k} w \right) \right)^{\frac{1}{q}}. \tag{3.10}
\end{aligned}$$

The assertion now follows from (3.6) and (3.10).  $\square$

*Remark 3.2.* The assumption  $a > 108$  is merely technical, as already noted below (2.1).

We are now in a position to prove a discrete characterization of (1.3).

**Theorem 3.3.** *Let  $0 < p, q, r < \infty$  and  $u, \delta, v, w$  be weights on  $(0, L)$ . Let  $\varphi$  be given by (1.5). Assume that there is  $t_0 \in (0, L)$  such that  $0 < \varphi(t_0) < \infty$ . Let  $\{x_k\}_{k=N}^M \in CS(\varphi, U^p, a)$  with  $a > 108$ . Set*

$$A(x_{k-1}, x_k) = \sup_{h \in \mathfrak{M}^+(0, L)} \frac{\left( \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}}}{\int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}}} \tag{3.11}$$

and

$$B(x_{k-1}, x_k) = \sup_{h \in \mathfrak{M}^+(0, L)} \frac{\left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{1}{r}}}{\int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}}} \tag{3.12}$$

for  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M$ . Denote by  $C_i$ ,  $i = 1, 2, 3, 4$ , the optimal constants in the inequalities:

$$\left( \sum_{k=N+1}^M a_k^q A(x_{k-1}, x_k)^q \right)^{\frac{1}{q}} \leq C_1 \left( \sum_{k=N+1}^M a_k^p \right)^{\frac{1}{p}}; \tag{3.13}$$

$$\left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r B(x_{i-1}, x_i)^r \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \leq C_2 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}}; \tag{3.14}$$

$$\left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r \varphi(x_i)^{-\frac{r}{p}} \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \leq C_3 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}}; \tag{3.15}$$

and

$$\left( \sum_{k=N+1}^{M-1} a_k^q \varphi(x_k)^{-\frac{q}{p}} \left( \int_{x_{k-1}}^{x_k} w \right) \right)^{\frac{1}{q}} \leq C_4 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}} \tag{3.16}$$

for every sequence  $\{a_k\}_{k=N+1}^M$  of nonnegative numbers. Then the  $C$  defined by (1.4) satisfies

$$C \approx C_1 + C_2 + C_3 + C_4,$$

in which the equivalence constants depend only on  $p, q, r$  and  $a$ .

**Proof.** In view of Lemma 3.1, it is sufficient to show that  $\widetilde{C}_i \approx C_i$ ,  $i = 1, 2, 3, 4$ , with the equivalence constants depending only on the parameters  $p, q, r$  and  $a$ .

First, we shall show that  $\widetilde{C}_1 \approx C_1$ . Assume that  $\widetilde{C}_1 < \infty$ . Consequently,  $A(x_{k-1}, x_k) < \infty$  for every  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M$ . Hence there are functions  $h_k \in \mathfrak{M}^+(0, L)$ ,  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M$ , supported in  $[x_{k-1}, x_k]$  such that

$$\int_{x_{k-1}}^{x_k} h_k \varphi^{\frac{1}{p}} = 1 \quad (3.17)$$

and

$$\left( \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h_k \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \geq \frac{1}{2} A(x_{k-1}, x_k).$$

Plugging  $h = \sum_{i=N+1}^M a_i h_i$ , where  $\{a_i\}_{i=N+1}^M$  is a sequence of nonnegative numbers, in (3.1), we obtain

$$\begin{aligned} \text{LHS (3.1)} &= \left( \sum_{k=N+1}^M \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} \sum_{i=N+1}^M a_i h_i \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ &= \left( \sum_{k=N+1}^M a_k^q \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h_k \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right)^{\frac{1}{q}} \\ &\gtrsim \left( \sum_{k=N+1}^M a_k^q A(x_{k-1}, x_k)^q \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\text{RHS (3.1)} = \widetilde{C}_1 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \sum_{i=N+1}^M a_i h_i \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} = \widetilde{C}_1 \left( \sum_{k=N+1}^M a_k^p \right)^{\frac{1}{p}}.$$

Therefore

$$\left( \sum_{k=N+1}^M a_k A(x_{k-1}, x_k)^q \right)^{\frac{1}{q}} \lesssim \widetilde{C}_1 \left( \sum_{k=N+1}^M a_k^p \right)^{\frac{1}{p}};$$

hence  $C_1 \lesssim \widetilde{C}_1$ . On the other hand, assume that  $C_1 < \infty$ . Let  $h \in \mathfrak{M}^+(0, L)$ . Using (3.11) and (3.13) with  $\{a_k\}_{k=N+1}^M$  defined as

$$a_k = \int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}}, \quad (3.18)$$

we obtain

$$\text{LHS (3.1)}$$

$$\begin{aligned}
&= \left( \sum_{k=N+1}^M a_k^q \left( \int_{x_{k-1}}^{x_k} \left( \frac{1}{\Delta(t)} \int_{x_{k-1}}^t \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{q}{r}} w(t) dt \right) \left( \int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}} \right)^{-q} \right)^{\frac{1}{q}} \\
&\leq \left( \sum_{k=N+1}^M a_k^q A(x_{k-1}, x_k)^q \right)^{\frac{1}{q}} \leq C_1 \left( \sum_{k=N+1}^M a_k^p \right)^{\frac{1}{p}} \\
&= C_1 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Hence  $\widetilde{C}_1 \leq C_1$ .

Second, we shall show that  $\widetilde{C}_2 \approx C_2$ . Assume that  $\widetilde{C}_2 < \infty$ . Consequently, for every  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ ,  $B(x_{k-1}, x_k) < \infty$ . Hence there are functions  $h_k \in \mathfrak{M}^+(0, L)$ ,  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ , supported in  $[x_{k-1}, x_k]$  and satisfying (3.17) such that

$$\left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} h \right)^r \delta(s) ds \right)^{\frac{1}{r}} \geq \frac{1}{2} B(x_{k-1}, x_k).$$

Testing (3.2) with  $h = \sum_{i=N+1}^{M-1} a_i h_i$ , where  $\{a_i\}_{i=N+1}^M$  is a sequence of nonnegative numbers, we get

$$\begin{aligned}
\text{LHS (3.2)} &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} \sum_{j=N+1}^{M-1} a_j h_j \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\
&= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} h_i \right)^r \delta(s) ds \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\
&\gtrsim \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r B(x_{i-1}, x_i)^r \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}}.
\end{aligned}$$

Plainly,

$$\text{RHS (3.2)} = \widetilde{C}_2 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \sum_{i=N+1}^{M-1} a_i h_i \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} = \widetilde{C}_2 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}}.$$

Therefore

$$\left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r B(x_{i-1}, x_i)^r \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \lesssim \widetilde{C}_2 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}},$$



which implies  $C_2 \lesssim \widetilde{C}_2$ . Assume now that  $C_2 < \infty$ . It is easy to see that  $\widetilde{C}_2 \leq C_2$ . Indeed, thanks to (3.14) with  $\{a_k\}_{k=N+1}^M$  defined by (3.18) and (3.12), we have

$$\begin{aligned} \text{LHS (3.2)} &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r \left( \int_{x_{i-1}}^{x_i} \left( \int_s^{x_i} h \right)^r \delta(s) ds \right) \left( \int_{x_{i-1}}^{x_i} h \varphi^{\frac{1}{p}} \right)^{-r} \right)^{\frac{q}{r}} \right. \\ &\quad \left. \times \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\leq \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r B(x_{i-1}, x_i)^r \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \leq C_2 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}} \\ &\leq C_2 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \varphi^{\frac{1}{p}} h \right)^p \right)^{\frac{1}{p}} \end{aligned}$$

for every  $h \in \mathfrak{M}^+(0, L)$ .

Next, we turn our attention to the equivalence  $\widetilde{C}_3 \approx C_3$ . Assume that  $\widetilde{C}_3 < \infty$ . Note that

$$\sup_{h \in \mathfrak{M}^+(0, L)} \frac{\int_{x_k}^{x_{k+1}} h}{\int_{x_k}^{x_{k+1}} h \varphi^{\frac{1}{p}}} = \sup_{t \in (x_k, x_{k+1})} \varphi(t)^{-\frac{1}{p}} = \varphi(x_k)^{-\frac{1}{p}} \quad (3.19)$$

for every  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ , owing to the saturation of Hölder's inequality and the monotonicity of  $\varphi$ . Consequently, there are functions  $h_k \in \mathfrak{M}^+(0, L)$ ,  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ , supported in  $[x_k, x_{k+1}]$  such that

$$\int_{x_k}^{x_{k+1}} h_k \varphi^{\frac{1}{p}} = 1 \quad (3.20)$$

and

$$\int_{x_k}^{x_{k+1}} h_k \geq \frac{1}{2} \varphi(x_k)^{-\frac{1}{p}}. \quad (3.21)$$

By plugging  $h = \sum_{n=N+1}^{M-1} a_n h_n$ , where  $\{a_n\}_{n=N+1}^{M-1}$  is a sequence of nonnegative numbers, in (3.3), we obtain

$$\begin{aligned} \text{LHS (3.3)} &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k \int_{x_j}^{x_{j+1}} \sum_{n=N+1}^{M-1} a_n h_n \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \right. \\ &\quad \left. \times \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\gtrsim \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k a_j \varphi(x_j)^{-\frac{1}{p}} \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \end{aligned}$$

$$\geq \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r \varphi(x_i)^{-\frac{r}{p}} \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}}$$

and

$$\begin{aligned} \text{RHS (3.3)} &= \widetilde{C}_3 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} \sum_{n=N+1}^{M-1} a_n h_n \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\ &= \widetilde{C}_3 \left( \sum_{k=N}^{M-1} \left( \int_{x_k}^{x_{k+1}} \sum_{n=N+1}^{M-1} a_n h_n \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\ &= \widetilde{C}_3 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$\left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k a_i^r \varphi(x_i)^{-\frac{r}{p}} \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \lesssim \widetilde{C}_3 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}},$$

and so  $C_3 \lesssim \widetilde{C}_3$ . Assume now that  $C_3 < \infty$ . Let  $h \in \mathfrak{M}^+(0, L)$ , and test (3.15) with  $\{a_k\}_{k=N+1}^{M-1}$  defined as

$$a_k = \varphi(x_k)^{\frac{1}{p}} \sum_{j=k}^{M-1} b_j \varphi(x_j)^{-\frac{1}{p}},$$

where

$$b_j = \int_{x_j}^{x_{j+1}} h \varphi^{\frac{1}{p}}. \quad (3.22)$$

We have

$$\begin{aligned} \text{LHS (3.15)} &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \varphi(x_i)^{\frac{1}{p}} \sum_{j=i}^{M-1} b_j \varphi(x_j)^{-\frac{1}{p}} \right)^r \varphi(x_i)^{-\frac{r}{p}} \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \right. \\ &\quad \times \left. \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\geq \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k b_j \varphi(x_j)^{-\frac{1}{p}} \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ &\geq \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k b_j \left( \int_{x_j}^{x_{j+1}} h \right) \left( \int_{x_j}^{x_{j+1}} h \varphi^{\frac{1}{p}} \right)^{-1} \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \right. \\ &\quad \times \left. \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \end{aligned}$$

$$= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k \int_{x_j}^{x_{j+1}} h \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}}$$

in which we used (3.19) in the last inequality, and

$$\text{RHS (3.15)} = C_3 \left( \sum_{k=N+1}^{M-1} \varphi(x_k) \left( \sum_{j=k}^{M-1} b_j \varphi(x_j)^{-\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \approx C_3 \left( \sum_{k=N+1}^{M-1} b_k^p \right)^{\frac{1}{p}}, \quad (3.23)$$

in which we used (2.4) with  $\{\varrho_k\}_{k=N+1}^{M-1} = \{\varphi(x_k)\}_{k=N+1}^{M-1}$ ; moreover, the equivalence constants depend only on  $p$  and  $a$ . It follows that

$$\begin{aligned} & \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=N+1}^k \left( \sum_{j=i}^k \int_{x_j}^{x_{j+1}} h \right)^r \left( \int_{x_{i-1}}^{x_i} \delta \right) \right)^{\frac{q}{r}} \left( \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right) \right)^{\frac{1}{q}} \\ & \lesssim C_3 \left( \sum_{k=N+1}^{M-1} b_k^p \right)^{\frac{1}{p}} = C_3 \left( \sum_{k=N+1}^{M-1} \left( \int_{x_k}^{x_{k+1}} h \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \leq C_3 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}}; \end{aligned}$$

hence  $\widetilde{C}_3 \lesssim C_3$ .

Last, we shall show that  $\widetilde{C}_4 \approx C_4$ . Assume that  $\widetilde{C}_4 < \infty$ . Thanks to (3.19) again, there are functions  $h_k \in \mathfrak{M}^+(0, L)$ ,  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ , supported in  $[x_k, x_{k+1}]$  and satisfying (3.20) and (3.21). Let  $\{a_k\}_{k=N+1}^{M-1}$  be a sequence of nonnegative numbers. Inserting  $h = \sum_{j=N+1}^{M-1} a_j h_j$  in (3.4), we obtain

$$\begin{aligned} \text{LHS (3.4)} &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \sum_{j=N+1}^{M-1} a_j h_j \right)^q \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \\ &\gtrsim \left( \sum_{k=N+1}^{M-1} \left( \sum_{i=k}^{M-1} a_i \varphi(x_i)^{-\frac{1}{p}} \right)^q \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \\ &\geq \left( \sum_{k=N+1}^{M-1} a_k^q \varphi(x_k)^{-\frac{q}{p}} \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\text{RHS (3.4)} = \widetilde{C}_4 \left( \sum_{k=N}^{M-1} \left( \int_{x_k}^{x_{k+1}} \sum_{i=N+1}^{M-1} a_i h_i \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} = \widetilde{C}_4 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}}.$$

Hence

$$\left( \sum_{k=N+1}^{M-1} a_k^q \varphi(x_k)^{-\frac{q}{p}} \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \lesssim \widetilde{C}_4 \left( \sum_{k=N+1}^{M-1} a_k^p \right)^{\frac{1}{p}},$$

and so  $C_4 \lesssim \widetilde{C}_4$ . Now, the proof will be finished once we show that  $\widetilde{C}_4 \lesssim C_4$ . Assume that  $C_4 < \infty$ . Let  $h \in \mathfrak{M}^+(0, L)$ , and consider the sequence  $\{\varphi(x_k)^{\frac{1}{p}} \sum_{j=k}^{M-1} b_j \varphi(x_j)^{-\frac{1}{p}}\}_{j=N+1}^{M-1}$ ,

where  $\{b_j\}_{j=N+1}^{M-1}$  is defined by (3.22). Plugging it in (3.16), we get

$$\begin{aligned} \text{LHS (3.16)} &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{j=k}^{M-1} b_j \varphi(x_j)^{-\frac{1}{p}} \right)^q \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \\ &\geq \left( \sum_{k=N+1}^{M-1} \left( \sum_{j=k}^{M-1} b_j \left( \int_{x_j}^{x_{j+1}} h \right) \left( \int_{x_j}^{x_{j+1}} h \varphi^{\frac{1}{p}} \right)^{-1} \right)^q \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \\ &= \left( \sum_{k=N+1}^{M-1} \left( \sum_{j=k}^{M-1} \int_{x_j}^{x_{j+1}} h \right)^q \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}}, \end{aligned}$$

in which we used (3.19), and

$$\text{RHS (3.16)} \approx C_4 \left( \sum_{k=N+1}^{M-1} b_k^p \right)^{\frac{1}{p}} = C_4 \left( \sum_{k=N+1}^{M-1} \left( \int_{x_k}^{x_{k+1}} h \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}},$$

in which we used the same argument as in (3.23). It follows that

$$\begin{aligned} \left( \sum_{k=N+1}^{M-1} \left( \sum_{j=k}^{M-1} \int_{x_j}^{x_{j+1}} h \right)^q \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} &\lesssim C_4 \left( \sum_{k=N+1}^{M-1} \left( \int_{x_k}^{x_{k+1}} h \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\ &\leq C_4 \left( \sum_{k=N+1}^M \left( \int_{x_{k-1}}^{x_k} h \varphi^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}}, \end{aligned}$$

which finishes the proof.  $\square$

*Remark 3.4.* For future reference, note that, thanks to the following equivalent expression for optimal constants in (continuous) Hardy inequalities with weights (see [5] for  $r \geq 1$  and [25] for  $r < 1$ ), we have

$$B(x_{k-1}, x_k) \approx \begin{cases} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} & \text{if } r \geq 1, \\ \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{1-r}{r}} & \text{if } r < 1, \end{cases}$$

for every  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M$ , in which the equivalence constants depend only on  $r$ .

**Theorem 3.5.** *Let  $0 < p \leq q < \infty$ ,  $0 < r < \infty$  and  $u, \delta, v, w$  be weights on  $(0, L)$ . Let  $\varphi$  be the function defined by (1.5). Assume that there is  $t_0 \in (0, L)$  such that  $0 < \varphi(t_0) < \infty$ . Let  $\{x_k\}_{k=N}^M \in CS(\varphi, U^p, a)$  with  $a > 108$ . Let  $C$  be given by (1.4).*

(i) If  $p \leq q$ ,  $p \leq r$ ,  $1 \leq q$ ,  $1 \leq r$ , then  $C \approx C_{1,1} + C_{1,2} + C_{3,1} + C_{4,1}$ , where

$$\begin{aligned} C_{1,1} &:= \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{1}{r}} \varphi(s)^{-\frac{1}{p}}, \\ C_{1,2} &:= \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}}, \end{aligned}$$

$$C_{3,1} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}},$$

and

$$C_{4,1} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}}.$$

(ii) If  $p \leq r < 1 \leq q$ , then  $C \approx C_{1,2} + C_{1,3} + C_{3,2} + C_{4,1}$ , where

$$C_{1,3} := \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \times \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^s \delta \right)^{\frac{r}{1-r}} \delta(s) \varphi(s)^{-\frac{r}{p(1-r)}} ds \right)^{\frac{1-r}{r}},$$

and

$$C_{3,2} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{1-r}{r}}.$$

(iii) If  $1 \leq r < p \leq q$ , then  $C \approx C_{1,1} + C_{1,2} + C_{3,3} + C_{4,1}$ , where

$$C_{3,3} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k \sup_{t \in (x_{i-1}, x_i)} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{p}{p-r}} \varphi(t)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}}.$$

(iv) If  $r < p \leq q$ ,  $r < 1 \leq q$ , then  $C \approx C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1}$ , where

$$C_{3,4} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \times \left( \sum_{i=N+1}^k \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{p(1-r)}{p-r}} \right)^{\frac{p-r}{pr}}.$$

(v) If  $p \leq q < 1 \leq r$ , then  $C \approx C_{1,4} + C_{1,5} + C_{3,1} + C_{4,1}$ , where

$$C_{1,4} := \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} w(t) \Delta(t)^{-\frac{q}{r}} \times \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r(1-q)}} \varphi(s)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}},$$

and

$$C_{1,5} := \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{q}{1-q}} \right)^{\frac{1-q}{q}}$$

$$\times \Delta(t)^{-\frac{q}{r}} w(t) \left( \int_{x_{k-1}}^t \delta \right)^{\frac{q}{r}} \varphi(t)^{-\frac{q}{p(1-q)}} dt \Big)^{\frac{1-q}{q}}.$$

(vi) If  $p \leq q < 1$ ,  $p \leq r < 1$ , then  $C \approx C_{1,5} + C_{1,6} + C_{3,2} + C_{4,1}$ , where

$$C_{1,6} := \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(t)^{-\frac{q}{r}} w(t) \right. \\ \left. \times \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^s \delta \right)^{\frac{r}{1-r}} \delta(s) \varphi(s)^{-\frac{r}{p(1-r)}} ds \right)^{\frac{q(1-r)}{r(1-q)}} dt \right)^{\frac{1-q}{q}}.$$

(vii) If  $r < p \leq q < 1$ , then  $C \approx C_{1,5} + C_{1,6} + C_{3,4} + C_{4,1}$ .

**Proof.** Owing to Theorem 3.3, we have

$$C \approx C_1 + C_2 + C_3 + C_4, \quad (3.24)$$

in which  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  are the optimal constants in (3.13), (3.14), (3.15) and (3.16), respectively.

First, we shall find equivalent expressions for  $C_1$ . By the standard argument, sometimes referred to as the Landau theorem (e.g., [8]\*Lemma 1.4.1),

$$C_1 = \sup_{N+1 \leq k \leq M} A(x_{k-1}, x_k),$$

where the quantities  $A(x_{k-1}, x_k)$  are defined by (3.11). By [13]\*Theorem A we have

$$C_1 = \sup_{N+1 \leq k \leq M} A(x_{k-1}, x_k) \approx \begin{cases} C_{1,1} + C_{1,2} & \text{if } 1 \leq \min\{q, r\}; \\ C_{1,2} + C_{1,3} & \text{if } r < 1 \leq q; \\ C_{1,4} + C_{1,5} & \text{if } q < 1 \leq r; \\ C_{1,5} + C_{1,6} & \text{if } \max\{q, r\} < 1. \end{cases} \quad (3.25)$$

Second, we shall find equivalent expressions for  $C_2$ . Using (2.9) with

$$b_k = B(x_{k-1}, x_k)^r, \quad k \in \mathbb{Z}, N+1 \leq k \leq M-1,$$

where the quantities  $B(x_{k-1}, x_k)$  are defined by (3.12), and

$$a_k = \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w, \quad k \in \mathbb{Z}, N+1 \leq k \leq M-1,$$

gives us

$$C_2 \approx \begin{cases} \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} B(x_{k-1}, x_k) & \text{if } p \leq \min\{q, r\}; \\ \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k B(x_{i-1}, x_i)^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} & \text{if } r < p \leq q. \end{cases}$$

Combining that with Remark 3.4, we obtain

$$C_2 \approx \begin{cases} C_{3,1} & \text{if } p \leq \min\{q, r\}, 1 \leq r; \\ C_{3,3} & \text{if } 1 \leq r < p \leq q; \\ C_{3,2} & \text{if } p \leq \min\{q, r\}, r < 1; \\ C_{3,4} & \text{if } r < p \leq q, r < 1. \end{cases} \quad (3.26)$$

Next, we shall turn our attention to  $C_3$ . Using (2.9) with  $b_k = \varphi(x_k)^{-\frac{r}{p}} \int_{x_{k-1}}^{x_k} \delta$  and  $a_k = \int_{x_k}^{x_{k+1}} \Delta^{-\frac{q}{r}} w$ ,  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ , we infer that

$$C_3 \approx \begin{cases} C_{2,1} & \text{if } p \leq \min\{q, r\}; \\ C_{2,2} & \text{if } r < p \leq q, \end{cases} \quad (3.27)$$

where

$$C_{2,1} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^{x_k} \delta \right)^{\frac{1}{r}} \varphi(x_k)^{-\frac{1}{p}},$$

$$C_{2,2} := \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k \left( \int_{x_{i-1}}^{x_i} \delta \right)^{\frac{p}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}}.$$

Now, by the same argument as in the case  $C_1$ , we have

$$C_4 = \sup_{N+1 \leq k \leq M-1} \left( \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} = C_{4,1}. \quad (3.28)$$

Finally, the assertion follows from (3.24) combined with (3.25), (3.26), (3.27) and (3.28) upon making a few simple observations. Namely

$$C_{2,1} \leq C_{3,1} \lesssim C_{3,2} \quad (3.29)$$

and

$$C_{2,2} \leq C_{3,3} \lesssim C_{3,4}. \quad (3.30)$$

□

#### 4. ANTIDISCRETIZATION

We start with a technical lemma, which will prove useful later.

**Lemma 4.1.** *Let  $0 < r < p < \infty$ . Let  $\varphi$  and  $\sigma$  be functions defined by (1.5) and (1.6), respectively. Assume that there is  $t_0 \in (0, L)$  such that  $0 < \varphi(t_0) < \infty$ . Let  $\{x_k\}_{k=N}^M \in CS(\varphi, U^p, a)$ .*

(i) *Let  $i \in \mathbb{Z}$ ,  $N+2 \leq i \leq M$  and  $y \in [x_{i-1}, x_i]$ . Let  $h \in Q_U(0, y)$ . We have*

$$\int_{x_{i-1}}^y \sigma(t) h(t)^{\frac{pr}{p-r}} dt \lesssim h(y)^{\frac{pr}{p-r}} \varphi(y)^{-\frac{r}{p-r}} \quad \text{if } i \in \mathcal{Z}_1, \quad (4.1)$$

and

$$\int_{x_{i-1}}^y \sigma(t) h(t)^{\frac{pr}{p-r}} dt \lesssim h(x_{i-1})^{\frac{pr}{p-r}} \varphi(x_{i-1})^{-\frac{r}{p-r}} \quad \text{if } i \in \mathcal{Z}_2. \quad (4.2)$$

(ii) *If  $N > -\infty$ , then  $N+1 \in \mathcal{Z}_2$  and, for every  $y \in (0, x_{N+1}]$  and  $h \in Q_U(0, y)$ ,*

$$\int_0^y \sigma(t) h(t)^{\frac{pr}{p-r}} dt \lesssim \sup_{t \in (0, y]} h(t)^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}}. \quad (4.3)$$

(iii) Let  $k \in \mathbb{Z}$ ,  $N + 1 \leq k \leq M$ . If  $h \in Q_U(0, x_k)$ , then

$$\begin{aligned} \sum_{i=N+1}^{k-1} h(x_i)^{\frac{pr}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}} &\lesssim \int_0^{x_k} \sigma(t) h(t)^{\frac{pr}{p-r}} dt \\ &\lesssim \sum_{i=N+1}^k h(x_i)^{\frac{pr}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}} + \sup_{t \in (0, x_{N+1})} h(t)^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}}. \end{aligned} \quad (4.4)$$

The multiplicative constants in this lemma depend only on  $a$ .

**Proof.** First, observe that

$$\varphi(t) = V(t) + U(t)^p \int_t^L U(s)^{-p} v(s) ds \quad \text{for every } t \in (0, \infty),$$

and recall that since  $\varphi \in Q_{U^p}(0, L)$ , for  $N + 2 \leq i \leq M$ ,

$$\varphi(y) \approx \varphi(x_{i-1}) \quad \text{for all } y \in [x_{i-1}, x_i] \text{ and every } i \in \mathcal{Z}_1,$$

and

$$\frac{\varphi}{U^p}(y) \approx \frac{\varphi}{U^p}(x_{i-1}) \quad \text{for all } y \in [x_{i-1}, x_i] \text{ and every } i \in \mathcal{Z}_2.$$

Now, if  $i \in \mathcal{Z}_1$ ,  $N + 2 \leq i \leq M$ , then, for each  $y \in [x_{i-1}, x_i]$ , we have

$$\begin{aligned} &\int_{x_{i-1}}^y \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p} v \right) U(t)^{p-1} u(t) h(t)^{\frac{pr}{p-r}} dt \\ &\leq h(y)^{\frac{pr}{p-r}} \int_{x_{i-1}}^y \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p} v \right) U(t)^{p-1} u(t) dt \\ &\leq h(y)^{\frac{pr}{p-r}} \int_{x_{i-1}}^y \varphi(t)^{-\frac{r}{p-r}-1} \left( \int_t^L U^{-p} v \right) U(t)^{p-1} u(t) dt \\ &\approx h(y)^{\frac{pr}{p-r}} \int_{x_{i-1}}^y d \left[ -\varphi^{-\frac{r}{p-r}} \right] \leq h(y)^{\frac{pr}{p-r}} \varphi(x_{i-1})^{-\frac{r}{p-r}} \approx h(y)^{\frac{pr}{p-r}} \varphi(y)^{-\frac{r}{p-r}}. \end{aligned}$$

On the other hand, if  $i \in \mathcal{Z}_2$ ,  $N + 2 \leq i \leq M$ , then, for each  $y \in [x_{i-1}, x_i]$ , we have

$$\begin{aligned} &\int_{x_{i-1}}^y \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p} v \right) U(t)^{p-1} u(t) h(t)^{\frac{pr}{p-r}} dt \\ &\leq \left( \frac{h(x_{i-1})}{U(x_{i-1})} \right)^{\frac{pr}{p-r}} \int_{x_{i-1}}^y \varphi^{-\frac{r}{p-r}-1} V U^{-1} u U^{\frac{pr}{p-r}} \\ &\approx \left( \frac{h(x_{i-1})}{U(x_{i-1})} \right)^{\frac{pr}{p-r}} \int_{x_{i-1}}^y d \left[ \left( \frac{\varphi}{U^p} \right)^{-\frac{r}{p-r}} \right] \leq \left( \frac{h(x_{i-1})}{U(x_{i-1})} \right)^{\frac{pr}{p-r}} \frac{\varphi(y)^{-\frac{r}{p-r}}}{U(y)^{-\frac{pr}{p-r}}} \\ &\approx h(x_{i-1})^{\frac{pr}{p-r}} \varphi(x_{i-1})^{-\frac{r}{p-r}}. \end{aligned}$$

Next, note that  $\lim_{t \rightarrow 0+} \varphi(t) = 0$  thanks to the fact that there is  $t_0 \in (0, L)$  such that  $\varphi(t_0) < \infty$ . Therefore, if  $N > -\infty$ , then  $\lim_{t \rightarrow 0+} \varphi(t)/U^p(t) > 0$ ; hence  $N + 1 \in \mathcal{Z}_2$ . Consequently, for every  $y \in (0, x_{N+1}]$ , we have

$$\int_0^y \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p} v \right) U(t)^{p-1} u(t) h(t)^{\frac{pr}{p-r}} dt$$



$$\begin{aligned}
&\leq \lim_{t \rightarrow 0^+} \left( \frac{h(t)}{U(t)} \right)^{\frac{pr}{p-r}} \int_0^y d \left[ \left( \frac{\varphi}{U^p} \right)^{-\frac{r}{p-r}} \right] \leq \lim_{t \rightarrow 0^+} \left( \frac{h(t)}{U(t)} \right)^{\frac{pr}{p-r}} \left( \frac{\varphi(y)}{U(y)^p} \right)^{-\frac{r}{p-r}} \\
&\approx \lim_{t \rightarrow 0^+} \left( \frac{h(t)}{U(t)} \right)^{\frac{pr}{p-r}} \lim_{t \rightarrow 0^+} \left( \frac{U(t)^p}{\varphi(t)} \right)^{\frac{r}{p-r}} \lesssim \sup_{t \in (0, y]} h(t)^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}}.
\end{aligned}$$

Finally, we shall prove (4.4). Note that it clearly holds if  $k = N + 1 > -\infty$  thanks to (4.3). Let  $k \in \mathbb{Z}$ ,  $N + 2 \leq k \leq M$ . By combining (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}
\int_0^{x_k} \sigma h^{\frac{pr}{p-r}} &\lesssim \int_0^{x_{N+1}} \sigma h^{\frac{pr}{p-r}} + \sum_{i \in \mathcal{Z}_1, N+2 \leq i \leq k} \int_{x_{i-1}}^{x_i} \sigma h^{\frac{pr}{p-r}} \\
&\quad + \sum_{i \in \mathcal{Z}_2, N+2 \leq i \leq k} \int_{x_{i-1}}^{x_i} \sigma h^{\frac{pr}{p-r}} \\
&\leq \sup_{t \in (0, x_{N+1})} h(t)^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}} + \sum_{i \in \mathcal{Z}_1, N+2 \leq i \leq k} h(x_i)^{\frac{pr}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}} \\
&\quad + \sum_{i \in \mathcal{Z}_2, N+2 \leq i \leq k} h(x_{i-1})^{\frac{pr}{p-r}} \varphi(x_{i-1})^{-\frac{r}{p-r}} \\
&\leq \sup_{t \in (0, x_{N+1})} h(t)^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}} + \sum_{i=N+1}^k h(x_i)^{\frac{pr}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\int_0^{x_k} \sigma h^{\frac{pr}{p-r}} &\gtrsim \sum_{i=N+1}^{k-1} \left( \int_{x_{i-1}}^{x_i} \sigma h^{\frac{pr}{p-r}} + \int_{x_i}^{x_{i+1}} \sigma h^{\frac{pr}{p-r}} \right) \\
&\geq \sum_{i=N+1}^{k-1} \left( \left( \frac{h(x_i)}{U(x_i)} \right)^{\frac{pr}{p-r}} \left( \int_{x_i}^L U^{-p} v \right) \int_{x_{i-1}}^{x_i} d \left[ (U^{-p} \varphi)^{-\frac{r}{p-r}-1} \right] \right. \\
&\quad \left. + h(x_i)^{\frac{pr}{p-r}} V(x_i) \int_{x_i}^{x_{i+1}} d \left[ -\varphi^{-\frac{r}{p-r}-1} \right] \right) \\
&= \sum_{i=N+1}^{k-1} \left( \left( \frac{h(x_i)}{U(x_i)} \right)^{\frac{pr}{p-r}} \left( \int_{x_i}^L U^{-p} v \right) \left( (U^{-p} \varphi)(x_i)^{-\frac{r}{p-r}-1} - (U^{-p} \varphi)(x_{i-1})^{-\frac{r}{p-r}-1} \right) \right. \\
&\quad \left. + h(x_i)^{\frac{pr}{p-r}} V(x_i) \left( \varphi(x_i)^{-\frac{r}{p-r}-1} - \varphi(x_{i+1})^{-\frac{r}{p-r}-1} \right) \right).
\end{aligned}$$

Since  $\{x_k\}_{k=N}^M \in CS(\varphi, U^p, a)$ ,  $\{(U^{-p} \varphi)^{-\frac{r}{p-r}}(x_i)\}_{i=N+1}^{M-1}$  and  $\{\varphi(x_i)^{-\frac{r}{p-r}-1}\}_{i=N+1}^{M-1}$  are strictly increasing and strictly decreasing, respectively. Therefore, we have

$$(U^{-p} \varphi)(x_i)^{-\frac{r}{p-r}-1} - (U^{-p} \varphi)(x_{i-1})^{-\frac{r}{p-r}-1} \geq (1 - a^{-\frac{r}{p-r}-1}) (U^{-p} \varphi)(x_i)^{-\frac{r}{p-r}-1}, \quad (4.5)$$

and

$$\varphi(x_i)^{-\frac{r}{p-r}-1} - \varphi(x_{i+1})^{-\frac{r}{p-r}-1} \geq (1 - a^{-\frac{r}{p-r}-1}) \varphi(x_i)^{-\frac{r}{p-r}-1}. \quad (4.6)$$

Note that  $a^{-r/(p-r)-1} \in (0, 1)$ . Then in view of (4.5) and (4.6) we get

$$\begin{aligned}
\int_0^{x_k} \sigma h^{\frac{pr}{p-r}} &\gtrsim \sum_{i=N+1}^{k-1} \left( \left( \frac{h(x_i)}{U(x_i)} \right)^{\frac{pr}{p-r}} \left( \int_{x_i}^L U^{-p\nu} \right) \varphi(x_i)^{-\frac{r}{p-r}-1} U(x_i)^{\frac{pr}{p-r}+p} \right. \\
&\quad \left. + h(x_i)^{\frac{pr}{p-r}} V(x_i) \varphi(x_i)^{-\frac{r}{p-r}-1} \right) \\
&= \sum_{i=N+1}^{k-1} h(x_i)^{\frac{pr}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}-1} \left( U(x_i)^p \left( \int_{x_i}^L U^{-p\nu} \right) + V(x_i) \right) \\
&\approx \sum_{i=N+1}^{k-1} h(x_i)^{\frac{pr}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}}.
\end{aligned}$$

□

We now apply the lemma above to two particular choices of  $h$ .

*Remark 4.2.* Let  $r, p, \varphi$  and  $\{x_k\}_{k=N}^M$  be as in Lemma 4.1. Let  $k \in \mathbb{Z}$ ,  $N+1 \leq k \leq M-1$ . In what follows, the multiplicative constants depend only on the discretization parameter  $a$ .

(i) Consider the function

$$h(t) = U(t) \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1}, \quad t \in (0, x_k).$$

Plainly  $h \in Q_U(0, x_k)$  and, by Lemma 4.1, we have

$$\begin{aligned}
&\sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\
&\lesssim \int_0^{x_k} \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p\nu} \right) U(t)^{p-1} u(t) U(t)^{\frac{pr}{p-r}} \\
&\quad \times \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} dt
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\lesssim \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\
&\quad + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} U(t)^{\frac{pr}{p-r}} \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\
&\lesssim \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\
&\quad + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} U(t)^{\frac{pr}{p-r}} \sup_{\tau \in (t, x_{N+1})} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\
&= \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\
&\quad + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \Delta(t)^{\frac{p}{p-r}}.
\end{aligned} \tag{4.8}$$

(ii) Assume, in addition, that  $r < 1$  and consider the function

$$h(t) = \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}}\} ds \right)^{\frac{1-r}{r}}, \quad t \in (0, x_k).$$

Plainly  $h \in Q_U(0, x_k)$  and, by Lemma 4.1, we have

$$\begin{aligned} & \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(x_i)^{\frac{r}{1-r}}\} ds \right)^{\frac{p(1-r)}{p-r}} \\ & \lesssim \int_0^{x_k} \sigma(t) \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}}\} ds \right)^{\frac{p(1-r)}{p-r}} dt \quad (4.9) \\ & \lesssim \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} \\ & \quad \times \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(x_i)^{\frac{r}{1-r}}\} ds \right)^{\frac{p(1-r)}{p-r}} \\ & + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \\ & \quad \times \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}}\} ds \right)^{\frac{p(1-r)}{p-r}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}}\} ds \right)^{\frac{p(1-r)}{p-r}} \\ & \approx \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \left( \Delta(t)^{\frac{1}{1-r}} + U(t)^{\frac{r}{1-r}} \int_t^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\ & \approx \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \Delta(t)^{\frac{p}{p-r}} \\ & \quad + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} U(t)^{\frac{pr}{p-r}} \left( \int_t^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\ & \approx \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \Delta(t)^{\frac{p}{p-r}} \\ & \quad + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} U(t)^{\frac{pr}{p-r}} \left( \int_t^{x_{N+1}} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\ & \quad + \varphi(x_{N+1})^{-\frac{r}{p-r}} U(x_{N+1})^{\frac{pr}{p-r}} \left( \int_{x_{N+1}}^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}}. \end{aligned}$$

Finally

$$\int_0^{x_k} \varphi(t)^{-\frac{r}{p-r}-2} V(t) \left( \int_t^L U^{-p} v \right) U(t)^{p-1} u(t)$$

$$\begin{aligned}
& \times \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min\{U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}}\} ds \right)^{\frac{p(1-r)}{p-r}} dt \\
& \lesssim \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} \left( \Delta(x_i)^{\frac{1}{1-r}} + U(x_i)^{\frac{r}{1-r}} \int_{x_i}^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\
& + \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{r}{p-r}} \left( \Delta(t)^{\frac{1}{1-r}} + U(t)^{\frac{r}{1-r}} \int_t^{x_{N+1}} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}}. \tag{4.10}
\end{aligned}$$

## 5. PROOF OF THE MAIN RESULT

**Proof of Theorem 1.1.** We start by fixing a covering sequence  $\{x_k\}_{k=N}^M \in CS(\varphi, U^p, a)$  with any  $a > 108$  (for example,  $a = 109$ ). In the entire proof, equivalence constants depend only on  $p, q, r$  (and on the completely immaterial choice of  $a > 108$ , see Remark 3.2). When proving a desired upper/lower bound on  $C$ , we always implicitly assume that the quantity on the right/left-hand side is finite.

(i) By Theorem 3.5, we have

$$C \approx C_{1,1} + C_{1,2} + C_{3,1} + C_{4,1}.$$

It is easy to see that

$$C_{1,1} \leq \sup_{N+1 \leq k \leq M} \sup_{s \in (x_{k-1}, x_k)} \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \Delta(s)^{\frac{1}{r}} \varphi(s)^{-\frac{1}{p}} = B_2, \tag{5.1}$$

$$C_{1,2} \leq \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \leq B_1, \tag{5.2}$$

$$C_{3,1} \leq B_2, \tag{5.3}$$

$$C_{4,1} \leq B_1. \tag{5.4}$$

Hence

$$C_{1,1} + C_{1,2} + C_{3,1} + C_{4,1} \lesssim B_1 + B_2. \tag{5.5}$$

As for the opposite inequality, note that

$$\begin{aligned}
B_1 &= \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} W(t)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \\
&\approx C_{4,1} + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}}, \tag{5.6}
\end{aligned}$$

where we used (2.8).

Next,

$$\begin{aligned}
B_1 &\approx C_{4,1} + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \Delta^{-\frac{q}{r}} w \Delta^{\frac{q}{r}} \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \\
&\approx C_{4,1} + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \\
& \leq C_{4,1} + C_{1,2} + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \\
& \lesssim C_{4,1} + C_{1,2} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{i=N+1}^k \int_{x_{i-1}}^{x_i} \delta \right)^{\frac{1}{r}} \\
& \approx C_{4,1} + C_{1,2} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \delta \right)^{\frac{1}{r}} \\
& \leq C_{4,1} + C_{1,2} + C_{3,1}, \tag{5.7}
\end{aligned}$$

where we used (2.8) again (note that the sequence  $\left\{ \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \right\}_{k=N+1}^{M-1}$  is strongly decreasing). As for  $B_2$ , we have

$$\begin{aligned}
B_2 & = \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \lesssim \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x-1}^t \delta \right)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \\
& \quad + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \left( \int_{x_{k-1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \lesssim C_{1,1} + C_{3,1} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{i=N+1}^k \int_{x_{i-1}}^{x_i} \delta \right)^{\frac{1}{r}} \\
& \approx C_{1,1} + C_{3,1} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \delta \right)^{\frac{1}{r}} \\
& \approx C_{1,1} + C_{3,1}, \tag{5.8}
\end{aligned}$$

where the first equivalence follows from (2.8) again. Thus we have

$$B_1 + B_2 \lesssim C_{1,1} + C_{1,2} + C_{3,1} + C_{4,1}, \tag{5.9}$$

which together with (5.5) gives

$$C \approx B_1 + B_2.$$

(ii) By Theorem 3.5, we have

$$C \approx C_{1,2} + C_{1,3} + C_{3,2} + C_{4,1}.$$

We start by establishing the desired upper estimate on  $C$ . In view of (5.2) and (5.4), we only need to prove suitable upper estimates on  $C_{1,3}$  and  $C_{3,2}$ .

Observe that for  $k \in \mathcal{Z}_1$  and  $t \in (x_{k-1}, x_k]$

$$\begin{aligned} & \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}} \\ & \approx \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) d\tau \right)^{\frac{1-r}{r}} \\ & \leq \varphi(x_k)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}}, \end{aligned} \quad (5.10)$$

and that for  $k \in \mathcal{Z}_2$  and  $t \in (x_{k-1}, x_k]$

$$\begin{aligned} & \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}} \\ & \approx U(x_{k-1}) \varphi(x_{k-1})^{-\frac{1}{p}} \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) U(\tau)^{-\frac{r}{1-r}} d\tau \right)^{\frac{1-r}{r}} \end{aligned} \quad (5.11)$$

$$\leq \sup_{s \in (x_{k-1}, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}}. \quad (5.12)$$

Consequently, using (5.10) and (5.12), we have

$$\begin{aligned} C_{1,3} & \lesssim \sup_{k \in \mathcal{Z}_1} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \\ & \quad + \sup_{k \in \mathcal{Z}_2} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (x_{k-1}, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ & \lesssim \sup_{k \in \mathcal{Z}_1} \varphi(x_k)^{-\frac{1}{p}} W(x_k)^{\frac{1}{q}} \\ & \quad + \sup_{k \in \mathcal{Z}_2} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ & \leq B_1 + B_3. \end{aligned} \quad (5.14)$$

Next, since  $U \varphi^{-\frac{1}{p}} \in Q_U(0, L)$ , [18]\*Lemma 3.5 applied to  $p \mapsto \frac{1-r}{r}$ ,  $q \mapsto U$ ,  $\varphi \mapsto U \varphi^{-\frac{1}{p}}$  and  $f \mapsto \Delta^{\frac{r}{1-r}} \delta$ , in which the symbols on the left-hand sides refer to those in the lemma, gives us

$$\begin{aligned} & \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_0^{x_k} \frac{\Delta(\tau)^{\frac{r}{1-r}} \delta(\tau)}{U(s)^{\frac{r}{1-r}} + U(\tau)^{\frac{r}{1-r}}} d\tau \right)^{\frac{1-r}{r}} \\ & \approx \sup_{N \leq i \leq k} U(x_i) \varphi(x_i)^{-\frac{1}{p}} \left( \int_0^{x_k} \frac{\Delta(\tau)^{\frac{r}{1-r}} \delta(\tau)}{U(x_i)^{\frac{r}{1-r}} + U(\tau)^{\frac{r}{1-r}}} d\tau \right)^{\frac{1-r}{r}} \end{aligned}$$

$$\approx \sup_{N+1 \leq i \leq k} \left( \int_{x_{i-1}}^{x_i} \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}}. \quad (5.15)$$

Consequently

$$\begin{aligned} C_{3,2} &\leq \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{N+1 \leq i \leq k} \left( \int_{x_{i-1}}^{x_i} \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\ &\approx \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_0^{x_k} \frac{\Delta(\tau)^{\frac{r}{1-r}} \delta(\tau)}{U(s)^{\frac{r}{1-r}} + U(\tau)^{\frac{r}{1-r}}} d\tau \right)^{\frac{1-r}{r}} \\ &\leq \sup_{N+1 \leq k \leq M-1} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (0, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_0^t \frac{\Delta(\tau)^{\frac{r}{1-r}} \delta(\tau)}{U(s)^{\frac{r}{1-r}} + U(\tau)^{\frac{r}{1-r}}} d\tau \right)^{\frac{1-r}{r}} \\ &\approx \sup_{N+1 \leq k \leq M-1} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, t)} \varphi(s)^{-\frac{1}{p}} \left( \int_0^s \Delta^{\frac{r}{1-r}} \delta \right)^{\frac{1-r}{r}} \\ &\quad + \sup_{N+1 \leq k \leq M-1} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (0, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &\approx B_2 + B_3. \end{aligned} \quad (5.16)$$

Altogether, we have

$$C_{1,2} + C_{1,3} + C_{3,2} + C_{4,1} \lesssim B_1 + B_2 + B_3. \quad (5.17)$$

On the other hand, we shall prove that  $B_1 + B_2 + B_3 \lesssim C_{1,2} + C_{1,3} + C_{3,2} + C_{4,1}$ . First, observe that

$$\begin{aligned} \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{1}{r}} \varphi(s)^{-\frac{1}{p}} &\approx \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \left( \int_{x_{k-1}}^{\tau} \delta \right)^{\frac{r}{1-r}} \delta(\tau) d\tau \right)^{\frac{1-r}{r}} \varphi(s)^{-\frac{1}{p}} \\ &\leq \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^{\tau} \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}}. \end{aligned} \quad (5.18)$$

Then it is clear that

$$C_{1,1} \lesssim C_{1,3} \quad (5.19)$$

and

$$C_{3,1} \lesssim C_{3,2}. \quad (5.20)$$

Therefore, using (5.9), we arrive at

$$B_1 + B_2 \lesssim C_{1,3} + C_{1,2} + C_{3,2} + C_{4,1}. \quad (5.21)$$

As for  $B_3$ , we have

$$\begin{aligned} B_3 &= \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &\approx \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (0, x_{k-1})} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &+ \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (x_{k-1}, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &+ \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &\lesssim \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} d\tau \right)^{\frac{1-r}{r}} \\ &+ \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (x_{k-1}, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}}. \end{aligned}$$

In view of (3.5), we have

$$\begin{aligned} &\sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &\lesssim \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_0^{x_k} \frac{\Delta(\tau)^{\frac{r}{1-r}} \delta(\tau)}{U(s)^{\frac{r}{1-r}} + U(\tau)^{\frac{r}{1-r}}} d\tau \right)^{\frac{1-r}{r}}. \quad (5.22) \end{aligned}$$

Then,

$$\begin{aligned} B_3 &\lesssim \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_0^{x_k} \frac{\Delta(\tau)^{\frac{r}{1-r}} \delta(\tau)}{U(s)^{\frac{r}{1-r}} + U(\tau)^{\frac{r}{1-r}}} d\tau \right)^{\frac{1-r}{r}} \end{aligned}$$



$$\begin{aligned}
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \sup_{s \in (x_{k-1}, t)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& =: \text{I} + \text{II}.
\end{aligned}$$

Now, for each  $i \in \mathbb{Z}$ ,  $N+2 \leq i \leq k$ , integration by parts yields

$$\begin{aligned}
& \left( \int_{x_{i-1}}^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \lesssim \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} + \left( \int_{x_{i-1}}^t \Delta(\tau)^{\frac{1}{1-r}} d \left[ -\varphi^{-\frac{r}{p(1-r)}}(\tau) \right] \right)^{\frac{1-r}{r}} \\
& \approx \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} + \Delta(x_{i-1})^{\frac{1}{r}} \varphi(x_{i-1})^{-\frac{1}{p}} \\
& \quad + \left( \int_{x_{i-1}}^t \left( \int_{x_{i-1}}^{\tau} \delta \right)^{\frac{1}{1-r}} d \left[ -\varphi^{-\frac{r}{p(1-r)}}(\tau) \right] \right)^{\frac{1-r}{r}} \\
& \lesssim \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} + \Delta(x_{i-1})^{\frac{1}{r}} \varphi(x_{i-1})^{-\frac{1}{p}} \\
& \quad + \left( \int_{x_{i-1}}^t \left( \int_{x_{i-1}}^{\tau} \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}}. \tag{5.23}
\end{aligned}$$

Combining (5.15) and (5.23) with  $t = x_i$ , we obtain

$$\begin{aligned}
\text{I} & \approx \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{N+1 \leq i \leq k} \left( \int_{x_{i-1}}^{x_i} \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\
& \lesssim \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{N+2 \leq i \leq k} \Delta(x_i)^{\frac{1}{r}} \varphi(x_i)^{-\frac{1}{p}} \\
& \quad + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{N+2 \leq i \leq k} \Delta(x_{i-1})^{\frac{1}{r}} \varphi(x_{i-1})^{-\frac{1}{p}} \\
& \quad + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \quad \times \sup_{N+1 \leq i \leq k} \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^{\tau} \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}} \\
& \leq \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{N+1 \leq i \leq k} \Delta(x_i)^{\frac{1}{r}} \varphi(x_i)^{-\frac{1}{p}} + C_{3,2} \\
& \approx \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{N+1 \leq i \leq k} \left( \int_{x_{i-1}}^{x_i} \delta \right)^{\frac{1}{r}} \varphi(x_i)^{-\frac{1}{p}} + C_{3,2} \\
& \leq C_{2,1} + C_{3,2}
\end{aligned}$$

$$\approx C_{3,2}, \quad (5.24)$$

where we used (2.8) and (3.29) in the last but one and the last equivalences, respectively. For future reference, note that we have showed that

$$\sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{s \in (0, x_k)} U(s) \varphi(s)^{-\frac{1}{p}} \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \lesssim C_{3,2}. \quad (5.25)$$

As for II, observe that, applying (5.23) with  $i = k$ , we have

$$\begin{aligned} & \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\ & \lesssim \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \\ & \quad + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \\ & \quad + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ & \quad \quad \times \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}} \\ & \lesssim C_{1,3} + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \\ & \quad + \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \\ & \lesssim C_{1,3} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^{x_k} \delta \right)^{\frac{1}{r}} \varphi(x_k)^{-\frac{1}{p}} \\ & \quad + \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ & \quad \quad \times \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{1-r}{r}} \\ & \leq C_{2,1} + 2C_{1,3} \lesssim C_{1,3} + C_{3,2}, \end{aligned} \quad (5.26)$$

where we used (2.8) and (3.29) again. Next, thanks to  $U \varphi^{-\frac{1}{p}}$  being nondecreasing and (5.26), we have

$$\text{II} \leq \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \lesssim C_{1,3} + C_{3,2}.$$

Thus, we have arrived at

$$B_3 \lesssim \text{I} + \text{II} \lesssim C_{3,2} + C_{1,3}. \quad (5.27)$$

Combining (5.21) and (5.27), we have

$$B_1 + B_2 + B_3 \lesssim C_{1,2} + C_{1,3} + C_{3,2} + C_{4,1},$$

which together with (5.17) yields

$$C \approx B_1 + B_2 + B_3.$$

(iii) By Theorem 3.5, we have

$$C \approx C_{1,1} + C_{1,2} + C_{3,3} + C_{4,1}.$$

As for the desired upper estimate on  $C$ , it is sufficient to show that  $C_{3,3} \lesssim B_2 + B_4$  owing to (5.1), (5.2), (5.4). To this end, one has

$$\begin{aligned} C_{3,3} &\approx \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \\ &\quad + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+2}^{k-1} \sup_{t \in (x_{i-1}, x_i)} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{p}{p-r}} \varphi(t)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}} \\ &\lesssim B_2 + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \sum_{i=N+2}^{k-1} \sup_{t \in (x_{i-1}, x_i)} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{p}{p-r}} \varphi(t)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}}. \end{aligned} \quad (5.28)$$

Observe that for  $N+2 \leq k \leq M-1$  we have

$$\begin{aligned} &\sum_{i=N+2}^{k-1} \sup_{t \in (x_{i-1}, x_i)} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{p}{p-r}} \varphi(t)^{-\frac{r}{p-r}} \leq \sum_{i=N+2}^{k-1} \sup_{t \in (x_{i-1}, x_i)} \Delta(t)^{\frac{p}{p-r}} \varphi(t)^{-\frac{r}{p-r}} \\ &\approx \sum_{\substack{i \in \mathcal{Z}_1 \\ N+2 \leq i \leq k-1}} \varphi(x_i)^{-\frac{r}{p-r}} \Delta(x_i)^{\frac{p}{p-r}} \\ &\quad + \sum_{\substack{i \in \mathcal{Z}_2 \\ N+2 \leq i \leq k-1}} \varphi(x_{i-1})^{-\frac{r}{p-r}} U(x_{i-1})^{\frac{pr}{p-r}} \sup_{t \in (x_{i-1}, x_i)} \Delta(t)^{\frac{p}{p-r}} U(t)^{-\frac{pr}{p-r}} \\ &\leq \sum_{\substack{i \in \mathcal{Z}_1 \\ N+2 \leq i \leq k-1}} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{t \in (x_i, x_{i+1})} \Delta(t)^{\frac{p}{p-r}} U(t)^{-\frac{pr}{p-r}} \\ &\quad + \sum_{\substack{i \in \mathcal{Z}_2 \\ N+2 \leq i \leq k-1}} \varphi(x_{i-1})^{-\frac{r}{p-r}} U(x_{i-1})^{\frac{pr}{p-r}} \sup_{t \in (x_{i-1}, x_i)} \Delta(t)^{\frac{p}{p-r}} U(t)^{-\frac{pr}{p-r}} \\ &\leq \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{t \in (x_i, x_k)} \Delta(t)^{\frac{p}{p-r}} U(t)^{-\frac{pr}{p-r}} \end{aligned}$$

$$\lesssim \int_0^{x_k} \sigma(t) U(t)^{\frac{pr}{p-r}} \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} dt,$$

where we used (4.7) in the last inequality. Inserting this to (5.28), we obtain

$$\begin{aligned} C_{3,3} &\lesssim B_2 + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \int_0^{x_k} \sigma(t) U(t)^{\frac{pr}{p-r}} \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} dt \right)^{\frac{p-r}{pr}} \\ &\leq B_2 + \sup_{N+2 \leq k \leq M-1} \sup_{x \in (x_k, x_{k+1})} \left( \int_x^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \int_0^x \sigma(t) U(t)^{\frac{pr}{p-r}} \sup_{\tau \in (t, x)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} dt \right)^{\frac{p-r}{pr}} \\ &\leq B_2 + B_4. \end{aligned}$$

It follows that  $C_{1,1} + C_{1,2} + C_{3,3} + C_{4,1} \lesssim B_1 + B_2 + B_4$ .

As for establishing the opposite inequality, note that

$$B_1 + B_2 \lesssim C_{1,1} + C_{1,2} + C_{3,1} + C_{4,1} \leq C_{1,1} + C_{1,2} + C_{3,3} + C_{4,1}$$

owing to (5.9). Therefore, it is sufficient to show that  $B_4 \lesssim C_{3,3} + C_{1,1}$ . To this end, we have

$$\begin{aligned} B_4 &= \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \int_0^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s, t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\ &\approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \int_0^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s, t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\ &+ \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_k} \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\ &\approx \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \int_0^{x_{k-1}} \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s, t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\ &+ \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& \times \left( \int_{x_{k-1}}^t \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_k} \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \approx \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_{k-1}} \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,x_{k-1})} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+2 \leq k \leq M} \left( \int_0^{x_{k-1}} \sigma(s)U(s)^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \quad \times \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \\
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_k} \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \leq \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_k} \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+2 \leq k \leq M} \left( \int_0^{x_{k-1}} \sigma(s)U(s)^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \quad \times \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \\
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s)U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}}
\end{aligned}$$

=: I + II + III.

In view of (4.8), we have

$$\begin{aligned} \mathbf{I} &\lesssim \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \\ &+ \left( \int_{x_{N+1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_{N+1})} \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}}. \end{aligned}$$

Note that

$$\left( \int_{x_{N+1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_{N+1})} \Delta(t)^{\frac{1}{r}} \varphi(t)^{-\frac{1}{p}} \leq C_{3,3} \quad (5.29)$$

and

$$\begin{aligned} &\sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\ &= \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{i \leq m \leq k-1} \sup_{\tau \in (x_m, x_{m+1})} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\ &\approx \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_{i+1})} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \\ &\leq \sum_{i=N+1}^{k-1} \sup_{\tau \in (x_i, x_{i+1})} \Delta(\tau)^{\frac{p}{p-r}} \varphi(\tau)^{-\frac{r}{p-r}}, \end{aligned}$$

where we used (2.5) in the equivalence.

Consequently,

$$\begin{aligned} \mathbf{I} &\lesssim C_{3,3} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \Delta(x_k)^{\frac{1}{r}} \\ &+ \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \sup_{\tau \in (x_i, x_k)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \\ &\leq C_{3,3} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} \Delta(x_i)^{\frac{p}{p-r}} \right)^{\frac{p-r}{pr}} \\ &+ \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^{k-1} \sup_{\tau \in (x_i, x_{i+1})} \Delta(\tau)^{\frac{p}{p-r}} \varphi(\tau)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}} \end{aligned}$$

$$\begin{aligned}
&\lesssim C_{3,3} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} \Delta(x_i)^{\frac{p}{p-r}} \right)^{\frac{p-r}{pr}} \\
&\quad + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^{k-1} \sup_{\tau \in (x_i, x_{i+1})} \left( \int_{x_i}^{\tau} \delta \right)^{\frac{p}{p-r}} \varphi(\tau)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}} \\
&\lesssim C_{3,3},
\end{aligned}$$

where we used (2.7) in the last inequality.

Clearly  $U \in Q_U(0, L)$ . Thus, applying Lemma 4.1 to  $h = U$ , we have, for every  $N + 2 \leq k \leq M$ ,

$$\begin{aligned}
\int_0^{x_{k-1}} \sigma(s) U(s)^{\frac{pr}{p-r}} ds &\lesssim \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} + \sup_{t \in (0, x_{N+1})} U(t)^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}} \\
&\approx \varphi(x_{k-1})^{-\frac{r}{p-r}} U(x_{k-1})^{\frac{pr}{p-r}} + U(x_{N+1})^{\frac{pr}{p-r}} \varphi(x_{N+1})^{-\frac{r}{p-r}} \\
&\lesssim \varphi(x_{k-1})^{-\frac{r}{p-r}} U(x_{k-1})^{\frac{pr}{p-r}}, \tag{5.30}
\end{aligned}$$

where we again used the fact that  $\{U^p(x_k)/\varphi(x_k)\}_{k=N+1}^{M-1}$  is strongly increasing in the equivalence. Thus,

$$\begin{aligned}
\text{II} &\lesssim \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} U(x_{k-1}) \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \\
&\approx \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
&\quad + \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} U(x_{k-1}) \\
&\quad \times \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^{\tau} \delta \right)^{\frac{1}{r}} U(\tau)^{-1} \\
&\leq \sup_{N+1 \leq k \leq M-1} \varphi(x_k)^{-\frac{1}{p}} \Delta(x_k)^{\frac{1}{r}} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
&\quad + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \varphi(\tau)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{\tau} \delta \right)^{\frac{1}{r}} \\
&\approx C_{2,1} + C_{1,1}, \tag{5.31}
\end{aligned}$$

where we used (2.8) in the last equivalence. Then using (3.29) we arrive at

$$\text{II} \lesssim C_{3,1} + C_{1,1} \leq C_{3,3} + C_{1,1}.$$

Finally,

$$\text{III} \approx \sup_{t \in (0, x_{N+1})} \left( \int_t^{x_{N+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s, t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}}$$

$$\begin{aligned}
& + \sup_{\substack{N+2 \leq k \leq M \\ k \in \mathcal{Z}_1}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{\substack{N+2 \leq k \leq M \\ k \in \mathcal{Z}_2}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}}.
\end{aligned}$$

For  $k \in \mathcal{Z}_1$  we have

$$\int_{x_{k-1}}^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \lesssim \varphi(t)^{-\frac{r}{p-r}} \Delta(t)^{\frac{p}{p-r}}$$

by (4.1). For  $k \in \mathcal{Z}_2$ , we have

$$\begin{aligned}
& \int_{x_{k-1}}^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \\
& \lesssim \varphi(x_{k-1})^{-\frac{r}{p-r}} U(x_{k-1})^{\frac{pr}{p-r}} \sup_{\tau \in (x_{k-1}, t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}}
\end{aligned}$$

by (4.2). Recall that, if  $N > -\infty$ , then  $N+1 \in \mathcal{Z}_2$  (see the proof of Lemma 4.1). Consequently, for every  $t \in (0, x_{N+1}]$ , arguing similarly as in the proof of (4.3), we have

$$\begin{aligned}
& \int_0^t \sigma(s) U(s)^{\frac{pr}{p-r}} \sup_{\tau \in (s,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} ds \\
& \leq \sup_{\tau \in (0,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \int_0^t d \left[ \left( \frac{\varphi}{U^p} \right)^{-\frac{r}{p-r}} \right] \\
& \lesssim \left( \sup_{\tau \in (0,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}} \right) \varphi(t)^{-\frac{r}{p-r}} U(t)^{\frac{pr}{p-r}} \\
& \approx \left[ \frac{U^p}{\varphi}(x_{N+1}) \right]^{\frac{r}{p-r}} \sup_{\tau \in (0,t)} \Delta(\tau)^{\frac{p}{p-r}} U(\tau)^{-\frac{pr}{p-r}}.
\end{aligned}$$

Then,

$$\begin{aligned}
\text{III} & \lesssim \sup_{t \in (0, x_{N+1})} \left( \int_t^{x_{N+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{N+1})^{-\frac{1}{p}} U(x_{N+1}) \sup_{\tau \in (0,t)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \\
& + \sup_{\substack{N+2 \leq k \leq M \\ k \in \mathcal{Z}_1}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \\
& + \sup_{\substack{N+2 \leq k \leq M \\ k \in \mathcal{Z}_2}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} U(x_{k-1})
\end{aligned}$$



$$\times \sup_{\tau \in (x_{k-1}, t)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1}.$$

Since  $N + 1 \in \mathcal{Z}_2$ , we have by using (5.31) combined with (3.29),

$$\begin{aligned} \text{III} &\lesssim \sup_{\substack{N+2 \leq k \leq M \\ k \in \mathcal{Z}_1}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \\ &\quad + \sup_{\substack{N+1 \leq k \leq M \\ k \in \mathcal{Z}_2}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \Delta(\tau)^{\frac{1}{r}} \varphi(\tau)^{-\frac{1}{p}} \\ &\leq \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \varphi(\tau)^{-\frac{1}{p}} \Delta(\tau)^{\frac{1}{r}} \\ &\approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \varphi(\tau)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{\tau} \delta \right)^{\frac{1}{r}} \\ &\quad + \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\lesssim C_{1,1} + C_{3,3}. \end{aligned}$$

For future reference, note that we have showed that

$$\sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \varphi(\tau)^{-\frac{1}{p}} \Delta(\tau)^{\frac{1}{r}} \lesssim C_{1,1} + C_{3,3}. \quad (5.32)$$

Thus, we have obtained

$$B_4 \lesssim \text{I} + \text{II} + \text{III} \lesssim C_{1,1} + C_{3,3}.$$

Hence, putting all things together, we have

$$C_{1,1} + C_{1,2} + C_{3,3} + C_{4,1} \approx B_1 + B_2 + B_4.$$

(iv) By Theorem 3.5, we have

$$C \approx C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1}.$$

Moreover, thanks to (5.2), (5.14) and (5.4), to establish the desired upper bound on  $C$ , it is sufficient to show that  $C_{3,4} \lesssim B_2 + B_3 + B_5$ . To this end, note that

$$\begin{aligned}
C_{3,4} &\approx \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{1-r}{r}} \\
&\quad + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
&\quad \times \left( \sum_{i=N+2}^{k-1} \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{p(1-r)}{p-r}} \right)^{\frac{p-r}{pr}} \\
&= C_{3,2} + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
&\quad \times \left( \sum_{i=N+2}^{k-1} \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{p(1-r)}{p-r}} \right)^{\frac{p-r}{pr}} \\
&\lesssim B_2 + B_3 + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
&\quad \times \left( \sum_{i=N+2}^{k-1} \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{p(1-r)}{p-r}} \right)^{\frac{p-r}{pr}}
\end{aligned} \tag{5.33}$$

where we used (5.16) in the last inequality. For any  $k \in \mathbb{Z}$  satisfying  $N+2 \leq k \leq M-1$ , we have

$$\begin{aligned}
&\sum_{i=N+2}^{k-1} \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^t \delta \right)^{\frac{r}{1-r}} \delta(t) \varphi(t)^{-\frac{r}{p(1-r)}} dt \right)^{\frac{p(1-r)}{p-r}} \\
&\leq \sum_{i=N+2}^{k-1} \left( \int_{x_{i-1}}^{x_i} \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{p(1-r)}{p-r}} \\
&\approx \sum_{\substack{i \in \mathcal{Z}_1 \\ N+2 \leq i \leq k-1}} \varphi(x_i)^{-\frac{r}{p-r}} \left( \int_{x_{i-1}}^{x_i} \Delta^{\frac{r}{1-r}} \delta \right)^{\frac{p(1-r)}{p-r}} \\
&\quad + \sum_{\substack{i \in \mathcal{Z}_2 \\ N+2 \leq i \leq k-1}} \varphi(x_{i-1})^{-\frac{r}{p-r}} U(x_{i-1})^{\frac{pr}{p-r}} \left( \int_{x_{i-1}}^{x_i} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{\substack{i \in \mathcal{Z}_1 \\ N+2 \leq i \leq k-1}} \varphi(x_i)^{-\frac{r}{p-r}} \Delta(x_i)^{\frac{p}{p-r}} \\
&\quad + \sum_{\substack{i \in \mathcal{Z}_2 \\ N+1 \leq i \leq k-2}} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \left( \int_{x_i}^{x_{i+1}} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\
&\lesssim \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} \left[ \Delta(x_i)^{\frac{1}{r}} + U(x_i) \left( \int_{x_i}^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} \\
&\lesssim \int_0^{x_k} \sigma(t) \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \min \left\{ U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}} \right\} ds \right)^{\frac{p(1-r)}{p-r}} dt,
\end{aligned}$$

where we used (4.9) in the last inequality. Plugging this into (5.33), we obtain

$$\begin{aligned}
C_{3,4} &\lesssim B_2 + B_3 \\
&\quad + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_k} \sigma(t) \left( \int_0^{x_k} \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \right. \right. \\
&\quad \quad \quad \left. \left. \times \min \left\{ U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}} \right\} ds \right)^{\frac{p(1-r)}{p-r}} dt \right)^{\frac{p-r}{pr}} \\
&\lesssim B_2 + B_3 \\
&\quad + \sup_{N+2 \leq k \leq M-1} \sup_{x \in (x_k, x_{k+1})} \left( \int_x^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^x \sigma(t) \left( \int_0^x \Delta(s)^{\frac{r}{1-r}} \delta(s) U(s)^{-\frac{r}{1-r}} \right. \right. \\
&\quad \quad \quad \left. \left. \times \min \left\{ U(s)^{\frac{r}{1-r}}, U(t)^{\frac{r}{1-r}} \right\} ds \right)^{\frac{p(1-r)}{p-r}} dt \right)^{\frac{p-r}{pr}} \\
&\lesssim B_2 + B_3 + B_5. \tag{5.34}
\end{aligned}$$

Altogether, we obtain

$$C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1} \lesssim B_1 + B_2 + B_3 + B_5.$$

As for the opposite inequality, note that

$$B_1 + B_2 \lesssim C_{1,2} + C_{1,3} + C_{3,2} + C_{4,1} \leq C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1}$$

owing to (5.9) combined with (5.19) and (5.20), and that

$$B_3 \lesssim C_{1,3} + C_{3,2} \leq C_{1,3} + C_{3,4}$$

thanks to (5.27). Consequently,

$$B_1 + B_2 + B_3 \lesssim C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1}.$$

As for  $B_5$ , we have

$$B_5 \approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}}$$

$$\begin{aligned}
& \times \left( \int_0^t \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^t \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_k} \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \approx \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_{k-1}} \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_k} \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \approx \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_{k-1}} \sigma \Delta^{\frac{p}{p-r}} \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_{k-1}} \sigma(s) U(s)^{\frac{pr}{p-r}} \left( \int_s^{x_{k-1}} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} ds \right)^{\frac{p-r}{pr}}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_{k-1}} \sigma U^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_k} \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& \leq \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{x_k} \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& + \sup_{N+2 \leq k \leq M} \left( \int_0^{x_{k-1}} \sigma U^{\frac{pr}{p-r}} \right)^{\frac{p-r}{pr}} \\
& \quad \times \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_{k-1}}^t \sigma(s) \left[ \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\
& =: \text{I} + \text{II} + \text{III}.
\end{aligned}$$

In view of (4.10), we have

$$\text{I} \lesssim \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k \varphi(x_i)^{-\frac{r}{p-r}} \left[ \Delta(x_i)^{\frac{1}{r}} + \right. \right.$$

$$\begin{aligned}
& + U(x_i) \left( \int_{x_i}^{x_k} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \left[ \frac{pr}{p-r} \right]^{\frac{p-r}{pr}} ds \\
& + \left( \int_{x_{N+1}}^L \Delta_{\frac{r}{1-r}}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \\
& + \left( \int_{x_{N+1}}^L \Delta_{\frac{r}{1-r}}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{1}{p}} U(t) \left( \int_t^{x_{N+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}}.
\end{aligned}$$

Note that

$$\left( \int_{x_{N+1}}^L \Delta_{\frac{r}{1-r}}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \lesssim C_{3,3} \lesssim C_{3,4}.$$

owing to (5.29) and (5.18). Moreover,

$$\begin{aligned}
& \left( \int_{x_{N+1}}^L \Delta_{\frac{r}{1-r}}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_{N+1})} \varphi(t)^{-\frac{1}{p}} U(t) \left( \int_t^{x_{N+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& \leq \left( \int_{x_{N+1}}^L \Delta_{\frac{r}{1-r}}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^{x_{N+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\
& \leq C_{3,4}.
\end{aligned}$$

Furthermore, note that

$$\begin{aligned}
& \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \left( \int_{x_i}^{x_k} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\
& = \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \left( \sum_{m=i}^{k-1} \int_{x_m}^{x_{m+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\
& \approx \sum_{i=N+1}^{k-1} \varphi(x_i)^{-\frac{r}{p-r}} U(x_i)^{\frac{pr}{p-r}} \left( \int_{x_i}^{x_{i+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \\
& \leq \sum_{i=N+2}^k \left( \int_{x_{i-1}}^{x_i} \Delta_{1-r}^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{p(1-r)}{p-r}} \\
& \lesssim \sum_{i=N+2}^k \Delta(x_i)^{\frac{p}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}} + \sum_{i=N+2}^k \Delta(x_{i-1})^{\frac{p}{p-r}} \varphi(x_{i-1})^{-\frac{r}{p-r}} \\
& + \sum_{i=N+2}^k \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^{\tau} \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{p(1-r)}{p-r}},
\end{aligned}$$

where we used (2.4) in the equivalence and (5.23) with  $t = x_i$  in the last step. Consequently, we have

$$\begin{aligned} \text{I} &\lesssim C_{3,4} + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \sum_{i=N+1}^k \Delta(x_i)^{\frac{p}{p-r}} \varphi(x_i)^{-\frac{r}{p-r}} \right)^{\frac{p-r}{pr}} \\ &\quad + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\ &\quad \times \left( \sum_{i=N+2}^k \left( \int_{x_{i-1}}^{x_i} \left( \int_{x_{i-1}}^{\tau} \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{p(1-r)}{p-r}} \right)^{\frac{p-r}{pr}}. \end{aligned}$$

Now, taking (2.7) combined with (3.30) into consideration we arrive at

$$I \lesssim C_{3,4}.$$

Recall that

$$\int_0^{x_{k-1}} \sigma U^{\frac{pr}{p-r}} \lesssim \varphi(x_{k-1})^{-\frac{r}{p-r}} U(x_{k-1})^{\frac{pr}{p-r}} \quad \text{for } k \in \mathbb{Z}, N+2 \leq k \leq M$$

thanks to (5.30). Next, applying (5.30), exploiting the monotonicity of  $U\varphi^{-\frac{1}{p}}$  and using (5.26) in turn, we get,

$$\begin{aligned} \text{II} &\lesssim \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} U(x_{k-1}) \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ &\leq \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\ &\lesssim C_{1,3} + C_{3,4}. \end{aligned}$$

For every  $t \in (0, L)$ , set

$$\tilde{h}_t(s) = \Delta(s)^{\frac{1}{r}} + U(s) \left( \int_s^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \quad \text{for } s \in (0, t).$$

Note that  $\tilde{h}_t \in Q_U(0, t)$ . Finally,

$$\begin{aligned} \text{III} &\approx \sup_{t \in (0, x_{N+1})} \left( \int_t^{x_{N+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^t \sigma(s) \tilde{h}_t(s)^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\ &\quad + \sup_{\substack{k \in \mathbb{Z}_1 \\ N+2 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \sigma(s) \tilde{h}_t(s)^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}} \\ &\quad + \sup_{\substack{k \in \mathbb{Z}_2 \\ N+2 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \sigma(s) \tilde{h}_t(s)^{\frac{pr}{p-r}} ds \right)^{\frac{p-r}{pr}}. \end{aligned}$$

If  $k \in \mathcal{Z}_1$ ,  $N + 2 \leq k \leq M$ , then, thanks to (4.1),

$$\int_{x_{k-1}}^t \sigma(s) \tilde{h}_t(s)^{\frac{pr}{p-r}} ds \lesssim \varphi(t)^{-\frac{r}{p-r}} \Delta(t)^{\frac{p}{p-r}},$$

while, if  $k \in \mathcal{Z}_2$ ,  $N + 2 \leq k \leq M$ , then, owing to (4.2),

$$\begin{aligned} & \int_{x_{k-1}}^t \sigma(s) \tilde{h}_t(s)^{\frac{pr}{p-r}} ds \\ & \lesssim \varphi(x_{k-1})^{-\frac{r}{p-r}} \left[ \Delta(x_{k-1})^{\frac{1}{r}} + U(x_{k-1}) \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \right]^{\frac{pr}{p-r}}. \end{aligned}$$

Moreover, for every  $t \in (0, x_{N+1})$ , since  $N + 1 \in \mathcal{Z}_2$ , we have

$$\begin{aligned} \int_0^t \sigma(s) \tilde{h}_t(s)^{\frac{pr}{p-r}} ds & \lesssim \lim_{\tau \rightarrow 0^+} \left[ \frac{\tilde{h}_t(\tau)}{U(\tau)} \right]^{\frac{pr}{p-r}} \varphi(t)^{-\frac{r}{p-r}} U(t)^{\frac{pr}{p-r}} \\ & \approx \varphi(x_{N+1})^{-\frac{r}{p-r}} U(x_{N+1})^{\frac{pr}{p-r}} \left( \int_0^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{p(1-r)}{p-r}}. \end{aligned}$$

Consequently, using (5.32) combined with (5.19) and (5.26), we get

$$\begin{aligned} \text{III} & \lesssim \sup_{t \in (0, x_{N+1})} \left( \int_t^{x_{N+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{N+1})^{-\frac{1}{p}} U(x_{N+1}) \left( \int_0^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ & + \sup_{\substack{k \in \mathcal{Z}_1 \\ N+2 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \\ & + \sup_{\substack{k \in \mathcal{Z}_2 \\ N+2 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \Delta(x_{k-1})^{\frac{1}{r}} \\ & + \sup_{\substack{k \in \mathcal{Z}_2 \\ N+2 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \\ & \quad \times U(x_{k-1}) \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\ & \approx \sup_{\substack{k \in \mathcal{Z}_1 \\ N+2 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \\ & + \sup_{\substack{k \in \mathcal{Z}_2 \\ N+2 \leq k \leq M}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \Delta(x_{k-1})^{\frac{1}{r}} \\ & + \sup_{\substack{k \in \mathcal{Z}_2 \\ N+1 \leq k \leq M}} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \end{aligned}$$



$$\begin{aligned}
& \times U(x_{k-1}) \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& \lesssim \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, t)} \varphi(\tau)^{-\frac{1}{p}} \Delta(\tau)^{\frac{1}{r}} \\
& \quad + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\
& \lesssim C_{1,3} + C_{3,3} + \sup_{t \in (0, x_{N+1})} \left( \int_t^{x_{N+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_0^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\
& \quad + \sup_{N+2 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_{k-1}}^t \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{1-r}{r}} \\
& \lesssim C_{1,3} + C_{3,4}.
\end{aligned}$$

We finally arrive at

$$B_5 \lesssim \text{I} + \text{II} + \text{III} \lesssim C_{3,4} + C_{1,3}. \quad (5.35)$$

Putting all things together, we have

$$C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1} \lesssim B_1 + B_2 + B_3 + B_5 \lesssim C_{1,2} + C_{1,3} + C_{3,4} + C_{4,1}.$$

(v) By Theorem 3.5, we have

$$C \approx C_{1,4} + C_{1,5} + C_{3,1} + C_{4,1}.$$

We start by establishing the desired upper estimate. In view of (5.3) and (5.4), it is sufficient to prove suitable upper bounds on  $C_{1,4}$  and  $C_{1,5}$ .

For every  $t \in (x_{k-1}, x_k]$ , we have, if  $k \in \mathcal{Z}_1$ ,

$$\begin{aligned}
\sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r(1-q)}} \varphi(s)^{-\frac{q}{p(1-q)}} & \approx \varphi(x_k)^{-\frac{q}{p(1-q)}} \left( \int_{x_{k-1}}^t \delta \right)^{\frac{q}{r(1-q)}} \\
& \leq \varphi(x_k)^{-\frac{q}{p(1-q)}} \Delta(t)^{\frac{q}{r(1-q)}},
\end{aligned}$$

and, if  $k \in \mathcal{Z}_2$ ,

$$\begin{aligned}
& \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r(1-q)}} \varphi(s)^{-\frac{q}{p(1-q)}} \\
& \approx \left( \frac{U(x_{k-1})^p}{\varphi(x_{k-1})} \right)^{\frac{q}{p(1-q)}} \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r(1-q)}} U(s)^{-\frac{q}{1-q}}.
\end{aligned}$$

Then

$$C_{1,4} \lesssim \sup_{\substack{k \in \mathcal{Z}_1 \\ N+1 \leq k \leq M}} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(t)^{-\frac{q}{r}} w(t) \Delta(t)^{\frac{q}{r(1-q)}} dt \right)^{\frac{1-q}{q}}$$

$$\begin{aligned}
& + \sup_{\substack{k \in \mathcal{Z}_2 \\ N+1 \leq k \leq M}} \frac{U(x_{k-1})}{\varphi(x_{k-1})^{\frac{1}{p}}} \left( \int_{x_{k-1}}^{x_k} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(t)^{-\frac{q}{r}} w(t) \right. \\
& \quad \left. \times \sup_{s \in (x_{k-1}, t)} \Delta(s)^{\frac{q}{r(1-q)}} U(s)^{-\frac{q}{1-q}} dt \right)^{\frac{1-q}{q}} \\
& \lesssim \sup_{N+1 \leq k \leq M} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} d \left[ - \left( \int_t^{x_k} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \quad + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \frac{U(t)}{\varphi(t)^{\frac{1}{p}}} \left( \int_t^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \quad \left. \times \sup_{\tau \in (t, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} ds \right)^{\frac{1-q}{q}} \\
& \leq B_1 + B_6.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
C_{1,5} & = \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{q}{1-q}} \right. \\
& \quad \left. \times \Delta(t)^{-\frac{q}{r}} w(t) \left( \int_{x_{k-1}}^t \delta \right)^{\frac{q}{r}} \varphi(t)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}} \\
& \leq \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t w \right)^{\frac{q}{1-q}} w(t) \varphi(t)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}}.
\end{aligned}$$

Applying [18]\*Lemma 3.5 with  $p \mapsto \frac{1-q}{q}$ ,  $\varphi \mapsto U\varphi^{-\frac{1}{p}}$ ,  $\varrho \mapsto U$  and  $f \mapsto W^{\frac{q}{1-q}} w$ , we have

$$\begin{aligned}
C_{1,5} & \leq \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}} w \varphi^{-\frac{q}{p(1-q)}} \right)^{\frac{1-q}{q}} \\
& \approx \sup_{t \in (0, L)} \varphi(t)^{-\frac{1}{p}} U(t) \left( \int_0^L \frac{W(s)^{\frac{q}{1-q}} w(s)}{U(t)^{\frac{q}{1-q}} + U(s)^{\frac{q}{1-q}}} ds \right)^{\frac{1-q}{q}} \\
& \approx \sup_{t \in (0, L)} \varphi(t)^{-\frac{1}{p}} W(t)^{\frac{1}{q}} + \sup_{t \in (0, L)} \varphi(t)^{-\frac{1}{p}} U(t) \left( \int_t^L W^{\frac{q}{1-q}} w U^{-\frac{q}{1-q}} \right)^{\frac{1-q}{q}} \\
& = B_1 + B_7. \tag{5.36}
\end{aligned}$$

Putting all these upper estimates together, we have

$$C \approx C_{1,4} + C_{1,5} + C_{3,1} + C_{4,1} \lesssim B_1 + B_2 + B_6 + B_7.$$

Conversely, combining the estimate

$$\begin{aligned} C_{1,2} &\approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_{x_{k-1}}^t \left( \int_{x_{k-1}}^\tau \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{q}{1-q}} \right. \\ &\quad \times \left. \Delta(\tau)^{-\frac{q}{r}} w(\tau) \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{q}{r}} d\tau \right)^{\frac{1-q}{q}} \varphi(t)^{-\frac{1}{p}} \\ &\leq C_{1,5} \end{aligned}$$

with (5.7), we have

$$B_1 \lesssim C_{4,1} + C_{1,2} + C_{3,1} \lesssim C_{4,1} + C_{1,5} + C_{3,1}. \quad (5.37)$$

Next, coupling the estimate

$$\begin{aligned} C_{1,1} &\approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \left( \int_t^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\ &\quad \times \sup_{s \in (x_{k-1}, t)} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{1}{r}} \varphi(s)^{-\frac{1}{p}} \\ &\leq C_{1,4} \end{aligned}$$

with (5.8), we obtain

$$B_2 \lesssim C_{1,1} + C_{3,1} \lesssim C_{1,4} + C_{3,1}. \quad (5.38)$$

Next,

$$\begin{aligned} B_6 &\approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\ &\quad \times \left( \int_t^{x_k} \sup_{\tau \in (t, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\ &\quad + \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\ &\quad \times \left( \int_{x_k}^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \sup_{\tau \in (x_k, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} ds \right)^{\frac{1-q}{q}} \\ &\quad + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \right) \\ &=: \text{I} + \text{II} + \text{III}. \end{aligned}$$

Observe that

$$\sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right)$$

$$\begin{aligned}
& \times \sup_{\tau \in (x_{k-1}, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} \varphi(\tau)^{-\frac{q}{p(1-q)}} ds \Big)^{\frac{1-q}{q}} \\
\approx & \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \times \sup_{\tau \in (x_{k-1}, s)} \left( \int_{x_{k-1}}^{\tau} \delta \right)^{\frac{q}{r(1-q)}} \varphi(\tau)^{-\frac{q}{p(1-q)}} ds \Big)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \\
& \times \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\
\approx & C_{1,4} + \sup_{N+1 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
\lesssim & C_{1,4} + C_{3,1}, \tag{5.39}
\end{aligned}$$

where we used (5.38) in the last inequality. Now, integrating by parts and using (5.38), we obtain

$$\begin{aligned}
\text{I} & \lesssim \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} \varphi(t)^{-\frac{1}{p}} \Delta(t)^{\frac{1}{r}} \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\
& \times \left( \int_t^{x_k} \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} d \left[ \sup_{\tau \in (t, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
\lesssim & C_{1,4} + C_{3,1} + \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\
& \times \left( \int_t^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \sup_{\tau \in (t, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} ds \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \sup_{\tau \in (t, x_k)} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1}.
\end{aligned}$$

Next, monotonicity of  $U \varphi^{-\frac{1}{p}}$ , (5.38) and (5.39) gives

$$\begin{aligned}
\text{I} & \lesssim C_{1,4} + C_{3,1} \\
& + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \sup_{\tau \in (x_{k-1}, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} \varphi(\tau)^{-\frac{q}{p(1-q)}} ds \Big)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, x_k)} \Delta(\tau)^{\frac{1}{r}} \varphi(\tau)^{-\frac{1}{p}} \\
& \lesssim C_{1,4} + C_{3,1}.
\end{aligned}$$

We shall now deal with II. Note that

$$\begin{aligned}
\text{II} & \approx \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \times \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \sup_{\tau \in (x_k, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \approx \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \times \left( \sum_{i=k+1}^{M-1} \sup_{\tau \in (x_k, x_i)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} \int_{x_i}^{x_{i+1}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \times \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \sup_{\tau \in (x_i, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& =: \text{II}_1 + \text{II}_2.
\end{aligned}$$

Using (2.6) and (5.39), we get

$$\begin{aligned}
\text{II}_1 & \lesssim \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{i=k+1}^{M-1} \left( \sum_{m=k+1}^i \sup_{\tau \in (x_{m-1}, x_m)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} \right) \right. \\
& \quad \left. \times \int_{x_i}^{x_{i+1}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& = \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{m=k+1}^{M-1} \sup_{\tau \in (x_{m-1}, x_m)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} \right. \\
& \quad \left. \times \int_{x_m}^L d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \approx \sup_{N+1 \leq k \leq M-2} \left( \int_{x_{k+1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \sup_{\tau \in (x_k, x_{k+1})} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \\
& \lesssim C_{1,4} + C_{3,1}.
\end{aligned}$$

On the other hand, applying (2.6) and integrating by parts twice, we obtain

$$\begin{aligned}
\text{II}_2 &\approx \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
&\quad \times \left( \int_{x_k}^{x_{k+1}} \sup_{\tau \in (x_k, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
&\lesssim \sup_{N+1 \leq k \leq M-1} \varphi(x_k)^{-\frac{1}{p}} \Delta(x_k)^{\frac{1}{r}} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
&\quad + \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k+1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_k, x_{k+1})} \Delta(\tau)^{\frac{1}{r}} U(\tau)^{-1} \\
&\quad + \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_k}^{x_{k+1}} \left( \int_s^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
&\quad \quad \left. \times \sup_{\tau \in (x_k, s)} \Delta(\tau)^{\frac{q}{r(1-q)}} U(\tau)^{-\frac{q}{1-q}} ds \right)^{\frac{1-q}{q}}.
\end{aligned}$$

Thus, (5.38) and (5.39) give  $\text{II}_2 \lesssim C_{1,4} + C_{3,1}$ . Consequently, we have  $\text{II} \lesssim C_{1,4} + C_{3,1}$ . Furthermore, using (5.38) once again, we obtain

$$\text{III} \leq \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{\tau \in (x_{k-1}, x_k)} \Delta(\tau)^{\frac{1}{r}} \varphi(\tau)^{-\frac{1}{p}} \leq C_{1,4} + C_{3,1}.$$

Altogether, we have

$$B_6 \approx \text{I} + \text{II} + \text{III} \lesssim C_{1,4} + C_{3,1}.$$

Lastly, note that for  $N+2 \leq k \leq M$  and  $y \in (x_{k-1}, L)$ , by integrating by parts, we have

$$\begin{aligned}
&\left( \int_{x_{k-1}}^y W^{\frac{q}{1-q}} w \varphi^{-\frac{q}{p(1-q)}} \right)^{\frac{1-q}{q}} \\
&\lesssim W(y)^{\frac{1}{q}} \varphi(y)^{-\frac{1}{p}} + \left( \int_{x_{k-1}}^y W(t)^{\frac{1}{1-q}} d \left[ -\varphi(t)^{-\frac{q}{p(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
&\lesssim W(y)^{\frac{1}{q}} \varphi(y)^{-\frac{1}{p}} + W(x_{k-1})^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \\
&\quad + \left( \int_{x_{k-1}}^y \left( \int_{x_{k-1}}^t w \right)^{\frac{q}{1-q}} w(t) \varphi(t)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}}. \tag{5.40}
\end{aligned}$$

Finally, using (5.36) and (5.40), we have

$$B_7 \lesssim \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}} w \varphi^{-\frac{q}{p(1-q)}} \right)^{\frac{1-q}{q}}$$

$$\begin{aligned}
&\approx \left( \int_0^{x_{N+1}} W^{\frac{q}{1-q}} w \varphi^{-\frac{q}{p(1-q)}} \right)^{\frac{1-q}{q}} + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} W^{\frac{q}{1-q}} w \varphi^{-\frac{q}{p(1-q)}} \right)^{\frac{1-q}{q}} \\
&\lesssim \sup_{N+2 \leq k \leq M} W(x_k)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} + \sup_{N+2 \leq k \leq M} W(x_{k-1})^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \\
&\quad + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t w \right)^{\frac{q}{1-q}} w(t) \varphi(t)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}}.
\end{aligned}$$

It is clear that, using (2.8), we have

$$\sup_{N+2 \leq k \leq M} W(x_{k-1})^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \approx \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-2}}^{x_{k-1}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} = C_{4,1}, \quad (5.41)$$

and

$$\sup_{N+2 \leq k \leq M} W(x_k)^{\frac{1}{q}} \varphi(x_k)^{-\frac{1}{p}} \lesssim \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t w \right)^{\frac{q}{1-q}} w(t) \varphi(t)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}}. \quad (5.42)$$

Then, in view of (5.41), (5.42) and integrating by parts, we get

$$\begin{aligned}
B_7 &\lesssim C_{4,1} + \sup_{N+1 \leq k \leq M} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}} \\
&\quad + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \varphi(t)^{-\frac{q}{p(1-q)}} d \left[ \left( \int_{x_{k-1}}^t w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}}.
\end{aligned}$$

Using (5.6) and (5.37) we obtain

$$\begin{aligned}
B_7 &\lesssim B_1 + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \varphi(t)^{-\frac{q}{p(1-q)}} d \left[ \left( \int_{x_{k-1}}^t w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
&\lesssim C_{4,1} + C_{1,5} + C_{3,1} + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \varphi(t)^{-\frac{q}{p(1-q)}} d \left[ \left( \int_{x_{k-1}}^t w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}}. \quad (5.43)
\end{aligned}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
&\sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \varphi(t)^{-\frac{q}{p(1-q)}} d \left[ \left( \int_{x_{k-1}}^t w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
&\approx \sup_{N+1 \leq k \leq M} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} w \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} d \left[ -\varphi(t)^{-\frac{q}{p(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{1}{1-q}} d \left[ -\varphi(t)^{-\frac{q}{p(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
& \lesssim B_1 + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \varphi(x_{k-1})^{-\frac{1}{p}} \\
& + \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r}} ds \right)^{\frac{q}{1-q}} \right. \\
& \quad \left. \times \Delta(t)^{-\frac{q}{r}} w(t) \left( \int_{x_{k-1}}^t \delta \right)^{\frac{q}{r}} \varphi(t)^{-\frac{q}{p(1-q)}} dt \right)^{\frac{1-q}{q}} \\
& \leq B_1 + B_2 + C_{1,5} \\
& \lesssim C_{4,1} + C_{1,4} + C_{3,1} + C_{1,5}.
\end{aligned}$$

Note that we used (5.37) and (5.38) in the last inequality. Plugging the last estimate in (5.43), we arrive at

$$B_7 \lesssim C_{1,4} + C_{3,1} + C_{1,5} + C_{4,1}. \quad (5.44)$$

Putting all things together, we have

$$B_1 + B_2 + B_6 + B_7 \lesssim C_{4,1} + C_{1,5} + C_{3,1} + C_{1,4} \lesssim B_1 + B_2 + B_6 + B_7;$$

consequently

$$C \approx B_1 + B_2 + B_6 + B_7.$$

(vi) By Theorem 3.5, we have

$$C \approx C_{1,5} + C_{1,6} + C_{3,2} + C_{4,1}.$$

Using (5.10) and (5.11) we have

$$\begin{aligned}
C_{1,6} & \lesssim \sup_{N+1 \leq k \leq M} \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(t)^{\frac{q^2}{r(1-q)}} w(t) dt \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M} \sup_{\tau \in (x_{k-1}, x_k)} U(\tau) \varphi(\tau)^{-\frac{1}{p}} \left( \int_{\tau}^{x_k} \left( \int_t^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \right. \\
& \quad \left. \times \Delta(t)^{-\frac{q}{r}} w(t) \left( \int_{\tau}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} dt \right)^{\frac{1-q}{q}} \\
& \lesssim \sup_{N+1 \leq k \leq M} \varphi(x_k)^{-\frac{1}{p}} W(x_k)^{\frac{1}{q}}
\end{aligned}$$



$$\begin{aligned}
& + \sup_{N+1 \leq k \leq M} \sup_{\tau \in (x_{k-1}, x_k)} U(\tau) \varphi(\tau)^{-\frac{1}{p}} \left( \int_{\tau}^L \left( \int_t^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \right. \\
& \quad \left. \times \Delta(t)^{-\frac{q}{r}} w(t) \left( \int_{\tau}^t \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} dt \right)^{\frac{1-q}{q}} \\
& = B_1 + B_8.
\end{aligned} \tag{5.45}$$

Combining that with (5.4), (5.16) and (5.36), we arrive at

$$C_{1,5} + C_{1,6} + C_{3,2} + C_{4,1} \lesssim B_1 + B_2 + B_3 + B_7 + B_8.$$

As for the opposite inequality, note that

$$B_1 + B_2 \lesssim C_{1,4} + C_{1,5} + C_{3,1} + C_{4,1} \lesssim C_{1,6} + C_{1,5} + C_{3,2} + C_{4,1} \tag{5.46}$$

thanks to (5.37) and (5.38) combined with (5.20) and (5.18). Using the same argument as in (5.18) it can be easily shown that  $C_{1,3} \lesssim C_{1,6}$ . Consequently, observe that

$$B_3 \lesssim C_{1,3} + C_{3,2} \lesssim C_{1,6} + C_{3,2} \tag{5.47}$$

owing to (5.27), and that

$$B_7 \lesssim C_{1,4} + C_{3,1} + C_{1,5} + C_{4,1} \lesssim C_{1,6} + C_{3,2} + C_{1,5} + C_{4,1} \tag{5.48}$$

thanks to (5.44) combined with (5.20) and (5.18).

Next, integration by parts yields

$$\begin{aligned}
B_8 & \approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\
& \quad \times \left( \int_t^{x_k} \left( \int_t^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-1} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\
& \quad \times \left( \int_{x_k}^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_t^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \lesssim \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\
& \quad \times \left( \int_t^{x_k} \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} d \left[ \left( \int_t^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\
& \quad \times \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \int_{x_k}^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \\
& \quad \times \left( \int_t^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} d \left[ \left( \int_t^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& + \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \int_{x_k}^L \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& =: \text{I} + \text{II} + \text{III}.
\end{aligned}$$

As for II, using (5.25), we obtain

$$\text{II} \lesssim \sup_{N+1 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \lesssim C_{3,2}.$$

We shall now turn our attention to I. Integrating by parts, applying (5.23) and the monotonicity of  $\varphi$ , we have

$$\begin{aligned}
\text{I} & \approx \sup_{N+1 \leq k \leq M} \sup_{t \in (x_{k-1}, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \quad \left. \times \left( \int_t^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \leq \sup_{N+1 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \quad \left. \times \left( \int_{x_{k-1}}^s \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \approx \left( \int_0^{x_{N+1}} \left( \int_s^{x_{N+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_0^s \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \quad \left. \times \left( \int_{x_{k-1}}^s \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \lesssim C_{1,6} + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \quad \left. \times \Delta(s)^{\frac{q}{r(1-q)}} \varphi(s)^{-\frac{q}{p(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \quad + \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\
& \quad + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \right. \\
& \quad \left. \times \left( \int_{x_{k-1}}^s \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) \varphi(\tau)^{-\frac{r}{p(1-r)}} d\tau \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \lesssim C_{1,6} + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \Delta(s)^{\frac{q}{r(1-q)}} \varphi(s)^{-\frac{q}{p(1-q)}} d \left[ - \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \quad + \sup_{N+2 \leq k \leq M} \varphi(x_{k-1})^{-\frac{1}{p}} \Delta(x_{k-1})^{\frac{1}{r}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \lesssim C_{1,6} + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^s \delta \right)^{\frac{q}{r(1-q)}} \varphi(s)^{-\frac{q}{p(1-q)}} \right. \\
& \quad \left. \times \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\
& \quad + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \lesssim C_{1,6} + \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^s \left( \int_{x_{k-1}}^\tau \delta \right)^{\frac{r}{1-r}} \delta(\tau) d\tau \right)^{\frac{q(1-r)}{r(1-q)}} \right. \\
& \quad \left. \times \varphi(s)^{-\frac{q}{p(1-q)}} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \leq 2C_{1,6} + \sup_{N+2 \leq k \leq M} \Delta(x_{k-1})^{\frac{1}{r}} \varphi(x_{k-1})^{-\frac{1}{p}} \left( \int_{x_{k-1}}^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \lesssim C_{1,6} + C_{1,5} + C_{3,2} + C_{4,1},
\end{aligned}$$

where we used the fact that  $B_2 \lesssim C_{1,6} + C_{1,5} + C_{3,2} + C_{4,1}$  thanks to (5.46) in the last inequality.

For future reference, note that we also showed that

$$\begin{aligned}
& \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) \left( \int_{x_{k-1}}^s \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{q(1-r)}{r(1-q)}} ds \right)^{\frac{1-q}{q}} \\
& \lesssim C_{1,6} + C_{1,5} + C_{3,2} + C_{4,1}. \tag{5.49}
\end{aligned}$$

Finally, as for III, we have

$$\begin{aligned}
\text{III} & \approx \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \approx \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \sum_{i=k}^{M-1} \int_{x_i}^{x_{i+1}} \left( \int_{x_i}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \sum_{i=k+1}^{M-1} \left( \int_{x_k}^{x_i} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \int_{x_i}^{x_{i+1}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& =: \text{III}_1 + \text{III}_2.
\end{aligned}$$

Using (2.6), integration by parts, (5.49) and (5.25), we obtain

$$\begin{aligned}
\text{III}_1 & \approx \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \lesssim \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
& \times \left( \int_{x_k}^{x_{k+1}} \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} d \left[ \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
& \approx \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \int_{x_k}^{x_{k+1}} \left( \int_s^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} d \left[ \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \right] \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_{k+1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \left( \int_{x_k}^{x_{k+1}} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& \lesssim \sup_{N+1 \leq k \leq M-1} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^s \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \left( \int_s^{x_{k+1}} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\
& + \sup_{N+1 \leq k \leq M-2} \left( \int_{x_{k+1}}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_{k+1}} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& \leq \sup_{N+2 \leq k \leq M} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^s \Delta^{\frac{r}{1-r}} \delta \varphi^{-\frac{r}{p(1-r)}} \right)^{\frac{q(1-r)}{r(1-q)}} \right. \\
& \quad \left. \times \left( \int_s^{x_k} \Delta^{-\frac{q}{r}} w \right)^{\frac{q}{1-q}} \Delta(s)^{-\frac{q}{r}} w(s) ds \right)^{\frac{1-q}{q}} \\
& + \sup_{N+2 \leq k \leq M-1} \left( \int_{x_k}^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_k} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& \lesssim C_{1,6} + C_{1,5} + C_{4,1} + C_{3,2},
\end{aligned}$$

and

$$\begin{aligned}
\text{III}_2 &= \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{i=k+1}^{M-1} \left( \sum_{j=k+1}^i \int_{x_{j-1}}^{x_j} \Delta^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \right. \\
& \quad \left. \times \int_{x_i}^{x_{i+1}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}}.
\end{aligned}$$

If  $q \leq r$ , then  $\frac{q(1-r)}{r(1-q)} \leq 1$ , and so, applying (2.6) and (5.25) again, we have

$$\text{III}_2 \leq \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{i=k+1}^{M-1} \int_{x_i}^{x_{i+1}} d \left[ - \left( \int_s^L \Delta^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)$$

$$\begin{aligned}
& \times \sum_{j=k+1}^i \left( \int_{x_{j-1}}^{x_j} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \frac{1-q}{q} \\
& = \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \sum_{j=k+1}^{M-1} \left( \int_{x_{j-1}}^{x_j} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} \int_{x_j}^L d \left[ - \left( \int_s^L \Delta_{1-r}^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{1-q}{q}} \\
& \approx \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_k}^{x_{k+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \left( \int_{x_{k+1}}^L \Delta_{1-r}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \leq \sup_{N+1 \leq k \leq M-2} \left( \int_{x_{k+1}}^L \Delta_{1-r}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \sup_{t \in (0, x_k)} U(t) \varphi(t)^{-\frac{1}{p}} \left( \int_t^{x_{k+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \\
& \lesssim C_{3,2}.
\end{aligned}$$

On the other hand, if  $r < q$ , then  $\frac{q(1-r)}{r(1-q)} > 1$  and we can apply the Minkowski inequality. Using that and the similar arguments as above, we arrive at

$$\begin{aligned}
\text{III}_2 & \leq \sup_{N+1 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \sum_{j=k+1}^{M-1} \left( \int_{x_{j-1}}^{x_j} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right) \right. \\
& \quad \times \left. \left( \sum_{i=j}^{M-1} \int_{x_i}^{x_{i+1}} d \left[ - \left( \int_s^L \Delta_{1-r}^{-\frac{q}{r}} w \right)^{\frac{1}{1-q}} \right] \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
& = \sup_{N+2 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \\
& \quad \times \left( \sum_{j=k+1}^{M-1} \left( \int_{x_{j-1}}^{x_j} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right) \left( \int_{x_j}^L \Delta_{1-r}^{-\frac{q}{r}} w \right)^{\frac{r}{q(1-r)}} \right)^{\frac{1-r}{r}} \\
& \approx \sup_{N+2 \leq k \leq M-2} U(x_k) \varphi(x_k)^{-\frac{1}{p}} \left( \int_{x_k}^{x_{k+1}} \Delta_{1-r}^{\frac{r}{1-r}} \delta U^{-\frac{r}{1-r}} \right)^{\frac{1-r}{r}} \left( \int_{x_{k+1}}^L \Delta_{1-r}^{-\frac{q}{r}} w \right)^{\frac{1}{q}} \\
& \lesssim C_{3,2}.
\end{aligned}$$

The estimates above yields

$$\text{III} \lesssim C_{1,6} + C_{1,5} + C_{4,1} + C_{3,2};$$

hence, altogether,

$$B_8 \lesssim C_{1,6} + C_{1,5} + C_{4,1} + C_{3,2}. \quad (5.50)$$

Finally, putting all things together, we obtain

$$B_1 + B_2 + B_3 + B_7 + B_8 \lesssim C_{1,5} + C_{1,6} + C_{3,2} + C_{4,1} \lesssim B_1 + B_2 + B_3 + B_7 + B_8.$$

(vii) By Theorem 3.5, we have

$$C \approx C_{1,5} + C_{1,6} + C_{3,4} + C_{4,1}.$$

Now, (5.36), (5.45), (5.34) and (5.4) combined all together yield the desired upper estimate on  $C_{1,5} + C_{1,6} + C_{3,4} + C_{4,1}$ .

Finally, we obtain the opposite inequality by combining (5.46), (5.47), (5.35), (5.48) and (5.50) upon observing that  $C_{3,2} \leq C_{3,4}$  and  $C_{1,3} \lesssim C_{1,6}$ .  $\square$

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*Email address*, A. Gogatishvili: [gogatish@math.cas.cz](mailto:gogatish@math.cas.cz)

ORCID: 0000-0003-3459-0355

*Email address*, Z. Mihula: [mihulzde@fel.cvut.cz](mailto:mihulzde@fel.cvut.cz)

ORCID: 0000-0001-6962-7635

*Email address*, L. Pick: [pick@karlin.mff.cuni.cz](mailto:pick@karlin.mff.cuni.cz)

ORCID: 0000-0002-3584-1454

*Email address*, H. Turčinová: [turcinova@karlin.mff.cuni.cz](mailto:turcinova@karlin.mff.cuni.cz)

ORCID: 0000-0002-5424-9413

*Email address*, T. Ünver: [tugceunver@kku.edu.tr](mailto:tugceunver@kku.edu.tr)

ORCID: 0000-0003-0414-8400

INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC

CZECH TECHNICAL UNIVERSITY IN PRAGUE, FACULTY OF ELECTRICAL ENGINEERING, DEPARTMENT OF MATHEMATICS, TECHNICKÁ 2, 166 27 PRAHA 6, CZECH REPUBLIC; AND DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, 186 75 PRAHA 8, CZECH REPUBLIC

DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, 186 75 PRAHA 8, CZECH REPUBLIC

DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, 186 75 PRAHA 8, CZECH REPUBLIC

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, KIRIKKALE UNIVERSITY, 71450 YAH-SIHAN, KIRIKKALE, TURKEY