INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Holmstedt's formula for the K-functional: The limit case $\theta_0 = \theta_1$

Irshaad Ahmed
Alberto Fiorenza
Amiran Gogatishvili

Preprint No. 60-2022

PRAHA 2022

Holmstedt's formula for the K-functional: the limit case $\theta_0 = \theta_1$

Irshaad Ahmed[1], **Alberto Fiorenza**[2] and **Amiran Gogatishvili**[3]

[1]Department of Mathematics, Sukkur IBA University, Sukkur, Pakistan.
irshaad.ahmed@iba-suk.edu.pk
[2]Universitá di Napoli Federico II, Dipartimento di Architettura, via Monteoliveto, 3, 80134 - Napoli, Italy and Consiglio Nazionale delle Ricerche, Istituto per le Applicazioni del Calcolo "Mauro Picone", Sezione di Napoli, via Pietro Castellino, 111, 80131 - Napoli, Italy.
fiorenza@unina.it
[3]Institute of Mathematics of the Czech Academy of Sciences - Žitná, 115 67 Prague 1, Czech Republic.
gogatish@math.cas.cz

Abstract

We consider K-interpolation spaces involving slowly varying functions, and derive necessary and sufficient conditions for a Holmstedt-type formula to be held in the limiting case $\theta_0 = \theta_1 \in \{0, 1\}$. We also study the case $\theta_0 = \theta_1 \in (0, 1)$. Applications are given to Lorentz-Karamata spaces, generalized gamma spaces and Besov spaces.

Key words: K-functional, slowly varying functions, K-interpolation spaces, Holmstedt's formula, reiteration

MSC 2020: 46B70, 46E30, 41A65, 26D15

1 Introduction

Let (A_0, A_1) be a compatible couple of quasi-normed spaces. For each $f \in A_0 + A_1$ and $t > 0$, the Peetre's K-functional is defined by

$$\begin{aligned} K(t, f) &= K(t, f; A_0, A_1) \\ &= \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : \ f_0 \in A_0, \ f_1 \in A_1, \ f = f_0 + f_1\}. \end{aligned}$$

Let $0 < q \leq \infty$, $0 \leq \theta \leq 1$, and let b be a slowly varying function on $(0, \infty)$. The K-interpolation space $\bar{A}_{\theta,q;b} = (A_0, A_1)_{\theta,q;b}$ is formed of those $f \in A_0 + A_1$ for which the quasi-norm

$$\|f\|_{\bar{A}_{\theta,q;b}} = \|t^{-\theta-1/q} b(t) K(t, f)\|_{q,(0,\infty)}$$

is finite; see [22]. If $b \equiv 1$ and $(\theta, q) \in ([0, 1] \times [1, \infty]) \setminus (\{0, 1\} \times [1, \infty])$, then we recover the classical real interpolation spaces $\bar{A}_{\theta,q}$ (see [5, 6, 27]). Let

$0 < q_0, q_1 \leq \infty$. The celebrated classical Holmstedt's formula states that, for all for $f \in A_0 + A_1$ and for all $t > 0$, we have

$$K(t^{\theta_1-\theta_0}, f; \bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1}) \approx \|u^{-\theta_0-1/q_0} K(u,f)\|_{q_0,(0,t)} + t^{\theta_1-\theta_0}\|u^{-\theta_1-1/q_1} K(u,f)\|_{q_1,(t,\infty)},$$

provided $0 < \theta_0 < \theta_1 < 1$ (see [25, Theorem 2.1]). In the limiting case $\theta = 0$ or $\theta = 1$, the classical space $\bar{A}_{\theta,q}$ contains only zero element unless $q = \infty$. However, the limiting K-interpolation spaces $\bar{A}_{0,q;b}$ and $\bar{A}_{1,q;b}$ do make sense (for all $q \in (0,\infty]$) under appropriate conditions on b. In the non-limiting case $0 < \theta_0 < \theta_1 < 1$, we have the following straightforward extension of the classical Holmstedt's formula (see [22, Theorem 3.1]):

$$K(w(t), f; \bar{A}_{\theta_0,q_0;b_0}, \bar{A}_{\theta_1,q_1;b_1}) \approx \|u^{-\theta_0-1/q_0} b_0(u) K(u,f)\|_{q_0,(0,t)} + w(t)\|u^{-\theta_1-1/q_1} b_1(u) K(u,f)\|_{q_1,(t,\infty)},$$

where b_0 and b_1 are slowly varying functions and $w(t) = t^{\theta_1-\theta_0} b_0(t)/b_1(t)$. The limiting case when $\theta_0 = 0$ and $\theta_1 = 1$ is contained in [3, Example 5]. However, the limiting case $\theta_0 = \theta_1 \in [0,1]$ still remains open for general slowly varying functions b_0 and b_1. The main goal of the current paper is to fill this gap.

Let us describe our main results. To this end, let

$$\rho(t) = \frac{\|u^{-1/q_0} b_0(u)\|_{q_0,(t,\infty)}}{\|u^{-1/q_1} b_1(u)\|_{q_1,(t,\infty)}}, \quad t > 0$$

and, for each $\epsilon > 0$, put

$$\rho_\epsilon(t) = \frac{\|u^{-1/q_0} b_0(u)\|^{1+\epsilon}_{q_0,(t,\infty)}}{\|u^{-1/q_1} b_1(u)\|_{q_1,(t,\infty)}}.$$

In the limiting case $\theta_0 = \theta_1 = 0$, there are two distinguishing cases: $q_0 \neq q_1$ and $q_0 = q_1$. In the case when $q_0 \neq q_1$, we establish that the following version of Holmstedt's formula

$$K(\rho(t), f; \bar{A}_{0,q_0;b_0}, \bar{A}_{0,q_1;b_1}) \approx \|u^{-1/q_0} b_0(u) K(u,f)\|_{q_0,(0,t)} + \rho(t)\|u^{-1/q_1} b_1(u) K(u,f)\|_{q_1,(t,\infty)},$$

holds for all for $f \in A_0 + A_1$ and for all $t > 0$ provided the following condition is met: ρ_ϵ is equivalent to a non-decreasing function for some $\epsilon > 0$. This condition also turns out to be necessary under the additional assumption that the given couple (A_0, A_1) is K-surjective (see Definition 3.1 below). When $q_0 = q_1 < \infty$, the previous estimate holds if ρ is increasing and the couple (A_0, A_1) is K-surjective, and when $q_0 = q_1 = \infty$ the previous estimate holds under the natural condition that ρ is increasing. The corresponding Holmstedt's formulae for the symmetric counterpart limiting case $\theta_0 = \theta_1 = 1$ follows immediately, by the usual symmetry argument, from the limiting case $\theta_0 = \theta_1 = 0$.

Finally, in the limiting case $\theta_0 = \theta_1 \in (0,1)$ we further have two distinguishing cases: while in the case $q_0 = q_1$, a version of Holmstedt's formula does exist, there exists no analogue of Holmstedt's formula in the case $q_0 \neq q_1$ if the given couple (A_0, A_1) is K-surjective.

The reader is referred to recent works [1,3,9–11,15–20] for other generalized versions of Holmstedt's formula.

The paper is organised as follows. All the Holmstedt's formulae mentioned above are contained in Section 3. The necessary background is collected in Section 2. Section 4 contains the reiteration formulae, and some concrete examples of these reiteration formulae are included in the final section 5.

2 Background material

2.1 Notation

We write $A \lesssim B$ or $B \gtrsim A$ for two non-negative quantities A and B to mean that $A \leq cB$ for some positive constant c which is independent of appropriate parameters involved in A and B. If both the estimates $A \lesssim B$ and $B \lesssim A$ hold, we simply put $A \approx B$. We let $\|\cdot\|_{q,(a,b)}$ denote the standard L^q-quasi-norm on an interval $(a,b) \subset \mathbb{R}$. We write $X \hookrightarrow Y$ for two quasi-normed spaces X and Y to mean that X is continuously embedded in Y.

2.2 Slowly varying functions

Let $b : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Following [22], we say b is slowly varying on $(0,\infty)$ if for every $\varepsilon > 0$, there are positive functions g_ε and $g_{-\varepsilon}$ on $(0,\infty)$ such that g_ε is non-decreasing and $g_{-\varepsilon}$ is non-increasing, and we have

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \ \text{ and } \ t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \ \text{ for all } \ t \in (0,\infty).$$

We denote the class of all slowly varying functions by SV. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. Define

$$\ell^{\mathbb{A}}(t) = \begin{cases} (1 - \ln t)^{\alpha_0}, & 0 < t \leq 1, \\ \\ (1 + \ln t)^{\alpha_\infty}, & t > 1, \end{cases}$$

Then $\ell^{\mathbb{A}} \in SV$. In addition, the class SV contains compositions of appropriate log-functions, $\exp|\log t|^\alpha$ with $\alpha \in (0,1)$, etc.

We collect in next Proposition some elementary properties of slowly varying functions, which will be used in the sequel time and again without explicit mention. The proofs of these assertions can be carried out as in [22, Lemma 2.1] or [12, Proposition 3.4.33].

Proposition 2.1 *Given b, b_1, $b_2 \in SV$, the following assertions hold:*

(i) $b_1b_2 \in SV$ *and* $b^r \in SV$ *for each* $r \in \mathbb{R}$.

(ii) *If* $g(t) \approx h(t)$, $t > 0$, *then* $b(g(t)) \approx b(h(t))$, $t > 0$.

(iii) *If* $\alpha > 0$, *then*

$$\|u^{\alpha-1}b(u)\|_{1,(0,t)} \approx t^{\alpha}b(t), \quad t > 0.$$

(iv) *If* $\alpha > 0$, *then*

$$\|u^{-\alpha-1}b(u)\|_{1,(t,\infty)} \approx t^{-\alpha}b(t), \quad t > 0.$$

(v) *Assume that*

$$\|u^{-1}b(u)\|_{1,(1,\infty)} < \infty,$$

and set

$$\tilde{b}(t) = \|u^{-1}b(u)\|_{1,(t,\infty)}, \quad t > 0.$$

Then $\tilde{b} \in SV$, *and* $b(t) \lesssim \tilde{b}(t)$, $t > 0$.

2.3 K-interpolation spaces

Let A_0 and A_1 be two quasi-normed spaces. We say (A_0, A_1) is a compatible couple if A_0 and A_1 are continuously embedded in the same Hausdorff topological vector space. For each $f \in A_0 + A_1$ and $t > 0$, the K-functional is defined by

$$\begin{aligned} K(t,f) &= K(t,f;A_0,A_1) \\ &= \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f_0 \in A_0,\ f_1 \in A_1,\ f = f_0 + f_1\}. \end{aligned}$$

Note that $K(t,f)$ is, as a function of t, non-decreasing on $(0,\infty)$, while $K(t,f)/t$ is, as a function of t, non-increasing on $(0,\infty)$,

Let $0 < q \le \infty$, $0 \le \theta \le 1$, and let $b \in SV$. The K-interpolation space $\bar{A}_{\theta,q;b} = (A_0, A_1)_{\theta,q;b}$ is formed of those $f \in A_0 + A_1$ for which the quasi-norm

$$\|f\|_{\bar{A}_{\theta,q;b}} = \|t^{-\theta-1/q}b(t)K(t,f;A_0,A_1)\|_{q,(0,\infty)}$$

is finite; see [22]. If $b = \ell^{\mathbb{A}}$, then we obtain the K-interpolation spaces $\bar{A}_{\theta,q;\mathbb{A}}$ considered in [13] and [14]. If $\mathbb{A} = (0,0)$ and $(\theta,q) \in ([0,1] \times [1,\infty]) \setminus (\{0,1\} \times [1,\infty])$, then we recover the classical K-interpolation spaces $\bar{A}_{\theta,q}$ (see [5,6,27]).

It is not hard to check that for $\theta \in (0,1)$ the spaces $\bar{A}_{\theta,q;b}$ are intermediate, without any condition on b and q, for the couple (A_0, A_1), that is,

$$A_0 \cap A_1 \hookrightarrow \bar{A}_{\theta,q;b} \hookrightarrow A_0 + A_1.$$

However, while working with the limiting spaces $\bar{A}_{0,q;b}$ and $\bar{A}_{1,q;b}$, we have to impose an appropriate condition on b and q. For convenience, let us introduce two notations. We say $b \in SV_{0,q}$ if $b \in SV$ and

$$\|u^{-1/q}b(u)\|_{q,(1,\infty)} < \infty.$$

And we say $b \in SV_{1,q}$ if $b(1/t) \in SV_{0,q}$. For $b \in SV_{0,q}$ (or $b \in SV_{1,q}$), the space $\bar{A}_{0,q;b}$ (or $\bar{A}_{1,q;b}$) is intermediate for the couple (A_0, A_1) (see [22, Proposition 2.5]).

2.4 Weighted inequalities

First let us recall that a function $\phi : (0,\infty) \to (0,\infty)$ is called quasi-concave if both ϕ and $t\phi(1/t)$ are non-decreasing.

Theorem 2.2 *[23, Theorem 5.1] Let $0 < p, q < \infty$, and let v and w be positive functions on $(0,\infty)$. Consider the inequality*

$$\left(\int_0^\infty [h(s)w(s)]^q \, \frac{ds}{s}\right)^{1/q} \le C \left(\int_0^\infty [h(s)v(s)]^p \, \frac{ds}{s}\right)^{1/p}. \tag{2.1}$$

(a) Let $0 < p \le q < \infty$. Then the inequality (2.1) holds for all quasi-concave functions h on $(0,\infty)$ if and only if

$$A_1 = \sup_{x>0} \frac{\left(\int_0^x s^q w^q(s) \frac{ds}{s} + x^q \int_x^\infty w^q(s) \frac{ds}{s}\right)^{1/q}}{\left(\int_0^x s^p v^p(s) \frac{ds}{s} + x^p \int_x^\infty v^p(s) \frac{ds}{s}\right)^{1/p}} < \infty.$$

Moreover, $C = A_1$ is the best constant.

(b) Let $0 < q < p < \infty$. Then the inequality (2.1) holds for all quasi-concave functions h on $(0,\infty)$ if and only if

$$A_2 = \left(\int_0^\infty \frac{\left(\int_0^x s^q w^q(s) \frac{ds}{s} + x^q \int_x^\infty w^q(s) \frac{ds}{s}\right)^{\frac{q}{p-q}}}{\left(\int_0^x s^p v^p(s) \frac{ds}{s} + x^p \int_x^\infty v(s)^p \frac{ds}{s}\right)^{\frac{q}{p-q}}} x^q w^q(x) \frac{dx}{x}\right)^{1/q-1/p} < \infty.$$

Moreover, $C = A_2$ is the best constant.

Corollary 2.3 *Let $0 < p, q < \infty$, and let $v \in SV_{0,p}$ and $w \in SV_{0,q}$.*

(a) Let $0 < p \le q < \infty$. Then the inequality (2.1) holds for all quasi-concave functions h on $(0,\infty)$ if and only if

$$A_3 = \sup_{x>0} \frac{\left(\int_x^\infty w^q(s) \frac{ds}{s}\right)^{1/q}}{\left(\int_x^\infty v^p(s) \frac{ds}{s}\right)^{1/p}} < \infty.$$

Moreover, $C = A_3$ is the best constant.

(b) Let $0 < q < p < \infty$. Then the inequality (2.1) holds for all quasi-concave functions h on $(0,\infty)$ if and only if

$$A_4 = \left(\int_0^\infty \frac{\left(\int_x^\infty w^q(s) \frac{ds}{s}\right)^{\frac{q}{p-q}}}{\left(\int_x^\infty v(s)^p \frac{ds}{s}\right)^{\frac{q}{p-q}}} w^q(x) \frac{dx}{x}\right)^{1/q-1/p} < \infty.$$

Moreover, $C = A_4$ is the best constant.

Proof. The proof immediately follows from the previous theorem in view of the assertions (iv) and (vi) in Proposition 2.1. ■

Corollary 2.4 *Let $0 < p, q < \infty$, and let $v \in SV_{0,p}$ and $w \in SV_{0,q}$. Consider the inequality*

$$\left(\int_0^\infty \left[h(s)w(s)\chi_{(0,t)}(s)\right]^q \frac{ds}{s}\right)^{1/q} \lesssim \phi(t) \left(\int_0^\infty \left[h(s)v(s)\right]^p \frac{ds}{s}\right)^{1/p}. \tag{2.2}$$

(a) Let $0 < p \le q < \infty$. Then the inequality (2.2) holds for all quasi-concave functions h on $(0, \infty)$ if and only if we have

$$\sup_{0<x<t} \frac{\left(\int_x^\infty w^q(s)\frac{ds}{s}\right)^{1/q}}{\left(\int_x^\infty v^p(s)\frac{ds}{s}\right)^{1/p}} \lesssim \phi(t), \quad t > 0. \tag{2.3}$$

(b) Let $0 < q < p < \infty$. Then the inequality (2.2) holds for all quasi-concave functions h on $(0, \infty)$ if and only if we have

$$\left(\int_0^t \frac{\left(\int_x^\infty w^q(s)\frac{ds}{s}\right)^{\frac{q}{p-q}}}{\left(\int_x^\infty v(s)^p\frac{ds}{s}\right)^{\frac{q}{p-q}}} w^q(x)\frac{dx}{x}\right)^{1/q-1/p} \lesssim \phi(t), \quad t > 0. \tag{2.4}$$

Corollary 2.5 *Let $0 < p, q < \infty$, and let $v \in SV_{0,p}$ and $w \in SV_{0,q}$. Consider the inequality*

$$\left(\int_0^\infty \left[h(s)w(s)\chi_{(t,\infty)}(s)\right]^q \frac{ds}{s}\right)^{1/q} \lesssim \psi(t) \left(\int_0^\infty \left[h(s)v(s)\right]^p \frac{ds}{s}\right)^{1/p}. \tag{2.5}$$

(a) Let $0 < p \le q < \infty$. Then the inequality (2.5) holds for all quasi-concave functions h on $(0, \infty)$ if and only if we have

$$\sup_{x>t} \frac{\left(\int_x^\infty w^q(s)\frac{ds}{s}\right)^{1/q}}{\left(\int_x^\infty v^p(s)\frac{ds}{s}\right)^{1/p}} \lesssim \psi(t), \quad t > 0. \tag{2.6}$$

(b) Let $0 < q < p < \infty$. Then the inequality (2.5) holds for all quasi-concave functions h on $(0, \infty)$ if and only if we have

$$\left(\int_t^\infty \frac{\left(\int_x^\infty w^q(s)\frac{ds}{s}\right)^{\frac{q}{p-q}}}{\left(\int_x^\infty v(s)^p\frac{ds}{s}\right)^{\frac{q}{p-q}}} w^q(x)\frac{dx}{x}\right)^{1/q-1/p} \lesssim \psi(t), \quad t > 0. \tag{2.7}$$

Theorem 2.6 *[1, Lemma 3.2] Let $1 < \alpha < \infty$, and assume that w and ϕ are positive functions on $(0, \infty)$. Put*

$$v(t) = (w(t))^{1-\alpha} \left(\phi(t) \int_t^\infty w(u)du\right)^\alpha.$$

Then

$$\int_0^\infty \left(\int_0^t \phi(u)h(u)du\right)^\alpha w(t)dt \lesssim \int_0^\infty h^\alpha(t)v(t)dt$$

holds for all positive functions h on $(0,\infty)$.

An appropriate change of variable gives us the following variant of the previous theorem.

Theorem 2.7 *Let $1<\alpha<\infty$, and assume that w and ϕ are positive functions on $(0,\infty)$. Put*

$$v(t) = (w(t))^{1-\alpha}\left(\phi(t)\int_0^t w(u)du\right)^\alpha.$$

Then

$$\int_0^\infty \left(\int_t^\infty \phi(u)h(u)du\right)^\alpha w(t)dt \lesssim \int_0^\infty h^\alpha(t)v(t)dt$$

holds for all positive functions h on $(0,\infty)$.

Theorem 2.8 *[1, Lemma 3.3] Let $0<\alpha<1$, and assume that w and ϕ are positive functions on $(0,\infty)$. Put*

$$v(t) = \phi(t)\left(\int_0^t \phi(u)du\right)^{\alpha-1}\int_t^\infty w(u)du.$$

Then

$$\int_0^\infty \left(\int_t^\infty \phi(u)h(u)du\right)^\alpha w(t)dt \lesssim \int_0^\infty h^\alpha(t)v(t)dt$$

holds for all positive, non-increasing functions h on $(0,\infty)$.

Again an appropriate change of variable gives us the following variant of the previous theorem.

Theorem 2.9 *Let $0<\alpha<1$, and assume that w and ϕ are positive functions on $(0,\infty)$. Put*

$$v(t) = \phi(t)\left(\int_t^\infty \phi(u)du\right)^{\alpha-1}\int_0^t w(u)du.$$

Then

$$\int_0^\infty \left(\int_t^\infty \phi(u)h(u)du\right)^\alpha w(t)dt \lesssim \int_0^\infty h^\alpha(t)v(t)dt$$

holds for all positive, non-decreasing functions h on $(0,\infty)$.

Theorem 2.10 *[24, Theorem 3.3 (b)] Let $0<\alpha\le 1$. Assume that w and v are non-negative functions on $(0,\infty)$, and ψ is a non-negative function on $(0,\infty)\times(0,\infty)$. Then*

$$\int_0^\infty \left(\int_0^\infty \psi(t,u)h(u)du\right)^\alpha w(t)dt \lesssim \int_0^\infty h^\alpha(t)v(t)dt \tag{2.8}$$

holds for all non-negative, non-decreasing functions h *on* $(0,\infty)$ *if and only if*

$$\int_0^\infty \left(\int_x^\infty \psi(t,u)du\right)^\alpha w(t)dt \lesssim \int_x^\infty v(t)dt \tag{2.9}$$

holds for all $x>0$.

3 Holmstedt-type formulae

This section contains our main results. In order to describe our results, we need the following class of compatible couples.

Definition 3.1 *[26, p. 217] We say a compatible couple* (A_0,A_1) *of quasi-normed spaces is* K*-surjective if for every quasi-concave function* ϕ*, there exists* $f\in A_0+A_1$ *such that*

$$\phi(t)\approx K(t,f),\ \ t>0.$$

3.1 The case $\theta_0=\theta_1=0$

Let $0<q_j\le\infty$ and $b_j\in SV_{0,q_j}$ $(j=0,1)$. Put

$$\rho(t)=\frac{\|u^{-1/q_0}b_0(u)\|_{q_0,(t,\infty)}}{\|u^{-1/q_1}b_1(u)\|_{q_1,(t,\infty)}},$$

$$I(t,f)=\|u^{-1/q_0}b_0(u)K(u,f;A_0,A_1)\|_{q_0,(0,t)},$$

and

$$J(t,f)=\|u^{-1/q_1}b_1(u)K(u,f;A_0,A_1)\|_{q_1,(t,\infty)}.$$

Moreover, let $\epsilon>0$ and set

$$\rho_\epsilon(t)=\frac{\|u^{-1/q_0}b_0(u)\|^{1+\epsilon}_{q_0,(t,\infty)}}{\|u^{-1/q_1}b_1(u)\|_{q_1,(t,\infty)}}.$$

Theorem 3.2 *Let* $0<q_0,q_1\le\infty$*,* $q_0\ne q_1$*, and* $b_j\in SV_{0,q_j}$ *(j=0,1). Assume that the following condition is met:*

$$\rho_\epsilon \textit{ is equivalent to a non-decreasing function for some } \epsilon>0. \tag{3.1}$$

Then, for all $f\in A_0+A_1$ *and all* $t>0$*, we have*

$$K(\rho(t),f;\bar{A}_{0,q_0;b_0},\bar{A}_{0,q_1;b_1})\approx I(t,f)+\rho(t)J(t,f). \tag{3.2}$$

Moreover, the condition (3.1) is also necessary provided that the given couple (A_0,A_1) *is* K*-surjective.*

Proof. First assume that the condition (3.1) is met. According to the estimate (2.30) in [1, Theorem 2.3], we have the following estimate from below:

$$K(\rho(t), f; \bar{A}_{0,q_0;b_0}, \bar{A}_{0,q_1;b_1}) \lesssim I(t,f) + \rho(t)J(t,f) + \rho(t)b_1(t)K(t,f) \\ + \|u^{-1/q_0} b_0(u)\|_{q_0,(t,\infty)} K(t,f). \tag{3.3}$$

Since $t \mapsto K(t,f)$ is non-decreasing, we obtain

$$J(t,f) \geq K(t,f)\|u^{-1/q_1} b_1(u)\|_{q_1,(t,\infty)}, \tag{3.4}$$

whence we get

$$\rho(t)J(t,f) \geq \|u^{-1/q_0} b_0(u)\|_{q_0,(t,\infty)} K(t,f). \tag{3.5}$$

Moreover, by Proposition 2.1 (vi), (3.4) gives

$$J(t,f) \gtrsim K(t,f) b_1(t). \tag{3.6}$$

Now, in view of (3.5) and (3.6), the estimate " $\lesssim$ " in (3.2) follows from (3.3). In order to establish the converse estimate " $\gtrsim$ ", it will suffice to show that the following estimates

$$I(t,f) \lesssim \|f_0\|_{\bar{A}_{0,q_0;b_0}} + \rho(t)\|f_1\|_{\bar{A}_{0,q_1;b_1}},$$

and

$$\rho(t)J(t,f) \lesssim \|f_0\|_{\bar{A}_{0,q_0;b_0}} + \rho(t)\|f_1\|_{\bar{A}_{0,q_1;b_1}},$$

hold for an arbitrary decomposition $f = f_0 + f_1$ with $f_j \in A_j$ $(j = 0,1)$. As $K(u,f) \lesssim K(u,f_0) + K(u,f_1)$, we have

$$I(t,f) \lesssim I(t,f_0) + I(t,f_1),$$

and

$$J(t,f) \lesssim J(t,f_0) + J(t,f_1).$$

Clearly, $I(t,f_0) \leq \|f_0\|_{\bar{A}_{0,q_0;b_0}}$ and $J(t,f_1) \leq \|f_1\|_{\bar{A}_{0,q_1;b_1}}$. Therefore, it remains to show that

$$I(t,f_1) \lesssim \rho(t)\|f_1\|_{\bar{A}_{0,q_1;b_1}}, \tag{3.7}$$

and

$$\rho(t)J(t,f_0) \lesssim \|f_0\|_{\bar{A}_{0,q_0;b_0}}. \tag{3.8}$$

Since

$$\|f_j\|_{\bar{A}_{0,q_j;b_j}} \geq K(x,f_j)\|u^{-1/q_j} b_j(u)\|_{q_j,(x,\infty)}, \quad x > 0, \tag{3.9}$$

it follows that the estimate (3.7) holds if the following condition is met:

$$\left\| \frac{x^{-1/q_0} b_0(x)}{\|s^{-1/q_1} b_1(s)\|_{q_1,(x,\infty)}} \right\|_{q_0,(0,t)} \lesssim \rho(t), \tag{3.10}$$

while the estimate (3.8) holds if the following condition is met:

$$\left\| \frac{x^{-1/q_1} b_1(x)}{\|s^{-1/q_0} b_0(s)\|_{q_0,(x,\infty)}} \right\|_{q_1,(t,\infty)} \lesssim 1/\rho(t). \tag{3.11}$$

Next let us derive the estimate (3.10) using the condition (3.1). We consider only the case when $q_0 < \infty$ since the case $q_0 = \infty$ is analogous and easier. Observe that

$$\begin{aligned} & \int_0^t \frac{b_0^{q_0}(x)}{\left(\int_x^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{q_0/q_1}} \frac{dx}{x} \\ \lesssim\ & \frac{\left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{1+\epsilon}}{\left(\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{q_0/q_1}} \int_0^t \left(\int_x^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{-\epsilon-1} b_0^{q_0}(x)\frac{dx}{x} \\ \approx\ & \frac{\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}}{\left(\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{q_0/q_1}}, \end{aligned}$$

whence we get (3.10). Next we show that (3.11) also follows from the condition (3.1). Again we consider only the case when $q_1 < \infty$. This time we observe that

$$\begin{aligned} & \int_t^\infty \frac{b_1^{q_1}(x)}{\left(\int_x^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{q_1/q_0}} \frac{dx}{x} \\ \lesssim\ & \frac{\left(\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}}{\left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{q_1/q_0}} \int_t^\infty \left(\int_x^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{\frac{-1}{1+\epsilon}} b_1^{q_1}(x)\frac{dx}{x} \\ \approx\ & \frac{\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}}{\left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{q_1/q_0}}, \end{aligned}$$

whence we get (3.11). The proof of the sufficiency of the condition (3.1) is complete.

Next assume that the estimate (3.2) holds for all $A_0 + A_1$ and $t > 0$. We distinguish two cases: $q_1 < q_0$ and $q_0 < q_1$. First we treat the case $q_1 < q_0$. Taking a particular decomposition $f = f + 0$, $f \in \bar{A}_{0,q_0;b_0}$ and $0 \in \bar{A}_{1,q_1;b_1}$, we obtain

$$\rho(t)J(t,f) \lesssim \|f\|_{\bar{A}_{0,q_0;b_0}},$$

from which, according to Corollary 2.5 (b), it follows that

$$\left(\int_t^\infty \frac{\left(\int_x^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{\frac{q_1}{q_0-q_1}}}{\left(\int_x^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{\frac{q_1}{q_0-q_1}}} b_1^{q_1}(x)\frac{dx}{x} \right)^{1/q_1-1/q_0} \lesssim 1/\rho(t). \tag{3.12}$$

Next we introduce the operator

$$(Qw)(t) = \int_t^\infty w(x) \left(\int_x^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{-1} b_1^{q_1}(x)\frac{dx}{x}.$$

Set temporarily

$$\sigma = \left[\frac{1}{\rho}\right]^{\frac{q_1 q_0}{q_0 - q_1}}$$

and

$$B(t) = \int_t^\infty b_1^{q_1}(s)\frac{ds}{s}.$$

For each $k \in \mathbb{N}$, define $Q^{k+1} = Q(Q^k)$. Then

$$(Q^{k+1}w)(t) = \frac{1}{k!}\int_t^\infty w(x)\ln^k\left[\frac{B(t)}{B(x)}\right][B(x)]^{-1}\, b_1^{q_1}(x)\frac{dx}{x}, \quad k \in \mathbb{N}. \tag{3.13}$$

Moreover we see that (3.12) translates into

$$(Q\sigma)(t) \le c\sigma(t)$$

for some constant $c > 0$. Therefore, we have

$$(Q^{k+1}\sigma)(t) \le c^{k+1}\sigma(t), \quad k \in \mathbb{N},$$

which, in view of (3.13), leads us to

$$\frac{1}{k!}\int_t^\infty \sigma(x)\ln^k\left[\frac{B(t)}{B(x)}\right][B(x)]^{-1}\, b_1^{q_1}(x)\frac{dx}{x} \le c^{k+1}\sigma(t), \quad k \in \mathbb{N}. \tag{3.14}$$

We choose $\epsilon > 0$ such that $\max(\epsilon c, \epsilon) < 1$. Then by (3.14), we obtain

$$\int_t^\infty \sigma(x)\sum_{k=0}^\infty \frac{\ln^k\left[\frac{B(t)}{B(x)}\right]^\epsilon}{k!}[B(x)]^{-1}\, b_1^{q_1}(x)\frac{dx}{x} \le c\sum_{k=0}^\infty (\epsilon c)^k\sigma(t),$$

whence we get

$$\int_t^\infty \sigma(x)\left[\frac{B(t)}{B(x)}\right]^\epsilon [B(x)]^{-1}\, b_1^{q_1}(x)\frac{dx}{x} \le \frac{c}{1-\epsilon c}\sigma(t),$$

or,

$$\int_t^\infty \sigma(x)\,[B(x)]^{-\epsilon-1}\, b_1^{q_1}(x)\frac{dx}{x} \le \frac{c}{1-\epsilon c}\sigma(t)\,[B(t)]^{-\epsilon}. \tag{3.15}$$

Now the converse estimate

$$\int_t^\infty \sigma(x)\,[B(x)]^{-\epsilon-1}\, b_1^{q_1}(x)\frac{dx}{x} \gtrsim \sigma(t)\,[B(t)]^{-\epsilon} \tag{3.16}$$

holds as well. Indeed,

$$\begin{aligned}\int_t^\infty \sigma(x)\,[B(x)]^{-\epsilon-1}\, b_1^{q_1}(x)\frac{dx}{x} &\gtrsim \left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{\frac{q_1}{q_1-q_0}}\int_t^\infty [B(x)]^{\frac{q_1 q_0}{q_0-q_1}-\epsilon-1}\, b_1^{q_1}(x)\frac{dx}{x}\\ &\approx \left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{\frac{q_1}{q_1-q_0}}[B(x)]^{\frac{q_1 q_0}{q_0-q_1}-\epsilon},\end{aligned}$$

whence we get (3.1). From (3.15) and (3.1), it follows that

$$\int_t^\infty \sigma(x)\,[B(x)]^{-\epsilon-1}\, b_1^{q_1}(x)\frac{dx}{x} \approx \sigma(t)\,[B(t)]^{-\epsilon}$$

which shows that $t \mapsto \sigma(t)\,[B(t)]^{-\epsilon}$ is equivalent to a non-increasing function. That is,

$$t \mapsto \left[\frac{1}{\rho(t)}\right]^{\frac{q_1 q_0}{q_0-q_1}} \left(\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{-\epsilon}$$

is equivalent to a non-increasing function, or,

$$t \mapsto \rho(t)\left(\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{\frac{\epsilon(q_0-q_1)}{q_0 q_1}}$$

is equivalent to a non-decreasing function. It follows that

$$t \mapsto \rho(t)\left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{\frac{\epsilon(q_0-q_1)}{q_0(q_0-\epsilon(q_0-q_1))}}$$

is equivalent to a non-decreasing function since

$$\rho(t)\left(\int_t^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{\frac{\epsilon(q_0-q_1)}{q_0(q_0-\epsilon(q_0-q_1))}} = \left(\rho(t)\left(\int_t^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{\frac{\epsilon(q_0-q_1)}{q_0 q_1}}\right)^{\frac{q_0}{q_0-\epsilon(q_0-q_1)}}.$$

Thus the condition (3.1) is valid in the case when $q_1 < q_0$. As for the case $q_0 < q_1$, we take a particular decomposition $f = 0 + f$, $0 \in \bar{A}_{0,q_0;b_0}$ and $f \in \bar{A}_{1,q_1;b_1}$ to get

$$I(t,f) \lesssim \rho(t)\|f\|_{\bar{A}_{0,q_1;b_1}},$$

from which, according to Corollary 2.4 (b), it follows that

$$\left(\int_0^t \frac{\left(\int_x^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{\frac{q_0}{q_1-q_0}}}{\left(\int_x^\infty b_1^{q_1}(s)\frac{ds}{s}\right)^{\frac{q_0}{q_1-q_0}}} b_0^{q_0}(x)\frac{dx}{x}\right)^{1/q_0-1/q_1} \lesssim \rho(t). \tag{3.17}$$

This time we introduce the operator

$$(Pw)(t) = \int_0^t w(x)\left(\int_x^\infty b_0^{q_0}(s)\frac{ds}{s}\right)^{-1} b_1^{q_0}(x)\frac{dx}{x},$$

and using a similar argument as in the case $q_1 < q_0$ we can conclude that the condition (3.1) is also valid in the case when $q_0 < q_1$. This completes the proof of the theorem. ■

Remark 3.3 The argument for the estimate " $\lesssim$ " in (3.2) also works when $q_0 = q_1$. Moreover, the condition (3.1) is only required for the estimate " $\gtrsim$ " in (3.2).

Remark 3.4 The converse estimate in (3.11) holds trivially.

Next we treat the case $q_0 = q_1$.

Theorem 3.5 *Let $0 < p \leq \infty$ and $b_j \in SV_{0,p}$. Assume that ρ is increasing. In the case $p < \infty$, assume additionally that the given couple (A_0, A_1) is K-surjective. Then, for all $f \in A_0 + A_1$ and all $t > 0$, we have*

$$K(\rho(t), f; \bar{A}_{0,p;b_0}, \bar{A}_{0,p;b_1}) \approx I(t,f) + \rho(t)J(t,f). \tag{3.18}$$

Proof. In view of Remark 3.3, it remains to derive the converse estimate " $\gtrsim$ " in (3.18). First we consider the case $p = \infty$. In this case we can assume, without loss of generality, that b_0 and b_1 are non-increasing functions. Therefore, we have $\rho = b_0/b_1$. The desired estimate " $\gtrsim$ " follows from the estimate (2.35) in [1]. Next we turn to the case $0 < p < \infty$. As in the previous theorem, we need to show that the estimates (3.7) and (3.8) hold (with $q_0 = q_1 = p$) for an arbitrary decomposition $f = f_0 + f_1$ with $f_j \in A_j$ $(j = 0, 1)$. According to Corollary 2.4 (a), the estimate (3.7) holds if the following condition is met:

$$\sup_{0<x<t} \rho(x) \lesssim \rho(t), \tag{3.19}$$

while, according to Corollary 2.5 (a), the estimate (3.8) holds if the following condition is met:

$$\sup_{x>t} 1/\rho(x) \lesssim 1/\rho(t). \tag{3.20}$$

But both (3.19) and (3.20) hold trivially in view of the fact that ρ is increasing. The proof of the theorem is complete. ■

3.2 The case $\theta_0 = \theta_1 = 1$

The case $\theta_0 = \theta_1 = 1$ is symmetric counterpart of the case $\theta_0 = \theta_1 = 1$, and the corresponding estimates can be derived immediately from the estimates in the previous subsection using the same symmetry argument as in the proof of Theorem 4.3 in [22].

In order to formulate the results, we introduce some further notation. Let $0 < q_j \leq \infty$ and $b_j \in SV_{1,q_j}$ $(j = 0, 1)$. Put

$$\eta(t) = \frac{\|u^{-1/q_0} b_0(u)\|_{q_0,(0,t)}}{\|u^{-1/q_1} b_1(u)\|_{q_1,(0,t)}},$$

$$I_1(t,f) = \|u^{-1-1/q_0} b_0(u) K(u,f;A_0,A_1)\|_{q_0,(0,t)},$$

and

$$J_1(t,f) = \|u^{-1-1/q_1} b_1(u) K(u,f;A_0,A_1)\|_{q_1,(t,\infty)}.$$

Moreover, let $\epsilon > 0$ and set

$$\eta_\epsilon(t) = \frac{\|u^{-1/q_0} b_0(u)\|^{1+\epsilon}_{q_0,(0,t)}}{\|u^{-1/q_1} b_1(u)\|_{q_1,(0,t)}}.$$

Theorem 3.6 *Let $0 < q_0, q_1 \le \infty$, $q_0 \neq q_1$, and $b_j \in SV_{1,q_j}$ (j=0,1). Assume that the following condition is met:*

$$\eta_\epsilon \text{ is equivalent to a non-decreasing function for some } \epsilon > 0. \tag{3.21}$$

Then, for all $f \in A_0 + A_1$ and all $t > 0$, we have

$$K(\eta(t), f; \bar{A}_{1,q_0;b_0}, \bar{A}_{1,q_1;b_1}) \approx I_1(t,f) + \rho(t) J_1(t,f). \tag{3.22}$$

Moreover, the condition (3.21) is also necessary provided that the given couple (A_0, A_1) is K-surjective.

Theorem 3.7 *Let $0 < p \le \infty$ and $b_j \in SV_{1,p}$. Assume that η is increasing. In the case $p < \infty$, assume additionally that the given couple (A_0, A_1) is K-surjective. Then, for all $f \in A_0 + A_1$ and all $t > 0$, we have*

$$K(\eta(t), f; \bar{A}_{1,p;b_0}, \bar{A}_{1,p;b_1}) \approx I_1(t,f) + \eta(t) J_1(t,f). \tag{3.23}$$

3.3 The case $\theta_0 = \theta_1 \in (0,1)$

First we treat the case $q_0 \neq q_1$.

Theorem 3.8 *Let $0 < \theta < 1$, $0 < q_0 \neq q_1 \le \infty$, and let $b_0, b_1 \in SV$. Assume that the given couple (A_0, A_1) is K-surjective. Then there exists no positive function w on $(0,\infty)$ such that the following estimate holds*

$$\begin{aligned} K(w(t), f; \bar{A}_{\theta,q_0;b_0}, \bar{A}_{\theta,q_1;b_1}) \quad &\gtrsim \quad \left(\int_0^t s^{-\theta q_0} K^{q_0}(s,f) \frac{ds}{s} \right)^{1/q_0} \\ &\quad + w(t) \left(\int_t^\infty s^{-\theta q_1} K^{q_1}(s,f) \frac{ds}{s} \right)^{1/q_1} \end{aligned}$$

for all $f \in A_0 + A_1$ and for all $t > 0$.

Proof. We give the argument only in the case $0 < q_0 < q_1 \le \infty$ since the argument in the other case $0 < q_1 < q_0 \le \infty$ is similar. We assume, on the contrary, that there exists such a positive function w on $(0,\infty)$. Taking a particular decomposition $f = f + 0$, $f \in \bar{A}_{\theta,q_0;b_0}$ and $0 \in \bar{A}_{\theta,q_1;b_1}$, we obtain

$$w(t) \| s^{-\theta - 1/q} b_1(s) K(s,f) \|_{q_1,(t,\infty)} \lesssim \|f\|_{\bar{A}_{\theta,q_0;b_0}}, \tag{3.24}$$

while taking a particular decomposition $f = 0 + f$, $0 \in \bar{A}_{\theta,q_0;b_0}$ and $f \in \bar{A}_{\theta,q_1;b_1}$, we obtain

$$\left(\int_0^t s^{-\theta q_0} b_0^{q_0}(s) K^{q_0}(s,f) \frac{ds}{s} \right)^{1/q_0} \lesssim w(t) \|f\|_{\bar{A}_{\theta,q_1;b_1}}. \tag{3.25}$$

First let $q_1 < \infty$. Now, according to Corollary 2.5 (a), it follows from (3.24) that

$$w(t) \lesssim \frac{b_0(t)}{b_1(t)}, \quad t > 0, \tag{3.26}$$

and while, according to Corollary 2.4 (b), it follows from (3.25) that

$$\left(\int_0^t \left[\frac{b_0(s)}{b_1(s)}\right]^{\frac{q_0q_1}{q_0-q_1}} \frac{ds}{s}\right)^{1/q_1-1/q_0} \lesssim w(t), \quad t>0. \tag{3.27}$$

Finally, combining (3.26) and (3.27) yields

$$\left(\int_0^t \left[\frac{b_0(s)}{b_1(s)}\right]^{\frac{q_0q_1}{q_0-q_1}} \frac{ds}{s}\right)^{1/q_1-1/q_0} \lesssim \frac{b_0(t)}{b_1(t)}, \quad t>0,$$

which is not possible since b_0 and b_1 are slowly varying functions. Next we turn to the case $q_1=\infty$. Choose $q_0<r<\infty$. Then, in view of the well-known embedding $\bar{A}_{\theta,r;b_1} \hookrightarrow \bar{A}_{\theta,\infty;b_1}$, (3.25) gives

$$\left(\int_0^t s^{-\theta q_0} b_0^{q_0}(s) K^{q_0}(s,f)\frac{ds}{s}\right)^{1/q_0} \lesssim w(t)\|f\|_{\bar{A}_{\theta,r;b_1}},$$

from which, using Corollary 2.4 (b), it follows that

$$\left(\int_0^t \left[\frac{b_0(s)}{b_1(s)}\right]^{\frac{q_0r}{q_0-r}} \frac{ds}{s}\right)^{1/r-1/q_0} \lesssim w(t), \quad t>0. \tag{3.28}$$

Next we choose an $\epsilon>0$ so that both $\theta+\epsilon$ and $\theta-\epsilon$ lie in the interval $(0,1)$. Next choose a $f\in A_0+A_1$ such that we have

$$K(s,f)=\begin{cases} s^{\theta+\epsilon}, & 0<s<t,\\ t^{2\epsilon}s^{\theta-\epsilon}, & s\geq t.\end{cases}$$

Then we again get (3.26) from (3.24). This time combining (3.26) and (3.28) yields

$$\left(\int_0^t \left[\frac{b_0(s)}{b_1(s)}\right]^{\frac{q_0r}{q_0-r}} \frac{ds}{s}\right)^{1/r-1/q_0} \lesssim \frac{b_0(t)}{b_1(t)}, \quad t>0,$$

which is again not possible. The proof is complete. ■

Next in the case $q_0=q_1$, a version of Holmstedt's formula does exit.

Theorem 3.9 *Let $0<\theta<1$, $0<q\leq\infty$. Let b_0 and b_1 be slowly varying functions such that $\rho=b_0/b_1$ is non-decreasing. Then for all $f\in A_0+A_1$ and for all $t>0$, we have*

$$\begin{aligned} K(\rho(t),f;\bar{A}_{\theta,q;b_0},\bar{A}_{\theta,q;b_1}) \quad &\approx \quad \|u^{-\theta-1/q}b_0(u)K(u,f)\|_{p,(0,t)} \\ &\quad +\rho(t)\|u^{-\theta-1/p}b_1(u)K(u,f)\|_{p,(t,\infty)}. \end{aligned}$$

Proof. The proof follows immediately from the estimates (2.30) and (2.35) in [1, Theorem 2.3]. ■

4 Reiteration

Theorem 4.1 *Let $0 < q_0, q_1, q < \infty$, $0 < \theta < 1$, $b_j \in SV_{0,q_j}$ (j=0,1), and $b \in SV$. Assume that ρ is increasing on $(0, \infty)$ with $\lim_{t\to 0^+} \rho(t) = 0$ and $\lim_{t\to\infty} \rho(t) = \infty$. If $q_0 \neq q_1$, assume additionally that the condition (3.1) is met, while if $q_0 = q_1$, assume additionally that the given couple (A_0, A_1) is K-surjective and that the following two-sided estimate holds:*

$$\frac{\rho'(t)}{\rho(t)} \approx \frac{t^{-1} b_1^{q_1}(t)}{\int_t^\infty b_1^{q_1}(u)\frac{du}{u}}, \quad t > 0.$$

Put

$$\tilde{b}(t) = [\rho(t)]^{(1-\theta)} b(\rho(t)) [b_1(t)]^{q_1/q} \left(\int_t^\infty b_1^{q_1}(u) \frac{du}{u} \right)^{1/q_1 - 1/q}.$$

Then

$$\left(\bar{A}_{0,q_0;b_0}, \bar{A}_{0,q_1;b_1}\right)_{\theta,q;b} = \bar{A}_{0,q;\tilde{b}}. \tag{4.1}$$

Proof. We consider only the case $q_0 \neq q_1$; the other case $q_0 = q_1$ being similar. Let $f \in A_0 + A_1$, and set $X = \left(\bar{A}_{0,q_0;b_0}, \bar{A}_{0,q_1;b_1}\right)_{\theta,q;b}$, , $Y = \bar{A}_{0,q;\tilde{b}}$ and

$$\frac{1}{\sigma(t)} = \left\| \frac{x^{-1/q_1} b_1(x)}{\|s^{-1/q_0} b_0(s)\|_{q_0,(x,\infty)}} \right\|_{q_1,(t,\infty)}.$$

Then, in view of (3.11) along with Remark 3.4, we have $\sigma \approx \rho$. Thus, by Theorem 3.2, we get

$$\|f\|_X^q \approx I_1 + I_2,$$

where

$$I_1 = \int_0^\infty [\sigma(t)]^{-\theta q} b^q(\sigma(t)) \left(\int_0^t b_0^{q_0}(u) K^{q_0}(u,f) \frac{du}{u} \right)^{q/q_0} \frac{\sigma'(t)}{\sigma(t)} dt,$$

and

$$I_2 = \int_0^\infty [\sigma(t)]^{(1-\theta)q} b^q(\sigma(t)) \left(\int_t^\infty b_1^{q_1}(u) K^{q_1}(u,f) \frac{du}{u} \right)^{q/q_1} \frac{\sigma'(t)}{\sigma(t)} dt.$$

We can compute that

$$\frac{\sigma'(t)}{\sigma(t)} \approx \frac{t^{-1} b_1^{q_1}(t)}{\int_t^\infty b_1^{q_1}(u)\frac{du}{u}}.$$

First we show that $I_2 \approx \|f\|_Y^q$. Now $I_2 \geq \|f\|_Y^q$ is a simple consequence of the fact that $u \mapsto K(u,f)$ is non-decreasing. In order to establish the converse estimate $I_2 \lesssim \|f\|_Z^q$, we distinguish three cases: $q = q_1$, $q > q_1$ and $q < q_1$. The case $q = q_1$ simply follows from Fubini's theorem. Next the case $q > q_1$ follows from Theorem 2.7, while the case $q < q_1$ follows from Theorem 2.9. Thus, it remains to show that $I_1 \lesssim \|f\|_Y^q$. Again we distinguish three cases: $q = q_0$,

$q > q_0$ and $q < q_0$. The case $q = q_0$ follows from Fubini's theorem, while the case $q > q_0$ follows from Theorem 2.6 in view of the following estimate

$$\frac{b_0^{q_0}(t)}{\int_t^\infty b_0^{q_0}(u)\frac{du}{u}} \lesssim \frac{b_1^{q_1}(t)}{\int_t^\infty b_1^{q_1}(u)\frac{du}{u}}, \quad t > 0,$$

which is a simple consequence of our assumption that ρ is increasing on $(0, \infty)$. As for the case $q < q_0$, we apply Theorem 2.10 with $\alpha = q/q_0$, $h(t) = K(t, f)$, $w(t) = t^{-1}[\tilde{b}(t)]^q$, $\psi(t, u) = u^{-1} b_0^{q_0} \chi_{(0,t)}(u)$ and

$$v(t) = t^{-1}[\rho(t)]^{q(1-\theta)} b^q(\rho(t))[b_1(t)]^{q_1} \left(\int_t^\infty b_1^{q_1}(u)\frac{du}{u} \right)^{q/q_1 - 1}.$$

We observe that

$$\begin{aligned}
\int_0^\infty \left(\int_x^\infty \psi(t,u)du \right)^\alpha w(t)dt &= \int_x^\infty \left(\int_x^t b_0^{q_0}(u)\frac{du}{u} \right)^{q/q_0} [\tilde{b}(t)]^q \frac{dt}{t} \\
&\leq \left(\int_x^\infty b_0^{q_0}(u)\frac{du}{u} \right)^{q/q_0} \int_x^\infty [\tilde{b}(t)]^q \frac{dt}{t} \\
&\approx \left(\int_x^\infty b_0^{q_0}(u)\frac{du}{u} \right)^{q/q_0} [\rho(t)]^{-\theta q} b^q(\rho(t)),
\end{aligned}$$

and

$$\begin{aligned}
\int_x^\infty v(t)dt &\gtrsim [\rho(t)]^{q(1-\theta)} b^q(\rho(t)) \int_x^\infty [b_1(t)]^{q_1} \left(\int_t^\infty b_1^{q_1}(u)\frac{du}{u} \right)^{q/q_1 - 1} \frac{dt}{t} \\
&\approx \left(\int_x^\infty b_0^{q_0}(u)\frac{du}{u} \right)^{q/q_0} [\rho(t)]^{-\theta q} b^q(\rho(t)).
\end{aligned}$$

Thus, $I_1 \lesssim \|f\|_Y^q$ holds. The proof of the theorem is complete. ■

Remark 4.2 We have left out the cases $\theta = 0$ and $\theta = 1$ since in these cases no simplification takes place and the resulting interpolation spaces involve the K-interpolation spaces of type $\mathcal{L}$ and $\mathcal{R}$ (see, for instance, [22]). We leave the details to the reader.

Remark 4.3 We refer the reader to a recent reiteration formula [2, Theorem 5.8] which deals with the case $q_0 = q_1$ (without K-surjective assumption) for general weights (under certain appropriate conditions) and for ordered couples (A_0, A_1) in the sense that $A_1 \hookrightarrow A_0$.

The reiteration theorem corresponding to the limiting case $\theta_0 = \theta_1 = 1$ reads as follows.

Theorem 4.4 *Let $0 < q_0, q_1, q < \infty$, $0 < \theta < 1$, $b_j \in SV_{1,q_j}$ (j=0,1), and $b \in SV$. Assume that η is increasing on $(0,\infty)$ with $\lim_{t\to 0^+} \eta(t) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$. If $q_0 \neq q_1$, assume additionally that the condition (3.21) is met, while if $q_0 = q_1$, assume additionally that the given couple (A_0, A_1) is K-surjective and that the following two-sided estimate holds:*

$$\frac{\eta'(t)}{\eta(t)} \approx \frac{t^{-1} b_1^{q_1}(t)}{\int_0^t b_1^{q_1}(u)\frac{du}{u}}, \quad t > 0.$$

Put

$$\hat{b}(t) = [\eta(t)]^{\theta} b(\eta(t)) [b_1(t)]^{q_1/q} \left(\int_0^t b_1^{q_1}(u) \frac{du}{u} \right)^{1/q_1 - 1/q}.$$

Then

$$\left(\bar{A}_{1,q_0;b_0}, \bar{A}_{1,q_1;b_1}\right)_{\theta,q;b} = \bar{A}_{1,q;\hat{b}}. \tag{4.2}$$

Remark 4.5 The previous two reiteration theorems have already obtained in the special case when b_j $(j = 0, 1)$ are logarithmic functions (see [8, Corollary 1]) or broken logarithmic functions (see [14, Corollaries 7.8 and 7.11]).

Remark 4.6 Let $0 < \theta, \eta < 1$ and $0 < q \leq \infty$. The characterization of the interpolation spaces

$$(\bar{A}_{\theta,q;b_0}, \bar{A}_{\theta,q;b_1})_{\theta,r;b}$$

will involve the K-interpolation spaces of type $\mathcal{L}$ and $\mathcal{R}$. We once again leave the elementary details to the reader.

5 Concrete examples

5.1 Lorentz-Karamata spaces

Let (Ω, μ) be a σ-finite measure space. Let f^* denotes the non-increasing rearrangement of a μ-measurable function f on Ω (see, for instance, [5]).

Definition 5.1 *[22] Let $0 < p, q \leq \infty$ and $b \in SV$. The Lorentz-Karamata space $L_{p,q;b}$ consists of all μ-measurable functions f on Ω such that the quasi-norm*

$$\|f\|_{L_{p,q;b}} = \|t^{1/p-1/q} b(t) f^*(t)\|_{q,(0,\infty)}$$

is finite.

For $b = \ell^{\mathbb{A}}$, the spaces $L_{p,q;b}$ coincide with the spaces $L_{p,q;\mathbb{A}}$ from [13] and [14]. When $b \equiv 1$, the spaces $L_{p,q;b}$ become the Lorentz spaces $L^{p,q}$, which coincide with the classical Lebesgue spaces L^p for $p = q$.

We give an application of Theorem 4.4 to the interpolation of Lorentz-Karamata spaces $L_{p,q;b}$ in the critical case when $p = \infty$. To this end, we characterize $L_{\infty,q;b}$ as limiting K-interpolation spaces for the couple (L^1, L^∞).

Lemma 5.2 *Let $0 < q < \infty$ and $b \in SV_{1,q}$. Then*

$$L_{\infty,q;b} = (L^1, L^\infty)_{1,q;b}.$$

Proof. Put $X = (L^1, L^\infty)_{0,q;b}$ and $Y = L_{\infty,q;b}$, and let $f \in L^1 + L^\infty$. Since (see [6, Theorem 5.2.1])

$$K(t, f; L^1, L^\infty) = \int_0^t f^*(u)du, \quad t > 0,$$

it turns out that

$$\|f\|_X = \left(\int_0^\infty t^{-q} b^q(t) \left(\int_0^t f^*(u)du\right)^q \frac{dt}{t}\right)^{1/q}.$$

Now the estimate $\|f\|_X \geq \|f\|_Y$ follows immediately in view of the fact that f^* is non-increasing. On the other hand, the converse estimate $\|f\|_X \lesssim \|f\|_Y$ follows from Theorem 2.6 (in the case $q > 1$), Theorem 2.8 (in the case $q < 1$) and Fubini's theorem (in the case $q = 1$). The proof is complete. ■

Theorem 5.3 *Let $0 < q_0, q_1, q < \infty$, $0 < \theta < 1$, $b_j \in SV_{1,q_j}$ (j=0,1), and $b \in SV$. Assume that η is increasing on $(0, \infty)$ with $\lim\limits_{t\to 0^+} \eta(t) = 0$ and $\lim\limits_{t\to\infty} \eta(t) = \infty$. If $q_0 \neq q_1$, assume additionally that the condition (3.21) is met, while if $q_0 = q_1$, assume additionally that the following two-sided estimate holds:*

$$\frac{\eta'(t)}{\eta(t)} \approx \frac{t^{-1} b_1^{q_1}(t)}{\int_0^t b_1^{q_1}(u) \frac{du}{u}}, \quad t > 0.$$

Then

$$(L_{\infty,q_0;b_0}, L_{\infty,q_1;b_1})_{\theta,q;b} = L_{\infty,q;\hat{b}},$$

where

$$\hat{b}(t) = [\eta(t)]^\theta b(\eta(t)) [b_1(t)]^{q_1/q} \left(\int_0^t b_1^{q_1}(u) \frac{du}{u}\right)^{1/q_1 - 1/q}.$$

Proof. Take $A_0 = L^1$ and $A_1 = L^\infty$, and apply Theorem 4.4 to obtain

$$\left((L^1, L^\infty)_{1,q_0;b_0}, (L^1, L^\infty)_{1,q_1;b_1}\right)_{\theta,q;b} = (L^1, L^\infty)_{1,q;\hat{b}}.$$

Now it remains to apply Lemma 5.2. ■

5.2 Generalized gamma spaces

Definition 5.4 *([21]) Let $0 < q, r \leq \infty$, $0 < p < \infty$, $b \in SV_{0,q}$ and $w \in SV$ The generalized gamma space $\Gamma(r, q, p; b, w) = \Gamma(r, q, p; b, w)(\Omega)$ consists of all those real-valued Lebesgue measurable functions f on Ω, for which the quasi-norm*

$$\|f\|_{\Gamma(r,q,p;b,w)} = \left\| t^{-1/q} b(t) \| \tau^{1/p - 1/r} w(\tau) f^*(\tau) \|_{r,(0,t)} \right\|_{q,(0,\infty)}$$

is finite.

Theorem 5.5 *Let $0 < q_0 \neq q_1 < \infty$, $0 < q, p, r < \infty$, $0 < \theta < 1$, $b_j \in SV_{0,q_j}$ (j=0,1), and $b, w \in SV$. Assume that the condition (3.1) is met, and assume that ρ is increasing with $\lim_{t\to 0^+} \rho(t) = 0$ and $\lim_{t\to\infty} \rho(t) = \infty$. Then*

$$(\Gamma(r, q_0, p; b_0, w), \Gamma(r, q_1, p; b_1, w))_{\theta,q;b} = \Gamma(r, q, p; \tilde{b}, w),$$

where

$$\tilde{b}(t) = [\rho(t)]^{(1-\theta)} b(\rho(t))[b_1(t)]^{q_1/q} \left(\int_t^\infty b_1^{q_1}(u) \frac{du}{u} \right)^{1/q_1 - 1/q}.$$

Proof. The proof follows from Theorem 4.1 and the following interpolation formula (for $j = 0, 1$)

$$\Gamma(r, q_j, p; b_j, w) = (L_{p,r;w}, L^\infty)_{0,q_j;b_j}.$$

■

5.3 Homogeneous Besov spaces

Let E be a rearrangement invariant Banach function space on $\mathbb{R}^n$ as in [5], and let $\omega_E(f, t) = \sup_{|h|\le t} \|\Delta_h f\|_E$ is the modulus of continuity of $f \in E$ (see, for example, [4]).

Definition 5.6 *([4]) Let $0 < q \le \infty$ and $b \in SV_{0,q}$. The homogeneous Besov space $B^{0,b}_{E,q}$ consists of those functions $f \in E$ for which the semi-quasi-norm*

$$\|f\|_{B^{0,b}_{E,q}} = \|t^{-1/q} b(t)(t)\omega_E(f, t)\|_{q,(0,\infty)}$$

is finite.

It is well-known (see, for instance, [7]) that

$$K(f, t; E, W^1 E) \approx \omega_E(f, t), \quad t > 0,$$

where $W^1 E$ is the Sobolev space built over E with a norm $\|f\|_{W^1 E} = \||D^1 f|\|_E$. Here $|D^1 f| = \sum_{|\alpha|=1} |D^\alpha f|$. Then it follows immediately that

$$(E, W^1 E)_{0,b,q} = B^{0,b}_{E,q}. \tag{5.1}$$

Remark 5.7 We observe that Theorem 4.1 also holds when the compatible couple quasi-normed spaces is replaced by a compatible couple of semi-quasi-normed spaces.

Theorem 5.8 *Let $0 < q_0 \neq q_1 < \infty$, $0 < q < \infty$, $0 < \theta < 1$, $b_j \in SV_{0,q_j}$ (j=0,1), and $b \in SV$. Assume that the condition (3.1) is met, and assume that ρ is increasing with $\lim_{t\to 0^+} \rho(t) = 0$ and $\lim_{t\to\infty} \rho(t) = \infty$. Then*

$$(B^{0,b_0}_{E,q_0}, B^{0,b_1}_{E,q_1})_{\theta,q;b} = B^{0,\tilde{b}}_{E,q},$$

where

$$\tilde{b}(t) = [\rho(t)]^{(1-\theta)} b(\rho(t))[b_1(t)]^{q_1/q} \left(\int_t^\infty b_1^{q_1}(u) \frac{du}{u} \right)^{1/q_1 - 1/q}.$$

Proof. The proof follows from Theorem 4.1 and the interpolation formula (5.1). ■

Acknowledgement

The third author, A. Gogatishvili started work on this project during his visit to J.M. Rakotoson to the Laboratory of Mathematics of the University of Poitiers, France, in April 2022. He thanks for his generous hospitality and helpful atmosphere during the visit. He has been partially supported by the RVO:67985840 Czech Republic and by Shota Rustaveli National Science Foundation of Georgia (SRNSFG), grant no: FR21-12353.

The authors are very grateful to Professor Fernando Cobos for pointing out the Definition 3.1 to us.

References

[1] I. Ahmed, D. E. Edmunds, W. D. Evans and G. E. Karadzhov, Reiteration theorems for the K-interpolation method in limiting cases, *Math. Nachr.* **284** (2011), 421-442.

[2] I. Ahmed, A. Fiorenza and M. R. Formica, Interpolation of Generalized Gamma Spaces in a critical case, *J. Fourier Anal. Appl.* **28**, 54 (2022). https://doi.org/10.1007/s00041-022-09947-1

[3] I. Ahmed, G. E. Karadzhov and A. Raza, General Holmstedt's formulae for the K-functional, *J. Funct. Spaces* vol. 2017, Article ID 4958073, 9 pages, 2017. https://doi.org/10.1155/2017/4958073

[4] Z. Bashir and G. E. Karadzhov, Optimal embeddings of genearized Besov spaces, *Eurasian Math J.* **2** (2011), 5-31.

[5] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[6] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.

[7] Y. A. Brudnyǐ and N. Y. Krugljak, *Interpolation Functors and Interpolation Spaces, Volume I*, North-Holland, Amsterdam, 1991.

[8] R. Ya. Doktorski, Reiteration relations of the real interpolation method, *Soviet Math. Dokl.* **44** (1992), 665-669.

[9] R. Ya. Doktorski, Reiteration formulae for the real interpolation method including $\mathcal{L}$ or $\mathcal{R}$ limiting spaces, *J. Funct. Spaces* vol. 2020, Article ID 6685993, 15 pages, 2020. https://doi.org/10.1155/2020/6685993

[10] R. Ya. Doktorski, Some reiteration theorems for $\mathcal{R}$, $\mathcal{L}$, $\mathcal{RR}$, $\mathcal{RL}$, $\mathcal{LR}$, and $\mathcal{LL}$ limiting interpolation spaces, *J. Funct. Spaces* Vol. 2021, Article ID 8513304, 31 pages, 2021. https://doi.org/10.1155/2021/8513304

[11] R. Ya. Doktorski, P. Fernández-Martínez and T. Signes, Reiteration theorem for $\mathcal{R}$ and $\mathcal{L}$-spaces with the same parameter, *J. Math. Anal. Appl.* **508** (2022): 125846.

[12] D. E. Edmunds and W. D. Evans, *Hardy operators, Function Spaces and Embeddings,* Springer, Berlin, 2004.

[13] W. D. Evans and B. Opic, Real interpolation with logarithmic functors and reiteration, *Canad. J. Math.* **52** (2000), 920-960.

[14] W. D. Evans, B. Opic and L. Pick, Real interpolation with logarithmic functors, *J. Inequal. Appl.* **7** (2002), 187-269.

[15] P. Fernández-Martínez and T. Signes, Real interpolation with symmetric spaces and slowly varying functions, *Q. J. Math.* **63** (2012), 133-164.

[16] P. Fernández-Martínez and T. Signes, Reiteration theorems with extreme values of parameters, *Ark. Mat.* **52** (2014), 227-256.

[17] P. Fernández-Martínez and T. Signes, Limit cases of reiteration theorems, *Math. Nachr.* **288** (2015), 25-47.

[18] P. Fernández-Martínez and T. Signes, General reiteration theorems for $\mathcal{L}$ and $\mathcal{R}$ classes: Case of right $\mathcal{R}$-spaces and left $\mathcal{L}$-spaces, *Miditerr. J. Math.* **19**, 193 (2020).

[19] P. Fernández-Martínez and T. Signes, General reiteration theorems for $\mathcal{R}$ and $\mathcal{L}$ classes: Case of left $\mathcal{R}$-spaces and right $\mathcal{L}$-spaces, *J. Math. Anal. Appl.* **494** (2021) 124649.

[20] P. Fernández-Martínez and T. Signes, General reiteration theorems for $\mathcal{R}$ and $\mathcal{L}$ classes: mixed interpolation of $\mathcal{R}$ and $\mathcal{L}$-spaces, *Positivity* **26**, 47 (2022). https://doi.org/10.1007/s11117-022-00888-z

[21] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani and J. M. Rakotoson, Characterization of interpolation between Grand, small or classical Lebesgue spaces, *Nonlinear Anal.* **177** (2018), 422-453.

[22] A. Gogatishvili, B. Opic and W. Trebels, Limiting reiteration for real interpolation with slowly varying functions, *Math. Nachr.* **278** (2005), 86-107.

[23] A. Gogatishvili and L. Pick, Discretization and anti-discretization of rearrangement-invariant norms, *Publ. Mat.* **47** (2003), 311-358.

[24] H. Heinig and L. Maligranda, Weighted inequalities for monotone and concave functions, *Studia Math.* **116** (1995), 133-165.

[25] T. Holmstedt, Interpolation of quasi-normed spaces, *Math. Scand.* **26** (1970), 177-199.

[26] P. Nilsson, Interpolation of Calderón and Ovčinnikov pairs, *Ann. Mat. Pura App.* **134** (1983), 201–232. https://doi.org/10.1007/BF01773505

[27] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.