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LOW MACH NUMBER LIMIT ON PERFORATED DOMAINS FOR THE EVOLUTIONARY NAVIER–STOKES–FOURIER SYSTEM

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Abstract. We consider the Navier–Stokes–Fourier system describing the motion of a compressible, viscous and heat-conducting fluid on a domain perforated by tiny holes. First, we identify a class of dissipative solutions to the Oberbeck-Boussinesq approximation as a low Mach number limit of the primitive system. Secondly, by proving the weak–strong uniqueness principle, we obtain strong convergence to the target system on the lifespan of the strong solution.

1. Introduction

The aim of this work is to study the asymptotic analysis of the scaled Navier–Stokes–Fourier system in a domain perforated with tiny holes. More precisely, we consider the physical situation corresponding to the *low stratification* of a fluid, i.e. the equations describing the motion of a compressible viscous fluid are scaled by a small Mach number $\mathrm{Ma} = \varepsilon^m$ and Froude number $\mathrm{Fr} = \varepsilon^{m/2}$ for a fixed integer m; in addition, we suppose that the fluid is confined to a bounded spatial domain perforated by balls of radius ε^α and having mutual distance ε^β for some positive α and β such that $\alpha > \beta$. We keep other characteristic numbers Strouhal number, Reynolds number and Péclet number as unity. Our goal consists in analyzing what happens when we let ε go to zero.

In the absence of holes, the problem reduces to a classical asymptotic analysis problem in a fixed domain, mainly the low Mach number limit, which is also referred to as the incompressible limit in the context of compressible systems in the literature. The first approach, proposed by Kleinarman and Majda [18], is based on classical or strong solutions of the compressible system and proves that the limit is an incompressible system. This approach has been followed by Alazard [5] to analyze the low Mach number limit for the Navier-Stokes-Fourier system. On the other hand, based on global-in-time weak solutions, Lions and Masmoudi [19], and Desjardins et al. [9] studied the low Mach number limit for the compressible Navier-Stokes system and they obtained the incompressible Navier-Stokes system as a limit. This approach has also been extended to the Navier-Stokes-Fourier system. We refer to the monograph of Feireisl and Novotný [15], where different multiscale problems (like, $\mathrm{Ma} = \mathrm{Fr}$ and $\sqrt{\mathrm{Ma}} = \mathrm{Fr}$) are addressed. These multiple scalings explain the stratification of fluid.

On the other hand, for a fixed Mach and Froude number, the problem coincides with the homogenization problem for fluid dynamics, which aims to describe the macroscopic behavior of microscopically heterogeneous systems. In general, the limiting behavior depends on the size and mutual distance of holes, that is, the relation between the radius of holes ε^α and mutual distance ε^β. For incompressible stationary Stokes and Navier-Stokes problems with periodically distributed holes, in his seminal works Allaire (in [3], [4], see also Tartar [30]) proved that in the case of "large" holes, that is, $\beta = 1$ and $1 \leq \alpha < 3$, the limit system is governed by the Darcy law, while for "tiny" holes, that is, $\beta = 1$ and $\alpha > 3$, the limit system remains the same as the original one. The critical case $\beta = 1$ and $\alpha = 3$ leads to Brinkmann's law. Similar results hold in the context of evolutionary Stokes and incompressible Navier-Stokes systems, as shown by Mikelić [25] and Feireisl, Namlyeyeva, and Nečasová [11]. All of the above results are in three dimensions.

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In the case of compressible fluids, the situation is more complex than its incompressible counterpart. For the barotropic Navier-Stokes system with Strouhal number proportional to ε^2 and the diameter of the holes proportional to their mutual distance (i.e., "large" holes with $\alpha = \beta = 1$), the problem was considered by Masmoudi [23] who deduced that the limit system is the porous medium equation with the nonlinear Darcy's law. For the heat-conducting fluid (Navier-Stokes-Fourier) system with the same α, Feireisl, Novotný, and Takahashi [12] achieved similar results. Recently, the case of tiny holes ($\alpha > 3$ and $\beta = 1$) has been studied in several papers, and the limit problem was identified as the same as in the perforated domain in three dimensions. Along with the mutual distance and diameters of the holes, the results also depend on the adiabatic exponent γ. For the steady compressible Navier-Stokes equations, Feireisl and Lu [14] considered $\gamma > 3$, while Diening, Feireisl, and Lu [10] considered $\gamma > 2$. Lu and Schwarzacher [22] studied the evolutionary compressible Navier-Stokes equations and proved that the presence of tiny holes is negligible for $\gamma > 6$, which was recently improved to $\gamma > 3$ by Oschmann and Pokorný [28]. Lu and Pokorný [21] proved that the size of holes is negligible in the context of the stationary Navier-Stokes-Fourier system, while the same result was achieved by Pokorný and Skříšovský [29] for the evolutionary case, considering a pressure of the type $p(\varrho, \vartheta) = \varrho^\gamma + \varrho\vartheta + \vartheta^4$ with $\beta = 1$, $\alpha > 7$, and $\gamma > 6$. Recently, Oschmann and Pokorný [28] improved the above results for the evolutionary compressible Navier-Stokes and Navier-Stokes-Fourier systems to $\beta = 1$, $\alpha > 3$, and $\gamma > 3$. For the Navier-Stokes system, the challenging situation with dimension two was considered by Nečasová and Pan [27] for $\gamma > 2$ and by Nečasová and Oschmann [26] for $\gamma > 1$. Recently, Bella and Oschmann [7] considered the case of randomly perforated domains with the random size of holes.

For the low Mach number limit of the compressible Navier-Stokes equation in a perforated domain, Höfer, Kowalczyk and Schwarzacher [17] recover Darcy's law as a limit of the system by considering $\beta = 1$ and $4m > 3(\gamma + 2)(\alpha - 1)$, where the adiabatic exponent $\gamma \geq 2$. They also consider the Strouhal number to be proportional to $\varepsilon^{3-\alpha}$ and the Froude number to be 1. Very recently, Bella, Feireisl, and Oschmann [6] proved that in the case of tiny holes ($\alpha \geq 3$ and $\beta = 1$) and under the hypothesis $\frac{2m}{\gamma} > \alpha$ with the adiabatic exponent $\gamma > \frac{3}{2}$, weak solutions of the compressible Navier-Stokes equation converge to a dissipative solution of the incompressible Navier-Stokes system for well-prepared initial data. Eventually, the use of the weak-strong uniqueness property ensures the convergence of weak solutions of the primitive system towards the strong solution for the target system, at least in the interval of existence of the strong solution.

To the best of the authors' knowledge, this is the first time that the low Mach number limit and the homogenization of the spatial domain have been performed simultaneously for the Navier-Stokes-Fourier system, enabling the consideration of general forms for pressure. Following the idea proposed in previous work [6], we consider the weak solution for the Navier-Stokes-Fourier system and take the limit as $\epsilon \to 0$ to obtain a dissipative solution of the Oberbeck-Boussinesq system for well-prepared initial data. Subsequently, we apply the weak-strong uniqueness property to ensure convergence to the strong solution of the target system, at least in the interval of existence od the latter. The two main ingredients we use are based on the restriction operator constructed by Diening et al. [10] and a suitable extension operator for state variables, mainly for temperature, as suggested by Lu and Pokorný [21] in Sobolev spaces, and later extended by Pokorny and Skříšovský[29] in time dependent Sobolev spaces.

1.1. **Primitive system.** Let us consider the scaled Navier–Stokes–Fourier system with small Mach number $\mathrm{Ma} = \varepsilon^m$ and Froude number $\mathrm{Fr} = \sqrt{\mathrm{Ma}} = \varepsilon^{m/2}$, with the integer $m \geq 1$ fixed; specifically, we will consider

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^{2m}} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon^m} \varrho \nabla_x G, \tag{1.2}$$

$$\partial_t\big(\varrho e(\varrho, \vartheta)\big) + \mathrm{div}_x\big(\varrho e(\varrho, \vartheta)\mathbf{u}\big) + \mathrm{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) = \varepsilon^{2m}\ \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}. \tag{1.3}$$

Here the unknown variables are the density $\varrho = \varrho(t, x)$, the velocity $\mathbf{u} = \mathbf{u}(t, x)$ and the absolute temperature $\vartheta = \vartheta(t, x)$ of the fluid, while the pressure $p = p(\varrho, \vartheta)$ and the internal energy $e = e(\varrho, \vartheta)$ are related to a third quantity, the entropy $s = s(\varrho, \vartheta)$, through Gibb's relation

$$\vartheta D s = D e + p D \left(\frac{1}{\varrho} \right). \tag{1.4}$$

Due to the aforementioned relation, equation (1.3) can be equivalently rewritten as

$$\partial_t\big(\varrho s(\varrho,\vartheta)\big)+\mathrm{div}_x\big(\varrho s(\varrho,\vartheta)\mathbf{u}\big)+\mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\right)$$
$$=\frac{1}{\vartheta}\left(\varepsilon^{2m}\,\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right).$$

We suppose that the fluid is Newtonian, meaning that the viscous stress tensor $\mathbb{S}=\mathbb{S}(\vartheta,\nabla_x\mathbf{u})$ is given by

$$\mathbb{S}(\vartheta,\nabla_x\mathbf{u})=\mu(\vartheta)\left(\nabla_x\mathbf{u}+\nabla_x^\top\mathbf{u}-\frac{2}{3}(\mathrm{div}_x\mathbf{u})\mathbb{I}\right)+\eta(\vartheta)(\mathrm{div}_x\mathbf{u})\mathbb{I}, \tag{1.5}$$

with the shear viscosity $\mu=\mu(\vartheta)$ and the bulk viscosity $\eta=\eta(\vartheta)$ coefficients depending on temperature. Similarly, we suppose that the heat flux $\mathbf{q}=\mathbf{q}(\vartheta,\nabla_x\vartheta)$ is determined by Fourier's law,

$$\mathbf{q}(\vartheta,\nabla_x\vartheta)=-\kappa(\vartheta)\nabla_x\vartheta, \tag{1.6}$$

with the heat conductivity coefficient $\kappa=\kappa(\vartheta)$. Finally, $G=G(x)$ is a given potential, usually identified with the gravitational one.

1.2. **Perforated domain.** We study the scaled Navier–Stokes–Fourier system (1.1)–(1.6) on $(0,T)\times\Omega_\varepsilon$, where the time $T>0$ can be chosen arbitrarily large while Ω_ε denotes a domain perforated with many tiny holes; specifically, we assume

$$\Omega_\varepsilon:=\Omega\setminus\bigcup_{n=1}^{N(\varepsilon)}\overline{B}_{\varepsilon,n}, \tag{1.7}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded $C^{2,\nu}$-domain and $\{B_{\varepsilon,n}\}_{n=1}^{N(\varepsilon)}$ are the balls

$$B_{\varepsilon,n}:=B(x_{\varepsilon,n},\varepsilon^\alpha)$$

centred at $x_{n,\varepsilon}$ and radius ε^α. Moreover, we suppose that the balls $\{B_{\varepsilon,n}\}_{n=1}^{N(\varepsilon)}$ have mutual distance ε^β, $1\leq\beta<\alpha$. More precisely, defining

$$D_{\varepsilon,n}:=B\left(x_{\varepsilon,n},\varepsilon^\alpha+\frac{1}{2}\varepsilon^\beta\right)$$

we require that the balls $D_{\varepsilon,n}$ are mutually disjoint. The latter condition gives an upper limit on the number of holes as

$$N(\varepsilon)\simeq\frac{3}{4\pi}|\Omega|\left(\varepsilon^\alpha+\frac{1}{2}\varepsilon^\beta\right)^{-3}\lesssim\varepsilon^{-3\beta}. \tag{1.8}$$

Note, however, that we do not assume any periodicity for the distribution of the holes.

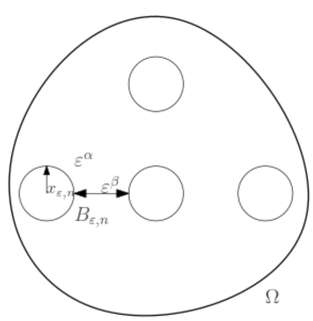

FIGURE 1. An example of perforated domain

We consider the homogeneous Dirichlet and Neumann boundary conditions for the velocity $\mathbf{u}$ and the temperature ϑ, respectively; specifically,

$$\mathbf{u}|_{\partial\Omega_\varepsilon}=0,\quad\nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}=0. \tag{1.9}$$

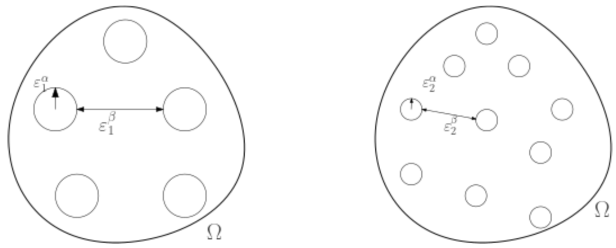

FIGURE 2. Perforated domains with $\varepsilon_1 > \varepsilon_2$

1.3. **Constitutive relations.** In order to motivate the existence of global-in-time weak solutions to system (1.1)–(1.6) some extra assumptions are necessary. Motivated by [15], we assume

$$p(\varrho,\vartheta) = p_{\rm m}(\varrho,\vartheta) + p_{\rm rad}(\vartheta), \quad \text{with} \quad p_{\rm m}(\varrho,\vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right),\ p_{\rm rad}(\vartheta) = \frac{a}{3}\vartheta^4, \tag{1.10}$$

$$e(\varrho,\vartheta) = e_{\rm m}(\varrho,\vartheta) + e_{\rm rad}(\varrho,\vartheta), \quad \text{with} \quad e_{\rm m}(\varrho,\vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right),\ e_{\rm rad}(\varrho,\vartheta) = \frac{a}{\varrho}\vartheta^4, \tag{1.11}$$

$$s(\varrho,\vartheta) = s_{\rm m}(\varrho,\vartheta) + s_{\rm rad}(\varrho,\vartheta), \quad \text{with} \quad s_{\rm m}(\varrho,\vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right),\ s_{\rm rad}(\varrho,\vartheta) = \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{1.12}$$

where $a>0$, $P \in C^1[0,\infty)\cap C^3(0,\infty)$ satisfies

$$P(0)=0, \quad P'(Z)>0 \text{ for } Z\geq 0, \quad 0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \leq c \text{ for } Z \geq 0, \tag{1.13}$$

and

$$\mathcal{S}'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}. \tag{1.14}$$

Consequently, the function $Z \mapsto P(Z)/Z^{\frac{5}{3}}$ is decreasing and we assume

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{1.15}$$

Furthermore, we suppose that the transport coefficients μ, η and κ are continuously differentiable functions of temperature ϑ satisfying

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \tag{1.16}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta), \tag{1.17}$$

$$0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3). \tag{1.18}$$

for all $\vartheta \geq 0$, with $\underline{\mu}, \overline{\mu}, \overline{\eta}, \underline{\kappa}, \overline{\kappa}$ positive constants. Finally, we suppose that the potential $G \in W^{1,\infty}(\Omega)$ has zero mean,

$$\int_\Omega G \ {\rm d}x = 0. \tag{1.19}$$

1.4. **Well-prepared initial data.** We suppose that

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} := \overline{\varrho} + \varepsilon^m \varrho^{(1)}_{0,\varepsilon}, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \quad \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} := \overline{\vartheta} + \varepsilon^m \vartheta^{(1)}_{0,\varepsilon}, \tag{1.20}$$

where $\varrho^{(1)}_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta^{(1)}_{0,\varepsilon}$ are measurable functions and $\overline{\varrho}, \overline{\vartheta}$ are positive constants. Moreover, in order to get uniform bounds on Ω and to guarantee the extension of the field equations to the whole domain, we suppose that $[\varrho^{(1)}_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta^{(1)}_{0,\varepsilon}]$ are extended by zero on $\Omega\setminus\Omega_\varepsilon$; more precisely, we denote

$$\widetilde{\varrho}^{(1)}_{0,\varepsilon} := \begin{cases} \varrho^{(1)}_{0,\varepsilon} & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega\setminus\Omega_\varepsilon, \end{cases} \quad \widetilde{\mathbf{u}}_{0,\varepsilon} := \begin{cases} \mathbf{u}_{0,\varepsilon} & \text{in } \Omega_\varepsilon, \\ \mathbf{0} & \text{in } \Omega\setminus\Omega_\varepsilon, \end{cases} \quad \widetilde{\vartheta}^{(1)}_{0,\varepsilon} := \begin{cases} \vartheta^{(1)}_{0,\varepsilon} & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega\setminus\Omega_\varepsilon, \end{cases} \tag{1.21}$$

and

$$[\widetilde{\varrho}_{0,\varepsilon}, \widetilde{\vartheta}_{0,\varepsilon}] := [\overline{\varrho}, \overline{\vartheta}] + \varepsilon^m [\widetilde{\varrho}^{(1)}_{0,\varepsilon}, \widetilde{\vartheta}^{(1)}_{0,\varepsilon}].$$

In addition, we suppose that

$$\int_\Omega \widetilde{\varrho}_{0,\varepsilon}^{(1)} \, \mathrm{d}x = \int_\Omega \widetilde{\vartheta}_{0,\varepsilon}^{(1)} \, \mathrm{d}x = 0 \quad \text{for all } \varepsilon > 0, \tag{1.22}$$

and

$$\widetilde{\varrho}_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \quad \text{weakly-}(*) \text{ in } L^\infty(\Omega) \text{ and a.e. in } \Omega, \tag{1.23}$$

$$\widetilde{\mathbf{u}}_{0,\varepsilon} \to \mathbf{u}_0 \quad \text{weakly-}(*) \text{ in } L^\infty(\Omega;\mathbb{R}^3) \text{ and a.e. in } \Omega, \tag{1.24}$$

$$\widetilde{\vartheta}_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \quad \text{weakly-}(*) \text{ in } L^\infty(\Omega) \text{ and a.e. in } \Omega; \tag{1.25}$$

moreover, in order to get the maximal regularity for the dissipative solution of the target system, we suppose

$$\vartheta_0^{(1)} \in W^{2-\frac{2}{p},p}(\Omega) \quad \text{with} \quad p = \frac{5}{4}. \tag{1.26}$$

Finally, we suppose that limiting initial data $\varrho_0^{(1)}, \vartheta_0^{(1)}$ are *well-prepared*, meaning that they satisfy the following relation:

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} = \overline{\varrho} G. \tag{1.27}$$

We restrict ourselves to the consideration of well-prepared data only. The problem is more interesting for ill-prepared data, where the presence of Rossby-acoustic waves play an important role in the analysis of singular limits. We wish to consider it in our future works.

1.5. **Target system.** Our goal is to show that the low Mach number asymptotic limit on a perforated domain leads to the *Oberbeck-Boussinesq approximation*

$$\mathrm{div}_x \mathbf{U} = 0, \tag{1.28}$$

$$\overline{\varrho}\left[\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla_x)\mathbf{U}\right] + \nabla_x \Pi - \mu(\overline{\vartheta})\Delta_x \mathbf{U} = -A\Theta\nabla_x G, \tag{1.29}$$

$$\overline{\varrho} c_p \left[\partial_t \Theta + \mathbf{U}\cdot\nabla_x \Theta\right] - \kappa(\overline{\vartheta})\Delta_x \Theta = \overline{\vartheta} A \nabla_x G \cdot \mathbf{U} \tag{1.30}$$

on the homogenized domain. Here, $\overline{\varrho}, \overline{\vartheta}$ are the positive constants introduced in Section 1.4 while the positive constant A is defined as

$$A := \overline{\varrho}\; a(\overline{\varrho},\overline{\vartheta}), \tag{1.31}$$

where a denotes the coefficient of thermal extension given by

$$a(\varrho,\vartheta) := \frac{1}{\varrho} \frac{\partial_\vartheta p}{\partial_\varrho p}(\varrho,\vartheta), \tag{1.32}$$

and c_p is the specific heat at constant pressure evaluated in $(\overline{\varrho},\overline{\vartheta})$,

$$c_p := \frac{\partial e}{\partial \vartheta}(\overline{\varrho},\overline{\vartheta}) + a(\overline{\varrho},\overline{\vartheta}) \frac{\overline{\vartheta}}{\overline{\varrho}} \frac{\partial p}{\partial \vartheta}(\overline{\varrho},\overline{\vartheta}). \tag{1.33}$$

Moreover, the functions $[\mathbf{U},\Theta]$ inherit the same boundary conditions of $[\mathbf{u},\theta^{(1)}]$; more precisely, we suppose

$$\mathbf{U}|_{\partial\Omega} = 0, \quad \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.34}$$

Remark 1.1. We point out that if the couple $[\mathbf{U},\Theta]$ is a strong solution of system (1.28)–(1.34), it is easy to check that $\mathrm{div}_x \nabla_x^\top \mathbf{U} = \nabla_x \mathrm{div}_x \mathbf{U} = 0$. Therefore, the viscosity term appearing in (1.29) can be equivalently written as

$$\mu(\overline{\vartheta})\mathrm{div}_x(\nabla_x \mathbf{U} + \nabla_x^\top \mathbf{U});$$

the latter will be preferred when introducing the concept of dissipative solution, cf. Definition 2.4 below.

1.6. **Notation.** To avoid confusion, we fix the notation that will be used throughout the paper.

Given two positive quantities A, B, we write

- $A \simeq B$ if there exist positive constants c_1, c_2 such that $c_1 A \leq B \leq c_2 A$;
- $A \lesssim B$ if there exists a positive constant c such that $A \leq cB$.

Moreover, given $Q \subseteq \mathbb{R}^N$, $N \geq 1$, an open set, X a Banach space and $M \geq 1$, we denote with

- $\mathcal{D}(Q;X) = C_c^\infty(Q;X)$ the space of functions belonging to $C^\infty(Q;X)$ and having compact support in Q;
- $\mathcal{D}'(Q;\mathbb{R}^M) = [C_c^\infty(Q;\mathbb{R}^M)]^*$ the space of distributions;
- $\mathcal{M}(Q;\mathbb{R}^M) = \big[\overline{C_c(Q;\mathbb{R}^M)}^{\|\cdot\|_\infty}\big]^*$ the space of vector-valued Radon measures. If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then $\mathcal{M}(\overline{\Omega}) = [C(\overline{\Omega})]^*$.
- $\mathcal{M}^+(Q)$ the space of positive Radon measures;
- $\mathcal{M}^+(Q;\mathbb{R}^{N\times N}_{\text{sym}})$ the space of tensor–valued Radon measures $\mathfrak{R}$ such that $\mathfrak{R} : (\xi \otimes \xi) \in \mathcal{M}^+(Q)$ for all $\xi \in \mathbb{R}^d$, and with components $\mathfrak{R}_{i,j} = \mathfrak{R}_{j,i}$;
- $L^p(Q;X)$, with $1 \leq p \leq \infty$, the Lebesgue space defined on Q and ranging in X;
- $W^{k,p}(Q;\mathbb{R}^M)$, with $1 \leq p \leq \infty$ and k a positive integer, the Sobolev space defined on Q;
- $W^{s,p}(Q;\mathbb{R}^M)$, with $1 \leq p \leq \infty$ and $s \in (0,1)$, the Sobolev-Slobodeckii space defined on Q.

Structure of the paper. The plan for the paper is as follows.

- In Section 2, we recall the definition of weak solution for the Navier-Stokes-Fourier system, cf. Definition 2.1, and provide the definition of dissipative solution for the Oberbeck-Boussinesq system, cf. Definition 2.4. Subsequenlty, we state our main result, cf. Theorem 2.5.
- Section 3 is devoted to the extension of the state variables defined on the perforated domain Ω_ε to the whole domain Ω, and to the derivation of all the necessary uniform estimates.
- In Section 4, we extend the validity of the field equations to the homogenized domain Ω.
- Section 5 is dedicated to the limit passage, leading to the concept of dissipative solution for the target system, cf. Proposition 5.2.
- In Section 6, we prove the weak-strong uniqueness principle for the Oberbeck-Boussinesq system, cf. Theorem 6.2.
- We postpone to the Appendix the construction of suitable extension and restriction operators, cf. Propositions A.2 and B.1 respectively.

2. Concepts of solution and main result

2.1. **Weak solution.** We start providing the definition of weak solution to the Navier–Stokes-Fourier system, whose existence was proved in [15, Theorem 3.1].

Definition 2.1 (Weak solution of the Navier–Stokes–Fourier system on perforated domains)**.** Let $\Omega \subset \mathbb{R}^3$ be a bounded $C^{2,\nu}$-domain. Moreover, let the thermodynamic variables p, e, s satisfy hypotheses (1.10)–(1.15) and the transport coefficients μ, η, κ satisfy conditions (1.16)–(1.18). For any fixed $\varepsilon > 0$, we say that the trio of functions

$$\begin{aligned}
&\varrho_\varepsilon \in C_{\text{weak}}([0,T]; L^{\frac{5}{3}}(\Omega_\varepsilon)),\\
&\mathbf{u}_\varepsilon \in L^2(0,T; W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)),\\
&\vartheta_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon)) \cap L^\infty(0,T; L^4(\Omega_\varepsilon))
\end{aligned}$$

is a *weak solution* of the scaled Navier–Stokes–Fourier system (1.1)–(1.6) in $(0,T) \times \Omega_\varepsilon$, where Ω_ε is the perforated domain given by (1.7), with the boundary conditions (1.9) and initial data (1.20) if the following holds.

(i) *Weak formulation of the continuity equation.* The integral identity

$$\left[\int_{\Omega_\varepsilon} \varrho_\varepsilon \varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_\varepsilon} [\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi]\ \mathrm{d}x\mathrm{d}t \tag{2.1}$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1_c([0,T] \times \overline{\Omega}_\varepsilon)$, with

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon} \quad \text{a.e. in } \Omega_\varepsilon.$$

(ii) *Weak formulation of the renormalized continuity equation.* For any function

$$b \in C^1[0,\infty),\ b' \in C_c[0,\infty)$$

the integral identity

$$\begin{aligned}\left[\int_{\Omega_\varepsilon} b(\varrho_\varepsilon)\varphi(t,\cdot)\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_{\Omega_\varepsilon} \left[b(\varrho_\varepsilon)\partial_t\varphi + b(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \varphi\right] \mathrm{d}x\mathrm{d}t \\ &+ \int_0^\tau \int_{\Omega_\varepsilon} \varphi\ \big(b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon\big)\mathrm{div}_x \mathbf{u}_\varepsilon\ \mathrm{d}x\mathrm{d}t\end{aligned} \tag{2.2}$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1_c([0,T] \times \overline{\Omega}_\varepsilon)$.

(iii) *Weak formulation of the momentum equation.* The integral identity

$$\begin{aligned}\left[\int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi}(t,\cdot)\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_{\Omega_\varepsilon} \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \left[(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon)\right] : \nabla_x \boldsymbol{\varphi}\right] \mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{\varepsilon^m} \int_0^\tau \int_{\Omega_\varepsilon} \left(\frac{1}{\varepsilon^m} p(\varrho_\varepsilon, \vartheta_\varepsilon)\mathrm{div}_x \boldsymbol{\varphi} + \varrho_\varepsilon \nabla_x G \cdot \boldsymbol{\varphi}\right) \mathrm{d}x\mathrm{d}t\end{aligned} \tag{2.3}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}_\varepsilon; \mathbb{R}^3)$, $\boldsymbol{\varphi}|_{\partial\Omega_\varepsilon} = 0$, with

$$(\varrho_\varepsilon \mathbf{u}_\varepsilon)(0,\cdot) = \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \quad \text{a.e. in } \Omega_\varepsilon.$$

(iv) *Weak formulation of the entropy equality.* There exists a non-negative measure

$$\mathfrak{S}_\varepsilon \in \mathcal{M}([0,T] \times \overline{\Omega}_\varepsilon),$$

such that the integral identity

$$\begin{aligned}&-\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\varphi(0,\cdot)\mathrm{d}x \\ &\quad = \int_0^T \int_{\Omega_\varepsilon} \left[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\big) - \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x \varphi\right] \mathrm{d}x\mathrm{d}t \\ &\quad + \varepsilon^{2m} \int_0^T \int_{\Omega_\varepsilon} \frac{\varphi}{\vartheta_\varepsilon} \left[\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^{2m}} \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} |\nabla_x \vartheta_\varepsilon|^2\right] \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\overline{\Omega}_\varepsilon} \varphi\ \mathrm{d}\mathfrak{S}_\varepsilon,\end{aligned} \tag{2.4}$$

holds for any $\varphi \in C^1_c([0,T) \times \overline{\Omega}_\varepsilon)$.

(v) *Energy inequality.* The integral identity

$$\begin{aligned}&\int_{\Omega_\varepsilon} \left(\frac{\varepsilon^{2m}}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon^m \varrho_\varepsilon G\right)(t,\cdot)\ \mathrm{d}x \\ &\quad = \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^{2m}}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon^m \varrho_{0,\varepsilon} G\right) \mathrm{d}x\end{aligned} \tag{2.5}$$

holds for a.e. $t \in (0,T)$.

Remark 2.2. Even if we are dealing with functions defined only almost everywhere on $(0,T)$, the left-hand sides of equations (2.1)–(2.3) are well-defined since the density ϱ_ε and the momentum $\mathbf{m}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$ are weakly continuous in time.

Remark 2.3. Defining the *Helmholtz function* $H_{\overline{\vartheta}} = H_{\overline{\vartheta}}(\varrho, \vartheta)$ as

$$H_{\overline{\vartheta}}(\varrho, \vartheta) := \varrho\big(e(\varrho, \vartheta) - \overline{\vartheta} s(\varrho, \vartheta)\big),$$

combining (1.22), (2.4) and (2.5), it is easy to show that the integral equality

$$\begin{aligned}&\int_{\Omega_\varepsilon}\left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^{2m}}\left(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon)-(\varrho_\varepsilon-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\right)-\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon^m}G\right](\tau,\cdot)\,\mathrm{d}x\\&+\int_0^\tau\int_{\Omega_\varepsilon}\frac{\overline{\vartheta}}{\vartheta_\varepsilon}\left(\mathbb{S}(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon):\nabla_x\mathbf{u}_\varepsilon+\frac{1}{\varepsilon^{2m}}\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}|\nabla_x\vartheta_\varepsilon|^2\right)\mathrm{d}x\mathrm{d}t+\frac{\overline{\vartheta}}{\varepsilon^{2m}}\mathfrak{S}_\varepsilon([0,\tau]\times\overline{\Omega}_\varepsilon)\\&=\int_{\Omega_\varepsilon}\left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2+\frac{1}{\varepsilon^{2m}}\left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon})-(\varrho_{0,\varepsilon}-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\right)-\frac{\varrho_{0,\varepsilon}-\overline{\varrho}}{\varepsilon^m}G\right]\mathrm{d}x\end{aligned}\tag{2.6}$$

holds for a.e. $\tau\in(0,T)$, where $\overline{\varrho},\overline{\vartheta}$ are the positive constants appearing in the definition of the initial density and temperature in (1.20).

2.2. **Dissipative solution.** Inspired by [2], we will refer to the concept of dissipative solutions, i.e. solutions that satisfy the target system in the weak sense but with extra defect terms appearing in the equations and in the energy inequality. The motivation of the following definition will be clarified in the proof of Proposition 5.2, when performing the passage to the limit.

Definition 2.4 (Dissipative solution of the Oberbeck-Boussinesq system)**.** Let $\Omega\subset\mathbb{R}^3$ be a bounded $C^{2,\nu}$-domain. We say that the couple of functions

$$\begin{aligned}\mathbf{u}&\in C_{\text{weak}}([0,T];L^p(\Omega;\mathbb{R}^3))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),\\ \vartheta^{(1)}&\in C([0,T];W^{2-\frac{2}{p},p}(\Omega))\cap W^{1,p}(0,T;L^p(\Omega))\cap L^p(0,T;W^{2,p}(\Omega)),\end{aligned}\qquad p=\frac{5}{4}\tag{2.7}$$

is a *dissipative solution* of the Oberbeck-Boussinesq system (1.28)–(1.34) in $[0,T]\times\Omega$ with initial data $[\mathbf{u}_0,\vartheta_0^{(1)}]$ if the following holds.

(i) *Incompressibility.* Equation (1.28) holds a.e. on $(0,T)\times\Omega$ for $\mathbf{U}=\mathbf{u}$.

(ii) *Incompressible Navier-Stokes system.* There exists a positive measure

$$\mathfrak{R}\in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{3\times3}_{\text{sym}}))$$

such that the integral identity

$$\begin{aligned}\overline{\varrho}\left[\int_\Omega\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\mathrm{d}x\right]_{t=0}^{t=\tau}&=\overline{\varrho}\int_0^\tau\int_\Omega[\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}-(\mathbf{u}\cdot\nabla_x)\mathbf{u}\cdot\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t\\&-\int_0^\tau\int_\Omega\left[\mu(\overline{\vartheta})(\nabla_x\mathbf{u}+\nabla_x^\top\mathbf{u}):\nabla_x\boldsymbol{\varphi}+A\vartheta^{(1)}\nabla_xG\cdot\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t\\&+\int_0^\tau\int_{\overline{\Omega}}\nabla_x\boldsymbol{\varphi}:\mathrm{d}\mathfrak{R}\,\mathrm{d}t\end{aligned}\tag{2.8}$$

holds for any $\tau\in[0,T]$ and any $\boldsymbol{\varphi}\in C_c^1([0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\boldsymbol{\varphi}|_{\partial\Omega}=0$ such that $\mathrm{div}_x\boldsymbol{\varphi}=0$, with

$$\mathbf{u}(0,\cdot)=\mathbf{u}_0\quad\text{a.e. in }\Omega.$$

(iii) *Heat equation with insulated boundary.* Equation (1.30) holds a.e. on $(0,T)\times\Omega$ for $\Theta=\vartheta^{(1)}$, $\mathbf{U}=\mathbf{u}$, with $\nabla_x\vartheta^{(1)}\cdot\mathbf{n}|_{\partial\Omega}=0$ in the sense of traces and $\vartheta^{(1)}(0,\cdot)=\vartheta_0^{(1)}$ a.e. in Ω.

(iv) *Energy inequality.* There exists a positive measure

$$\mathfrak{E}\in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$$

such that the integral inequality

$$\begin{aligned}&\int_\Omega\left(\frac{1}{2}\overline{\varrho}|\mathbf{u}|^2+\frac{c_p}{2}\frac{\overline{\varrho}}{\overline{\vartheta}}\left|\vartheta^{(1)}\right|^2\right)(\tau,\cdot)\mathrm{d}x+\int_\Omega\mathrm{d}\mathfrak{E}(\tau)\\&+\frac{\mu(\overline{\vartheta})}{2}\int_0^\tau\int_\Omega|\nabla_x\mathbf{u}+\nabla_x^\top\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t+\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\int_0^\tau\int_\Omega|\nabla_x\vartheta^{(1)}|^2\,\mathrm{d}x\mathrm{d}t\\&\le\int_\Omega\left(\frac{1}{2}\overline{\varrho}|\mathbf{u}_0|^2+\frac{c_p}{2}\frac{\overline{\varrho}}{\overline{\vartheta}}\left|\vartheta_0^{(1)}\right|^2\right)\mathrm{d}x\end{aligned}\tag{2.9}$$

holds for a.e. $\tau\in(0,T)$.

(v) *Compatibility condition.* There holds

$$\mathrm{Tr}[\mathfrak{R}] \simeq \mathfrak{E}. \tag{2.10}$$

2.3. **Main result.** Having collected all the necessary ingredients, we are now ready to state our main result.

Theorem 2.5. *Let*

- *the constants $\alpha, \beta \geq 1$ and the integer m be fixed such that*

$$m > \alpha > 3\beta; \tag{2.11}$$

- *$\Omega \subset \mathbb{R}^3$ be a bounded $C^{2,\nu}$-domain and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of perforated domains defined by* (1.7)*;*
- *the thermodynamic variables p, e, s satisfy hypotheses* (1.10)–(1.15)*;*
- *the transport coefficients μ, η, κ satisfy conditions* (1.16)–(1.18)*;*
- *the potential G have zero mean* (1.19)*;*
- *$\{[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}]\}_{\varepsilon>0}$ be a family of initial data satisfying conditions* (1.20)–(1.27).

Moreover, let

- *$\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon]\}_{\varepsilon>0}$ be the family of weak solutions to the scaled Navier–Stokes–Fourier system on the perforated domains, emanating from $\{[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}]\}_{\varepsilon>0}$ in the sense of Definition 2.1;*
- *$\{[\widetilde{\varrho}_\varepsilon, \widetilde{\mathbf{u}}_\varepsilon, \widetilde{\vartheta}_\varepsilon]\}_{\varepsilon>0}$ be the family of their extensions to the homogenized domain Ω, specified in Section 3.1 below.*

Then there exists a positive time T^* such that, passing to suitable subsequences as the case may be,

$$\widetilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{U} \quad \text{in } L^2(0,T^*; W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.12}$$

$$\frac{\widetilde{\vartheta}_\varepsilon - \overline{\vartheta}}{\varepsilon^m} \rightharpoonup \Theta \quad \text{in } L^2(0,T^*; W^{1,2}(\Omega)), \tag{2.13}$$

where $[\mathbf{U}, \Theta]$ is the strong solution to the Oberbeck-Boussinesq system emanating from $[\mathbf{U}_0, \Theta_0] = [\mathbf{u}_0, \vartheta_0^{(1)}]$, with $\mathbf{u}_0, \vartheta_0^{(1)}$ the weak limits appearing in (1.24), (1.25)*, respectively.*

Remark 2.6. The positive time T^* appearing in (2.12), (2.13) denotes the maximal time of existence of strong solution to the Oberbeck-Boussinesq system (1.28)–(1.34), cf. Theorem (6.1).

Theorem 2.5 is a direct consequence of two results: first, we will show that the extended weak solutions of the Navier–Stokes-Fourier system converge to the dissipative solution of the Oberbeck–Boussinesq system, cf. Proposition 5.2; secondly, by proving the weak–strong uniqueness principle, we are able to conclude that the dissipative solution must coincide with the strong solution of the target system, as long as the latter exists, cf. Theorem 6.2.

3. Preparation

3.1. **Extension of functions.** In order to get the uniform bounds on the homogenized domain Ω and the correspondent convergences necessary to pass to the limit, we first need to properly extend all the quantities appearing in the system.

From now on, we will denote

$$\varrho_\varepsilon^{(1)} := \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^m}, \quad \vartheta_\varepsilon^{(1)} := \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^m}, \quad \ell_\varepsilon^{(1)} := \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon^m}. \tag{3.1}$$

We can simply extend $\left[\varrho_\varepsilon^{(1)}, \mathbf{u}_\varepsilon\right]$ by zero on $\Omega \setminus \Omega_\varepsilon$; more precisely, we consider

$$\widetilde{\varrho}_\varepsilon^{(1)} := \begin{cases} \varrho_\varepsilon^{(1)} & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}, \quad \widetilde{\mathbf{u}}_\varepsilon := \begin{cases} \mathbf{u}_\varepsilon & \text{in } \Omega_\varepsilon \\ \mathbf{0} & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}. \tag{3.2}$$

The extension of $\vartheta_\varepsilon^{(1)}$ and $\ell_\varepsilon^{(1)}$ is more delicate due to the Neumann boundary condition for the temperature: the extension by zero may not preserve the $W^{1,2}$-regularity. However, we may use the spatial extensions E_ε and P_ε constructed in Lemma A.1 and Proposition A.2, respectively:

$$\widetilde{\vartheta}_\varepsilon^{(1)} := \begin{cases} \vartheta_\varepsilon^{(1)} & \text{in } \Omega_\varepsilon \\ E_\varepsilon(\vartheta_\varepsilon^{(1)}) & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}, \quad \widetilde{\ell}_\varepsilon^{(1)} := \begin{cases} \ell_\varepsilon^{(1)} & \text{in } \Omega_\varepsilon \\ P_\varepsilon(\ell_\varepsilon^{(1)}) & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}. \tag{3.3}$$

Accordingly, we consider the following extensions

$$[\widetilde{\varrho}_\varepsilon, \widetilde{\vartheta}_\varepsilon, \widetilde{\ell}_\varepsilon] := [\overline{\varrho}, \overline{\vartheta}, \log(\overline{\vartheta})] + \varepsilon^m [\widetilde{\varrho}_\varepsilon^{(1)}, \widetilde{\vartheta}_\varepsilon^{(1)}, \widetilde{\ell}_\varepsilon^{(1)}].$$

Next, we introduce analogous quantities to (3.1) for the thermodynamic functions,

$$p_\varepsilon^{(1)} := \frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m}, \quad e_\varepsilon^{(1)} := \frac{e(\varrho_\varepsilon, \vartheta_\varepsilon) - e(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m}, \quad s_\varepsilon^{(1)} := \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m},$$

and for the heat conductivity coefficient,

$$\kappa_\varepsilon^{(1)} := \frac{\kappa(\vartheta_\varepsilon) - \kappa(\overline{\vartheta})}{\varepsilon^m},$$

extending them by zero on $\Omega \setminus \Omega_\varepsilon$:

$$[\widetilde{p}_\varepsilon^{(1)}, \widetilde{e}_\varepsilon^{(1)}, \widetilde{s}_\varepsilon^{(1)}, \widetilde{\kappa}_\varepsilon^{(1)}] := \begin{cases} [p_\varepsilon^{(1)}, e_\varepsilon^{(1)}, s_\varepsilon^{(1)}, \kappa_\varepsilon^{(1)}] & \text{in } \Omega_\varepsilon \\ \mathbf{0} & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}.$$

Proceeding as before, we consider the following extensions

$$[\widetilde{p}_\varepsilon, \widetilde{e}_\varepsilon, \widetilde{s}_\varepsilon, \widetilde{\kappa}_\varepsilon] := [p(\overline{\varrho}, \overline{\vartheta}), e(\overline{\varrho}, \overline{\vartheta}), s(\overline{\varrho}, \overline{\vartheta}), \kappa(\overline{\vartheta})] + \varepsilon^m [\widetilde{p}_\varepsilon^{(1)}, \widetilde{e}_\varepsilon^{(1)}, \widetilde{s}_\varepsilon^{(1)}, \widetilde{\kappa}_\varepsilon^{(1)}]; \tag{3.4}$$

notice, in particular, that $\widetilde{\varrho}_\varepsilon = \overline{\varrho}$ and $[\widetilde{p}_\varepsilon, \widetilde{e}_\varepsilon, \widetilde{s}_\varepsilon, \widetilde{\kappa}_\varepsilon] = [p(\overline{\varrho}, \overline{\vartheta}), e(\overline{\varrho}, \overline{\vartheta}), s(\overline{\varrho}, \overline{\vartheta}), \kappa(\overline{\vartheta})]$ on $\Omega \setminus \Omega_\varepsilon$. Finally, we let the non-negative measure $\mathfrak{S}_\varepsilon$ to be zero in $\Omega \setminus \Omega_\varepsilon$

$$\widetilde{\mathfrak{S}}_\varepsilon := \begin{cases} \mathfrak{S}_\varepsilon & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}.$$

3.2. **Essential and residual parts.** Following [15], we introduce the set of essential values $\mathcal{O}_{\text{ess}} \subset (0,\infty)^2$ together with its residual counterpart $\mathcal{O}_{\text{res}} \subset (0,\infty)^2$ as

$$\mathcal{O}_{\text{ess}} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \;\middle|\; \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho}, \; \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\},$$
$$\mathcal{O}_{\text{res}} := (0,\infty)^2 \setminus \mathcal{O}_{\text{ess}},$$

while the essential set $\mathcal{M}_{\text{ess}} \subset (0,T) \times \Omega_\varepsilon$ and its residual counterpart $\mathcal{M}_{\text{res}} \subset (0,T) \times \Omega_\varepsilon$ are defined as

$$\mathcal{M}_{\text{ess}} := \left\{ (t,x) \in (0,T) \times \Omega_\varepsilon \;\middle|\; \big(\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x)\big) \in \mathcal{O}_{\text{ess}} \right\},$$
$$\mathcal{M}_{\text{res}} := \big((0,T) \times \Omega_\varepsilon\big) \setminus \mathcal{M}_{\text{ess}}.$$

We point out that $\mathcal{O}_{\text{ess}}, \mathcal{O}_{\text{res}}$ are fixed subsets of $(0,\infty)^2$, while $\mathcal{M}_{\text{ess}}, \mathcal{M}_{\text{res}}$ are measurable subsets of the time-space cylinder $(0,T) \times \Omega_\varepsilon$ depending on $\varrho_\varepsilon, \vartheta_\varepsilon$. Moreover, in view of the extensions introduced in section 3.1, along with $\mathcal{M}_{\text{ess}}, \mathcal{M}_{\text{res}}$ it makes sense to consider a third set $\mathcal{M}_{\text{holes}}$ defined as

$$\mathcal{M}_{\text{holes}} := (0,T) \times (\Omega \setminus \Omega_\varepsilon).$$

Denoting with $\widetilde{h}_\varepsilon$ the extension of any measurable function h_ε defined on $(0,T) \times \Omega_\varepsilon$, it makes sense to write

$$\begin{gathered} \widetilde{h}_\varepsilon = h_\varepsilon \mathbb{1}_{(0,T)\times\Omega_\varepsilon} + \widetilde{h}_\varepsilon \mathbb{1}_{(0,T)\times(\Omega\setminus\Omega_\varepsilon)} := [h_\varepsilon]_{\text{ess}} + [h_\varepsilon]_{\text{res}} + [h_\varepsilon]_{\text{holes}}, \\ [h_\varepsilon]_{\text{ess}} := h_\varepsilon \mathbb{1}_{\mathcal{M}_{\text{ess}}}, \quad [h_\varepsilon]_{\text{res}} := h_\varepsilon \mathbb{1}_{\mathcal{M}_{\text{res}}} = h_\varepsilon - [h_\varepsilon]_{\text{ess}}, \quad [h_\varepsilon]_{\text{holes}} := \widetilde{h}_\varepsilon \mathbb{1}_{\mathcal{M}_{\text{holes}}} = \widetilde{h}_\varepsilon - h_\varepsilon \mathbb{1}_{(0,T)\times\Omega_\varepsilon}. \end{gathered} \tag{3.5}$$

3.3. **Uniform bounds.** We are now ready to establish the uniform bounds on the whole domain Ω.

Lemma 3.1. *Under the hypotheses of Theorem 2.5, the following uniform bounds hold.*

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}_{\rm holes}(t)| \le c\varepsilon^{3(\alpha-\beta)}, \tag{3.6}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}_{\rm res}(t)| \le c\varepsilon^{2m}, \tag{3.7}$$

$$\widetilde{\mathfrak{S}}_\varepsilon([0,T]\times\overline{\Omega}) \le c\varepsilon^{2m}, \tag{3.8}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left[\left\| [\varrho_\varepsilon(t)]_{\rm res} \right\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega)} + \left\| [\vartheta_\varepsilon(t)]_{\rm res} \right\|^4_{L^4(\Omega)} \right] \le c\varepsilon^{2m}, \tag{3.9}$$

$$\left\| \left(\left[\varrho_\varepsilon^{(1)} \right]_{\rm ess}, \left[\vartheta_\varepsilon^{(1)} \right]_{\rm ess} \right) \right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^2))} \le c, \tag{3.10}$$

$$\| \sqrt{\widetilde{\varrho}_\varepsilon} \widetilde{\mathbf{u}}_\varepsilon \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \le c, \tag{3.11}$$

$$\| \widetilde{\varrho}_\varepsilon \widetilde{\mathbf{u}}_\varepsilon \|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \le c, \tag{3.12}$$

$$\| \widetilde{\mathbf{u}}_\varepsilon \|_{L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))} \le c, \tag{3.13}$$

$$\left\| \left(\widetilde{\vartheta}_\varepsilon^{(1)}, \widetilde{\ell}_\varepsilon^{(1)} \right) \right\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^2))} \le c, \tag{3.14}$$

$$\left\| \left[\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon^m} \right]_{\rm res} \right\|_{L^\infty(0,T;L^1(\Omega))} \le c\varepsilon^m, \tag{3.15}$$

$$\left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon^m} \right]_{\rm res} \right\|_{L^2(0,T;L^{\frac{30}{23}}(\Omega))} \le c, \tag{3.16}$$

$$\left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon^m} \right]_{\rm res} \widetilde{\mathbf{u}}_\varepsilon \right\|_{L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3))} \le c, \tag{3.17}$$

$$\left\| \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon^m} \right) \right]_{\rm res} \right\|_{L^{\frac{14}{13}}(0,T;L^{\frac{14}{13}}(\Omega;\mathbb{R}^3))} \le c. \tag{3.18}$$

Proof. The uniform bounds (3.7)–(3.13) and (3.15)–(3.18) are a direct consequence of [15, Proposition 5.1] since all the involved quantities vanish on $\Omega\setminus\Omega_\varepsilon$. Bound (3.6) follows from (1.8), while (3.14) can be deduced from [15, Proposition 5.1, equations (5.52), (5.53)], namely

$$\left\| \left(\vartheta_\varepsilon^{(1)}, \ell_\varepsilon^{(1)} \right) \right\|_{L^2(0,T;W^{1,2}(\Omega_\varepsilon;\mathbb{R}^2))} \le c, \tag{3.19}$$

combined with estimates (A.2) and (A.6). □

4. Field equations on the homogenized domain

Before passing to the limit, along with the extension of all the quantities appearing in the primitive system (1.1)–(1.3), it is also necessary to extend the validity of the integral identities of Definition 2.1 to arbitrary test functions defined on the whole domain Ω: the latter is the purpose of this section.

4.1. **Continuity equation.**

Lemma 4.1. *Under the hypotheses of Theorem 2.5, the integral identity*

$$\left[\int_\Omega \widetilde{\varrho}_\varepsilon \varphi(t,\cdot)\ {\rm d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\widetilde{\varrho}_\varepsilon \partial_t\varphi + \widetilde{\varrho}_\varepsilon \widetilde{\mathbf{u}}_\varepsilon \cdot \nabla_x \varphi]\ {\rm d}x{\rm d}t, \tag{4.1}$$

holds for any $\tau\in[0,T]$ and any $\varphi\in C^1_c([0,T]\times\overline{\Omega})$.

Proof. Let $\varphi \in C^1_c([0,T]\times\overline{\Omega})$; then $\varphi|_{\overline{\Omega}_\varepsilon} \in C^1_c([0,T]\times\overline{\Omega}_\varepsilon)$ can be used as test function in the weak formulation of the continuity equation (2.1), obtaining

$$\left[\int_{\Omega_\varepsilon} \varrho_\varepsilon \varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \overline{\varrho}\left[\int_{\Omega\setminus\Omega_\varepsilon} \varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_\varepsilon} [\varrho_\varepsilon \partial_t\varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon\cdot\nabla_x\varphi]\ \mathrm{d}x\mathrm{d}t + \overline{\varrho}\int_0^\tau\int_{\Omega\setminus\Omega_\varepsilon} \partial_t\varphi\ \mathrm{d}x\mathrm{d}t.$$

Now, it is enough to use the fact that $\widetilde{\varrho}_\varepsilon = \widetilde{\varrho}_{0,\varepsilon} = \overline{\varrho}$ and $\widetilde{\mathbf{u}}_\varepsilon = \mathbf{0}$ on $\Omega\setminus\Omega_\varepsilon$ to get (4.1). □

4.2. **Momentum equation.** The extension of the weak formulation of the balance of momentum (2.3) is delicate due to the fact that the latter holds for test functions that vanish on the boundary of the perforated domain Ω_ε. Therefore, given an arbitrary test function defined on Ω, we need to apply a suitable restriction operator

$$\mathcal{R}_\varepsilon : W^{1,p}_0(\Omega;\mathbb{R}^3) \to W^{1,p}_0(\Omega_\varepsilon;\mathbb{R}^3)$$

for any $p\in(1,\infty)$, preserving the "divergence-free" property. The construction of the operator $\mathcal{R}_\varepsilon$ is postponed to the Appendix B

Lemma 4.2. *Under the hypotheses of Theorem 2.5, the integral identity*

$$\begin{aligned}\left[\int_\Omega \widetilde{\varrho}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon\cdot\boldsymbol{\varphi}(t,\cdot)\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega [\widetilde{\varrho}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon\cdot\partial_t\boldsymbol{\varphi} + (\widetilde{\varrho}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon\otimes\widetilde{\mathbf{u}}_\varepsilon):\nabla_x\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau\int_\Omega \left(\mathbb{S}(\widetilde{\vartheta}_\varepsilon,\nabla_x\widetilde{\mathbf{u}}_\varepsilon):\nabla_x\boldsymbol{\varphi} - \widetilde{\varrho}^{(1)}_\varepsilon\nabla_x G\cdot\boldsymbol{\varphi}\right)\mathrm{d}x\mathrm{d}t + \langle \boldsymbol{r}_{1,\varepsilon},\boldsymbol{\varphi}\rangle_{\mathcal{M},C},\end{aligned} \tag{4.2}$$

holds for any $\tau\in[0,T]$ and any $\boldsymbol{\varphi}\in C^1_c([0,T]\times\overline{\Omega};\mathbb{R}^3)$, $\boldsymbol{\varphi}|_{\partial\Omega}=0$ such that $\mathrm{div}_x\boldsymbol{\varphi}=0$, where the residual measure $\boldsymbol{r}_{1,\varepsilon}\in\mathcal{M}([0,T]\times\overline{\Omega};\mathbb{R}^3)$ satisfies

$$|\langle \boldsymbol{r}_{1,\varepsilon},\boldsymbol{\varphi}\rangle_{\mathcal{M},C}| \lesssim \varepsilon^{\gamma_1}\|\boldsymbol{\varphi}\|_{W^{1,\infty}_0((0,T)\times\Omega;\mathbb{R}^3)}, \tag{4.3}$$

with γ_1 the positive exponent defined in (4.11) *below.*

Proof. First, let $\psi\in C^\infty_c(0,T)$ and $\boldsymbol{\varphi}\in C^\infty_c(\Omega;\mathbb{R}^3)$ be such that $\mathrm{div}_x\boldsymbol{\varphi}=0$; then, we can use $\psi\,\mathcal{R}_\varepsilon(\boldsymbol{\varphi})$ as test function in the weak formulation of the balance of momentum, where $\mathcal{R}_\varepsilon$ is the linear operator constructed in Porposition B.1. Notice in particular that $\mathrm{div}_x[\psi\,\mathcal{R}_\varepsilon(\boldsymbol{\varphi})]=0$. Therefore, using the fact that

$$\int_\Omega \nabla_x G\cdot\boldsymbol{\varphi}\ \mathrm{d}x = -\int_\Omega G\ \mathrm{div}_x\boldsymbol{\varphi}\ \mathrm{d}x = 0,$$

we obtain

$$\int_0^T \psi'\int_\Omega \widetilde{\varrho}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^T\psi\int_\Omega\left[(\widetilde{\varrho}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon\otimes\widetilde{\mathbf{u}}_\varepsilon):\nabla_x\boldsymbol{\varphi} - \mathbb{S}(\widetilde{\vartheta}_\varepsilon,\nabla_x\widetilde{\mathbf{u}}_\varepsilon):\nabla_x\boldsymbol{\varphi} + \widetilde{\varrho}^{(1)}_\varepsilon\nabla_x G\cdot\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t = \sum_{k=1}^4 I_{\varepsilon,k}$$

where, from the decomposition $\varrho_\varepsilon = \overline{\varrho} + \varepsilon^m \varrho_\varepsilon^{(1)}$, we have

$$
\begin{aligned}
I_{\varepsilon,1} &:= \int_0^T \psi' \int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t \\
&= \overline{\varrho} \int_0^T \psi' \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t + \varepsilon^m \int_0^T \psi' \int_{\Omega_\varepsilon} \varrho_\varepsilon^{(1)} \mathbf{u}_\varepsilon \cdot \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t = I_{\varepsilon,1}^{(1)} + I_{\varepsilon,1}^{(2)} \\
I_{\varepsilon,2} &:= \int_0^T \psi \int_{\Omega_\varepsilon} (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t \\
&= \overline{\varrho} \int_0^T \psi \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t + \varepsilon^m \int_0^T \psi \int_{\Omega_\varepsilon} (\varrho_\varepsilon^{(1)} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t \\
&= I_{\varepsilon,2}^{(1)} + I_{\varepsilon,2}^{(2)}, \\
I_{\varepsilon,3} &:= -\int_0^T \psi \int_{\Omega_\varepsilon} \left[\mu(\vartheta_\varepsilon) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon - \frac{2}{d} (\mathrm{div}_x \mathbf{u}_\varepsilon) \mathbb{I}\right) + \eta(\vartheta_\varepsilon)(\mathrm{div}_x \mathbf{u}_\varepsilon) \mathbb{I}\right] : \nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t, \\
I_{\varepsilon,4} &:= \int_0^T \psi \int_{\Omega_\varepsilon} \varrho_\varepsilon^{(1)} \nabla_x G \cdot \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big] \, \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Moreover, using the fact that

$$
\begin{aligned}
\mu(\vartheta_\varepsilon) &\le \overline{\mu} \left(1 + \overline{\vartheta}\right) + \overline{\mu} \varepsilon^m \vartheta_\varepsilon^{(1)}, \\
\eta(\vartheta_\varepsilon) &\le \overline{\eta} \left(1 + \overline{\vartheta}\right) + \overline{\eta} \varepsilon^m \vartheta_\varepsilon^{(1)},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|I_{\varepsilon,3}| &\le \left(1 + \overline{\vartheta}\right) \int_0^T |\psi| \int_{\Omega_\varepsilon} \left[\overline{\mu} \left|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon - \frac{2}{d} (\mathrm{div}_x \mathbf{u}_\varepsilon) \mathbb{I}\right| + \overline{\eta} |\mathrm{div}_x \mathbf{u}_\varepsilon|\right] \left|\nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big]\right| \, \mathrm{d}x\mathrm{d}t, \\
&\quad + \varepsilon^m \int_0^T |\psi| \int_{\Omega_\varepsilon} \left[\overline{\mu} |\vartheta_\varepsilon^{(1)}| \left|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon - \frac{2}{d} (\mathrm{div}_x \mathbf{u}_\varepsilon) \mathbb{I}\right| + \overline{\eta} |\vartheta_\varepsilon^{(1)}| |\mathrm{div}_x \mathbf{u}_\varepsilon|\right] \left|\nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big]\right| \, \mathrm{d}x\mathrm{d}t \\
&\quad = I_{\varepsilon,3}^{(1)} + I_{\varepsilon,3}^{(2)},
\end{aligned}
$$

We can now use the uniform bounds established in Lemma 3.1 and estimates (B.18), (B.19) to deduce

$$
|I_{\varepsilon,1}^{(1)}| \lesssim \|\psi'\|_{L^2(0,T)} \|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^6(\Omega))} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)} \lesssim \left[\varepsilon^{3\beta} + \varepsilon^{\frac{3}{2}(\alpha-\beta)}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,\frac{6}{5}}(\Omega)}; \tag{4.4}
$$

$$
\begin{aligned}
|I_{\varepsilon,1}^{(2)}| &\lesssim \varepsilon^m \|\psi'\|_{L^2(0,T)} \|\widetilde{\varrho}_\varepsilon^{(1)} \widetilde{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^{\frac{30}{23}}(\Omega))} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^{\frac{30}{7}}(\Omega_\varepsilon)} \\
&\lesssim \left[\varepsilon^{m+\frac{2}{5}\beta} + \varepsilon^{m+\frac{3}{10}(\alpha-3\beta)}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,\frac{30}{13}}(\Omega)};
\end{aligned} \tag{4.5}
$$

$$
|I_{\varepsilon,2}^{(1)}| \lesssim \|\psi\|_{L^\infty(0,T)} \|\widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon\|_{L^1(0,T;L^3(\Omega))} \left\|\nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big]\right\|_{L^{\frac{3}{2}}(\Omega_\varepsilon)} \lesssim \left[\varepsilon^{\frac{1}{2}\beta} + \varepsilon^{\frac{1}{2}(\alpha-2\beta)}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,2}(\Omega)}; \tag{4.6}
$$

$$
\begin{aligned}
|I_{\varepsilon,2}^{(2)}| &\lesssim \varepsilon^m \|\psi\|_{L^\infty(0,T)} \|\widetilde{\varrho}_\varepsilon^{(1)} \widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon\|_{L^1(0,T;L^{\frac{15}{14}}(\Omega))} \left\|\nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big]\right\|_{L^{15}(\Omega_\varepsilon)} \\
&\lesssim \left[\varepsilon^{m+\frac{1}{10}\beta} + \varepsilon^{m-\frac{9}{10}\alpha}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,30}(\Omega)};
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
|I_{\varepsilon,3}^{(1)}| &\lesssim \|\psi\|_{L^2(0,T)} \|\nabla_x \widetilde{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \left\|\nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big]\right\|_{L^2(\Omega_\varepsilon)} \\
&\lesssim \left[\varepsilon^{\frac{3}{2}\frac{\alpha-3\beta}{5\alpha-9\beta}\beta} + \varepsilon^{\frac{1}{2}\frac{(2\alpha-3\beta)(\alpha-3\beta)}{5\alpha-9\beta}}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,p}(\Omega)} \quad \text{with} \quad p := 2 + \frac{\alpha-3\beta}{2\alpha-3\beta};
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
|I_{\varepsilon,3}^{(2)}| &\lesssim \varepsilon^m \|\psi\|_{L^\infty(0,T)} \|\widetilde{\vartheta}_\varepsilon^{(1)} \nabla_x \widetilde{\mathbf{u}}_\varepsilon\|_{L^1(0,T;L^{\frac{3}{2}}(\Omega))} \left\|\nabla_x \big[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\big]\right\|_{L^3(\Omega_\varepsilon)} \\
&\lesssim \left[\varepsilon^{m+\frac{1}{2}\beta} + \varepsilon^{m-\frac{1}{2}\alpha+\frac{2}{3}\beta}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,6}(\Omega)};
\end{aligned} \tag{4.9}
$$

$$
|I_{\varepsilon,4}| \lesssim \|\psi\|_{L^1(0,T)} \|\widetilde{\varrho}_\varepsilon^{(1)}\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^{\frac{5}{2}}(\Omega_\varepsilon)} \lesssim \left[\varepsilon^{\frac{3}{5}\beta} + \varepsilon^{\frac{3}{5}\alpha-\beta}\right] \|\boldsymbol{\varphi}\|_{W_0^{1,\frac{15}{8}}(\Omega)}. \tag{4.10}
$$

Due to hypothesis (2.11), the exponent for ε is positive in (4.4)-(4.10). Therefore, condition (4.3) is satisfied choosing

$$\gamma_1 := \min\left\{m - \frac{9}{10}\alpha,\ \frac{3}{2}\frac{\alpha - 3\beta}{5\alpha - 9\beta}\beta\right\}. \tag{4.11}$$

To conclude the proof, it is now enough to use a density argument. □

4.3. **Entropy equality.**

Lemma 4.3. *Under the hypotheses of Theorem 2.5, the integral equality*

$$\begin{aligned}
&-\int_\Omega \widetilde{\varrho}_{0,\varepsilon}\frac{s(\widetilde{\varrho}_{0,\varepsilon}, \widetilde{\vartheta}_{0,\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m}\varphi(0,\cdot)\mathrm{d}x \\
&\quad = \int_0^T \int_\Omega \left[\widetilde{\varrho}_\varepsilon \widetilde{s}_\varepsilon^{(1)}\big(\partial_t\varphi + \widetilde{\mathbf{u}}_\varepsilon\cdot\nabla_x\varphi\big) - \widetilde{\kappa}_\varepsilon\nabla_x\widetilde{\ell}_\varepsilon^{(1)}\cdot\nabla_x\varphi\right]\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon^m\int_0^T\int_\Omega \varphi\left(\widetilde{\vartheta}_\varepsilon^{-1}\mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x\widetilde{\mathbf{u}}_\varepsilon):\nabla_x\widetilde{\mathbf{u}}_\varepsilon + \widetilde{\kappa}_\varepsilon|\nabla_x\widetilde{\ell}_\varepsilon^{(1)}|^2\right)\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon^m}\int_0^T\int_{\overline{\Omega}}\varphi\ \mathrm{d}\widetilde{\mathfrak{S}}_\varepsilon
\end{aligned} \tag{4.12}$$

holds for any $\varphi \in C^1_c([0,T)\times\overline{\Omega})$.

Proof. Let $\varphi \in C^1_c([0,T)\times\overline{\Omega})$; then $\varphi|_{\overline{\Omega}_\varepsilon} \in C^1_c([0,T)\times\overline{\Omega}_\varepsilon)$ can be used as test function in the weak formulation of the entropy inequality (2.4), obtaining

$$\begin{aligned}
&-\int_\Omega \widetilde{\varrho}_{0,\varepsilon}\frac{s(\widetilde{\varrho}_{0,\varepsilon}, \widetilde{\vartheta}_{0,\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m}\varphi(0,\cdot)\mathrm{d}x \\
&\quad = \int_0^T \int_\Omega \left[\widetilde{\varrho}_\varepsilon \widetilde{s}_\varepsilon^{(1)}\big(\partial_t\varphi + \widetilde{\mathbf{u}}_\varepsilon\cdot\nabla_x\varphi\big) - \widetilde{\kappa}_\varepsilon\nabla_x\widetilde{\ell}_\varepsilon^{(1)}\cdot\nabla_x\varphi\right]\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon^m\int_0^T\int_\Omega \varphi\left(\widetilde{\vartheta}_\varepsilon^{-1}\mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x\widetilde{\mathbf{u}}_\varepsilon):\nabla_x\widetilde{\mathbf{u}}_\varepsilon + \widetilde{\kappa}_\varepsilon|\nabla_x\widetilde{\ell}_\varepsilon^{(1)}|^2\right)\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon^m}\int_0^T\int_{\overline{\Omega}}\varphi\ \mathrm{d}\widetilde{\mathfrak{S}}_\varepsilon \\
&\quad + \kappa(\overline{\vartheta})\int_0^T\int_{\Omega\setminus\Omega_\varepsilon}\nabla_x P_\varepsilon(\ell_\varepsilon^{(1)})\cdot\nabla_x\left[\varphi + \varepsilon^m P_\varepsilon(\ell_\varepsilon^{(1)})\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Using the fact that

$$\nabla_x P_\varepsilon(\ell_\varepsilon^{(1)})\cdot\mathbf{n} = \nabla_x\ell_\varepsilon^{(1)}\cdot\mathbf{n} = \frac{1}{\varepsilon^m\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

and $\Delta_x P_\varepsilon(\ell_\varepsilon^{(1)}) = 0$ in $\Omega\setminus\Omega_\varepsilon$, we obtain that the latter integral vanishes and therefore (4.12) holds. □

4.4. **Boussinesq relation.** Along with (2.1)-(2.4), when letting ε go to zero, we need to consider an additional integral identity known as *Boussinesq relation*, obtained by multiplying (2.3) by ε^m. Similarly to Section 4.2, we have to solve the problem of vanishing test function on the boundary of Ω_ε. Since in this context the L^∞-norms of the gradient of the test functions will be necessary, we cannot use the restriction $\mathcal{R}_\varepsilon$ constructed in the Appendix B. Instead, we multiply our arbitrary test function defined on the whole Ω by a suitable smooth and compactly supported function on Ω_ε, cf. equation (4.18).

Lemma 4.4. *Under the hypotheses of Theorem 2.5, the integral equality*

$$\int_0^T\int_\Omega\left(\widetilde{p}_\varepsilon^{(1)}\mathrm{div}_x\boldsymbol{\varphi} + \widetilde{\varrho}_\varepsilon\nabla_x G\cdot\boldsymbol{\varphi}\right)\ \mathrm{d}x\mathrm{d}t = \langle\boldsymbol{r}_{2,\varepsilon}, \boldsymbol{\varphi}\rangle_{\mathcal{D}',\mathcal{D}} \tag{4.13}$$

holds for any $\boldsymbol{\varphi}\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$*, where the residual distribution* $\boldsymbol{r}_{2,\varepsilon}\in\mathcal{D}'((0,T)\times\Omega;\mathbb{R}^3)$ *satisfies*

$$|\langle\boldsymbol{r}_{2,\varepsilon}, \boldsymbol{\varphi}\rangle_{\mathcal{D}',\mathcal{D}}| \lesssim \varepsilon^{\gamma_2}\|\boldsymbol{\varphi}\|_{W_0^{1,\infty}((0,T)\times\Omega;\mathbb{R}^3)}, \tag{4.14}$$

with γ_2 *the positive exponent defined in* (4.28) *below.*

Proof. We will first construct a proper cut-off function $\phi_{\varepsilon,n}$. To this end, let us consider a constant $\delta_0 > 1$ such that $B(x_{\varepsilon,n}, \delta_0\varepsilon^\alpha)\subset D_{\varepsilon,n}$, i.e. $(\delta_0 - 1)\varepsilon^\alpha < \frac{1}{2}\varepsilon^\beta$. We may apply Lemma B.2 choosing $x_0 = x_{\varepsilon,n}$, $r_1 = 1$, $r_2 = \delta_0$ and $a = \varepsilon^\alpha$ to get the existence of

$$\phi_{\varepsilon,n}\in C_c^\infty(B(x_{\varepsilon,n}, \delta_0\varepsilon^\alpha)), \quad 0\le\phi_{\varepsilon,n}\le 1, \quad \phi|_{\overline{B_{\varepsilon,n}}} = 1, \tag{4.15}$$

such that, for any $1 \le p \le \infty$,

$$\|\phi_{\varepsilon,n}\|_{L^p(\mathbb{R}^3)} \le c(\delta_0)\ \varepsilon^{\frac{3}{p}\alpha}, \tag{4.16}$$

$$\|\nabla_x \phi_{\varepsilon,n}\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} \le c(\delta_0)\ \varepsilon^{\left(\frac{3}{p}-1\right)\alpha}. \tag{4.17}$$

Let us now consider

$$g_\varepsilon(x) := 1 - \sum_{n=1}^{N(\varepsilon)} \phi_{\varepsilon,n}(x); \tag{4.18}$$

clearly $g_\varepsilon \in C_c^\infty(\Omega_\varepsilon)$, $0 \le g_\varepsilon \le 1$, and from (1.8) and estimates (4.16), (4.17), we can deduce that

$$\|1-g_\varepsilon\|_{L^p(\Omega)} \le N(\varepsilon)^{\frac{1}{p}} \|\phi_{\varepsilon,n}\|_{L^p(\mathbb{R}^3)} \le c(\delta_0)\varepsilon^{\frac{3(\alpha-\beta)}{p}}, \tag{4.19}$$

$$\|\nabla_x g_\varepsilon\|_{L^p(\Omega;\mathbb{R}^3)} \le N(\varepsilon)^{\frac{1}{p}} \|\nabla_x \phi_{\varepsilon,n}\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} \le c(\delta_0)\varepsilon^{\frac{3(\alpha-\beta)}{p}-\alpha}. \tag{4.20}$$

Let $\boldsymbol{\varphi} \in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$. Then we can multiply (2.3) by ε^m and use

$$\boldsymbol{\varphi}_\varepsilon(t,x) := g_\varepsilon(x)\boldsymbol{\varphi}(t,x)$$

as test function in the resulting integral identity, obtaining

$$\int_0^T \int_\Omega \left(\widetilde{p}_\varepsilon^{(1)} \mathrm{div}_x \boldsymbol{\varphi} + \widetilde{\varrho}_\varepsilon \nabla_x G \cdot \boldsymbol{\varphi} \right)\ \mathrm{d}x\mathrm{d}t = \sum_{k=1}^6 I_{\varepsilon,k},$$

where

$$\begin{aligned}
I_{\varepsilon,1} &:= -\varepsilon^m \int_0^T \int_{\Omega_\varepsilon} g_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi}\ \mathrm{d}t\mathrm{d}x, \\
I_{\varepsilon,2} &:= -\varepsilon^m \int_0^T \int_{\Omega_\varepsilon} (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : (\nabla_x g_\varepsilon \otimes \boldsymbol{\varphi} + g_\varepsilon \nabla_x \boldsymbol{\varphi})\ \mathrm{d}t\mathrm{d}x \\
I_{\varepsilon,3} &:= \varepsilon^m \int_0^T \int_{\Omega_\varepsilon} \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : (\nabla_x g_\varepsilon \otimes \boldsymbol{\varphi} + g_\varepsilon \nabla_x \boldsymbol{\varphi})\ \mathrm{d}t\mathrm{d}x, \\
I_{\varepsilon,4} &:= \int_0^T \int_{\Omega_\varepsilon} p_\varepsilon^{(1)}\ [-\nabla_x g_\varepsilon \cdot \boldsymbol{\varphi} + (1-g_\varepsilon)\mathrm{div}_x \boldsymbol{\varphi}]\ \mathrm{d}t\mathrm{d}x \\
&= \int_0^T \int_{\Omega_\varepsilon} [p_\varepsilon^{(1)}]_{\mathrm{ess}}\ [-\nabla_x g_\varepsilon \cdot \boldsymbol{\varphi} + (1-g_\varepsilon)\mathrm{div}_x \boldsymbol{\varphi}]\ \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\Omega_\varepsilon} [p_\varepsilon^{(1)}]_{\mathrm{res}}\ [-\nabla_x g_\varepsilon \cdot \boldsymbol{\varphi} + (1-g_\varepsilon)\mathrm{div}_x \boldsymbol{\varphi}]\ \mathrm{d}t\mathrm{d}x = I_{\varepsilon,4}^{(1)} + I_{\varepsilon,4}^{(2)}, \\
I_{\varepsilon,5} &:= \int_0^T \int_{\Omega_\varepsilon} (1-g_\varepsilon)\varrho_\varepsilon \nabla_x G \cdot \boldsymbol{\varphi}\ \mathrm{d}t\mathrm{d}x, \\
I_{\varepsilon,6} &:= \overline{\varrho} \int_0^T \int_{\Omega \setminus \Omega_\varepsilon} \nabla_x G \cdot \boldsymbol{\varphi}\ \mathrm{d}t\mathrm{d}x.
\end{aligned}$$

We can now use the uniform bounds established in Lemma 3.1 and estimates (4.19), (4.20) to get the following bounds.

$$|I_{\varepsilon,1}| \lesssim \varepsilon^m \|\widetilde{\varrho}_\varepsilon \widetilde{\mathbf{u}}_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega))} \|g_\varepsilon\|_{L^5(\Omega)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} \lesssim \varepsilon^m \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}; \tag{4.21}$$

$$|I_{\varepsilon,2}| \lesssim \varepsilon^m \|\widetilde{\varrho}_\varepsilon \widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon\|_{L^1(0,T;L^{\frac{15}{14}}(\Omega))} \|\nabla_x g_\varepsilon\|_{L^{15}(\Omega)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} \lesssim \varepsilon^{m-\frac{1}{5}(4\alpha+\beta)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}; \tag{4.22}$$

$$|I_{\varepsilon,3}| \lesssim \varepsilon^m \|\mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x \widetilde{\mathbf{u}}_\varepsilon)\|_{L^1(0,T;L^{\frac{3}{2}}(\Omega))} \|\nabla_x g_\varepsilon\|_{L^3(\Omega)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} \lesssim \varepsilon^{m-\beta} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}; \tag{4.23}$$

$$|I_{\varepsilon,4}^{(1)}| \lesssim \left\| [p_\varepsilon^{(1)}]_{\rm ess} \right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \left(\|\nabla_x g_\varepsilon\|_{L^2(\Omega)} + \|1-g_\varepsilon\|_{L^2(\Omega)} \right) \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} \lesssim \varepsilon^{\frac{1}{2}(\alpha-3\beta)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}; \tag{4.24}$$

$$|I_{\varepsilon,4}^{(2)}| \lesssim \left\| [p_\varepsilon^{(1)}]_{\rm res} \right\|_{L^\infty(0,T;L^1(\Omega_\varepsilon))} \left(\|\nabla_x g_\varepsilon\|_{L^\infty(\Omega)} + \|1-g_\varepsilon\|_{L^\infty(\Omega)} \right) \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} \lesssim \varepsilon^{m-\alpha} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}; \tag{4.25}$$

$$|I_{\varepsilon,5}| \lesssim \|G\|_{W^{1,\infty}(\Omega)} \|\widetilde{\varrho}_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \|1-g_\varepsilon\|_{L^{\frac{5}{2}}(\Omega)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} \lesssim \varepsilon^{\frac{6}{5}(\alpha-\beta)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}; \tag{4.26}$$

$$|I_{\varepsilon,6}| \lesssim \|G\|_{W^{1,\infty}(\Omega)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}} |\Omega \setminus \Omega_\varepsilon| \lesssim \varepsilon^{3(\alpha-\beta)} \|\boldsymbol{\varphi}\|_{W_0^{1,\infty}}. \tag{4.27}$$

Due to hypothesis (2.11), the exponent for ε is positive in (4.21)-(4.27). Therefore, condition (4.14) is satisfied choosing

$$\gamma_2 := \min \left\{ m-\alpha, \ \frac{\alpha-3\beta}{2} \right\}. \tag{4.28}$$

□

4.5. **Ballistic energy inequality.**

Lemma 4.5. *Under the hypotheses of Theorem 2.5 and defining*

$$\widetilde{H}_{\overline{\vartheta},\varepsilon} := \widetilde{\varrho}_\varepsilon \left(\widetilde{e}_\varepsilon - \overline{\vartheta} \ \widetilde{s}_\varepsilon \right),$$

the integral inequality

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{2} \widetilde{\varrho}_\varepsilon |\widetilde{\mathbf{u}}_\varepsilon|^2 + \frac{1}{\varepsilon^{2m}} \left(\widetilde{H}_{\overline{\vartheta},\varepsilon} - (\widetilde{\varrho}_\varepsilon - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) - \widetilde{\varrho}_\varepsilon^{(1)} G \right] (\tau, \cdot) \ {\rm d}x \\
&\quad + \int_0^\tau \int_\Omega \overline{\vartheta} \left(\widetilde{\vartheta}_\varepsilon^{-1} \mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x \widetilde{\mathbf{u}}_\varepsilon) : \nabla_x \widetilde{\mathbf{u}}_\varepsilon + \widetilde{\kappa}_\varepsilon \left| \nabla_x \widetilde{\ell}_\varepsilon^{(1)} \right|^2 \right) {\rm d}x{\rm d}t \\
&\leq \int_\Omega \left[\frac{1}{2} \widetilde{\varrho}_{0,\varepsilon} |\widetilde{\mathbf{u}}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\widetilde{\varrho}_{0,\varepsilon}, \widetilde{\vartheta}_{0,\varepsilon}) - (\widetilde{\varrho}_{0,\varepsilon} - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) - \widetilde{\varrho}_{0,\varepsilon}^{(1)} G \right] {\rm d}x,
\end{aligned} \tag{4.29}$$

holds for a.e. $\tau \in (0,T)$.

Proof. From (2.6) and the extensions define in Section 3.1, it is easy to deduce that

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{2} \widetilde{\varrho}_\varepsilon |\widetilde{\mathbf{u}}_\varepsilon|^2 + \frac{1}{\varepsilon^{2m}} \left(\widetilde{H}_{\overline{\vartheta},\varepsilon} - (\widetilde{\varrho}_\varepsilon - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) - \widetilde{\varrho}_\varepsilon^{(1)} G \right] (\tau, \cdot) \ {\rm d}x \\
&+ \int_0^\tau \int_\Omega \overline{\vartheta} \left(\widetilde{\vartheta}_\varepsilon^{-1} \mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x \widetilde{\mathbf{u}}_\varepsilon) : \nabla_x \widetilde{\mathbf{u}}_\varepsilon + \widetilde{\kappa}_\varepsilon \left| \nabla_x \widetilde{\ell}_\varepsilon^{(1)} \right|^2 \right) {\rm d}x{\rm d}t - \overline{\vartheta} \kappa(\overline{\vartheta}) \int_0^\tau \int_{\Omega \setminus \Omega_\varepsilon} |\nabla_x P_\varepsilon(\ell_\varepsilon^{(1)})|^2 \ {\rm d}x{\rm d}t \\
&\leq \int_\Omega \left[\frac{1}{2} \widetilde{\varrho}_{0,\varepsilon} |\widetilde{\mathbf{u}}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\widetilde{\varrho}_{0,\varepsilon}, \widetilde{\vartheta}_{0,\varepsilon}) - (\widetilde{\varrho}_{0,\varepsilon} - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) - \widetilde{\varrho}_{0,\varepsilon}^{(1)} G \right] {\rm d}x,
\end{aligned}$$

holds for a.e. $\tau \in (0,T)$. Repeating the passages of Lemma 4.3, we have that

$$\int_{\Omega \setminus \Omega_\varepsilon} |\nabla_x P_\varepsilon(\ell_\varepsilon^{(1)})|^2 \ {\rm d}x = 0.$$

Consequently, we get (4.29). □

5. Convergence

From the uniform bounds established in Lemma 3.1, we deduce the following convergences.

Lemma 5.1. *Under the hypotheses of Theorem 2.5, the following convergences hold for $\varepsilon \to 0$, passing to suitable subsequences as the case may be.*

$$\widetilde{\varrho}_\varepsilon \to \overline{\varrho} \quad \text{in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \tag{5.1}$$

$$\widetilde{\vartheta}_\varepsilon \to \overline{\vartheta} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \tag{5.2}$$

$$\widetilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{5.3}$$

$$\widetilde{\varrho}_\varepsilon \widetilde{\mathbf{u}}_\varepsilon \overset{*}{\rightharpoonup} \overline{\varrho}\mathbf{u} \quad \text{in } L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)), \tag{5.4}$$

$$\sqrt{\widetilde{\varrho}_\varepsilon}\widetilde{\mathbf{u}}_\varepsilon \overset{*}{\rightharpoonup} \sqrt{\overline{\varrho}}\mathbf{u} \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.5}$$

$$\widetilde{\varrho}_\varepsilon \widetilde{\mathbf{u}}_\varepsilon \otimes \widetilde{\mathbf{u}}_\varepsilon \rightharpoonup \overline{\varrho}\, \overline{\mathbf{u} \otimes \mathbf{u}} \quad \text{in } L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^{3\times 3})), \tag{5.6}$$

$$\mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x \widetilde{\mathbf{u}}_\varepsilon) \rightharpoonup \mathbb{S}(\overline{\vartheta}, \nabla_x \mathbf{u}) \quad \text{in } L^{\frac{5}{4}}(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^{3\times 3})), \tag{5.7}$$

$$\widetilde{p}_\varepsilon^{(1)} \rightharpoonup \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)} \quad \text{in } L^\infty(0,T;L^1(\Omega)). \tag{5.8}$$

$$\widetilde{\varrho}_\varepsilon \widetilde{s}_\varepsilon^{(1)} \rightharpoonup \overline{\varrho}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}\right) \quad \text{in } L^2(0,T;L^{\frac{30}{23}}(\Omega), \tag{5.9}$$

$$\widetilde{\varrho}_\varepsilon \widetilde{s}_\varepsilon^{(1)} \widetilde{\mathbf{u}}_\varepsilon \rightharpoonup \overline{\varrho}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}\right)\mathbf{u} \quad \text{in } L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3)), \tag{5.10}$$

$$\widetilde{\kappa}_\varepsilon \nabla_x \widetilde{\ell}_\varepsilon^{(1)} \rightharpoonup \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x \vartheta^{(1)} \quad \text{in } L^{\frac{14}{13}}(0,T;L^{\frac{14}{13}}(\Omega;\mathbb{R}^3)). \tag{5.11}$$

Proof. The main observation used throughout the proof is that the measures of the holes and of the "residual" subset tend to zero, as it can be deduced from (3.7) and (3.6); specifically,

$$\operatorname*{ess\,sup}_{t\in(0,T)} (|\mathcal{M}_{\text{res}}|, |\mathcal{M}_{\text{holes}}|) \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.12}$$

First of all, we have that

$$\widetilde{\varrho}_\varepsilon^{(1)} = \varrho_\varepsilon^{(1)} \mathbb{1}_{\Omega_\varepsilon} = \left[\varrho_\varepsilon^{(1)}\right]_{\text{ess}} + \left[\varrho_\varepsilon^{(1)}\right]_{\text{res}};$$

noticing that, from (3.7), (3.9), for a.e. $t \in (0,T)$

$$\left\| \left[\varrho_\varepsilon^{(1)}(t)\right]_{\text{res}} \right\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{3}} = \left\| \left[\frac{\varrho_\varepsilon(t) - \overline{\varrho}}{\varepsilon^m}\right]_{\text{res}} \right\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{3}} \leq \varepsilon^{-\frac{5}{3}m}\left(\|[\varrho_\varepsilon(t)]_{\text{res}}\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{3}} + \overline{\varrho}^{\frac{5}{3}}|\mathcal{M}_{\text{res}}(t)|\right) \leq c(\overline{\varrho})\varepsilon^{\frac{m}{3}},$$

using additionally (3.10), we can deduce, passing to suitable subsequences as the case may be,

$$\left[\varrho_\varepsilon^{(1)}\right]_{\text{ess}} \overset{*}{\rightharpoonup} \varrho^{(1)} \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.13}$$

$$\left[\varrho_\varepsilon^{(1)}\right]_{\text{res}} \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \tag{5.14}$$

implying, in particular,

$$\widetilde{\varrho}_\varepsilon^{(1)} \overset{*}{\rightharpoonup} \varrho^{(1)} \quad \text{in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)). \tag{5.15}$$

At this point, it is straightforward to deduce the strong convergence (5.1).

If we now use the decomposition (3.5), we can write

$$\widetilde{\vartheta}_\varepsilon^{(1)} = \vartheta_\varepsilon^{(1)}\mathbb{1}_{\Omega_\varepsilon} + E_\varepsilon(\vartheta_\varepsilon^{(1)})\mathbb{1}_{\Omega\setminus\Omega_\varepsilon} = \left[\vartheta_\varepsilon^{(1)}\right]_{\text{ess}} + \left[\vartheta_\varepsilon^{(1)}\right]_{\text{res}} + \left[\vartheta_\varepsilon^{(1)}\right]_{\text{holes}}.$$

From (3.7), (3.9) and the fact that $\|[\vartheta_\varepsilon]_{\text{res}}\|_{L^2(\Omega)} \leq \|[\vartheta_\varepsilon]_{\text{res}}\|_{L^4(\Omega)}^2$ as consequence of Hölder's inequality, we have for a.e. $t \in (0,T)$

$$\left\| \left[\vartheta_\varepsilon^{(1)}(t)\right]_{\text{res}} \right\|_{L^2(\Omega)}^2 = \left\| \left[\frac{\vartheta_\varepsilon(t) - \overline{\vartheta}}{\varepsilon^m}\right]_{\text{res}} \right\|_{L^2(\Omega)}^2 \leq \varepsilon^{-2m}\left(\|[\vartheta_\varepsilon(t)]_{\text{res}}\|_{L^4(\Omega)}^4 + \overline{\vartheta}^2|\mathcal{M}_{\text{res}}(t)|\right) \leq c(\overline{\vartheta});$$

therefore, using additionally (3.10), (5.12) and estimate (A.3), we obtain, passing to suitable subsequences as the case may be,

$$\left[\vartheta_\varepsilon^{(1)}\right]_{\rm ess} \overset{*}{\rightharpoonup} \vartheta^{(1)} \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.16}$$

$$\left[\vartheta_\varepsilon^{(1)}\right]_{\rm res} \overset{*}{\rightharpoonup} 0 \qquad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.17}$$

$$\left[\vartheta_\varepsilon^{(1)}\right]_{\rm holes} \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.18}$$

implying, in particular, that

$$\widetilde{\vartheta}_\varepsilon^{(1)} \overset{*}{\rightharpoonup} \vartheta^{(1)} \quad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{5.19}$$

Moreover, from (3.14) we recover that

$$\widetilde{\vartheta}_\varepsilon^{(1)} \rightharpoonup \vartheta^{(1)} \quad \text{in } L^2(0,T;W^{1,2}(\Omega)). \tag{5.20}$$

From (5.19), (5.20) it is straightforward to deduce the strong convergence (5.2).

Next, convergences (5.3)–(5.6) can be deduced from (3.11)–(3.13) and (5.1). Similarly, $\mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x \widetilde{\mathbf{u}}_\varepsilon) = \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon)\mathbb{1}_{\Omega_\varepsilon}$ and therefore, from the constitutive relations (1.16), (1.17) and convergences (5.2), (5.3) we can deduce (5.7).

We now point out that for any given function $f \in C^1(\overline{\mathcal{O}}_{\rm ess})$, denoting

$$f_\varepsilon^{(1)} := \frac{f(\varrho_\varepsilon, \vartheta_\varepsilon) - f(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m}, \tag{5.21}$$

due to convergences (5.13), (5.16), we recover that

$$\left[f_\varepsilon^{(1)}\right]_{\rm ess} \overset{*}{\rightharpoonup} \frac{\partial f(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial f(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)} \quad \text{in } L^\infty(0,T;L^2(\Omega)); \tag{5.22}$$

see [15, Proposition 5.2] for more details. Therefore, writing

$$\widetilde{p}_\varepsilon^{(1)} = p_\varepsilon^{(1)}\mathbb{1}_{\Omega_\varepsilon} = \left[p_\varepsilon^{(1)}\right]_{\rm ess} + \left[p_\varepsilon^{(1)}\right]_{\rm res},$$

where, from (3.7), (3.15) we have for a.e. $t \in (0,T)$

$$\left\| \left[p_\varepsilon^{(1)}(t)\right]_{\rm res} \right\|_{L^1(\Omega)} \le \left\| \left[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon)(t)}{\varepsilon^m}\right]_{\rm res} \right\|_{L^1(\Omega)} + \frac{p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^m}|\mathcal{M}_{\rm res}(t)| \le c(\overline{\varrho}, \overline{\vartheta})\varepsilon^m,$$

using additionally (5.22), we obtain (5.8). Similarly, we write

$$\widetilde{\varrho}_\varepsilon \widetilde{s}_\varepsilon^{(1)} = \varrho_\varepsilon s_\varepsilon^{(1)}\mathbb{1}_{\Omega_\varepsilon} = [\varrho_\varepsilon]_{\rm ess}\left[s_\varepsilon^{(1)}\right]_{\rm ess} + \left[\varrho_\varepsilon s_\varepsilon^{(1)}\right]_{\rm res};$$

from (3.7), (3.9) and (3.16) we get

$$\begin{aligned} &\left\| \left[\varrho_\varepsilon s_\varepsilon^{(1)}\right]_{\rm res} \right\|^2_{L^2(0,T;L^{\frac{30}{23}}(\Omega))} \\ &\lesssim \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon^m}\right]_{\rm res} \right\|^2_{L^2(0,T;L^{\frac{30}{23}}(\Omega))} + T\frac{s^2(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^{2m}} \operatorname*{ess\,sup}_{t\in(0,T)} \left(\|[\varrho_\varepsilon(t)]_{\rm res}\|^2_{L^{\frac{5}{3}}(\Omega)} |\mathcal{M}_{\rm res}(t)|^{\frac{1}{3}} \right) \\ &\le c(\overline{\varrho}, \overline{\vartheta})(1 + \varepsilon^{\frac{16}{15}m}); \end{aligned}$$

therefore, using additionally (5.1), (5.12) and (5.22), we have

$$[\varrho_\varepsilon]_{\rm ess}\left[s_\varepsilon^{(1)}\right]_{\rm ess} \overset{*}{\rightharpoonup} \overline{\varrho}\left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)} \right) \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.23}$$

$$\left[\varrho_\varepsilon s_\varepsilon^{(1)}\right]_{\rm res} \rightharpoonup 0 \quad \text{in } L^2(0,T;L^{\frac{30}{23}}(\Omega)). \tag{5.24}$$

We get, in particular, (5.9). In a similar way, we write

$$\widetilde{\varrho}_\varepsilon \widetilde{s}_\varepsilon^{(1)} \widetilde{\mathbf{u}}_\varepsilon = [\varrho_\varepsilon]_{\rm ess}\left[s_\varepsilon^{(1)}\right]_{\rm ess}\widetilde{\mathbf{u}}_\varepsilon + \left[\varrho_\varepsilon s_\varepsilon^{(1)}\right]_{\rm res}\widetilde{\mathbf{u}}_\varepsilon;$$

from (3.7), (3.9), (3.13) and (3.17) we get

$$\begin{aligned}
&\left\| \left[\varrho_\varepsilon s_\varepsilon^{(1)} \right]_{\rm res} \widetilde{\mathbf{u}}_\varepsilon \right\|^2_{L^2(0,T;L^{\frac{30}{29}}(\Omega))} \\
&\lesssim \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon^m} \right]_{\rm res} \widetilde{\mathbf{u}}_\varepsilon \right\|^2_{L^2(0,T;L^{\frac{30}{29}}(\Omega))} + \frac{s^2(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^{2m}} \operatorname*{ess\,sup}_{t\in(0,T)} \left(\| [\varrho_\varepsilon(t)]_{\rm res} \|^2_{L^{\frac{5}{3}}(\Omega)} |\mathcal{M}_{\rm res}(t)|^{\frac{2}{5}} \right) \| \widetilde{\mathbf{u}}_\varepsilon \|^2_{L^2(0,T;L^6(\Omega))} \\
&\leq c(\overline{\varrho}, \overline{\vartheta})(1 + \varepsilon^{\frac{6}{5}m}),
\end{aligned}$$

and hence, from (5.3), (5.12), (5.23), we obtain

$$[\varrho_\varepsilon]_{\rm ess} \left[s_\varepsilon^{(1)} \right]_{\rm ess} \widetilde{\mathbf{u}}_\varepsilon \overset{*}{\rightharpoonup} \overline{ \overline{\varrho} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) \mathbf{u} } \quad \text{in } L^2(0,T; L^{\frac{3}{2}}(\Omega; \mathbb{R}^3)), \tag{5.25}$$

$$\left[\varrho_\varepsilon s_\varepsilon^{(1)} \right]_{\rm res} \widetilde{\mathbf{u}}_\varepsilon \rightharpoonup 0 \quad \text{in } L^2(0,T; L^{\frac{30}{29}}(\Omega; \mathbb{R}^3)). \tag{5.26}$$

Moreover, we can write

$$\begin{aligned}
\widetilde{\kappa}_\varepsilon \nabla_x \widetilde{\ell}_\varepsilon^{(1)} &= \kappa(\vartheta_\varepsilon) \nabla_x \ell_\varepsilon^{(1)} \mathbb{1}_{\Omega_\varepsilon} + \kappa(\overline{\vartheta}) \nabla_x P_\varepsilon \left(\ell_\varepsilon^{(1)} \right) \mathbb{1}_{\Omega \setminus \Omega_\varepsilon} \\
&= \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm ess} \nabla_x \left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^m} \right) + \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon^m} \right) \right]_{\rm res} + \kappa(\overline{\vartheta}) \nabla_x [\ell_\varepsilon^{(1)}]_{\rm holes},
\end{aligned}$$

and thus, in virtue of (3.18), (3.19), (5.2), (5.12), (5.16), (5.20) and estimate (A.6), we get

$$\left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm ess} \nabla_x \left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^m} \right) \rightharpoonup \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \nabla_x \vartheta^{(1)} \quad \text{in } L^2(0,T; L^2(\Omega; \mathbb{R}^3)), \tag{5.27}$$

$$\left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon^m} \right) \right]_{\rm res} \rightharpoonup 0 \quad \text{in } L^{\frac{14}{13}}(0,T; L^{\frac{14}{13}}(\Omega; \mathbb{R}^3)), \tag{5.28}$$

$$\nabla_x [\ell_\varepsilon^{(1)}]_{\rm holes} \rightharpoonup 0 \quad \text{in } L^2(0,T; L^2(\Omega; \mathbb{R}^3)), \tag{5.29}$$

leading to (5.11).

Finally, as consequence of the Div-Curl Lemma [15, Proposition 3.3], we obtain

$$\overline{ \overline{\varrho} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) \mathbf{u} } = \overline{\varrho} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) \mathbf{u}$$

and hence (5.10); notice, in particular, that we can repeat the same passages performed in [15, Section 5.3.2, (iii)] since only the essential parts of the functions are involved. □

We are ready to let $\varepsilon \to 0$ in the weak formulations of the problem on the homogenized domain Ω and get the first result of our work.

Proposition 5.2. *Under the hypotheses of Theorem 2.5, passing to suitable subsequences as the case may be,*

$$\begin{aligned}
\widetilde{\mathbf{u}}_\varepsilon &\rightharpoonup \mathbf{u} && \text{in } L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)), \\
\widetilde{\vartheta}_\varepsilon^{(1)} &\rightharpoonup \vartheta^{(1)} && \text{in } L^2(0,T; W^{1,2}(\Omega)),
\end{aligned}$$

where $[\mathbf{u}, \vartheta^{(1)}]$ *is a dissipative solution to the Oberbeck-Boussinesq system emanating from* $[\mathbf{u}_0, \vartheta_0^{(1)}]$ *in the sense of Definition 2.4, with* $\mathbf{u}_0, \vartheta_0^{(1)}$ *the weak limits appearing in* (1.24), (1.25), *respectively.*

Proof. Passage to the limit in the continuity equation. In view of (1.23), (5.1) and (5.4), passing to the limit in (4.1), we obtain that

$$\int_0^\tau \int_\Omega \mathbf{u} \cdot \nabla_x \varphi \, {\rm d}x{\rm d}t = 0$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C_c^1([0,T] \times \overline{\Omega})$; in particular, we get that condition (i) of Definition 2.4 is satisfied. Additionally, if we divide (4.1) by ε^m and let $\varepsilon \to 0$, from (1.23), (5.3) and (5.15) we recover that

$$\left[\int_\Omega \varrho^{(1)}\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho^{(1)}\partial_t\varphi + \varrho^{(1)}\mathbf{u}\cdot\nabla_x\varphi\right]\mathrm{d}x\mathrm{d}t \tag{5.30}$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C_c^1([0,T]\times\overline{\Omega})$. Therefore, choosing properly the test function φ in (5.30), from (1.22) we can deduce that for a.e. $\tau \in (0,T)$

$$\int_\Omega \varrho^{(1)}(\tau,\cdot)\ \mathrm{d}x = 0. \tag{5.31}$$

Passage to the limit in the momentum equation. Convergence (5.4) can be strengthened to

$$\widetilde{\varrho}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon \to \overline{\varrho}\mathbf{u} \quad \text{in } C_{\text{weak}}([0,T]; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3));$$

hence, from (5.3) and (5.4) we can deduce the regularity class (2.7) for $\mathbf{u}$. Using additionally convergences (1.23),(1.24), (5.6), (5.7) and (5.15), we are ready to pass to the limit in (4.2), obtaining that

$$\begin{aligned}\overline{\varrho}\left[\int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\mathrm{d}x\right]_{t=0}^{t=\tau} &= \overline{\varrho}\int_0^\tau\int_\Omega \left(\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \overline{\mathbf{u}\otimes\mathbf{u}} : \nabla_x\boldsymbol{\varphi}\right)\mathrm{d}x\mathrm{d}t \\ &- \mu(\overline{\vartheta})\int_0^\tau\int_\Omega \left(\nabla_x\mathbf{u} + \nabla_x^\top\mathbf{u}\right) : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \varrho^{(1)}\nabla_x G\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t\end{aligned} \tag{5.32}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\overline{\Omega};\mathbb{R}^3)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$ such that $\mathrm{div}_x\boldsymbol{\varphi} = 0$. We now introduce the measure

$$\begin{aligned}&\mathfrak{R} \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{3\times3}_{\text{sym}})),\\ &\mathrm{d}\mathfrak{R} := \overline{\varrho}\left(\overline{\mathbf{u}\otimes\mathbf{u}} - \mathbf{u}\otimes\mathbf{u}\right)\mathrm{d}x,\end{aligned} \tag{5.33}$$

where the positivity of $\mathfrak{R}$ follows from the fact that for any $\boldsymbol{\xi} \in \mathbb{R}^3$ and any open set $\mathcal{B} \subset \Omega$ we have

$$\begin{aligned}(\overline{\mathbf{u}\otimes\mathbf{u}} - \mathbf{u}\otimes\mathbf{u}) : (\boldsymbol{\xi}\otimes\boldsymbol{\xi}) &= \lim_{\varepsilon\to0}\left[(\widetilde{\mathbf{u}}_\varepsilon\otimes\widetilde{\mathbf{u}}_\varepsilon) : (\boldsymbol{\xi}\otimes\boldsymbol{\xi})\right] - (\mathbf{u}\otimes\mathbf{u}) : (\boldsymbol{\xi}\otimes\boldsymbol{\xi}) \\ &= \lim_{\varepsilon\to0}|\widetilde{\mathbf{u}}_\varepsilon\cdot\boldsymbol{\xi}|^2 - |\mathbf{u}\cdot\boldsymbol{\xi}|^2 = \overline{|\mathbf{u}\cdot\boldsymbol{\xi}|^2} - |\mathbf{u}\cdot\boldsymbol{\xi}|^2\end{aligned}$$

in $\mathcal{D}'((0,T)\times\mathcal{B})$ and $\overline{|\mathbf{u}\cdot\boldsymbol{\xi}|^2} \le |\mathbf{u}\cdot\boldsymbol{\xi}|^2$ due to the convexity of the function $\mathbf{u}\mapsto|\mathbf{u}\cdot\boldsymbol{\xi}|^2$; see e.g. [13, Theorem 2.1.1]. Noticing that

$$\int_\Omega G\nabla_x G\cdot\boldsymbol{\varphi}\ \mathrm{d}x = \frac{1}{2}\int_\Omega \nabla_x|G|^2\cdot\boldsymbol{\varphi}\ \mathrm{d}x = -\frac{1}{2}\int_\Omega G^2\ \mathrm{div}_x\boldsymbol{\varphi}\ \mathrm{d}x = 0,$$

and, due to (1.28) $\mathrm{div}_x(\mathbf{u}\otimes\mathbf{u}) = \mathbf{u}\cdot\nabla_x\mathbf{u}$, (5.32) can be rewritten as

$$\begin{aligned}\overline{\varrho}\left[\int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\mathrm{d}x\right]_{t=0}^{t=\tau} &= \overline{\varrho}\int_0^\tau\int_\Omega \left[\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} - (\mathbf{u}\cdot\nabla_x)\mathbf{u}\cdot\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t - \mu(\overline{\vartheta})\int_0^\tau\int_\Omega (\nabla_x\mathbf{u} + \nabla_x^\top\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t \\ &+ \int_0^\tau\int_\Omega \left(\varrho^{(1)} - \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}G\right)\nabla_x G\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t.\end{aligned} \tag{5.34}$$

Passage to the limit in the entropy equation. Similarly, due to convergences (1.23), (1.25), (5.9)–(5.11), letting $\varepsilon\to0$ in (4.12) we obtain that

$$\begin{aligned}&-\int_\Omega \overline{\varrho}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)}\right)\varphi(0,\cdot)\mathrm{d}x \\ &\quad= \int_0^T\int_\Omega \left[\overline{\varrho}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\right)\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) - \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x\vartheta^{(1)}\cdot\nabla_x\varphi\right]\mathrm{d}x\mathrm{d}t\end{aligned} \tag{5.35}$$

holds for any $\varphi \in C_c^1([0,T)\times\overline{\Omega})$. In particular, choosing properly φ in (5.35), from (1.22) and (5.30) we can deduce that for a.e. $\tau \in (0,T)$

$$\int_\Omega \vartheta^{(1)}(\tau,\cdot)\ \mathrm{d}x = 0. \tag{5.36}$$

Passage to the limit in the Boussinesq equation. Letting $\varepsilon \to 0$ in (4.13), in virtue of convergences (5.1) and (5.8), we obtain that

$$\int_0^T \int_\Omega \left(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) \mathrm{div}_x \boldsymbol{\varphi} \; \mathrm{d}x\mathrm{d}t = \overline{\varrho} \int_0^T \int_\Omega \nabla_x G \cdot \boldsymbol{\varphi} \; \mathrm{d}x\mathrm{d}t$$

holds for any $\boldsymbol{\varphi} \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^3)$. We get in particular that

$$\nabla_x \left(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) = \overline{\varrho} \nabla_x G \quad \Rightarrow \quad \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} = \overline{\varrho} G + f(t).$$

If we integrate the previous identity over $(0,\tau) \times \Omega$ for any $\tau \in [0,T]$ we can deduce from (1.19), (5.31) and (5.36) that $f \equiv 0$. Therefore,

$$\varrho^{(1)} = -A \vartheta^{(1)} + \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} G, \tag{5.37}$$

where A is the constant defined in (1.31).

We can now substitute (5.37) into (5.34) and (5.35); we obtain that condition (ii) of Definition 2.4 is satisfied and

$$-\overline{\varrho} c_p \int_\Omega \vartheta_0^{(1)} \; \varphi(0,\cdot) \; \mathrm{d}x = \int_0^T \int_\Omega \left[\overline{\varrho} c_p \vartheta^{(1)} (\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) - (\kappa(\overline{\vartheta}) \nabla_x \vartheta^{(1)} + \overline{\vartheta} A G \mathbf{u}) \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t$$

holds for any $\varphi \in C_c^1([0,T) \times \overline{\Omega})$. In particular, we have used the fact that, from Gibb's relation (1.4),

$$\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} = -\frac{1}{\overline{\varrho}^2} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta},$$

and, since the initial data $(\varrho_0^{(1)}, \vartheta_0^{(1)})$ are well-prepared and satisfy (1.27),

$$c_p \vartheta_0^{(1)} = \overline{\vartheta} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} + a(\overline{\varrho}, \overline{\vartheta}) G \right).$$

Hence, $\vartheta^{(1)}$ satisfies the weak formulation of (1.30).

Next, by interpolation, from (5.3) and (5.4) we can deduce that $\mathbf{u} \in L^{\frac{10}{3}}(0,T; L^{\frac{10}{3}}(\Omega; \mathbb{R}^3))$, implying that

$$\mathbf{u} \cdot \nabla_x \vartheta^{(1)} \in L^p(0,T; L^p(\Omega)) \quad \text{with} \quad p = \frac{5}{4}.$$

Due to the additional assumption (1.26) on the initial temperature $\vartheta_0^{(1)}$, we can apply [15, Theorem 10.22] to the deduce the regularity class (2.7) for $\vartheta^{(1)}$. Consequently, condition (iii) of Definition 2.4 is satisfied.

Passage to the limit in the energy inequality. We first point out that

$$\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon = \frac{\mu(\vartheta_\varepsilon)}{2} \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon - \frac{2}{3} (\mathrm{div}_x \mathbf{u}_\varepsilon) \mathbb{I} \right|^2 + \eta(\vartheta_\varepsilon) |\mathrm{div}_x \mathbf{u}_\varepsilon|^2,$$

and therefore, from (5.2), (5.3), (5.20) and the lower semi-continuity of convex functions, for any $\psi \in C^1[0,T]$, $\psi \geq 0$ we have

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \int_0^T \int_\Omega \overline{\vartheta} \left[\frac{1}{\widetilde{\vartheta}_\varepsilon} \mathbb{S}(\widetilde{\vartheta}_\varepsilon, \nabla_x \widetilde{\mathbf{u}}_\varepsilon) : \nabla_x \widetilde{\mathbf{u}}_\varepsilon + \widetilde{\kappa}_\varepsilon \left| \nabla_x \widetilde{\ell}_\varepsilon^{(1)} \right|^2 \right] \psi \; \mathrm{d}x\mathrm{d}t \\
&\geq \liminf_{\varepsilon \to 0} \int_0^T \int_\Omega \overline{\vartheta} \left(\frac{1}{2} \left[\frac{\mu(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\mathrm{ess}} \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon - \frac{2}{3} (\mathrm{div}_x \mathbf{u}_\varepsilon) \mathbb{I} \right|^2 + \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon^2} \right]_{\mathrm{ess}} \left| \nabla_x \vartheta_\varepsilon^{(1)} \right|^2 \right) \psi \; \mathrm{d}x\mathrm{d}t \\
&\geq \frac{\mu(\overline{\vartheta})}{2} \int_0^T \int_\Omega |\nabla_x \mathbf{u} + \nabla_x^\top \mathbf{u}|^2 \; \psi \; \mathrm{d}x\mathrm{d}t + \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \int_0^T \int_\Omega |\nabla_x \vartheta^{(1)}|^2 \; \psi \; \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Introducing the measure

$$\begin{aligned}
&\mathfrak{E} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega})), \\
&\mathrm{d}\mathfrak{E} := \frac{1}{2} \overline{\varrho} \left(\overline{|\mathbf{U}|^2} - |\mathbf{U}|^2 \right) \mathrm{d}x,
\end{aligned} \tag{5.38}$$

and noticing that

$$\widetilde{H}_{\overline{\vartheta},\varepsilon} - (\widetilde{\varrho}_\varepsilon - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) = \begin{cases} H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega \setminus \Omega_\varepsilon, \end{cases}$$

we can pass to the limit in (4.29) and repeat the same passages as in [15, Section 5.5.4], obtaining that the energy inequality

$$\begin{aligned} &\int_\Omega \left[\frac{1}{2}\overline{\varrho}|\mathbf{u}|^2 + \frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\left|\varrho^{(1)}\right|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}}\frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\left|\vartheta^{(1)}\right|^2 - \varrho^{(1)}G\right](\tau,\cdot)\mathrm{d}x \\ &+\frac{\mu(\overline{\vartheta})}{2}\int_0^\tau\int_\Omega |\nabla_x\mathbf{u} + \nabla_x^\top\mathbf{u}|^2\ \mathrm{d}x\mathrm{d}t + \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\int_0^\tau\int_\Omega |\nabla_x\vartheta^{(1)}|^2\ \mathrm{d}x\mathrm{d}t + \int_\Omega \mathrm{d}\mathfrak{E}(\tau) \\ &\le \int_\Omega \left[\frac{1}{2}\overline{\varrho}|\mathbf{u}_0|^2 + \frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\left|\varrho_0^{(1)}\right|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}}\frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\left|\vartheta_0^{(1)}\right|^2 - \varrho_0^{(1)}G\right]\mathrm{d}x \end{aligned}$$

holds for a.e. $\tau \in (0,T)$. Substituting (1.27) and (5.37), we get that condition (iv) of Definition 2.4 is satisfied.

Finally, condition (v) of Definition 2.4 follows from the fact that $\mathrm{Tr}[\mathbf{u}\otimes\mathbf{u}] = |\mathbf{u}|^2$; this concludes the proof. □

6. Weak–strong uniqueness

Our goal in this section is to prove the *weak–strong uniqueness* principle for the target system: if the Oberbeck–Boussinesq approximation admits a strong solution, then it must coincide with the dissipative solution emanating from the same initial data.

We start recalling the following result on the local existence of strong solutions, cf. [8, Theorem 2.1]. Notice that in [8] the authors considered time-periodic solutions with small data; however, the proof, based on Galerkin approximation and uniform bounds, can be adapted to get local existence with large data. We also recall the recent result by Abbatiello and Feireisl [1] where the existence was proven considering non-local boundary conditions for the temperature.

Theorem 6.1 (Existence of strong solutions to the Oberbeck–Boussinesq system)**.** *There exists a positive time T^* and a trio of functions*

$$\mathbf{U} \in W^{1,2}(0,T^*;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T^*;W^{1,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T^*;W^{2,2}(\Omega;\mathbb{R}^3)), \tag{6.1}$$

$$\Theta \in W^{1,2}(0,T^*;W^{1,2}(\Omega)) \cap L^\infty(0,T^*;W^{2,2}(\Omega)) \cap L^2(0,T^*;W^{2,3}(\Omega)), \tag{6.2}$$

$$\Pi \in L^2(0,T;W^{1,2}(\Omega)) \tag{6.3}$$

satisfying the Oberbeck–Boussinesq system (1.28)–(1.34) *a.e. in* $(0,T^*)\times\Omega$*.*

Theorem 6.2 (Weak–strong uniqueness principle)**.** *Let* $[\mathbf{U},\Theta,\Pi]$ *be a strong solution of the Oberbeck-Boussinesq system* (1.28)–(1.34) *on* $[0,T^*]$*, the existence of which is guaranteed by Theorem 6.1. Let* $[\mathbf{u},\theta^{(1)}]$ *be a dissipative solution of the same system with dissipation defects* $\mathfrak{R},\mathfrak{E}$ *in the sense of Definition 2.4. If*

$$[\mathbf{U}(0,x),\Theta(0,x)] = [\mathbf{u}(0,x),\vartheta^{(1)}(0,x)] \quad \textit{for a.e. } x\in\Omega \tag{6.4}$$

then $\mathfrak{R}\equiv\mathfrak{E}\equiv 0$ *and*

$$[\mathbf{U}(t,x),\Theta(t,x)] = [\mathbf{u}(t,x),\vartheta^{(1)}(t,x)] \quad \textit{for a.e. } (t,x)\in(0,T^*)\times\Omega. \tag{6.5}$$

Proof. Let us define

$$E(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta) := \frac{1}{2}\left(\overline{\varrho}|\mathbf{u}-\mathbf{U}|^2 + \frac{\overline{\varrho}}{\overline{\vartheta}}c_p|\vartheta^{(1)}-\Theta|^2\right)$$

and for any $\tau\in[0,T^*]$ the spatial integral of it, known as *relative energy functional*,

$$\begin{aligned} \mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(\tau) &:= \int_\Omega E(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(\tau,\cdot)\ \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega\left(\overline{\varrho}|\mathbf{u}-\mathbf{U}|^2 + \frac{\overline{\varrho}}{\overline{\vartheta}}c_p|\vartheta^{(1)}-\Theta|^2\right)(\tau,\cdot)\ \mathrm{d}x. \end{aligned}$$

Clearly, $\mathcal{E}(\mathbf{u}, \vartheta^{(1)} \mid \mathbf{U}, \Theta)(\tau) \geq 0$ for any $\tau \in [0, T^*]$ and the equality holds if and only if (6.5) holds. Therefore, it is enough to show that

$$\mathfrak{E} \equiv 0, \quad \mathcal{E}(\mathbf{u}, \vartheta^{(1)} \mid \mathbf{U}, \Theta) \equiv 0 \quad \text{a.e. in } (0, T^*). \tag{6.6}$$

Let us at first suppose that $[\mathbf{U}, \Theta, \Pi]$ are smooth and compactly supported functions such that $\mathbf{U}|_{\partial\Omega} = 0$ and $\mathrm{div}_x \mathbf{U} = 0$. Then, $\boldsymbol{\varphi} = \mathbf{U}$ can be used as test function in the weak formulation (2.8), obtaining

$$\begin{aligned} \overline{\varrho} \left[\int_\Omega (\mathbf{u} \cdot \mathbf{U})(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} &= \overline{\varrho} \int_0^\tau \int_\Omega (\mathbf{u} \cdot \partial_t \mathbf{U} + (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \mathbf{U}) \,\mathrm{d}x\mathrm{d}t \\ &\quad - 2\mu(\overline{\vartheta}) \int_0^\tau \int_\Omega \mathbb{D}_x \mathbf{u} : \mathbb{D}_x \mathbf{U} \ \mathrm{d}x\mathrm{d}t - A \int_0^\tau \int_\Omega \vartheta^{(1)} \nabla_x G \cdot \mathbf{U} \ \mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R} \ \mathrm{d}t, \end{aligned} \tag{6.7}$$

where we have introduced symmetric velocity gradient, defined as

$$\mathbb{D}_x \mathbf{v} = \frac{\nabla_x \mathbf{v} + \nabla_x^\top \mathbf{v}}{2}.$$

Similarly, $\varphi = \Theta$ can be used as test function in the weak formulation of (1.30), obtaining

$$\begin{aligned} \frac{\overline{\varrho}}{\overline{\vartheta}} c_p \left[\int_\Omega (\vartheta^{(1)} \Theta)(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} &= \frac{\overline{\varrho}}{\overline{\vartheta}} c_p \int_0^\tau \int_\Omega \vartheta^{(1)} \left(\partial_t \Theta + \mathbf{u} \cdot \nabla_x \Theta \right) \mathrm{d}x\mathrm{d}t \\ &\quad - \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \int_0^\tau \int_\Omega \nabla_x \vartheta^{(1)} \cdot \nabla_x \Theta \ \mathrm{d}x\mathrm{d}t + A \int_0^\tau \int_\Omega \Theta \nabla_x G \cdot \mathbf{u} \ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{6.8}$$

Moreover, using $\varphi = |\mathbf{U}|^2, |\Theta|^2$ as test function in the weak formulation of the incompressibility condition (1.28), we have the following identity

$$\begin{aligned} \frac{1}{2} \left[\int_\Omega \left(\overline{\varrho} |\mathbf{U}|^2 + \frac{\overline{\varrho}}{\overline{\vartheta}} c_p |\Theta|^2 \right) (t, \cdot) \right]_{t=0}^{t=\tau} &= \overline{\varrho} \int_0^\tau \int_\Omega \left[\mathbf{U} \cdot \partial_t \mathbf{U} + (\mathbf{u} \cdot \nabla_x) \mathbf{U} \cdot \mathbf{U} \right] \ \mathrm{d}x\mathrm{d}t \\ &\quad + \frac{\overline{\varrho}}{\overline{\vartheta}} c_p \int_0^\tau \int_\Omega \Theta (\partial_t \Theta + \mathbf{u} \cdot \nabla_x \Theta) \ \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{6.9}$$

We can now subtract (6.7), (6.8) and sum (6.9) to the energy inequality (2.9), obtaining

$$\begin{aligned} &\left[\mathcal{E}(\mathbf{u}, \vartheta^{(1)} \mid \mathbf{U}, \Theta)(t) \right]_{t=0}^{t=\tau} + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \\ &\quad + 2\mu(\overline{\vartheta}) \int_0^\tau \int_\Omega \mathbb{D}_x \mathbf{u} : \mathbb{D}_x (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x\mathrm{d}t + \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \int_0^\tau \int_\Omega \nabla_x \vartheta^{(1)} \cdot \nabla_x (\vartheta^{(1)} - \Theta) \ \mathrm{d}x\mathrm{d}t \\ &\leq - \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{U}) \cdot (\overline{\varrho} [\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_x) \mathbf{U}] + A \Theta \nabla_x G) \,\mathrm{d}x\mathrm{d}t \\ &\quad - \frac{1}{\overline{\vartheta}} \int_0^\tau \int_\Omega (\vartheta^{(1)} - \Theta) \left[\overline{\varrho} c_p (\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta) - \overline{\vartheta} A \nabla_x G \cdot \mathbf{U} \right] \mathrm{d}x\mathrm{d}t \\ &\quad - \overline{\varrho} \int_0^\tau \int_\Omega \left[(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \right] \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x\mathrm{d}t - \frac{\overline{\varrho}}{\overline{\vartheta}} c_p \int_0^\tau \int_\Omega (\vartheta^{(1)} - \Theta) \nabla_x \Theta \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R} \ \mathrm{d}t. \end{aligned}$$

Next, we add to the previous inequality the following vanishing integrals,

$$\mu(\overline{\vartheta})\int_0^\tau\int_\Omega [(\mathbf{u}-\mathbf{U})\cdot\Delta_x\mathbf{U}+2\mathbb{D}_x(\mathbf{u}-\mathbf{U}):\mathbb{D}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t,$$
$$\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\int_0^\tau\int_\Omega \Big[(\vartheta^{(1)}-\Theta)\Delta_x\Theta+\nabla_x(\vartheta^{(1)}-\Theta)\cdot\nabla_x\Theta\Big]\,\mathrm{d}x\mathrm{d}t,$$
$$\int_0^\tau\int_\Omega (\mathbf{u}-\mathbf{U})\cdot\nabla_x\Pi\ \mathrm{d}x\mathrm{d}t,$$

getting finally the *relative energy inequality*,

$$\begin{aligned}
&\Big[\mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(t)\Big]_{t=0}^{t=\tau}+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau)\\
&\quad+2\mu(\overline{\vartheta})\int_0^\tau\int_\Omega|\mathbb{D}_x(\mathbf{u}-\mathbf{U})|^2\ \mathrm{d}x\mathrm{d}t+\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\int_0^\tau\int_\Omega|\nabla_x(\vartheta^{(1)}-\Theta)|^2\ \mathrm{d}x\mathrm{d}t\\
&\leq-\int_0^\tau\int_\Omega(\mathbf{u}-\mathbf{U})\cdot\big(\overline{\varrho}[\partial_t\mathbf{U}+(\mathbf{U}\cdot\nabla_x)\mathbf{U}]+\nabla_x\Pi-\mu(\overline{\vartheta})\Delta_x\mathbf{U}+A\Theta\nabla_xG\big)\,\mathrm{d}x\mathrm{d}t\\
&\quad-\frac{1}{\overline{\vartheta}}\int_0^\tau\int_\Omega(\vartheta^{(1)}-\Theta)\left[\overline{\varrho}c_p(\partial_t\Theta+\mathbf{U}\cdot\nabla_x\Theta)-\kappa(\overline{\vartheta})\Delta_x\Theta-\overline{\vartheta}A\nabla_xG\cdot\mathbf{U}\right]\mathrm{d}x\mathrm{d}t\\
&\quad-\overline{\varrho}\int_0^\tau\int_\Omega[(\mathbf{u}-\mathbf{U})\cdot\nabla_x]\,\mathbf{U}\cdot(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\mathrm{d}t-\frac{\overline{\varrho}}{\overline{\vartheta}}c_p\int_0^\tau\int_\Omega(\vartheta^{(1)}-\Theta)\nabla_x\Theta\cdot(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\mathbf{U}:\mathrm{d}\mathfrak{R}\ \mathrm{d}t.
\end{aligned}\tag{6.10}$$

The class of functions $[\mathbf{U},\Theta,\Pi]$ satisfying the relative energy inequality can be enlarged by a density argument, as long as all the involved integrals remain well-defined. In particular (6.10) holds for $[\mathbf{U},\Theta,\Pi]$ belonging to the regularity classes defined in (6.1)–(6.3).

If we additionally suppose that $[\mathbf{U},\Theta,\Pi]$ is a strong solution of (1.28)–(1.34) satisfying (6.4), we get that $\mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(0)$ and the first two integrals on the right-hand side of (6.10) vanish; in particular, (6.10) reduced to

$$\begin{aligned}
&\mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(\tau)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau)\\
&\quad+2\mu(\overline{\vartheta})\int_0^\tau\int_\Omega|\mathbb{D}_x(\mathbf{u}-\mathbf{U})|^2\ \mathrm{d}x\mathrm{d}t+\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\int_0^\tau\int_\Omega|\nabla_x(\vartheta^{(1)}-\Theta)|^2\ \mathrm{d}x\mathrm{d}t\\
&\leq-\overline{\varrho}\int_0^\tau\int_\Omega[(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})]:\nabla_x\mathbf{U}\ \mathrm{d}x\mathrm{d}t-\frac{\overline{\varrho}}{\overline{\vartheta}}c_p\int_0^\tau\int_\Omega(\vartheta^{(1)}-\Theta)\nabla_x\Theta\cdot(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\mathbf{U}:\mathrm{d}\mathfrak{R}\ \mathrm{d}t,
\end{aligned}\tag{6.11}$$

for a.e $\tau\in(0,T)$. Clearly,

$$\overline{\varrho}|(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})|\lesssim\frac{1}{2}\overline{\varrho}\,\mathrm{Tr}[(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})]=\frac{1}{2}\overline{\varrho}|\mathbf{u}-\mathbf{U}|^2,$$
$$\frac{\overline{\varrho}}{\overline{\vartheta}}c_p|(\mathbf{u}-\mathbf{U})(\vartheta^{(1)}-\Theta)|\lesssim\frac{1}{2}\overline{\varrho}|\mathbf{u}-\mathbf{U}|^2+\frac{1}{2}\frac{\overline{\varrho}}{\overline{\vartheta}}c_p|\vartheta^{(1)}-\Theta|^2,$$
$$|\mathfrak{R}|\lesssim\mathrm{Tr}[\mathfrak{R}]\lesssim\mathfrak{E},$$

where in the last inequality we have used the compatibility condition (2.10). Therefore, for a.e. $\tau\in(0,T)$ we obtain

$$\mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(\tau)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau)\leq c(\nabla_x\mathbf{U},\nabla_x\Theta)\int_0^\tau\left(\mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(t)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(t)\right)\mathrm{d}t.$$

Applying the Gronwall argument, we recover in particular that for a.e. $\tau\in(0,T)$

$$\mathcal{E}(\mathbf{u},\vartheta^{(1)}\mid\mathbf{U},\Theta)(\tau)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau)\leq 0.$$

Since the left-hand side of the previous inequality is the sum of two non-negative quantities, the only possibility is that (6.6) holds; we get the claim. □

6.1. **Proof of Theorem 2.5.** In Proposition 5.2, we have proven that, passing to suitable subsequences as the case may be,

$$[\widetilde{\mathbf{u}}_\varepsilon, \widetilde{\vartheta}^{(1)}_\varepsilon] \rightharpoonup [\mathbf{u}, \vartheta^{(1)}] \quad \text{in } L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^4)), \tag{6.12}$$

where $[\mathbf{u}, \vartheta^{(1)}]$ is a dissipative solution to the Oberbeck-Boussinesq system in the sense of Definition 2.4 with dissipation defects $\mathfrak{R}$, $\mathfrak{E}$ defined by (5.33), (5.38), respectively. From the fact that $[\mathbf{U}_0, \Theta_0] = [\mathbf{u}_0, \theta_0^{(1)}]$, Theorem 2.5 is therefore a straightforward corollary of Theorem 6.2.

Appendix A. The extension operator P_ε

The aim of this section is to construct the extension operator P_ε for the logarithmic part $\ell^{(1)}_\varepsilon$, cf. Section 3.1; in particular, we want the Neumann boundary condition to be preserved. Moreover, in order to easily extend the weak formulations of the entropy equality (2.4) and the ballistic energy inequality (2.6) to the homogenized domain Ω, we opted for a harmonic extension on the holes.

We begin recalling the following result, which can be found in [21, Lemma 4.1].

Lemma A.1. *Suppose Ω_ε is given by* (1.7)*. There exists an extension operator*

$$E_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to W^{1,2}(\Omega)$$

such that for each $\varphi \in W^{1,2}(\Omega_\varepsilon)$ and any $1 \le q \le \infty$ we have

$$E_\varepsilon(\varphi) = \varphi \quad \text{in } \Omega_\varepsilon, \tag{A.1}$$

$$\|E_\varepsilon(\varphi)\|_{W^{1,2}(\Omega)} \le c\|\varphi\|_{W^{1,2}(\Omega_\varepsilon)}, \tag{A.2}$$

$$\|E_\varepsilon(\varphi)\|_{L^q(\Omega)} \le c\|\varphi\|_{L^q(\Omega_\varepsilon)}, \tag{A.3}$$

where the positive constant c is independent of ε.

We are now ready to construct the operator P_ε.

Proposition A.2. *Suppose Ω_ε is given by* (1.7)*. There exists an extension operator*

$$P_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to W^{1,2}(\Omega)$$

such that for each $\varphi \in W^{1,2}(\Omega_\varepsilon)$ we have

$$P_\varepsilon(\varphi) = \varphi \quad \text{in } \Omega_\varepsilon, \tag{A.4}$$

$$\Delta_x P_\varepsilon(\varphi) = 0 \quad \text{in } \Omega \setminus \Omega_\varepsilon, \tag{A.5}$$

$$\|P_\varepsilon(\varphi)\|_{W^{1,2}(\Omega)} \le c\|\varphi\|_{W^{1,2}(\Omega_\varepsilon)}, \tag{A.6}$$

where the positive constant c is independent of ε.

Proof. Let $\varphi \in W^{1,2}(\Omega_\varepsilon)$ be fixed. From [21, Lemma 4.1], for any $\varepsilon > 0$ and any $n = 1, \dots, N(\varepsilon)$, there exists an extension operator

$$E_{\varepsilon,n} : W^{1,2}(B_{\varepsilon,n,\delta_0} \setminus B_{\varepsilon,n}) \to W^{1,2}(B_{\varepsilon,n})$$

such that

$$E_{\varepsilon,n}(\varphi) = \varphi \quad \text{in } B_{\varepsilon,n,\delta_0} \setminus B_{\varepsilon,n}, \tag{A.7}$$

$$\|\nabla_x E_{\varepsilon,n}(\varphi)\|_{L^2(B_{\varepsilon,n})} \le c\|\nabla_x \varphi\|_{L^2(B_{\varepsilon,n,\delta_0}\setminus B_{\varepsilon,n})}, \tag{A.8}$$

$$\|E_{\varepsilon,n}(\varphi)\|_{L^2(B_{\varepsilon,n})} \le c\|\varphi\|_{L^2(B_{\varepsilon,n,\delta_0}\setminus B_{\varepsilon,n})}, \tag{A.9}$$

Let us now consider the translation

$$\widetilde{E}_{\varepsilon,n}(\varphi)(y) := E_{\varepsilon,n}(\varphi)(x_{\varepsilon,n} + \varepsilon^\alpha y), \quad \text{for any } y \in B_1(0);$$

Clearly, $\widetilde{E}_{\varepsilon,n}(\varphi) \in W^{1,2}(B_1(0))$ and therefore, there exists a unique $\widetilde{P}_{\varepsilon,n}(\varphi) \in W^{1,2}(B_1(0))$ such that

$$\Delta_y \widetilde{P}_{\varepsilon,n}(\varphi) = 0 \quad \text{on } B_1(0), \tag{A.10}$$

$$\widetilde{P}_{\varepsilon,n}(\varphi) = \widetilde{E}_{\varepsilon,n}(\varphi) \quad \text{on } \partial B_1(0), \tag{A.11}$$

$$\|\nabla_y \widetilde{P}_{\varepsilon,n}(\varphi)\|_{L^2(B_1(0);\mathbb{R}^3)} \le c\|\nabla_y \widetilde{E}_{\varepsilon,n}(\varphi)\|_{L^2(B_1(0);\mathbb{R}^3)}, \tag{A.12}$$

$$\|\widetilde{P}_{\varepsilon,n}(\varphi)\|^2_{L^2(B_1(0))} \le c\left(\|\widetilde{E}_{\varepsilon,n}(\varphi)\|^2_{L^2(B_1(0))} + \|\nabla_y \widetilde{E}_{\varepsilon,n}(\varphi)\|^2_{L^2(B_1(0);\mathbb{R}^3)}\right); \tag{A.13}$$

cf. [24, Theorem 7.1.2] (more precisely, one should first solve the Laplace equation with shifted boundary data $\widetilde{E}_{\varepsilon,n}(\varphi) - [\widetilde{E}_{\varepsilon,n}(\varphi)]_{B_1(0)}$, where $[\widetilde{E}_{\varepsilon,n}(\varphi)]_{B_1(0)}$ denotes the mean value

$$[\widetilde{E}_{\varepsilon,n}(\varphi)]_{B_1(0)} := \frac{1}{|B_1(0)|}\int_{B_1(0)} \widetilde{E}_{\varepsilon,n}(\varphi)(y)\, \mathrm{d}y,$$

and shift one again the correspondent solution). Defining

$$P_{\varepsilon,n}(\varphi)(x) := \widetilde{P}_{\varepsilon,n}(\varphi)\left(\frac{x - x_{\varepsilon,n}}{\varepsilon^\alpha}\right) \quad \text{for any } x \in B_{\varepsilon,n},$$

we have that $P_{\varepsilon,n}(\varphi) \in W^{1,2}(B_{\varepsilon,n})$ and from (A.7), (A.10) and (A.11)

$$\begin{aligned} \Delta_x P_{\varepsilon,n}(\varphi) &= 0 \quad \text{on } B_{\varepsilon,n}, \\ P_{\varepsilon,n}(\varphi) &= \varphi \quad \text{on } \partial B_{\varepsilon,n}. \end{aligned}$$

Moreover, from (A.8) and (A.12)

$$\|\nabla_x P_{\varepsilon,n}(\varphi)\|^2_{L^2(B_{\varepsilon,n};\mathbb{R}^3)} = \varepsilon^\alpha \|\nabla_y \widetilde{P}_{\varepsilon,n}(\varphi)\|^2_{L^2(B_1(0);\mathbb{R}^3)} \le c\|\nabla_x \varphi\|^2_{L^2(B_{\varepsilon,n,\delta_0}\setminus B_{\varepsilon,n})},$$

while from (A.9) and (A.13) we have

$$\begin{aligned} \|P_{\varepsilon,n}(\varphi)\|^2_{L^2(B_{\varepsilon,n})} &= \varepsilon^{3\alpha}\|\widetilde{P}_{\varepsilon,n}(\varphi)\|^2_{L^2(B_1(0))} \\ &\le c\left(\|E_{\varepsilon,n}(\varphi)\|^2_{L^2(B_{\varepsilon,n})} + \varepsilon^{2\alpha}\|\nabla_x E_{\varepsilon,n}(\varphi)\|^2_{L^2(B_{\varepsilon,n})}\right) \\ &\le c\|\varphi\|^2_{W^{1,2}(B_{\varepsilon,n,\delta_0}\setminus B_{\varepsilon,n})}. \end{aligned}$$

It is therefore enough to define

$$P_\varepsilon(\varphi) := \begin{cases} \varphi & \text{on } \Omega_\varepsilon, \\ P_{\varepsilon,n}(\varphi) & \text{on } B_{\varepsilon,n} \text{ for any } n = 1, \dots, N(\varepsilon), \end{cases}$$

to get the claim. □

Appendix B. The restriction operator $\mathcal{R}_\varepsilon$

In this section we construct the linear operator $\mathcal{R}_\varepsilon$ that restricts to the perforated domain Ω_ε functions initially defined on the whole domain Ω. This type of operator plays a crucial role in the extension of the weak formulation of the balance of momentum (2.3) to the homogenized domain Ω, cf. Section 4.2.

Proposition B.1. *Let $p \in (1,\infty)$ be fixed and let Ω_ε the perforated domain defined by* (1.7). *For any $\varepsilon \in (0,1)$ there exists a linear operator*

$$\mathcal{R}_\varepsilon : W_0^{1,p}(\Omega;\mathbb{R}^3) \to W_0^{1,p}(\Omega_\varepsilon;\mathbb{R}^3)$$

such that for any $\boldsymbol{\varphi} \in W_0^{1,p}(\Omega;\mathbb{R}^3)$,

$$\|\mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{W_0^{1,p}(\Omega_\varepsilon)} \le c\left(1 + \varepsilon^{\frac{3(\alpha-\beta)}{p} - \alpha}\right)\|\boldsymbol{\varphi}\|_{W_0^{1,p}(\Omega)}, \tag{B.1}$$

where the positive constant c does not depend on ε. Moreover, if $\mathrm{div}_x \boldsymbol{\varphi} = 0$ then $\mathrm{div}_x \mathcal{R}_\varepsilon(\boldsymbol{\varphi}) = 0$.

The operator $\mathcal{R}_\varepsilon$ can be constructed implementing the technique developed by Allaire for the case $p = 2$, cf. [3, Section 2.2]. Notice that an analogous restriction operator with $\beta = 1$ was considered by Diening, Feireisl and Lu when constructing the inverse of the divergence operator on perforated domains, cf. [10, equation (3.12)], and more recently by Lu, cf. [20, Theorem 2.1].

We introduce

$$C_{\varepsilon,n} := D_{\varepsilon,n} \setminus \overline{B_{\varepsilon,n}}$$

In order to prove Proposition B.1, we need three technical lemmas.

Lemma B.2. *For any give $x_0 \in \mathbb{R}^3$, $0 < r_1 < r_2$, $a > 0$, there exists*

$$\varphi_a \in C_c^\infty\big(B(x_0, ar_2)\big), \quad 0 \le \varphi_a \le 1, \quad \varphi_a\big|_{\overline{B(x_0,ar_1)}} = 1, \tag{B.2}$$

such that for any $1 \le p \le \infty$,

$$\|\varphi_a\|_{L^p(\mathbb{R}^3)} \le c(r_2)\ a^{\frac{3}{p}}, \tag{B.3}$$

$$\|\nabla_x \varphi_a\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} \le c(r_1, r_2)\ a^{\frac{3}{p}-1}. \tag{B.4}$$

Proof. It is easy to construct

$$\varphi \in C_c^\infty\big(B(0, r_2)\big), \quad 0 \le \varphi \le 1, \quad \varphi\big|_{\overline{B(0,r_1)}} = 1.$$

Now it is enough to define

$$\varphi_a(x) := \varphi\left(\frac{x - x_0}{a}\right).$$

Clearly, conditions (B.2) are satisfied; moreover, for any $1 \le p \le \infty$, we have

$$\|\varphi_a\|_{L^p(\mathbb{R}^3)} = \|\varphi_a\|_{L^p(B(x_0,ar_2))} = a^{\frac{3}{p}} \|\varphi\|_{L^p(B(0,r_2))} = a^{\frac{3}{p}} \|\varphi\|_{L^p(\mathbb{R}^3)},$$

$$\|\nabla_x\varphi_a\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} = \|\nabla_x\varphi_a\|_{L^p(B(x_0,ar_2)\setminus B(x_0,ar_1))} = a^{\frac{3}{p}-1}\|\nabla_x\varphi\|_{L^p(B(0,r_2)\setminus B(0,r_1))} = a^{\frac{3}{p}-1}\|\nabla_x\varphi\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)}.$$

□

Lemma B.3. *Let $p \in (1, \infty)$ be fixed. For any $\varepsilon \in (0, 1)$ and any $n = 1, \dots, N(\varepsilon)$ there exists a linear operator*

$$\mathcal{L}_{\varepsilon,n} : W^{1,p}(D_{\varepsilon,n}; \mathbb{R}^3) \to W^{1,p}(C_{\varepsilon,n}; \mathbb{R}^3)$$

such that for any $\boldsymbol{\varphi} \in W^{1,p}(D_{\varepsilon,n}; \mathbb{R}^3)$

$$\mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) = \begin{cases} \boldsymbol{\varphi} & \text{on } \partial D_{\varepsilon,n}, \\ \mathbf{0} & \text{on } \partial B_{\varepsilon,n}, \end{cases} \tag{B.5}$$

and

$$\|\nabla_x \mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(C_{\varepsilon,n})} \le c\left(\|\nabla_x\boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} + \varepsilon^{\frac{3(\alpha-\beta)}{p}-\alpha}\|\boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})}\right), \tag{B.6}$$

where the positive constant c does not depend on ε or n.

Proof. Let δ_0, δ_1 be two positive fixed constants such that

$$0 < \delta_1 < 1 < \delta_0, \quad \frac{\delta_0}{\delta_1} < \frac{3}{2};$$

then, defining for any constant $\delta > 0$

$$B_{\varepsilon,n,\delta} := B(x_{\varepsilon,n}, \delta\varepsilon^\alpha),$$

$$D_{\varepsilon,n,\delta} := B\left(x_{\varepsilon,n}, \delta\Big(\varepsilon^\alpha + \frac{1}{2}\varepsilon^\beta\Big)\right),$$

we have the following inclusions:

$$B_{\varepsilon,n} \subset B_{\varepsilon,n,\delta_0} \subset D_{\varepsilon,n,\delta_1} \subset D_{\varepsilon,n}.$$

Next, we introduce two cut-off functions $\phi_{\varepsilon,n}, \psi_{\varepsilon,n}$; indeed, it is sufficient to apply Lemma B.2 choosing first $x_0 = x_{\varepsilon,n}$, $r_1 = 1$, $r_2 = \delta_1$, $a = \varepsilon^\alpha$, and subsequently $x_0 = x_{\varepsilon,n}$, $r_1 = \delta_1$, $r_2 = 1$, $a = \varepsilon^\alpha + \frac{1}{2}\varepsilon^\beta$ to get

$$\phi_{\varepsilon,n} \in C_c^\infty(B_{\varepsilon,n,\delta_0}), \quad 0 \le \phi_{\varepsilon,n} \le 1, \quad \phi\big|_{\overline{B_{\varepsilon,n}}} = 1, \quad \|\nabla_x\phi_{\varepsilon,n}\|_{L^\infty(\mathbb{R}^3)} \le c(\delta_0)\varepsilon^{-\alpha}. \tag{B.7}$$

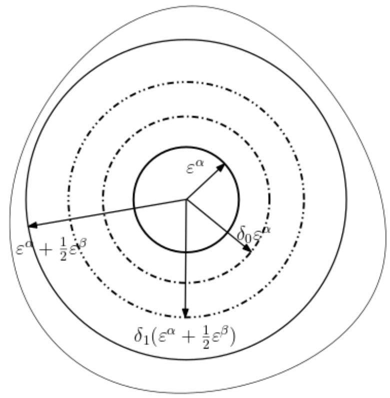

FIGURE 3. The inclusion: $B_{\varepsilon,n} \subset B_{\varepsilon,n,\delta_0} \subset D_{\varepsilon,n,\delta_1} \subset D_{\varepsilon,n}$

$$\psi_{\varepsilon,n} \in C_c^\infty(D_{\varepsilon,n}), \quad 0 \le \psi_{\varepsilon,n} \le 1, \quad \psi|_{\overline{D_{\varepsilon,n,\delta_1}}} = 1, \quad \|\nabla_x \psi_{\varepsilon,n}\|_{L^\infty(\mathbb{R}^3)} \le c(\delta_1)\varepsilon^{-\beta}. \tag{B.8}$$

We will denote

$$E_{\varepsilon,n} := D_{\varepsilon,n} \setminus \overline{D_{\varepsilon,n,\delta_1}}, \quad |E_{\varepsilon,n}| = c(\delta_1)\varepsilon^{3\beta}$$
$$F_{\varepsilon,n} := B_{\varepsilon,n,\delta_0} \setminus \overline{B_{\varepsilon,n}}, \quad |F_{\varepsilon,n}| = c(\delta_0)\varepsilon^{3\alpha}.$$

Letting

$$\langle h \rangle_B := \frac{1}{|B|}\int_B h \, \mathrm{d}x,$$

for any $\boldsymbol{\varphi} \in W^{1,p}(D_{\varepsilon,n};\mathbb{R}^3)$, we introduce

$$\boldsymbol{a}_{\varepsilon,n}(\boldsymbol{\varphi}) := (1-\phi_{\varepsilon,n})\left(\langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right), \tag{B.9}$$
$$\boldsymbol{b}_{\varepsilon,n}(\boldsymbol{\varphi}) := (1-\psi_{\varepsilon,n})\left(\boldsymbol{\varphi} - \langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right), \tag{B.10}$$

and the linear operator

$$\mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) := \boldsymbol{a}_{\varepsilon,n}(\boldsymbol{\varphi}) + \boldsymbol{b}_{\varepsilon,n}(\boldsymbol{\varphi}).$$

Notice that

- if $x \in \partial D_{\varepsilon,n}$ then $\phi_{\varepsilon,n}(x) = \psi_{\varepsilon,n}(x) = 0$ and $\mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) = \boldsymbol{\varphi}$;
- if $x \in \partial B_{\varepsilon,n}$ then $\phi_{\varepsilon,n}(x) = \psi_{\varepsilon,n}(x) = 1$ and $\mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) = \mathbf{0}$;

hence, (B.5) is satisfied. Furthermore, from (B.7), Jensen and Hölder inequalities we have

$$\begin{aligned}
\|\nabla_x \boldsymbol{a}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(F_{\varepsilon,n})} &= \left\|\nabla_x \phi_{\varepsilon,n} \otimes \langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right\|_{L^p(F_{\varepsilon,n})} \lesssim \|\nabla_x \phi_{\varepsilon,n}\|_{L^\infty(\mathbb{R}^3)} \left|\langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right| \ |F_{\varepsilon,n}|^{\frac{1}{p}} \\
&\le c(\delta_0)\varepsilon^{\left(\frac{3}{p}-1\right)\alpha} \frac{1}{|D_{\varepsilon,n}|}\int_{D_{\varepsilon,n}} |\boldsymbol{\varphi}| \, \mathrm{d}x \le c(\delta_0)\varepsilon^{\left(\frac{3}{p}-1\right)\alpha} |D_{\varepsilon,n}|^{\frac{1}{p'}-1} \|\boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} \\
&\le c(\delta_0)\varepsilon^{\frac{3(\alpha-\beta)}{p}-\alpha} \|\boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})}.
\end{aligned} \tag{B.11}$$

From Poincaré's inequality and the fact that $\varepsilon^\alpha < \varepsilon^\beta$, we have

$$\left\|\boldsymbol{\varphi} - \langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right\|_{L^p(D_{\varepsilon,n})} \lesssim \left(\varepsilon^\alpha + \frac{1}{2}\varepsilon^\beta\right) \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} \lesssim \varepsilon^\beta \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})}.$$

Therefore, from (B.8) we can estimate

$$\begin{aligned}
\|\nabla_x \boldsymbol{b}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(E_{\varepsilon,n})} &\le \left\|(1-\psi_{\varepsilon,n})\nabla_x\left(\boldsymbol{\varphi} - \langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right)\right\|_{L^p(D_{\varepsilon,n})} + \left\|\nabla_x \psi_{\varepsilon,n} \otimes \left(\boldsymbol{\varphi} - \langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right)\right\|_{L^p(D_{\varepsilon,n})} \\
&\lesssim \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} + \varepsilon^{-\beta}\left\|\boldsymbol{\varphi} - \langle \boldsymbol{\varphi} \rangle_{D_{\varepsilon,n}}\right\|_{L^p(D_{\varepsilon,n})} \\
&\lesssim \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})}.
\end{aligned} \tag{B.12}$$

Putting together (B.11) and (B.12), we finally obtain (B.6). □

Lemma B.4. *Let $p \in (1,\infty)$ be fixed. For any $\varepsilon \in (0,1)$ and any $n = 1,\dots,N(\varepsilon)$ there exists a linear operator*

$$\mathcal{S}_{\varepsilon,n} : W^{1,p}(D_{\varepsilon,n};\mathbb{R}^3) \to W^{1,p}(C_{\varepsilon,n};\mathbb{R}^3)$$

such that for any $\boldsymbol{\varphi} \in W^{1,p}(D_{\varepsilon,n};\mathbb{R}^3)$

$$\mathrm{div}_x \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) = \mathrm{div}_x \boldsymbol{\varphi} + \frac{1}{|C_{\varepsilon,n}|} \int_{B_{\varepsilon,n}} \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x, \tag{B.13}$$

$$\mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) = \begin{cases} \boldsymbol{\varphi} & \text{on } \partial D_{\varepsilon,n}, \\ \mathbf{0} & \text{on } \partial B_{\varepsilon,n}, \end{cases} \tag{B.14}$$

and

$$\|\nabla_x \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(C_{\varepsilon,n})} \leq c \left(\|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} + \varepsilon^{\frac{3(\alpha-\beta)}{p} - \alpha} \|\boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} \right), \tag{B.15}$$

where the positive constant c does not depend on ε or n.

Proof. Let $\boldsymbol{\varphi} \in W^{1,p}(D_{\varepsilon,n};\mathbb{R}^3)$ be fixed. Defining

$$\mathcal{F}(\boldsymbol{\varphi}) := \mathrm{div}_x \left[\boldsymbol{\varphi} - \mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi})\right] + \frac{1}{|C_{\varepsilon,n}|} \int_{B_{\varepsilon,n}} \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x,$$

where $\mathcal{L}_{\varepsilon,n}$ is the linear operator constructed in Lemma B.3, we have that $\mathcal{F}(\boldsymbol{\varphi}) \in L^p(C_{\varepsilon,n})$ and from (B.5),

$$\int_{C_{\varepsilon,n}} \mathcal{F}(\boldsymbol{\varphi}) \, \mathrm{d}x = \int_{D_{\varepsilon,n}} \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x - \int_{C_{\varepsilon,n}} \mathrm{div}_x \mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) \, \mathrm{d}x = \int_{\partial D_{\varepsilon,n}} \boldsymbol{\varphi} \cdot \mathbf{n} \, \mathrm{d}\sigma_x - \int_{\partial D_{\varepsilon,n}} \boldsymbol{\varphi} \cdot \mathbf{n} \, \mathrm{d}\sigma_x = 0.$$

Therefore, if we consider the standard Bogovskii operator

$$\mathcal{B}_{C_{\varepsilon,n}} : L^p_0(C_{\varepsilon,n}) \to W^{1,p}_0(C_{\varepsilon,n};\mathbb{R}^3),$$

there exists $\mathcal{B}_{C_{\varepsilon,n}}(\mathcal{F}(\boldsymbol{\varphi})) \in W^{1,p}_0(C_{\varepsilon,n};\mathbb{R}^3)$ such that $\mathrm{div}_x \mathcal{B}_{C_{\varepsilon,n}}(\mathcal{F}(\boldsymbol{\varphi})) = \mathcal{F}(\boldsymbol{\varphi})$ and

$$\begin{aligned} \|\nabla_x \mathcal{B}_{C_{\varepsilon,n}}(\mathcal{F}(\boldsymbol{\varphi}))\|_{L^p(C_{\varepsilon,n})} &\lesssim \left(1 + \frac{|D_{\varepsilon,n}|}{|C_{\varepsilon,n}|}\right) \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} + \|\mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(C_{\varepsilon,n})} \\ &\lesssim \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} + \|\mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(C_{\varepsilon,n})}, \end{aligned} \tag{B.16}$$

where we used the fact that

$$\frac{|D_{\varepsilon,n}|}{|C_{\varepsilon,n}|} = 1 + \frac{|B_{\varepsilon,n}|}{|C_{\varepsilon,n}|} \lesssim 1 + \varepsilon^{\alpha-\beta} \leq 2.$$

If we now consider the Stokes problem

$$\begin{aligned} \nabla_x q_{\varepsilon,n} - \Delta_x \mathbf{v}_{\varepsilon,n} &= -\Delta_x \left[\boldsymbol{\varphi} - \mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) - \mathcal{B}_{C_{\varepsilon,n}}(\mathcal{F}(\boldsymbol{\varphi}))\right], \\ \mathrm{div}_x \mathbf{v}_{\varepsilon,n} &= 0, \end{aligned}$$

on $C_{\varepsilon,n}$, with the boundary condition

$$\mathbf{v}_{\varepsilon,n} = 0 \quad \text{on } \partial C_{\varepsilon,n},$$

it is known, see e.g. [16], Theorem IV.6.1, point (b), that it admits a solution

$$(q_{\varepsilon,n}, \mathbf{v}_{\varepsilon,n}) \in L^p(C_{\varepsilon,n}) \times W^{1,p}_0(C_{\varepsilon,n};\mathbb{R}^3)$$

such that

$$\|\nabla_x \mathbf{v}_{\varepsilon,n}\|_{L^p(C_{\varepsilon,n})} \leq \left\|\nabla_x \left[\boldsymbol{\varphi} - \mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) - \mathcal{B}_{C_{\varepsilon,n}}(\mathcal{F}(\boldsymbol{\varphi}))\right]\right\|_{L^p(C_{\varepsilon,n})}. \tag{B.17}$$

It is sufficient to define

$$\mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) := \mathbf{v}_{\varepsilon,n} + \mathcal{L}_{\varepsilon,n}(\boldsymbol{\varphi}) + \mathcal{B}_{C_{\varepsilon,n}}(\mathcal{F}(\boldsymbol{\varphi}))$$

Indeed, equation (B.13) and the boundary conditions (B.14) are satisfied, as well as the compatibility condition,

$$\begin{aligned} \int_{C_{\varepsilon,n}} \mathrm{div}_x \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) \, \mathrm{d}x &= \int_{C_{\varepsilon,n}} \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x + \int_{B_{\varepsilon,n}} \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x = \int_{D_{\varepsilon,n}} \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x = \int_{\partial D_{\varepsilon,n}} \boldsymbol{\varphi} \cdot \mathbf{n} \, \mathrm{d}\sigma_x \\ &= \int_{\partial D_{\varepsilon,n}} \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) \cdot \mathbf{n} \, \mathrm{d}\sigma_x = \int_{\partial C_{\varepsilon,n}} \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) \cdot \mathbf{n} \, \mathrm{d}\sigma_x. \end{aligned}$$

Finally, combining (B.6), (B.16) and (B.17), we get (B.15). □

Proof of Proposition B.1. Given $\boldsymbol{\varphi} \in W^{1,p}(\Omega;\mathbb{R}^3)$, it is enough to define

$$\mathcal{R}_\varepsilon(\boldsymbol{\varphi}) := \begin{cases} \boldsymbol{\varphi} & \text{on } \Omega \setminus \bigcup_{n=1}^{N(\varepsilon)} D_{\varepsilon,n}, \\ \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) & \text{on } C_{n,\varepsilon},\ n=1,\dots,N(\varepsilon), \\ \mathbf{0} & \text{on } B_{n,\varepsilon},\ n=1,\dots,N(\varepsilon), \end{cases}$$

where $\mathcal{S}_{\varepsilon,n}$ is the linear operator constructed in Lemma B.4. Clearly $\mathcal{R}_\varepsilon$ is linear and it "glues well" on the boundary of each $C_{\varepsilon,n}$. Moreover, from (B.15), we have

$$\begin{aligned} \|\nabla_x \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^p(\Omega_\varepsilon)} &\le \|\nabla_x \boldsymbol{\varphi}\|_{L^p\left(\Omega\setminus\bigcup_{n=1}^{N(\varepsilon)} D_{\varepsilon,n}\right)} + \sum_{n=1}^{N(\varepsilon)} \|\nabla_x \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi})\|_{L^p(C_{\varepsilon,n})} \\ &\lesssim \|\nabla_x \boldsymbol{\varphi}\|_{L^p\left(\Omega\setminus\bigcup_{n=1}^{N(\varepsilon)} D_{\varepsilon,n}\right)} + \sum_{n=1}^{N(\varepsilon)} \|\nabla_x \boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})} + \varepsilon^{\frac{3(\alpha-\beta)}{p}-\alpha} \sum_{n=1}^{N(\varepsilon)} \|\boldsymbol{\varphi}\|_{L^p(D_{\varepsilon,n})}; \end{aligned}$$

from the fact that $D_{\varepsilon,n_1} \cap D_{\varepsilon,n_2} = \emptyset$ if $n_1 \neq n_2$, we get (B.1). Finally, if $\operatorname{div}_x \boldsymbol{\varphi} = 0$ on $D_{\varepsilon,n}$ for any $n = 1,\dots,N(\varepsilon)$, from (B.13) we obtain $\operatorname{div}_x \mathcal{S}_{\varepsilon,n}(\boldsymbol{\varphi}) = 0$ on $C_{\varepsilon,n}$ for any $n=1,\dots,N(\varepsilon)$. Consequently, if $\operatorname{div}_x \boldsymbol{\varphi} = 0$ on Ω, $\operatorname{div}_x \mathcal{R}_\varepsilon(\boldsymbol{\varphi}) = 0$ on Ω_ε. □

We conclude this section with the following result, which is extensively used in Section 4.2.

Lemma B.5. *Let $r, q \in (1,\infty)$ and $p \in \left[\frac{3}{2},\infty\right)$ be such that $p, q > r$. Then for any $\boldsymbol{\varphi} \in C_c^\infty(\Omega;\mathbb{R}^3)$*

$$\|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^r(\Omega_\varepsilon)} \lesssim \varepsilon^{3\beta\left(\frac{1}{r}-\frac{1}{p}\right)} \left(1 + \varepsilon^{\frac{3(\alpha-\beta)}{p}-\beta}\right) \|\boldsymbol{\varphi}\|_{W_0^{1,\frac{3p}{p+3}}(\Omega)}, \tag{B.18}$$

$$\left\|\nabla_x\left[\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\right]\right\|_{L^r(\Omega_\varepsilon)} \lesssim \varepsilon^{3\beta\left(\frac{1}{r}-\frac{1}{q}\right)} \left(1 + \varepsilon^{\frac{3(\alpha-\beta)}{q}-\alpha}\right) \|\boldsymbol{\varphi}\|_{W_0^{1,q}(\Omega)}, \tag{B.19}$$

Proof. From the proof of Proposition B.1 it is clear that $\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})$ vanishes everywhere with the exception of the disjoints sets $D_{\varepsilon,n}$, $n=1,\dots,N(\varepsilon)$. Therefore, form Hölder's inequality and the fact that $|D_{\varepsilon,n}| \simeq \varepsilon^{3\beta}$, we have

$$\begin{aligned} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^r(\Omega)} = \sum_{n=1}^{N(\varepsilon)} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^r(D_{\varepsilon,n})} &\le |D_{\varepsilon,n}|^{\frac{1}{r}-\frac{1}{p}} \sum_{n=1}^{N(\varepsilon)} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^p(D_{\varepsilon,n})} \\ &\lesssim \varepsilon^{3\beta\left(\frac{1}{r}-\frac{1}{p}\right)} \|\boldsymbol{\varphi} - \mathcal{R}_\varepsilon(\boldsymbol{\varphi})\|_{L^p(\Omega)}. \end{aligned}$$

Next, from the Sobolev embedding

$$W_0^{1,\frac{3p}{p+3}}(\Omega) \hookrightarrow L^p(\Omega), \quad \text{for } p \in \left[\frac{3}{2},\infty\right),$$

and using (B.1), we obtain (B.18); similarly, we get (B.19). □

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