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TINGLEY'S PROBLEM FOR COMBINATORIAL TSIRELSON SPACES

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ABSTRACT. We extend the existing results on surjective isometries of unit spheres in the Tsirelson space $T\left[\frac{1}{2}, \mathcal{S}_1\right]$ to the class $T[\theta, \mathcal{S}_\alpha]$ for any integer $\theta^{-1} \geqslant 2$ and $1 \leqslant \alpha < \omega_1$, where $\mathcal{S}_\alpha$ denotes the Schreier family of order α. This positively answers Tingley's problem for these spaces, which asks whether every surjective isometry between unit spheres can be extended to a surjective linear isometry of the entire space.

Furthermore, we improve the result stating that every linear isometry on $T[\theta, \mathcal{S}_1]$ $\left(\theta \in \left(0, \frac{1}{2}\right]\right)$ is determined by a permutation of the first $\lceil \theta^{-1} \rceil$ elements of the canonical unit basis, followed by a possible sign change of the corresponding coordinates and a sign change of the remaining coordinates. Specifically, we prove that only the first $\lfloor \theta^{-1} \rfloor$ elements can be permuted. This finding enables us to establish a sufficient condition for being a linear isometry in these spaces.

1. INTRODUCTION AND THE MAIN RESULT

In 1987, Tingley [22] proposed a question that has since become known as Tingley's problem:

Let X and Y be normed spaces with unit spheres $\mathbb{S}_X$ and $\mathbb{S}_Y$, respectively. Suppose that $U\colon \mathbb{S}_X \to \mathbb{S}_Y$ is a surjective isometry. Is there a linear isometry $\tilde{U}\colon X \to Y$ such that $\tilde{U}|_{\mathbb{S}_X} = U$?

Many authors have shown that Tingley's problem has a positive solution for surjective isometries of unit spheres in classical Banach spaces $\ell_p(\Gamma)$, $L_p(\mu)$ $(1 \leqslant p \leqslant \infty)$, and $C(\Omega)$ (see, *e.g.*, [6–11, 13–16, 19, 20, 25]). However, the general case remains open. Notable results in the search for a solution to Tingley's problem in specific spaces have been comprehensively documented in surveys by A. M. Peralta [18], G. G. Ding [12], X. Yang, and X. Zhao [26]. Recently, a positive solution to this isometric expansion problem has been found for 2-dimensional Banach spaces (see [2]); nevertheless, the answer remains unknown for higher dimensions. Positive solutions for certain subspaces of function algebras, including closed function algebras on locally compact Hausdorff spaces, have been presented in more recent studies (see [5]).

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The Tsirelson space T (the dual of the space constructed by Tsirelson [24], which was the first example of a space containing no isomorphic copies of c_0 or ℓ_p for $1 \leqslant p < \infty$) can be regarded as a special case of the double-parameter family of Banach spaces $T[\theta, \mathcal{S}_\alpha]$, where $\theta \in \left(0, \frac{1}{2}\right]$ and $1 \leqslant \alpha < \omega_1$, with $\mathcal{S}_\alpha$ being the Schreier family of order α, where α is a countable ordinal. For the sake of brevity, we use the term *combinatorial Tsirelson spaces* to refer to the members of this family, in line with the terminology used in previous articles such as [3, 17].

In [17] we have characterized linear isometries of combinatorial Tsirelson spaces. However, the methods employed assume linearity of the isometries throughout the entire space. We improve the main theorem from this article by proving the following first main result:

Theorem A. Let $\theta \in \left(0, \frac{1}{2}\right]$. Then $U\colon T\big[\theta, \mathcal{S}_1\big] \to T\big[\theta, \mathcal{S}_1\big]$ is a linear isometry if and only if

$$Ue_i = \begin{cases} \varepsilon_i e_{\pi(i)}, & 1 \leqslant i \leqslant \lfloor \theta^{-1} \rfloor \\ \varepsilon_i e_i, & i > \lfloor \theta^{-1} \rfloor \end{cases} \qquad (i \in \mathbb{N})$$

for some $\{-1, 1\}$-valued sequence $(\varepsilon_i)_{i=1}^{\infty}$ and a permutation π of $\big\{1, 2, \ldots, \lfloor \theta^{-1} \rfloor\big\}$.

Then, following the approach of [21], where the authors determine the surjective isometries of the unit spheres of Tsirelson space $T[\frac{1}{2}, \mathcal{S}_1]$ and the modified Tsirelson space T_M and answer Tingley's problem affirmatively in these spaces, we establish the subsequent main Theorem.

Theorem B. Let $\theta^{-1} \geqslant 2$ be an integer and let $U\colon \mathbb{S}_{T[\theta, \mathcal{S}_\alpha]} \to \mathbb{S}_{T[\theta, \mathcal{S}_\alpha]}$ be surjective isometry. If $\alpha = 1$, then

$$U\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{\theta^{-1}} \varepsilon_i a_i e_{\pi(i)} + \sum_{i=\theta^{-1}+1}^{\infty} \varepsilon_i a_i e_i$$

and if $1 < \alpha < \omega_1$, then

$$U\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{\infty} \varepsilon_i a_i e_i,$$

for every $\sum_{i=1}^{\infty} a_i e_i \in \mathbb{S}_{T[\theta, \mathcal{S}_\alpha]}$, where $(\varepsilon_i)_{i=1}^{\infty}$ is a $\{-1, 1\}$-valued sequence and π is a permutation of $\big\{1, 2, \ldots, \theta^{-1}\big\}$.

This result together with Theorem A get an affirmative answer to the Tingley's problem in combinatorial Tsirelson spaces $T\big[\theta, \mathcal{S}_\alpha\big]$ for an integer $\theta^{-1} \geqslant 2$ and $1 \leqslant \alpha < \omega_1$.

2. Preliminaries

2.1. **Combinatorial spaces.** Let us denote by $(e_i)_{i=1}^\infty$ the standard unit vector basis of c_{00} and by $[\mathbb{N}]^{<\omega}$ the family of finite subsets of $\mathbb{N}$. We adopt the following notation for sets $F_1, F_2 \in [\mathbb{N}]^{<\omega}$: $F_1 < F_2$ means that $\max F_1 < \min F_2$, and we say that F_1 and F_2 are *consecutive* in this case. Additionally, we use the notation $F_1 < n$ instead of $F_1 < \{n\}$ for $n \in \mathbb{N}$.

Definition 1. A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *regular*, whenever it is simultaneously

- *hereditary* $\big(F \in \mathcal{F}$ and $G \subset F \implies G \in \mathcal{F}\big)$;
- *spreading* $\big(\{l_1, l_2, \dots, l_n\} \in \mathcal{F}$ and $l_i \leqslant k_i \implies \{k_1, k_2, \dots, k_n\} \in \mathcal{F}\big)$;
- *compact* as a subset of the Cantor set $\{0,1\}^{\mathbb{N}}$ via the natural identification of $F \in \mathcal{F}$ with
$$\chi_F = \sum_{i \in F} e_i \in \{0,1\}^{\mathbb{N}}.$$

The simplest examples of regular families include
$$\mathcal{A}_n := \big\{F \in [\mathbb{N}]^{<\omega} \colon |F| \leqslant n\big\} \quad (n \in \mathbb{N})$$
i.e., for a given $n \in \mathbb{N}$, the family of subsets of $\mathbb{N}$ having at most n elements. The family of Schreier sets is defined using these families in the following manner.

Definition 2. Given a countable ordinal α, we define inductively the Schreier family of order α as follows:

- $\mathcal{S}_0 := \mathcal{A}_1$;
- if α is a successor ordinal, *i.e.*, $\alpha = \beta + 1$ for some $\beta < \omega_1$, then
$$\mathcal{S}_\alpha := \left\{ \bigcup_{i=1}^d S_\beta^i \colon d \leqslant S_\beta^1 < S_\beta^2 < \dots < S_\beta^d,\ \big\{S_\beta^i\big\}_{i=1}^d \subset \mathcal{S}_\beta \text{ and } d \in \mathbb{N} \right\} \cup \big\{\emptyset\big\};$$
- if α is a non-zero limit ordinal and $(\alpha_n)_{n=1}^\infty$ is a fixed strictly increasing sequence of successor ordinals converging to α with $\mathcal{S}_{\beta_n} \subset \mathcal{S}_{\beta_{n+1}}$ for all $n \in \mathbb{N}$, where $\alpha_n = \beta_n + 1$ for all $n \in \mathbb{N}$, we set
$$\mathcal{S}_\alpha := \big\{S_{\alpha_n} \in [\mathbb{N}]^{<\omega} \colon S_{\alpha_n} \in \mathcal{S}_{\alpha_n},\ n \leqslant \min S_{\alpha_n} \text{ for some n}\big\} \cup \big\{\emptyset\big\}.$$

We emphasize that in the case where α is a limit ordinal, we require the sequence $(\alpha_n)_{n=1}^\infty$ cofinal in α to comprise successor ordinals as needed in the proof of Theorem B. We can assume, and we will, that $S_{\alpha_n} \subset S_{\alpha_{n+1}}$ for all $n \in \mathbb{N}$, which will also be employed in the proof of Theorem B. Indeed, repeating the proof of [4, Proposition 3.2.] in the case of Schreier families $\big\{S_\xi\big\}_{\xi < \omega_1}$ which are multiplicative in the sense of [4] we can also derive the desired result for Schreier families. Elements belonging to $\mathcal{S}_\alpha$ are called $\mathcal{S}_\alpha$*-sets*. The fact that these families are regular is well-established; see [4][Proposition 3.2] or [23].

2.2. **Combinatorial Tsirelson spaces.** For a regular family $\mathcal{F}$ and $\theta \in \big(0, \frac{1}{2}\big]$, we define the Banach space $T[\theta, \mathcal{F}]$ specializing it later to a combinatorial Tsirelson space $T[\theta, \mathcal{S}_\alpha]$ for some countable ordinal α.

For a vector $x = (a_1, a_2, \dots, a_n) \in c_{00}$ and a finite set $E \subset \mathbb{N}$, we employ the symbol Ex to represent the projection of x onto the space $[e_i : i \in E]$, given by

$$E\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i \in E} a_i e_i. \tag{2.1}$$

We denote by $\|\cdot\|_0$ the supremum norm on c_{00}. Suppose that for some $n \in \mathbb{N}$ the norm $\|\cdot\|_n$ has been defined. Let

$$\|x\|_{n+1} = \max\big\{\|x\|_n, \|x\|_{T_n}\big\} \quad (n \in \mathbb{N}),$$

where

$$\|x\|_{T_n} = \sup\left\{\theta \sum_{i=1}^{d} \big\|E_i x\big\|_n : E_1 < \dots < E_d,\ d \in \mathbb{N},\ \{E_i\}_{i=1}^d \subset [\mathbb{N}]^{<\omega},\ \{\min E_i\}_{i=1}^d \in \mathcal{F}\right\}.$$

We define the norm $\|x\|_{\theta,\mathcal{F}} := \sup_{n \in \mathbb{N}} \|x\|_n$ and denote by $T[\theta, \mathcal{F}]$ the completion of c_{00} with respect to it.

A proof by induction demonstrates that this norm is bounded above by the ℓ_1-norm and is given by the following implicit formula for $x \in T[\theta, \mathcal{F}]$:

$$\|x\|_{\theta,\mathcal{F}} = \max\big\{\|x\|_\infty, \|x\|_T\big\}, \tag{2.2}$$

where

$$\|x\|_T = \sup\left\{\theta \sum_{i=1}^{d} \big\|E_i x\big\|_{\theta,\mathcal{F}} : E_1 < \dots < E_d,\ d \in \mathbb{N},\ \{E_i\}_{i=1}^d \subset [\mathbb{N}]^{<\omega},\ \{\min E_i\}_{i=1}^d \in \mathcal{F}\right\}.$$

It can be readily deduced from the definition that the unit vectors $(e_i)_{i=1}^\infty$ form an 1-unconditional basis of the space $T[\theta, \mathcal{S}_\alpha]$ for a countable ordinal α.

For $x_1, x_2 \in c_{00}$, we write $x_1 < x_2$ whenever $\operatorname{supp} x_1 < \operatorname{supp} x_2$ and for $n \in \mathbb{N}$ we streamline the notation of $\operatorname{supp} x_1 < n$ to $x_1 < n$.

We adopt the following convention in this paper: we say that the norm of an element $x \in T[\theta, \mathcal{F}]$ is given by sets $E_1 < E_2 < \dots < E_d$ for some $d \in \mathbb{N}$ (with $\{\min E_i\}_{i=1}^d \in \mathcal{F}$) precisely when

$$\|x\|_{\theta,\mathcal{F}} = \theta \cdot \sum_{i=1}^{d} \big\|E_i x\big\|_{\theta,\mathcal{F}}.$$

To be concise, we write $\|\cdot\|$ instead of $\|\cdot\|_{\theta,\mathcal{S}_\alpha}$, where $\theta\in\big(0,\frac{1}{2}\big]$, $1\leqslant\alpha<\omega_1$.

3. Linear isometries on $T[\theta,\mathcal{S}_1]$ spaces for $\theta\in\big(0,\frac{1}{2}\big]$

For further considerations, let us fix $\theta\in\big(0,\frac{1}{2}\big]$ and let $\lfloor\theta^{-1}\rfloor$ and $\lceil\theta^{-1}\rceil$ be the floor and the ceiling of θ^{-1}, respectively. Note that we do not yet require θ^{-1} to be an integer. Fix a countable ordinal $\alpha\geqslant 1$. Throughout this paper we use $\mathbb{S}_{\mathbb{T}[\theta,\mathcal{S}_\alpha]}$to denote the unit sphere of $T[\theta,\mathcal{S}_\alpha]$.

In [17, Theorem A] we have obtained the following description of linear isometries on combinatorial Tsirelson spaces.

Theorem 3. *Let $\theta\in\big(0,\frac{1}{2}\big]$. If $U\colon T\big[\theta,\mathcal{S}_1\big]\to T\big[\theta,\mathcal{S}_1\big]$ is a linear isometry, then*

$$Ue_i=\begin{cases}\varepsilon_i e_{\pi(i)}, & 1\leqslant i\leqslant\lceil\theta^{-1}\rceil\\ \varepsilon_i e_i, & i>\lceil\theta^{-1}\rceil\end{cases}\qquad(i\in\mathbb{N})$$

for some $\{-1,1\}$-valued sequence $(\varepsilon_i)_{i=1}^\infty$ and a permutation π of $\big\{1,2,\ldots,\lceil\theta^{-1}\rceil\big\}$.

Armed with this result, we are now ready to prove Theorem A.

Proof. Suppose that $U\colon T\big[\theta,\mathcal{S}_1\big]\to T\big[\theta,\mathcal{S}_1\big]$ is a linear isometry.If θ^{-1} is an integer, there is nothing to prove, so assume that this is not the case. Then $\lceil\theta^{-1}\rceil-1=\lfloor\theta^{-1}\rfloor$ and $\lceil\theta^{-1}\rceil>\theta^{-1}$.

It is enough to show that for any $i\neq\lceil\theta^{-1}\rceil$ holds $\pi(i)\neq\lceil\theta^{-1}\rceil$.

Let $i\in\{1,2,\ldots,\lceil\theta^{-1}\rceil-1\}$ and suppose for the contrary that $\pi(i)=\lceil\theta^{-1}\rceil$, *i.e.*, $Ue_i=\varepsilon_i e_{\lceil\theta^{-1}\rceil}$. Then, by Theorem 3, for any indices $\lceil\theta^{-1}\rceil<j_1<j_2<\cdots<j_{\lceil\theta^{-1}\rceil-1}$ we have

$$\left\|Ue_i-\sum_{k=1}^{\lceil\theta^{-1}\rceil-1}Ue_{j_k}\right\|=\left\|\varepsilon_i e_{\lceil\theta^{-1}\rceil}-\sum_{k=1}^{\lceil\theta^{-1}\rceil-1}\varepsilon_{j_k}e_{j_k}\right\|=\theta\cdot\lceil\theta^{-1}\rceil>1.$$

On the other hand, since U is a linear isometry, we obtain

$$\left\|Ue_i-\sum_{k=1}^{\lceil\theta^{-1}\rceil-1}Ue_{j_k}\right\|=\left\|Ue_i-U\left(\sum_{k=1}^{\lceil\theta^{-1}\rceil-1}e_{j_k}\right)\right\|=\left\|e_i-\sum_{k=1}^{\lceil\theta^{-1}\rceil-1}e_{j_k}\right\|=1.$$

This contradiction finishes the proof that the isometry has the desired form.

Let $U\colon T\big[\theta,\mathcal{S}_1\big]\to T\big[\theta,\mathcal{S}_1\big]$ be of the form

$$Ue_i=\begin{cases}\varepsilon_i e_{\pi(i)}, & 1\leqslant i\leqslant\lfloor\theta^{-1}\rfloor\\ \varepsilon_i e_i, & i>\lfloor\theta^{-1}\rfloor\end{cases}\qquad(i\in\mathbb{N}).$$

We will show that U is an isometry. If θ^{-1} is an integer, then the proof is in [1, Theorem 4.1], so assume that this is not the case. Then $\lfloor\theta^{-1}\rfloor<\theta^{-1}$. We will show that it is impossible that the norm of some $x \in T\big[\theta, \mathcal{S}_1\big]$ is given by certain sets $d \leqslant E_1 < E_2 < \cdots < E_d$ for some $d \in \mathbb{N}$ with $d \leqslant \lfloor \theta^{-1} \rfloor$. Assume not. Then

$$\|x\| = \theta \cdot \sum_{i=1}^{d} \big\|E_i x\big\|. \tag{3.1}$$

and

$$\theta \cdot \sum_{i=1}^{d} \big\|E_i x\big\| \leqslant \theta \cdot d \cdot \|x\| < \|x\|.$$

Hence (3.1) cannot hold; a contradiction.

□

4. Isometries on $\mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$ for an integer $\theta^{-1} \geqslant 2$ and $1 \leqslant \alpha < \omega_1$

To prove Theorem B we need a series of lemmas; the proofs emulate that of [21].

Lemma 4. *Let $u, v \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$. Then for $\alpha = 1$ we have*

(1) $\min(\|u+y\|, \|u-y\|) \leqslant 1$ *for all* $y \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$ *if and only if* $u \in \{\pm e_1, \pm e_2, \dots, \pm e_{\lfloor \theta^{-1} \rfloor}\}$*;*

(2) *If* $v \geqslant \lfloor \theta^{-1} \rfloor + 1$ *and* $\min(\|v+y\|, \|v-y\|) \leqslant \theta \cdot (\lfloor \theta^{-1} \rfloor + 1)$ *for all* $y \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$*, then v has one of the following forms:*

 (a) $|v_{\lfloor \theta^{-1} \rfloor + 1}| = 1$ *with* $|v_i| \leqslant \theta$ *for all* $i \neq \lfloor \theta^{-1} \rfloor + 1$*;*

 (b) $v = \varepsilon e_m + a e_{\lfloor \theta^{-1} \rfloor + 1}$ *for some* $m \geqslant \lfloor \theta^{-1} \rfloor + 2$*, some* $\varepsilon \in \{-1, 1\}$ *and some* $|a| \leqslant \theta$*,*

and for $\alpha > 1$ holds

(3) $\min(\|u+y\|, \|u-y\|) \leqslant 1$ *for all* $y \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$ *if and only if* $u = \pm e_1$*;*

(4) *If* $v > 1$ *and* $\min(\|v+y\|, \|v-y\|) \leqslant \theta \cdot (\lfloor \theta^{-1} \rfloor + 1)$ *for all* $y \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$*, then* $v = \pm e_m$ *for some* $m > 1$*.*

Proof. (1) Since the implication ($\Leftarrow$) is trivial, we only need to prove the implication ($\Rightarrow$). Assume that $\min(\|u+y\|, \|u-y\|) \leqslant 1$ for all $y \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$. We will show that v has only one non-zero coordinate. Indeed, suppose to the contrary that $u_n \neq 0$ and $u_m \neq 0$ for some $n, m \in \mathbb{N}, n \neq m$. Define $y := \operatorname{sgn} u_n e_n - \operatorname{sgn} u_m e_m$. Then $y \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$. Since

$$\|u+y\| \geqslant 1 + |u_n| > 1$$

and

$$\|u-y\| \geqslant 1 + |u_m| > 1$$

we get a contradiction.

Since $\|u\| = 1$, so $u = \pm e_i$ for some $i \in \mathbb{N}$. Suppose that $i \geqslant \lfloor \theta^{-1} \rfloor + 1$ and take any indices $i < j_1 < j_2 < \ldots < j_{\lfloor \theta^{-1} \rfloor}$. Then

$$\left\| \sum_{k=1}^{\lfloor \theta^{-1} \rfloor} e_{j_k} \right\| = \max\{1, \theta \cdot \lfloor \theta^{-1} \rfloor\} = 1$$

and

$$\left\| u + \sum_{k=1}^{\lfloor \theta^{-1} \rfloor} e_{j_k} \right\| = \left\| u - \sum_{k=1}^{\lfloor \theta^{-1} \rfloor} e_{j_k} \right\| = \theta \cdot (\lfloor \theta^{-1} \rfloor + 1) > 1.$$

This contradiction ends the proof that $u = \pm e_i$ for some $i \leqslant \lfloor \theta^{-1} \rfloor$.

(2) We will show that $\|v\|_\infty = 1$. Indeed, suppose to the contrary that $\|v\|_\infty < 1$. Take $\varepsilon = \frac{\theta^{-1} - \lfloor \theta^{-1} \rfloor + 1 - \|v\|_\infty}{4} > 0$. Since $\|v\|_T = 1$, there exist sets $d \leqslant E_1 < E_2 < \cdots < E_d$ for which

$$\sum_{i=1}^{d} \big\| E_i v \big\| > \theta^{-1} - \varepsilon.$$

Choose indices $E_d < j_1 < j_2$ such that $|v_{j_1}| + |v_{j_2}| < \varepsilon$.

Suppose that the set E_1 has more than 2 elements. Let F be the set consisting of 2 smallest numbers from the set E_1 and define $E_0 := E_1 \setminus F$. Note that $\|Fv\| \leqslant \|Fv\|_\infty \leqslant \|v\|_\infty$ and $E_0 \geqslant d + 2$. Taking $E_0 < E_2 < E_3 < \ldots < E_d < \{j_1\} < \{j_2\}$ we obtain

$$\begin{aligned}
\min(\|v + e_{j_1} + e_{j_2}\|, \|v - e_{j_1} - e_{j_2}\|\} &\geqslant \theta\Big(\big\| E_0 v \big\| + \sum_{i=2}^{d} \big\| E_i v \big\| + 1 - |v_{j_1}| + 1 - |v_{j_2}| \Big) \\
&\geqslant \theta\Big(\sum_{i=1}^{d} \big\| E_i v \big\| - \big\| F v \big\| + 1 - |v_{j_1}| + 1 - |v_{j_2}| \Big) \\
&\geqslant \theta\Big(\sum_{i=1}^{d} \big\| E_i v \big\| - \|v\|_\infty + 1 - |v_{j_1}| + 1 - |v_{j_2}| \Big) \\
&> \theta\big(\theta^{-1} - \varepsilon - \|v\|_\infty + 2 - \varepsilon \big) \\
&> \theta(\lfloor \theta^{-1} \rfloor + 1).
\end{aligned}$$

Since $\|e_{j_1} + e_{j_2}\| = \max\{1, 2\theta\} = 1$ we get a contradiction.

Suppose now that the set E_1 has at most 2 elements. Then $\|E_1 v\| \leqslant \|E_1 v\|_\infty \leqslant \|v\|_\infty$ and $E_2 \geqslant d+1$. Repeating the similar reasoning for the sets $E_2 < E_3 < \ldots < E_d < \{j_1\} < \{j_2\}$ we again obtain a contradiction.

Since $\|v\|_\infty = 1$, so $|v_m| = 1$ for some $m \in \mathbb{N}$. Observe that for any $i \neq m$

$$\|v + v_m e_m - \operatorname{sgn}\, v_i e_i\| \geqslant \|v + v_m e_m - \operatorname{sgn}\, v_i e_i\|_\infty = 2$$

and

$$\|v - \big(v_m e_m - \text{sgn } v_i e_i\big)\| \geqslant \|v - \big(v_m e_m - \text{sgn } v_i e_i\big)\|_\infty = 1 + |v_i|.$$

Hence, by assumption $\theta(\lfloor\theta^{-1}\rfloor + 1) \geqslant 1 + |v_i|$, so $|v_i| \leqslant \theta$ for $i \neq m$. If $m = \lfloor\theta^{-1}\rfloor + 1$, then v is of the form (a). Suppose that $m \geqslant \lfloor\theta^{-1}\rfloor + 2$. It is sufficient to show that $v_i = 0$ for all $i \neq m$ such that $i \geqslant \lfloor\theta^{-1}\rfloor + 2$.

Take any indices $\max\{m, i\} < j_1 < j_2 < \ldots < j_{\lfloor\theta^{-1}\rfloor}$. Then

$$\begin{aligned}\theta \cdot (\lfloor\theta^{-1}\rfloor + 1) &\geqslant \min\left(\left\|v + \sum_{k=1}^{\lfloor\theta^{-1}\rfloor} e_{j_k}\right\|, \left\|v - \sum_{k=1}^{\lfloor\theta^{-1}\rfloor} e_{j_k}\right\|\right) \\ &\geqslant \theta\left(|v_m| + |v_i| + \sum_{k=1}^{\lfloor\theta^{-1}\rfloor}(1 - |v_{j_k}|)\right) \\ &\geqslant \theta \cdot (\lfloor\theta^{-1}\rfloor + 1) + \theta\left(|v_i| - \sum_{k=1}^{\lfloor\theta^{-1}\rfloor}|v_{j_k}|\right)\end{aligned}$$

Since $\lim_{k\to\infty} v_{j_k} = 0$, so $v_i = 0$ for all $i \neq m$ such that $i \geqslant \lfloor\theta^{-1}\rfloor + 2$ and thus we get the form (b).

The proof of (3) and (4) is similar to the proofs of (1) and (2), respectively. Indeed, it is enough to take indices $j_1 < j_2 < \ldots < j_{\lfloor\theta^{-1}\rfloor}$ with additional assumption: $j_1 > \lfloor\theta^{-1}\rfloor$. □

Lemma 5. *Let $x \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$. Then $\|x + e_n\| = 2$ if and only if $x(n) = 1$.*

Proof. We omit the proof of implication ($\Leftarrow$) because it is trivial. Assume that $\|x + e_n\| = 2$. It is enough to show that norm of vector $x + e_n$ is the supremum norm.

Take any sets $d \leqslant E_1 < E_2 < \cdots < E_d$. We may assume that $n \in E_{i_0}$ for some $i_0 \in \{1, 2, \ldots, d\}$. Indeed, if this is not the case, we will not get a norm of vector $x + e_n$ greater than 1, because $\|x\| = 1$.

Since $E_i e_n = 0$ for $i \neq i_0$ and $\|x\| = 1$ we obtain

$$\begin{aligned}\theta \sum_{i=1}^{d} \|E_i(x + e_n)\| &\leqslant \theta \sum_{i=1}^{d}(\|E_i x\| + \|E_i e_n\||) \\ &= \theta \sum_{i=1}^{d} \|E_i x\| + \theta\|E_{i_0} e_n\| \\ &\leqslant \theta(\theta^{-1} + 1) < 2.\end{aligned}$$

□

The proof of the subsequent lemma is analogous to the proof of [21, Lemma 2.3]. Nevertheless, we include it here for the reader's convenience.

Lemma 6. *If* $U\colon \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]} \to \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$ *is an isometry satisfying* $-U(\mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}) \subset U(\mathbb{S}_{T[\theta,\mathcal{S}_\alpha]})$, *then* $-U(e_i) = U(-e_i)$ *for* $i \in \mathbb{N}$.

Proof. By the assumption $-U(e_i) = U(x_i)$ for some $x_i \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$. We will show that $x_i = -e_i$. Since $\|U(x_i)\| = 1$, so

$$\|x_i - e_i\| = \|U(x_i) - U(e_i)\| = \|U(x_i) + U(x_i)\| = 2.$$

By Lemma 5 we obtain $x_i(i) = -1$. Again, by assumption, there exists $x_j, y_j \in \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$, where $j \neq i$, such that $-U(e_j) = U(x_j)$ and $-U(-e_j) = U(y_j)$. Similarly, we obtain $x_j(j) = -1$ and $y_j(j) = 1$. Hence

$$|x_i(j) + 1| \leqslant \|x_i - x_j\| = \|U(x_i) - U(x_j)\| = \|e_i - e_j\| = 1.$$

From the other side

$$|x_i(j) - 1| \leqslant \|x_i - y_j\| = \|U(x_i) - U(y_j)\| = \|e_i + e_j\| = 1.$$

this means that $x_i(j) = 0$ for any $j \neq i$, so $x_i = -e_i$. □

Lemma 7. *Let* $\theta^{-1} \geqslant 2$ *be an integer and let* $U\colon \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]} \to \mathbb{S}_{T[\theta,\mathcal{S}_\alpha]}$ *be surjective isometry. If* $\alpha = 1$ *then*

$$Ue_i = \begin{cases} \varepsilon_i e_{\pi(i)}, & 1 \leqslant i \leqslant \theta^{-1} \\ \varepsilon_i e_i, & i > \theta^{-1} \end{cases} \qquad (i \in \mathbb{N}),$$

and if $\alpha > 1$ *then* $Ue_i = \varepsilon_i e_i$, *where* $(\varepsilon_i)_{i=1}^{\infty}$ *is some* $\{-1, 1\}$*-valued sequence and* π *is a permutation of* $\{1, 2, \ldots, \theta^{-1}\}$.

Proof. Case 1. Let $\alpha = 1$.
Step 1. Fix $1 \leqslant i \leqslant \theta^{-1}$.

For any $y \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$ there exists $x \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$ such that $U(x) = y$. Since U is isometry, so

$$\|U(e_i) - y\| = \|U(e_i) - U(x)\| = \|e_i - x\|.$$

By Lemma 6 we obtain

$$\|U(e_i) + y\| = \| - U(-e_i) + U(x)\| = \|e_i + x\|.$$

Hence

$$\min\{\|U(e_i) + y\|, \|U(e_i) - y\|\} = \min\{\|e_i + x\|, \|e_i - x\|\} \leqslant 1.$$

Thus, by Lemma 4 (1) for each i there is index $\pi(i) \in \{1, 2, \ldots, \theta^{-1}\}$ so that $U(e_i) = \pm e_{\pi(i)}$. Note that

$$1 = \|e_i \pm e_j\| = \big\|U(e_i) \pm U(e_j)\big\| = \|e_{\pi(i)} \pm e_{\pi(j)}\|$$

for any $j \neq i$ in $\{1, 2, \ldots, \theta^{-1}\}$. Therefore $\pi(j) \neq \pi(i)$ for $j \neq i$, so π is the desired permutation.

Step 2. Let $i > \theta^{-1}$. We will show that there is $\varepsilon_i \in \{-1, 1\}$ such that

$$U(e_i) = \varepsilon_i e_{\sigma(i)},$$

for some permutation σ of set $\mathbb{N} \setminus \{1, 2, \dots, \theta^{-1}\}$.

Note that

$$1 = \|e_i \pm e_j\| = \big\|U(e_i) \pm U(e_j)\big\| = \|U(e_i) \pm e_{\pi(j)}\|$$

for any j in $\{1, 2, \dots, \theta^{-1}\}$, so $U(e_i) > \theta^{-1}$. Since for any $x \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$ we have

$$\min\{\|e_i - x\|, \|e_i + x\|\} \leqslant \theta + 1$$

and since U is surjective, so

$$\min\{\|U(e_i) - y\|, \|U(e_i) + y\|\} \leqslant \theta + 1$$

for any $y \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$.

By the Lemma 4 (b) there are $\sigma, \tilde{\sigma} \colon \mathbb{N} \setminus \{1, 2, \dots, \theta^{-1}\} \to \mathbb{N} \setminus \{1, 2, \dots, \theta^{-1}\}$ such that

$$\big|\big(U(e_i)\big)(\sigma(i))\big| = 1 \quad \text{and} \quad \big|\big(U^{-1}(e_i)\big)(\tilde{\sigma}(i))\big| = 1, \tag{4.1}$$

for all $i > \theta^{-1}$. We claim that for all $k, i > \theta^{-1}$ with $k \neq i$ we have

$$\big(U(e_k)\big)(\sigma(i)) = 0 \quad \text{and} \quad \big(U^{-1}(e_k)\big)(\tilde{\sigma}(i)) = 0. \tag{4.2}$$

Indeed,

$$1 = \|e_i \pm e_k\| = \|U(e_i) \pm U(e_k)\| \geqslant \big|1 \pm \big(U(e_k)\big)(\sigma(i))\big|,$$

for $k \neq i$ in $\mathbb{N} \setminus \{1, 2, \dots, \theta^{-1}\}$, and similarly for U^{-1}, so the conclusion follows. In particular σ and $\tilde{\sigma}$ are injective.

We will show that there exists $l > \theta^{-1}$ such that $\big|\big(U(e_l)\big)(\theta^{-1} + 1)\big| = 1$.

If $\big|\big(U(e_{\theta^{-1}+1})\big)(\theta^{-1} + 1)\big| = 1$ then the thesis is fulfilled, so suppose that this is not the case. Since θ^{-1} is an integer, so by Lemma 4 (b) we have $U(e_{\theta^{-1}+1}) = a e_{\theta^{-1}+1} + \varepsilon e_m$ for some $m > \theta^{-1} + 1$, some $|a| \leqslant \theta$ and some $\varepsilon \in \{-1, 1\}$. Then

$$1 > \|U(e_{\theta^{-1}+1}) - \varepsilon e_m\| = \|e_{\theta^{-1}+1} - \varepsilon \cdot U^{-1}(e_m)\|. \tag{4.3}$$

Moreover, by (4.1), we have $|\big(U^{-1}(e_m)\big)(\tilde{\sigma}(m))| = 1$. If $\tilde{\sigma}(m) > \theta^{-1} + 1$ then

$$\|e_{\theta^{-1}+1} - \varepsilon \cdot U^{-1}(e_m)\| \geqslant \|e_{\theta^{-1}+1} - \varepsilon \cdot U^{-1}(e_m)\|_\infty \geqslant 1,$$

so we obtain a contradiction with (4.3).

This means that $\tilde{\sigma}(m) = \theta^{-1} + 1$, *i.e.*, $\big|\big(U^{-1}(e_m)\big)(\theta^{-1} + 1)\big| = 1$. Hence from (4.2) we have $\big(U^{-1}(e_{\theta^{-1}+1})\big)(\theta^{-1} + 1) = 0$.

This together with Lemma 4 (b) yields $U^{-1}(e_{\theta^{-1}+1}) = \tilde{\varepsilon} e_{\tilde{\sigma}(\theta^{-1}+1)}$ for some $\tilde{\varepsilon} \in \{-1, 1\}$.

So $U(e_{\tilde{\sigma}(\theta^{-1}+1)}) = \tilde{\varepsilon} e_{\theta^{-1}+1}$, hence $\tilde{\sigma}(\theta^{-1} + 1)$ is the l we are looking for.

This together with (4.2) gives us $\big(U(e_i)\big)(\theta^{-1}+1)=0$ for any $i\neq l$ with $i>\theta^{-1}$. By Lemma 4 (b) we obtain

$$U(e_i)=\varepsilon_i e_{\sigma(i)} \tag{4.4}$$

for all $i\neq l$ with $i>\theta^{-1}$ and some $\{-1,1\}$-valued sequence $(\varepsilon_i)_{\substack{i=\theta^{-1}+1,\\ i\neq l}}^{\infty}$. Hence

$$U^{-1}(e_{\sigma(i)})=\varepsilon_i e_i$$

for such i and $(\varepsilon_i)_{\substack{i=\theta^{-1}+1,\\ i\neq l}}^{\infty}$. This together with (4.1) means that $\tilde{\sigma}=\sigma^{-1}$, so σ is surjective. Since for every $p\in\operatorname{supp}(U(e_l))\setminus\sigma(l)$ there exists $i\neq l$ with $\sigma(i)=p$, *i.e.*, $\big|\big(U(e_i)\big)(p)\big|=1$, by (4.2) we have $\big(U(e_l)\big)(p)=0$. Hence $U(e_l)=\varepsilon_l e_{\sigma(l)}$ for some $\varepsilon_l\in\{-1,1\}$. This together with (4.4) gives as conclusion.

Step 3. We will show that σ from *Step 2* is an identity.

Define

$$x_k:=k^{-1}\cdot\theta^{-1}\cdot(e_k,e_{k+1},\dots,e_{2k-1})$$

for $k>\theta^{-1}$. Then

$$\|U(x_k)+U(e_i)\|=\|x_k+e_i\|=\|x_k-e_i\|=\|U(x_k)-U(e_i)\|$$

for all $i\notin\operatorname{supp}x_k$. By *Step 2* and since $\|U(x_k)\|=1$ it must be $\big(U(x_k)\big)(\sigma(i))=0$ for all $i\notin\operatorname{supp}x_k$, so

$$\operatorname{supp}U(x_k)\subseteq\{\sigma(k),\sigma(k+1),\dots,\sigma(2k-1)\}. \tag{4.5}$$

We claim that $\sigma(k)\geqslant k$ for any $k>\theta^{-1}$.

Suppose that $\sigma(k)<k$. Then by (4.5) there is $i\in\{k,k+1,\dots,2k-1\}$ such that

$$\big|\big(U(x_k)\big)(\sigma(i))\big|\geqslant(k-1)^{-1}\cdot\theta^{-1}.$$

Indeed, if not, then we obtain a contradiction, because

$$\|U(x_k)\|<\max\big\{(k-1)^{-1}\cdot\theta^{-1},\ \theta\cdot(k-1)\cdot(k-1)^{-1}\cdot\theta^{-1}\big\}=1.$$

Since $\sigma(\theta^{-1}+1)\geqslant\theta^{-1}+1$, assume firstly that $k\in\{\theta^{-1}+2,\theta^{-1}+3,\dots,2\theta^{-1}\}$. Then

$$\begin{aligned}1+(k-1)^{-1}\cdot\theta^{-1}&\leqslant\big\|U(x_k)+\operatorname{sgn}\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big\|\\&=\big\|x_k+\operatorname{sgn}\big(\big(U(x_k)\big)(\sigma(i))\big)U^{-1}(e_{\sigma(i)})\big\|\\&\leqslant\|x_k+e_i\|=\max\big\{k^{-1}\cdot\theta^{-1}+1,\theta\cdot\big(k^{-1}\cdot\theta^{-1}\cdot k+1\big)\big\}\\&=1+k^{-1}\cdot\theta^{-1},\end{aligned}$$

which cannot hold. Hence for $\theta^{-1}<k\leqslant2\theta^{-1}$ we have $\sigma(k)\geqslant k$.

Assume that $k > 2\theta^{-1}$. Then

$$\begin{aligned}\big\|U(x_k) - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big\| &\geqslant \|x_k - e_i\| \\ &= \max\big\{1 - k^{-1}\cdot\theta^{-1},\ \theta\big(k^{-1}\cdot\theta^{-1}(k-2)+1\big)\big\} \\ &= 1 - 2k^{-1} + \theta > 1 = \|U(x_k)\|.\end{aligned} \tag{4.6}$$

This means that $1 - \big|\big(U(x_k)\big)(\sigma(i))\big| > \big|\big(U(x_k)\big)(\sigma(i))\big|$, so $1 - 2\big|\big(U(x_k)\big)(\sigma(i))\big| > 0$. Since for any finite set $E_j \subset \mathbb{N}$, where $j \in \mathbb{N}$ we have

$$\begin{aligned}&\big\|E_j\big(U(x_k) - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big)\big\| \\ &\qquad\leqslant \big\|E_j\big(U(x_k) - 2\big(U(x_k)\big)(\sigma(i))e_{\sigma(i)}\big)\big\| \\ &\qquad+ \big\|E_j\big(2\big(U(x_k)\big)(\sigma(i))e_{\sigma(i)} - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big)\big\| \\ &\qquad= \|E_jU(x_k)\| \\ &\qquad+ \big\|E_j\big(2\big(U(x_k)\big)(\sigma(i))e_{\sigma(i)} - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big)\big\|,\end{aligned}$$

so multiplying by θ both sides of the above inequality and taking the supremum over all consecutive sets $d < E_1 < E_2 < \cdots < E_d$ for some $d \in \mathbb{N}$, we obtain

$$\begin{aligned}&\big\|\big(U(x_k) - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big\| \\ &\qquad\leqslant \|U(x_k)\| \\ &\qquad+ \theta\cdot\big\|E_{j_0}\big(2\big(U(x_k)\big)(\sigma(i))e_{\sigma(i)} - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big)\big\|\end{aligned}$$

for some $j_0 \in \{1, 2, \ldots, d\}$. Hence

$$\begin{aligned}\big\|U(x_k) - \text{sgn }\big(\big(U(x_k)\big)(\sigma(i))\big)e_{\sigma(i)}\big\| &\leqslant \|U(x_k)\| + \theta\big(1 - 2\big|\big(U(x_k)\big)(\sigma(i))\big|\big) \\ &= 1 + \theta - 2\theta\big|\big(U(x_k)\big)(\sigma(i))\big| \\ &\leqslant 1 + \theta - 2\theta(k-1)^{-1}\cdot\theta^{-1}.\end{aligned}$$

which contradicts (4.6).

Doing the same for U^{-1} instead of U we obtain $\sigma^{-1}(\sigma(k)) \geqslant \sigma(k)$, so $k \geqslant \sigma(k)$, hence $\sigma(k) = k$ for all $k > \theta^{-1}$. This ends the proof for $\alpha = 1$.

Case 2. Suppose that $\alpha = \beta + 1$ for some $\beta < \omega_1$.

The proof that $U(e_1) = \varepsilon_1 e_1$, where $\varepsilon_1 \in \{-1, 1\}$ and for any $i > 1$ there is $\varepsilon_i \in \{-1, 1\}$ such that

$$U(e_i) = \varepsilon_i e_{\sigma(i)}$$

for some permutation σ of set $\{2, 3, \ldots\}$ is similar to the previous case and much simpler, so we omit it. We will show that σ is an identity.

Fix $k > 1$ and suppose that $t := \sigma(k) < k$.

Note that every $\mathcal{S}_\alpha$-set whose minimum is k is the union of at most k many $\mathcal{S}_\beta$-sets, so the idea of the proof of this case is to choose the indices $j_1 < j_2 < \cdots < j_m$, for some

$m \in \mathbb{N}$, so that they creates k many consecutive $\mathcal{S}_\beta$-sets. At the same time, we must ensure that the set

$$\{\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_m)\}$$

associated with these indices was not $\mathcal{S}_\alpha$-set. We proceed as follows. Choose indices $j_1 = k, j_2 > \max\{k, \theta^{-1}\}$. Then take the next index $j_3 > \max\{j_2, \sigma(j_2)\}$, in sequence $j_4 = \max\{j_3, \sigma(j_3)\}$ and so on.

Following this procedure, we may choose a maximal $\mathcal{S}_\beta$-set created from the indices $j_2 < j_3 < \cdots < j_{m_1}$, for some $m_1 \in \mathbb{N}$. At the same time, we get the indices

$$\{\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_{m_1})\}$$

so that

$$\sigma(j_1) < j_2 \leqslant \max\{ j_2, \sigma(j_2) \} < j_3 \leqslant \cdots$$
$$\cdots \leqslant \max\{ j_{m_1-1}, \sigma(j_{m_1-1}) \} < j_{m_1}.$$

Similarly, we may find the second maximal $\mathcal{S}_\beta$-set with minimum j_{m_1+1} greater than $\max\{j_{m_1}, \sigma(j_{m_1})\}$, obtaining indices

$$\sigma(j_1) < j_2 \leqslant \max\{ j_2, \sigma(j_2) \} < j_3 \leqslant \cdots$$
$$\cdots \leqslant \max\{ j_{m_2-1}, \sigma(j_{m_2-1}) \} < j_{m_2}.$$

for some $m_2 \in \mathbb{N}$.

Proceeding analogously, we finally arrive at indices

$$j_2 < j_3 < \cdots < j_m,$$

for some $m \in \mathbb{N}$, that form a union of t maximal $\mathcal{S}_\beta$-sets, so we got the conclusion because we may choose $\mathcal{S}_\beta$-sets

$$k \leqslant S_\beta^1 < S_\beta^2 < \cdots < S_\beta^{t+1}, \tag{4.7}$$

where

- $S_\beta^1 = \{j_1\}$,
- $S_\beta^2 = \{j_2, j_3, \ldots, j_{m_1}\}$,
- $\vdots$
- $S_\beta^{t+1} = \{j_{m_t+1}, j_{m_t+2}, \ldots, j_m\}$.

By the above construction,

$$\tilde{S}_m := \{\sigma(j_1), \sigma(j_2), \sigma(j_3), \ldots \sigma(j_m)\}$$

is not $\mathcal{S}_\beta$-set because the Schreier family (of order β) is spreading (see Definition 1). Then we define

$$x_k := \theta^{-1} \cdot m^{-1} \cdot \sum_{i=1}^{m} e_{j_i}.$$

As in (4.5) we have

$$\operatorname{supp} U(x_n) \subseteq \tilde{S}_m.$$

To complete the proof, it is enough to replace each k with m in *Step 3* of *Case 1*. Note that

$$\left|\big(U(x_k)\big)(\sigma(i))\right| \geqslant (m-1)^{-1} \cdot \theta^{-1}$$

holds for some $i \in \{j_1, j_2, \dots, j_m\}$ as we ensured that $m > \theta^{-1} + 1$.
Case 3: Suppose that α is a limit ordinal.

We proceed as in Case 2 for $\alpha = \alpha_t$, where $(\alpha_i)_{i=1}^\infty$ is a fixed strictly increasing sequence of successor ordinals converging to α with $\mathcal{S}_{\beta_i} \subset \mathcal{S}_{\beta_n}$ for $i \leqslant n$, where $\alpha_n := \beta_n + 1$ for each $n \in \mathbb{N}$, choosing suitable sequence $(j_i)_{i=1}^m$. Indeed, $\mathcal{S}_{\beta_t}$-sets $k \leqslant S^1_{\beta_t} < S^2_{\beta_t} < \cdots < S^{t+1}_{\beta_t}$, where

- $S^1_{\beta_t} = \{j_1\}$,
- $S^2_{\beta_t} = \{j_2, j_3, \dots, j_{m_1}\}$,
- $\vdots$
- $S^{t+1}_{\beta_t} = \{j_{m_t+1}, j_{m_t+2}, \dots, j_m\}$,

give rise to an $\mathcal{S}_\alpha$-set (even an $\mathcal{S}_{\alpha_t}$-set). Moreover, the set

$$\tilde{S}_m := \big\{\sigma(j_1), \sigma(j_2), \dots \sigma(j_m)\big\}$$

is not $\mathcal{S}_{\alpha_t}$-set by the spreading property of S_{β_n}. Hence $\tilde{S}_m \notin \mathcal{S}_\alpha$ as we ensured that $\mathcal{S}_{\beta_i} \subset \mathcal{S}_{\beta_n}$ for $i \leqslant n$. Indeed, suppose $\tilde{S}_m \in \mathcal{S}_\alpha$. Then $\tilde{S}_m \in \mathcal{S}_{\alpha_j}$ for some $j \leqslant t$, i.e. $\tilde{S}_m$ is the union of at most j-many successive $\mathcal{S}_{\beta_j}$-sets, i.e. $\mathcal{S}_{\beta_t}$-sets by the assumption on $(\beta_i)_i$. This means that $\tilde{S}_m \in \mathcal{S}_{\alpha_t}$; a contradiction. □

We are now ready to prove Theorem A.

Proof. Fix $\alpha = 1$. Let $\theta^{-1} \geqslant 2$ be an integer and let

$$Ue_i = \begin{cases} \hat{\varepsilon}_i e_{\pi(i)}, & 1 \leqslant i \leqslant \theta^{-1} \\ \hat{\varepsilon}_i e_i, & i > \theta^{-1} \end{cases} \qquad (i \in \mathbb{N}),$$

where $(\hat{\varepsilon}_i)_{i=1}^\infty$ is some $\{-1,1\}$-valued sequence and π is a permutation of $\big\{1, 2, \dots, \theta^{-1}\big\}$.

Define $\hat{\pi}(i)$ as $\pi(i)$ for $1 \leqslant i \leqslant \theta^{-1}$ and $\hat{\pi}(i) = i$ for $i > \theta^{-1}$. For $i \in \mathbb{N}$ let us set $\varepsilon_i := \big(U(e_i)\big)\big(\hat{\pi}(i)\big)$.

Fix $x = \sum_{i=1}^{\infty} a_i e_i \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$ and take $y = \sum_{i=1}^{\infty} b_i e_i \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$ such that $U(x) = y$. If a_i is nonzero and $b_i = 0$ then we use the convention that sgn $(b_i) = 1$. Take

$$y_j = \sum_{i=1, i\neq\hat{\pi}(j)}^{\infty} \theta b_i e_i - \varepsilon_j \mathrm{sgn}\ (a_j) e_{\hat{\pi}(j)}$$

and

$$z_j = \sum_{i=1, i\neq\hat{\pi}(j)}^{\infty} \theta b_i e_i - \mathrm{sgn}\ (b_{\hat{\pi}(j)}) e_{\hat{\pi}(j)}.$$

Then $\|y_j\| = 1$ and $\|y - z_j\| = 1 + |b_{\hat{\pi}(j)}|$. Let $x_j \in \mathbb{S}_{T[\theta,\mathcal{S}_1]}$ be such that $U(x_j) = y_j$. We obtain

$$\|x_j - \mathrm{sgn}\ (a_j)e_j\| = \|U(x_j) - \mathrm{sgn}\ (a_j)U(e_j)\| = \|y_j - \mathrm{sgn}\ (a_j)\varepsilon_j e_{\hat{\pi}(j)}\| = 2.$$

So by Lemma 5 we have $x_j(j) = -\mathrm{sgn}\ (a_j)$. This yields

$$1 + |b_{\hat{\pi}(j)}| = \|y - z_j\| \geqslant \|y - y_j\| = \|U(x) - U(x_j)\| = \|x - x_j\| \geqslant 1 + |a_j|.$$

Hence

$$|b_{\hat{\pi}(j)}| \geqslant |a_j|. \tag{4.8}$$

Note that $\varepsilon_i = \big(U^{-1}(e_{\hat{\pi}(i)})\big)(i)$ and $U^{-1}(e_i) = \varepsilon_{\hat{\pi}^{-1}(i)} e_{\hat{\pi}^{-1}(i)}$. Similarly, we define

$$u_j = \sum_{i=1, i\neq\hat{\pi}^{-1}(j)}^{\infty} \theta a_i e_i - \varepsilon_{\hat{\pi}^{-1}(j)} \mathrm{sgn}\ (b_j) e_{\hat{\pi}^{-1}(j)}$$

and

$$v_j = \sum_{i=1, i\neq\hat{\pi}^{-1}(j)}^{\infty} \theta a_i e_i - \mathrm{sgn}\ (a_{\hat{\pi}^{-1}(j)}) e_{\hat{\pi}^{-1}(j)}.$$

Then

$$\|U(u_j) - \mathrm{sgn}\ (b_j)e_j\| = \|u_j - \mathrm{sgn}\ (b_j)\varepsilon_{\hat{\pi}^{-1}(j)} e_{\hat{\pi}^{-1}(j)}\| = 2.$$

So $\big(U(u_j)\big)(j) = -\mathrm{sgn}\ (b_j)$. Hence

$$1 + |a_{\hat{\pi}^{-1}(j)}| = \|x - v_j\| \geqslant \|x - u_j\| = \|U(x) - U(u_j)\| = \|y - U(u_j)\| \geqslant 1 + |b_j|.$$

This means that $|a_{\hat{\pi}^{-1}(j)}| \geqslant |b_j|$, which together with (4.8) gives us $|a_{\hat{\pi}^{-1}(j)}| = |b_j|$. We moreover have $\|x - v_j\| = \|x - u_j\|$, so $\varepsilon_{\hat{\pi}^{-1}(j)} \mathrm{sgn}\ (b_j) = \mathrm{sgn}\ (a_{\hat{\pi}^{-1}(j)})$ and finally $b_{\hat{\pi}(j)} = \varepsilon_j a_j$ for $j \in \mathbb{N}$, hence the conclusion follows.

Fix $1 < \alpha < \omega_1$ and let $Ue_i = \hat{\varepsilon}_i e_i$, where $(\hat{\varepsilon}_i)_{i=1}^{\infty}$ is some $\{-1, 1\}$-valued sequence. The proof is exactly the same if we define $\hat{\pi}(i)$ as identity for any $i \in \mathbb{N}$. □

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