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Abstract

In this paper, we consider a flow of heat-conducting self-gravitating compressible fluid in a time-dependent domain. The flow is governed by the 3-D Navier-Stokes-Fourier-Poisson equations where the velocity is supposed to fulfill the full-slip boundary condition and the temperature on the boundary is given by a non-homogeneous Dirichlet condition. We establish the global-in-time weak solution to the system. Our approach is based on the penalization of the boundary behavior, viscosity, and the pressure in the weak formulation. Moreover, to accommodate the non-homogeneous boundary heat flux, we introduce the concept of *ballistic energy* in this work.

Keywords. Compressible fluids, Navier-Stokes-Fourier-Poisson equations, non-homogeneous boundary, time-dependent domain.

AMS subject classifications. 35D30, 35Q35, 76N10.

1 Introduction and general setting

1.1 Problem statement

In this article, we study the flow of a *viscous gaseous star* with the influence of self-gravitation in a time-dependent domain. It is by now well-understood that the stars may be considered as a compressible fluid (e.g., [24]) and their dynamics are very often shaped and controlled by self-gravitation and high-temperature radiation effects (see for instance [3, 8]). The mathematical model of such flows is governed by the 3-D compressible *Navier-Stokes-Fourier-Poisson* system, namely

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, \theta) = \mathrm{div}_x \mathbb{S} + \rho \nabla_x \Psi, \tag{1.2}$$

$$\partial_t(\rho e(\rho, \theta)) + \mathrm{div}_x(\rho e(\rho, \theta)\mathbf{u}) + \mathrm{div}_x \mathbf{q} + p(\rho, \theta)\mathrm{div}_x \mathbf{u} = \mathbb{S} : \nabla_x \mathbf{u}, \tag{1.3}$$

$$-\Delta_x \Psi = 4\pi g \rho, \quad \int_{\Omega_t} \psi \,\mathrm{d}x = 0, \tag{1.4}$$

The density ρ, velocity $\mathbf{u}$, and absolute temperature θ are three typical macroscopic quantities that describe the motion of the fluid, while $p(\rho, \theta)$ is the pressure, $s(\rho, \theta)$ is the specific entropy and $e(\rho, \theta)$ is the specific internal energy of the fluid, interrelated through the Gibb's equation

$$\theta D s = De + pD(1/\rho), \tag{1.5}$$

where D stands for the total differential of the corresponding functions.

The viscous stress tensor $\mathbb{S}$ satisfies the classical Newton rheological law

$$\mathbb{S}(\theta, \nabla_x \mathbf{u}) = \mu(\theta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\theta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.6}$$

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with shear viscosity coefficient $\mu(\theta) > 0$ and bulk viscosity coefficient $\eta(\theta) \geq 0$.

In momentum equation (1.2), $\rho \nabla_x \Psi$ is an external force acting on the fluid due to self-gravitation, and the gravitational potential Ψ (of the star) solves the Poisson equation given by (1.4) with the gravity $g > 0$.

Moreover, the heat flux $\mathbf{q}$ is determined by the Fourier's law,

$$\mathbf{q}(\theta, \nabla_x \theta) = -\kappa(\theta) \nabla_x \theta, \tag{1.7}$$

with the heat conductivity coefficient $\kappa(\theta) > 0$.

Description of the time-dependent domain. Let us consider a regular domain $\Omega_0 \subset \mathbb{R}^3$ occupied by the fluid at the initial time $t = 0$. Then, we investigate the domain Ω_t (w.r.t. time t) described by a given velocity field $\mathbf{V}(t, x)$ with $t \geq 0$ and $x \in \mathbb{R}^3$. More specifically, when $\mathbf{V}$ is regular enough, we solve the associated system of differential equations

$$\frac{d}{dt} \mathbf{X}(t, x) = \mathbf{V}(t, x), \quad t > 0, \quad \mathbf{X}(0, x) = x, \tag{1.8}$$

and set

$$\begin{cases} \Omega_\tau = \mathbf{X}(\tau, \Omega_0), \quad \Gamma_\tau := \partial \Omega_\tau, \quad \text{and} \\ Q_\tau := \cup_{t \in (0,\tau)} \{t\} \times \Omega_t. \end{cases} \tag{1.9}$$

We assume that the volume of the domain cannot degenerate in time, meaning that

$$\exists\, M_0 > 0 \ \text{ such that } \ |\Omega_\tau| \geq M_0 \quad \forall \tau \in [0, T]. \tag{1.10}$$

Moreover, we make the following assumption

$$\mathrm{div}_x \mathbf{V} = 0 \ \text{ in the neighborhood of } \Gamma_\tau \quad \forall \tau \in [0, T]. \tag{1.11}$$

The condition (1.11) is not restrictive. Indeed, as it has been indicated in [18, Remark 5.3], for a general $\mathbf{V} \in \mathcal{C}^1([0, T]; \mathcal{C}_c^3(\mathbb{R}^3, \mathbb{R}^3))$, one can find $w \in W^{1,\infty}(Q_T)$ such that $(\mathbf{V} - w)|_{\Gamma_\tau} = 0$ for all $\tau \in [0, T]$ such that $\mathrm{div}_x w = 0$ on some neighborhood of Γ_τ; see [11, Section 4.3.1]. We again refer [18, Remark 5.3] for more details about this point.

We further consider a reference domain $\mathcal{B}$ (will be specified later), smooth and bounded, such that for each $\tau \in [0, T]$, one has $\Omega_\tau \subset \mathcal{B}$.

Boundary conditions. Here we prescribe the boundary conditions for the original system (1.1) – (1.4).

- We first impose the Navier-slip boundary conditions

$$[\mathbb{S}\mathbf{n}]_{\tan} + \alpha [\mathbf{u} - \mathbf{V}]_{\tan} = 0, \quad \text{in } \Gamma_t, \text{ for any } t \in [0, T], \tag{1.12}$$

where $\alpha \geq 0$ represents a *friction* coefficient and $\mathbb{S}$ is the viscous stress tensor. For simplicity, we take $\alpha = 0$ which gives the *full-slip* condition. Furthermore, the impermeability condition for $\mathbf{u}$ is given by

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = 0, \quad \text{in } \Gamma_t, \text{ for any } t \in [0, T]. \tag{1.13}$$

- The fluid temperature on the lateral boundary of the domain is given by

$$\theta \Big|_{\bigcup_{t \in (0,T)} \left(\{t\} \times \Gamma_t\right)} = \theta_B, \tag{1.14}$$

where $\theta_B = \theta_B(t, x)$ is a strictly positive smooth function and which can be extended smoothly in the whole reference domain $\mathcal{B}$, more precisely the extended θ_B satisfies

$$\theta_B \in \mathcal{C}^1([0, T] \times \mathcal{B}), \tag{1.15}$$

where we use the same notation for this extension.

- The gravitational potential Ψ satisfies the Neumann boundary condition

$$\frac{\partial \Psi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_t \ \text{ for each } t \in [0, T]. \tag{1.16}$$

Initial conditions. The system (1.1) – (1.4) is also supplemented with the initial conditions

$$\rho(0,\cdot) = \rho_0 \in L^{\frac{5}{3}}(\Omega_0), \quad (\rho\mathbf{u})(0,\cdot) = (\rho\mathbf{u})_0 \quad \text{in } \Omega_0, \quad \theta(0,\cdot) = \theta_0 \quad \text{in } \Omega_0 \tag{1.17}$$

where we assume that the fluid density is zero outside the domain Ω_0, more precisely,

$$\rho_0 \geq 0 \ \text{ in } \Omega_0, \ \ \rho_0 \not\equiv 0 \ \text{ and } \ \rho_0 = 0 \ \text{ in } \ \mathbb{R}^3 \setminus \Omega_0. \tag{1.18}$$

Furthermore, it holds that $0 < \underline{\theta} \leq \theta_0 \leq \overline{\theta}$ for some positive constants $\underline{\theta}$ and $\overline{\theta}$, and

$$(\rho s)_0 = \rho_0 s(\rho_0, \theta_0) \in L^1(\Omega_0). \tag{1.19}$$

We further assume that

$$\mathcal{E}_0 := \int_{\Omega_0} \left(\frac{1}{2\rho_0} |(\rho\mathbf{u})_0|^2 + \rho_0 e(\rho_0, \theta_0) \right) < +\infty. \tag{1.20}$$

Our goal is to establish the global-in-time existence of weak solutions to the whole system (1.1) – (1.4) in the domain Q_T with the boundary conditions (1.12)-(1.16) and initial conditions (1.17)-(1.20).

1.2 Bibliographic comments and main goal of our work

The self-gravitating flows have wide applications in astrophysics and the theory of nuclear fluids. In that regard, we mention that the global in time weak solutions for the compressible barotropic self-gravitating fluids governed by *Navier-Stokes-Poisson* equations has been initially studied by B. Ducomet and E. Feireisl [7] in the fixed spatial domain. Later on, they established the existence theory of weak solutions for the compressible *Navier-Stokes-Fourier-Poisson* (in short N-S-F-P) system in [6]. On the other hand, B. Ducomet et al [5] considered a compressible N-S-F-P system describing a motion of a viscous heat-conducting rotating fluid on a thin domain $\Omega_\epsilon = \omega \times (0, \epsilon)$ with a 2D domain ω and positive ϵ. More precisely, the authors in [5] proved that the weak solutions in the 3D domain converge to the strong solutions of the 2D system as $\epsilon \to 0$ in the time interval where the strong solution exists.

In the context of compressible fluids in time dependent domains, we first address the work [11] by E. Feireisl et al where the existence of weak solutions of the barotropic compressible Navier-Stokes systems has been addressed on time dependent domains as prescribed in (1.8) – (1.10). Their approach is based on the penalization of the boundary behavior, viscosity and pressure in the weak formulation. Later on, the existence of weak solutions to the full *Navier-Stokes-Fourier* (in short N-S-F) systems in the time dependent domain has been treated by O. Kreml et al [19], see also [18]. We also note here that the compressible micropolar fluids on a time-dependent domain with slip boundary conditions was considered in [16]. Furthermore, the local-in-time existence of strong solutions to the compressible Navier-Stokes on the moving domains was given in [21]. The global well-posedness of compressible Navier–Stokes equations on a moving domain in the L^p-L^q frame was investigated in [20]. Recently, the authors in [23] studied the existence of a weak solution to a nonlinear fluid-structure interaction model with heat exchange where the shell is governed by linear thermoelasticity equations and encompasses a time-dependent domain that is filled with a fluid governed by the full N-S-F system. In this regard, we also mention the work [17], where the authors analyze a system governing the interaction between two compressible mutually noninteracting fluids and a shell of Koiter type that actually encompasses a *time dependent 3-D domain* filled by the fluids.

In fixed spatial domains, the existence theory of compressible barotropic Navier-Stokes systems was developed by P. L. Lions [22] and later it has been extended in [14] to a class of physically relevant pressure-density state equations. The existence of weak solutions to the full N-S-F system has been then established by E. Feireisl [9, 10] and by E. Feireisl and A. Novotný in [12].

The global in time weak solutions to the Navier-Stokes-Fourier system with nonhomogeneous Dirichlet data (for both velocity and temperature) in fixed spatial domain has been rigorously studied by N. Chaudhuri and E. Feireisl [2]; they also investigated the weak-strong uniqueness result for their system. To handle the nonhomogeneous Dirichlet data for temperature and boundary heat flux, the authors in [2] introduced the concept of *ballistic energy* (see also [1]).

In the present work, we study the weak existence theory for the compressible *Navier-Stokes-Fourier-Poisson* equations in time dependent domain where we allow the nonhomogeneous Dirichet condition for the temperature and *non-vanishing heat flux on the boundary*, which is certainly more physical in the viewpoint of the motion of nuclear fluids or gaseous stars, and to the best of our knowledge, this problem has not been considered in the literature yet. As a reason, we shall intensively use the concept of *ballistic energy* to accommodate the nonhomogeneous heat flux on the spatial boundary.

Before going to more details, we note down the constitutive relations that will be used to proceed our work.

1.3 Hypothesis

Motivated by [9, 10], let us now make the following set of assumptions.

• **Viscosity coefficients.** We consider the viscosity coefficients $\mu(\theta)$ and $\eta(\theta)$ to be continuously differentiable functions depending on the temperature θ, namely $\mu(\theta), \eta(\theta) \in \mathcal{C}^1([0,+\infty))$ and satisfy

$$\begin{aligned} &0 < \underline{\mu}(1+\theta) \le \mu(\theta) \le \overline{\mu}(1+\theta), \qquad \sup_{\theta\in[0,+\infty)} |\mu'(\theta)| \le \overline{m}, \\ &0 \le \underline{\eta}(1+\theta) \le \eta(\theta) \le \overline{\eta}(1+\theta). \end{aligned} \tag{1.21}$$

• **Heat conductive coefficient.** In accordance with the recent work [2] (see also [1]), we need a much stricter assumption on the heat conducting coefficient $\kappa(\theta)$ appearing in the Fourier's law (1.7). More precisely, we assume $\kappa(\theta) \in \mathcal{C}^1([0,+\infty))$ such that

$$0 < \underline{\kappa}(1+\theta^\alpha) \le \kappa(\theta) \le \overline{\kappa}(1+\theta^\alpha), \quad \text{for } \alpha > 6. \tag{1.22}$$

In above, all the quantities $\underline{\mu}$, $\overline{\mu}$, $\overline{m}$, $\underline{\eta}$, $\overline{\eta}$, $\underline{\kappa}$, $\overline{\kappa}$ are positive.

• **Constitutive relations for pressure, internal energy and entropy.** We consider the following constitutive relations for the pressure and the internal energy, namely

$$p(p,\theta) = p_M(\rho,\theta) + \frac{a}{3}\theta^4, \quad a > 0, \tag{1.23}$$

$$e(\rho,\theta) = e_M(\rho,\theta) + \frac{a}{\rho}\theta^4, \tag{1.24}$$

$$s(\rho,\theta) = s_M(\rho,\theta) + \frac{4a}{3\rho}\theta^3. \tag{1.25}$$

According to the hypothesis of thermodynamic stability, the molecular components satisfies

$$\frac{\partial p_M}{\partial \rho} > 0 \quad \forall\, \rho,\, \theta > 0, \tag{1.26}$$

and there exists some positive constant $c > 0$ such that

$$0 < \frac{\partial e_M}{\partial \theta} \le c \quad \forall\, \rho,\, \theta > 0. \tag{1.27}$$

Moreover, it holds that

$$\lim_{\theta\to 0^+} e_M(\rho,\theta) = \underline{e_M}(\rho) > 0 \quad \text{for any fixed } \ \rho > 0, \tag{1.28}$$

and

$$\left| \rho \frac{\partial e_M}{\partial \rho}(\rho,\theta) \right| \le c e_M(\rho,\theta) \quad \forall\, \rho,\, \theta > 0. \tag{1.29}$$

We also suppose that there is a function

$$P \in \mathcal{C}^1[0,\infty), \quad P(0)=0, \quad P'(0)>0, \tag{1.30}$$

and two positive constants $\underline{Z}, \overline{Z}$ such that

$$p_M(\rho,\theta) = \theta^{\frac{5}{2}} P\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) \quad \text{whenever } 0 < \frac{\rho}{\theta^{\frac{3}{2}}} \le \underline{Z}, \ \text{ or } \ \frac{\rho}{\theta^{\frac{3}{2}}} > \overline{Z}, \tag{1.31}$$

and satisfying the relation

$$p_M(\rho,\theta) = \frac{2}{3}\rho e_M(\rho,\theta), \quad \text{for } \frac{\rho}{\theta^{\frac{3}{2}}} > \overline{Z}. \tag{1.32}$$

Based on the above assumption, we derive that

$$\rho e(\rho,\theta) \ge a\theta^4 + \frac{3}{2} p_\infty \rho^{\frac{5}{3}}. \tag{1.33}$$

This estimate can be shown in the following explicit way. First, observe that $P'(Z) > 0$ for all $0 < Z < \underline{Z}$ or $Z > \overline{Z}$. Now, we extend P as a strictly increasing function in $[\underline{Z}, \overline{Z}]$ so that we have

$$P'(Z) > 0 \quad \forall Z > 0. \tag{1.34}$$

Next, by (1.29), (1.31) and (1.32) we infer that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{1.35}$$

Using (1.31), (1.32) and (1.35) one can deduce that

$$\lim_{\theta\to 0^+} e_M(\rho,\theta) = \frac{3}{2}\rho^{\frac{2}{3}} p_\infty. \tag{1.36}$$

Moreover, e_M is a strictly increasing function of θ in $(0,\infty)$ (see (1.27)) for any fixed ρ, which together with (1.24) and (1.36), we obtain the required estimate (1.33).

Further, in agreement with the Gibb's relation (1.5), the molecular component s_M of the entropy s satisfies

$$\frac{\partial s_M}{\partial \theta} = \frac{1}{\theta}\frac{\partial e_M}{\partial \theta} \quad \text{and} \quad \frac{\partial s_M}{\partial \rho} = -\frac{1}{\rho^2}\frac{\partial p_M}{\partial \theta}. \tag{1.37}$$

We set

$$s_M(\rho,\theta) = S(Z), \quad Z = \frac{\rho}{\theta^{3/2}}, \quad S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0 \tag{1.38}$$

in the degenerate area $\rho > \overline{Z}\theta^{3/2}$. We also require that the *Third law of thermodynamics* is satisfied,

$$\lim_{Z\to\infty} S(Z) = 0. \tag{1.39}$$

We refer [12, Chapter 2], for more details on the hypothesis (1.23) – (1.39).

2 Weak formulations

In this section, we shall prescribe the expected weak formulations for the system (1.1) – (1.4). Throughout the section, we assume that the density ρ remains "zero" outside the fluid domain Ω_t for each $t \in [0,T]$. The reason is that, we will eventually show that if the initial density $\rho_0 = 0$ outside Ω_0, then ρ will also vanish outside Ω_t for any $t \in (0,T]$ (see Section 5.4).

I. Continuity equation. It is convenient to consider the continuity equation in the whole physical space $\mathbb{R}^3$ provided the density is supposed to be zero outside the fluid domain Ω_t for each $t \in [0,T]$. Specifically, the weak formulation of the continuity equation (1.1) is supposed to be

$$-\int_0^T \int_{\Omega_t} (\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi) = \int_{\Omega_0} \rho_0(\cdot)\varphi(0,\cdot), \tag{2.1}$$

for any test function $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R}^3; \mathbb{R})$ with $\varphi(T,\cdot) = 0$. Of course, we assume that $\rho \geq 0$ a.e. in $\mathbb{R}^3$.

Moreover, the equation (1.1) will also be satisfied in the sense of renormalized solutions introduced by DiPerna and Lions [4]:

$$-\int_0^T \int_{\Omega_t} \rho B(\rho) \left(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi\right) + \int_0^\tau \int_{\Omega_t} b(\rho) \mathrm{div}_x \mathbf{u} \varphi = \int_{\Omega_0} \rho_0 B(\rho_0) \varphi(0,\cdot), \tag{2.2}$$

for any test function $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R}^3; \mathbb{R})$ with $\varphi(T,\cdot) = 0$, $b \in L^\infty \cap \mathcal{C}([0,+\infty))$ such that $b(0) = 0$ and $B(\rho) = B(1) + \displaystyle\int_1^\rho \frac{b(z)}{z^2}$.

II. Momentum equation. We write the expected weak formulation for the momentum equation as

$$\begin{aligned} -\int_0^T \int_{\Omega_t} \left(\rho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \rho[\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\rho,\theta) \mathrm{div}_x \boldsymbol{\varphi}\right) - \int_{\Omega_0} (\rho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0,\cdot) \\ = \int_0^T \int_{\Omega_t} \rho \nabla_x \Psi \cdot \boldsymbol{\varphi} - \int_0^T \int_{\Omega_t} \mathbb{S} : \nabla_x \boldsymbol{\varphi} \end{aligned} \tag{2.3}$$

for any test function $\boldsymbol{\varphi} \in \mathcal{C}^1(\overline{Q}_T; \mathbb{R}^3)$ satisfying

$$\boldsymbol{\varphi}(T,\cdot) = 0, \quad \text{in } \Omega_T, \text{ and } \quad (\boldsymbol{\varphi} \cdot \mathbf{n})|_{\Gamma_t} = 0, \quad \text{for any } t \in [0,T].$$

The impermeability condition will be then satisfied in the sense of trace,

$$(\mathbf{u}, \nabla_x \mathbf{u}) \in L^2(Q_T; \mathbb{R}^3), \quad (\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\Gamma_t} = 0, \quad \text{for any } t \in [0,T].$$

III. Entropy inequality. Using the Gibbs' equation (1.5), we deduce the entropy equation from (1.3), given by

$$\partial_t(\rho s) + \mathrm{div}_x(\rho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\theta}\right) = \frac{1}{\theta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\theta} \cdot \nabla_x \theta\right). \tag{2.4}$$

Based on the fact that the *a priori* bounds only provide the L^1 bound for the entropy production rate, the entropy equation is formulated by an inequality (see [12]).

$$\partial_t(\rho s) + \mathrm{div}_x(\rho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\theta}\right) \geq \frac{1}{\theta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\theta} \cdot \nabla_x \theta\right). \tag{2.5}$$

The weak formulation for (2.5) should be of the form:

$$\begin{aligned} -\int_0^T \int_{\Omega_t} \left(\rho s \partial_t \varphi + \rho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\theta} \cdot \nabla_x \varphi\right) - \int_{\Omega_0} (\rho s)_0 \varphi(0,\cdot) \\ \geq \int_0^T \int_{\Omega_t} \frac{\varphi}{\theta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\theta} \cdot \nabla_x \theta\right), \end{aligned} \tag{2.6}$$

for any test function $\varphi \in \mathcal{C}^1(\overline{Q}_T; \mathbb{R})$ with $\varphi \geq 0$, $\varphi(T,\cdot) = 0$ and $\varphi|_{\Omega_t} = 0$ for all $t \in [0,T]$.

IV. Poisson equation. The Poisson equation (1.4) will be considered in the whole space $\mathbb{R}^3$ provided $\rho = 0$ outside the domain Ω_t for each $t \in [0,T]$. Accordingly, the expected weak formulation for the Poisson equation will be

$$\int_0^T \int_{\Omega_t} \nabla_x \Psi \cdot \nabla_x \varphi = \int_0^T \int_{\Omega_t} \rho \varphi, \tag{2.7}$$

for any test function $\varphi \in \mathcal{C}^1(\overline{Q}_T;\mathbb{R})$.

V. Ballistic energy inequality. We note that the weak formulation of energy equation (1.3) cannot be directly used to define the weak solution to system (1.1)-(1.4), due to the absence of information about heat flux on the boundary. The method we adopted here is combining the weak formulation of entropy with the energy balance to get the ballistic energy.

Assuming all the quantities of concern are smooth and multiplying the momentum equation (1.2) by $(\mathbf{u}-\mathbf{V})$, then integrating by parts w.r.t. space variable and integrating the energy equation (1.3), one has by summing up (as well as using the continuity equation)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho e\right)+\int_{\Gamma_t}\mathbf{q}\cdot\mathbf{n}-\int_{\Omega_t}\rho\nabla_x\Psi\cdot(\mathbf{u}-\mathbf{V})\\&=-\int_{\Omega_t}\big(\rho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{V}-\mathbb{S}:\nabla_x\mathbf{V}+p\,\mathrm{div}_x\mathbf{V}\big)+\int_{\Omega_t}\partial_t(\rho\mathbf{u})\cdot\mathbf{V}.\end{aligned}\tag{2.8}$$

We now observe that

$$\begin{aligned}-\int_{\Omega_t}\rho\nabla_x\Psi\cdot\mathbf{u}=\int_{\Omega_t}\Psi\,\mathrm{div}_x(\rho\mathbf{u})=-\int_{\Omega_t}\Psi\,\partial_t\rho=\frac{1}{4\pi g}\int_{\Omega_t}\Psi\partial_t(\Delta_x\Psi)\\=-\frac{1}{8\pi g}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}|\nabla_x\Psi|^2,\end{aligned}\tag{2.9}$$

where no boundary integral will appear after the first integration by parts since the density is supposed to be "zero" outside the fluid domain Ω_t for any $t\in[0,T]$ (indeed, we shall discuss it later in Section 5.4). In the second integration by parts we simply use that $\dfrac{\partial\Psi}{\partial\mathbf{n}}=0$ on Γ_t for each $t\in[0,T]$.

Using (2.9), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho e-\frac{1}{8\pi g}|\nabla_x\Psi|^2\right)+\int_{\Gamma_t}\mathbf{q}\cdot\mathbf{n}\\&=-\int_{\Omega_t}\big(\rho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{V}-\mathbb{S}:\nabla_x\mathbf{V}+p\,\mathrm{div}_x\mathbf{V}\big)+\int_{\Omega_t}\partial_t(\rho\mathbf{u})\cdot\mathbf{V}-\int_{\Omega_t}\rho\nabla_x\Psi\cdot\mathbf{V}.\end{aligned}\tag{2.10}$$

The last term in the r.h.s. of the above equality can be computed as follows:

$$-\int_{\Omega_t}\rho(\nabla_x\Psi\cdot\mathbf{V})=\frac{1}{4\pi g}\int_{\Omega_t}\Delta_x\Psi(\nabla_x\Psi\cdot\mathbf{V})=-\frac{1}{4\pi g}\int_{\Omega_t}\nabla_x\Psi\cdot\nabla_x(\nabla_x\Psi\cdot\mathbf{V}),$$

thanks to the homogeneous Neumann boundary condition of Ψ, and eventually one has,

$$-\int_{\Omega_t}\rho(\nabla_x\Psi\cdot\mathbf{V})=-\frac{1}{8\pi g}\int_{\Omega_t}|\nabla_x\Psi|^2\mathrm{div}_x\mathbf{V}.\tag{2.11}$$

But from the Poisson equation (1.4) we have

$$\int_{\Omega_t}|\nabla_x\Psi|^2\le C(g)\|\rho\|_{L^{\frac{6}{5}}(\Omega_t)}\|\Psi\|_{L^6(\Omega_t)}\le\epsilon\|\Psi\|^2_{W^{1,2}(\Omega_t)}+\frac{C(g)}{\epsilon}\|\rho\|^2_{L^{\frac{6}{5}}(\Omega_t)},$$

for some constant $C(g)>0$ and for any $\epsilon>0$, thanks to the fact that $W^{1,2}(\Omega_t)\hookrightarrow L^6(\Omega_t)$ (as we are in dimension 3). Now, since $\displaystyle\int_{\Omega_t}\Psi\,\mathrm{d}x=0$, by using generalized Poincaré inequality (see Lemma A.1) we have

$$\begin{aligned}\|\Psi\|_{W^{1,2}(\Omega_t)}\le C(g)\|\rho\|_{L^{\frac{6}{5}}(\Omega_t)}\le C_g\|\rho\|^{\frac{7}{12}}_{L^1(\Omega_t)}\|\rho\|^{\frac{5}{12}}_{L^{\frac{5}{3}}(\Omega_t)}\le C(\rho_0,g)\Big(\int_{\Omega_t}\rho^{\frac{5}{3}}\Big)^{\frac{1}{4}}\\\le\Big(\int_{\Omega_t}\rho^{\frac{5}{3}}\Big)^{\frac{1}{2}}+C(\rho_0,g),\end{aligned}\tag{2.12}$$

where we also used the standard interpolation inequality.

This leads to the following two facts:

(i) We have

$$\left| \int_{\Omega_t} \rho \nabla_x \Psi \cdot \mathbf{V} \right| \le C(\mathbf{V}) \int_{\Omega_t} \rho^{\frac{5}{3}} + C(g, \mathbf{V}, \rho_0), \tag{2.13}$$

(recall that the fluid satisfies the mass conservation law, i.e., $\int_{\Omega_t} \rho \, \mathrm{d}x = \int_{\Omega_0} \rho_0 \, \mathrm{d}x$.)

(ii) The negative term $-\frac{1}{8\pi g}\|\nabla_x \Psi\|^2_{L^2(\Omega_t)}$ can be absorbed by the term $\rho e(\rho, \theta)$ in the equation (2.10). In fact, the estimate (2.12) can be written in a more precise way, namely

$$\|\Psi\|^2_{W^{1,2}(\Omega_t)} \le 6\pi g p_\infty \int_{\Omega_t} \rho^{\frac{5}{3}} + C(\rho_0, g, p_\infty),$$

so that using the lower bound of $\rho e(\rho, \theta)$ from (1.33), we have

$$\frac{1}{8\pi g}\|\nabla_x \Psi\|^2_{L^2(\Omega_t)} \le \frac{3p_\infty}{4} \int_{\Omega_t} \rho^{\frac{5}{3}} + C(\rho_0, g, p_\infty) \le \frac{1}{2}\rho e(\rho, \theta) + C(\rho_0, g, p_\infty), \tag{2.14}$$

and thus the required result follows.

However, the relation (2.10) cannot be used in the weak formulation as the boundary integral

$$\int_{\Gamma_t} \mathbf{q} \cdot \mathbf{n}$$

cannot be controlled in the formulation. To get rid of this, we shall multiply the entropy inequality (2.5) by some $\widetilde{\theta} \in \mathcal{C}^1(\overline{Q}_T; \mathbb{R})$ such that

$$\inf_{\overline{Q}_T} \widetilde{\theta} > 0 \quad \text{and} \quad \widetilde{\theta}(t, x) = \theta_B(t, x), \quad \text{for } (t, x) \in \cup_{t \in (0,T)} \big(\{t\} \times \Gamma_t\big), \tag{2.15}$$

and integrating by parts we have

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho s \widetilde{\theta} + \int_{\Omega_t} \left[\rho s \left(\partial_t \widetilde{\theta} + \mathbf{u} \cdot \nabla_x \widetilde{\theta} \right) + \frac{\mathbf{q}}{\theta} \cdot \nabla_x \widetilde{\theta} \right] - \int_{\Gamma_t} \frac{\widetilde{\theta}}{\theta} \mathbf{q} \cdot \mathbf{n} \\ \le - \int_{\Omega_t} \frac{\widetilde{\theta}}{\theta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\theta} \cdot \nabla_x \theta \right). \end{aligned} \tag{2.16}$$

Let us first add (2.10) and (2.16) and use the bound (2.14) to absorb the negative term $-\frac{1}{8\pi g}\|\nabla_x \Psi\|^2_{L^2(\Omega_t)}$ in terms of $\rho e(\rho, \theta)$. Then, using (1.14), (2.15) and replacing $\mathbf{q} = -\kappa(\theta)\nabla_x \theta$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e - \rho s \widetilde{\theta} \right) + \int_{\Omega_t} \frac{\widetilde{\theta}}{\theta} \left(\mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\theta)}{\theta} |\nabla_x \theta|^2 \right) \\ &\le - \int_{\Omega_t} \big(\rho[\mathbf{u} \otimes \mathbf{u}] : \nabla_x \mathbf{V} - \mathbb{S} : \nabla_x \mathbf{V} + p \, \mathrm{div}_x \mathbf{V} \big) + \int_{\Omega_t} \partial_t (\rho \mathbf{u}) \cdot \mathbf{V} \\ &\quad - \int_{\Omega_t} \rho \nabla_x \Psi \cdot \mathbf{V} - \int_{\Omega_t} \left[\rho s \left(\partial_t \widetilde{\theta} + \mathbf{u} \cdot \nabla_x \widetilde{\theta} \right) - \frac{\kappa(\theta)}{\theta} \nabla_x \theta \cdot \nabla_x \widetilde{\theta} \right], \end{aligned} \tag{2.17}$$

where we observe that (2.17) does not contain the boundary heat flux and therefore, it is suitable for the weak formulation.

Till here, we get the ballistic energy for the system (1.1)-(1.4), and rewrite it in an explicit way as below:

$$
\begin{aligned}
&-\int_0^T \partial_t\psi \int_{\Omega_t} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e - \rho s\widetilde{\theta}\right) + \int_0^T \psi \int_{\Omega_t} \frac{\widetilde{\theta}}{\theta}\left(\mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\theta)}{\theta}|\nabla_x\theta|^2\right) \\
&\le \psi(0)\int_{\Omega_0}\left(\frac{1}{2}\frac{|(\rho\mathbf{u})_0|^2}{\rho_0} + \rho_0 e_0(\rho_0,\theta_0) - \rho_0 s(\rho_0,\theta_0)\widetilde{\theta}(0,\cdot)\right) \\
&-\int_0^T \psi \int_{\Omega_t} \left(\rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\mathbf{V} - \mathbb{S} : \nabla_x \mathbf{V} + p\,\mathrm{div}_x\mathbf{V}\right) + \int_0^T \psi\int_{\Omega_t} \partial_t(\rho\mathbf{u})\cdot\mathbf{V} \\
&\quad - C\int_0^T \psi \int_{\Omega_t} \rho\nabla_x\Psi\cdot\mathbf{V} - \int_0^T \psi \int_{\Omega_t}\left[\rho s\left(\partial_t\widetilde{\theta} + \mathbf{u}\cdot\nabla_x\widetilde{\theta}\right) - \frac{\kappa(\theta)}{\theta}\nabla_x\theta\cdot\nabla_x\widetilde{\theta}\right],
\end{aligned}
\tag{2.18}
$$

for any $\psi \in \mathcal{C}_c^1([0,T))$ with $\psi \ge 0$ and $\partial_t\psi \le 0$.

Definition 1. *We say that the trio $(\rho,\mathbf{u},\theta)$ is a weak solution of the problem* (1.1)–(1.4) *with boundary conditions* (1.12)–(1.16) *and initial conditions* (1.17)–(1.20) *if the following items hold:*

- $\rho \in L^\infty(0,T;L^{\frac{5}{3}}(\mathbb{R}^3))$, $\rho \ge 0$, $\rho \in L^q(Q_T)$ *with some certain* $q > 1$,
- $\mathbf{u}$, $\nabla_x\mathbf{u} \in L^2(Q_T)$, $\rho\mathbf{u} \in L^\infty(0,T;L^1(\mathbb{R}^3))$,
- $\theta > 0$ *a.e. on* Q_T, $\theta \in L^\infty(0,T;L^4(\mathbb{R}^3))$, $\theta, \nabla_x\theta, \log\theta, \nabla_x\log\theta \in L^2(Q_T)$,
- *relations* (2.1), (2.2), (2.3), (2.7), (2.18)*are satisfied.*

3 Main result

Here we state the main theorem of this paper:

Theorem 3.1. *Assume that $\Omega_0 \subset \mathbb{R}^3$ is a bounded domain of the class C^3, and suppose that $\mathbf{V} \in C^1([0,T];C_c^3(\mathbb{R}^3;\mathbb{R}^3))$ satisfying* (1.8) *and the hypothesis in subsection 1.3 are satisfied. Then the Naiver-Stokes-Fourier-Poisson* (1.1)–(1.4) *with boundary conditions* (1.12)–(1.16) *and initial conditions* (1.17)–(1.20) *admits a weak solution in the sense of Definition 1 on any finite time interval* $(0,T)$.

4 Penalized problem

Let us choose $R > 0$ such that

$$\mathbf{V}|_{[0,T]\times\{|x|>R\}} = 0, \quad \overline{\Omega}_0 \subset \{x\in\mathbb{R}^3 : |x| \le R\},$$

and then take the reference domain

$$\mathcal{B} := \{x \in \mathbb{R}^3 : |x| < 2R\}.$$

4.1 Mollification of the coefficients and initial data

• The viscosity coefficients are taken as

$$\mu_\omega(\theta) = f_\omega\mu(\theta) \in \mathcal{C}_c^\infty([0,T]\times\mathcal{B}), \tag{4.1}$$

$$\eta_\omega(\theta) = f_\omega\eta(\theta) \in \mathcal{C}_c^\infty([0,T]\times\mathcal{B}), \tag{4.2}$$

where the function $f_\omega \in \mathcal{C}_c^\infty([0,T]\times\mathcal{B})$ such that

$$
\begin{cases}
0 < \omega \le f_\omega \le 1 \quad \text{in } [0,T]\times\mathcal{B}, \text{ for } \omega > 0, \\
f_\omega(t,\cdot)|_{\Omega_\tau} = 1 \quad \text{for any } \tau \in [0,T], \\
\|f_\omega\|_{L^\infty((0,T)\times(\mathcal{B}\setminus\Omega_t))} \le c\,\omega.
\end{cases}
\tag{4.3}
$$

From the above definitions, it obviously holds that

$$\mu_\omega,\ \eta_\omega \to 0 \quad \text{a.e. in} \quad ((0,T)\times\mathcal{B})\setminus Q_T \quad \text{as} \ \ \omega\to 0. \tag{4.4}$$

• We also set the heat conductivity coefficient as follows:

$$\kappa_\nu(\theta,t,x)=\chi_\nu\kappa(\theta), \tag{4.5}$$

where $\chi_\nu \in L^\infty((0,T)\times\mathcal{B})$ such that

$$\chi_\nu = 1 \quad \text{in } Q_T \ \ \text{and} \ \ \chi_\nu=\nu \ \ \text{in} \ ((0,T)\times\mathcal{B})\setminus Q_T \ \ \text{for } \nu>0. \tag{4.6}$$

• Similarly, we introduce a variable coefficient $a_\xi := a_\xi(t,x)$ representing the radiative parts of the pressure, internal energy and entropy, given by

$$a_\xi(t,x)=\chi_\xi a, \tag{4.7}$$

with $\chi_\xi \in L^\infty((0,T)\times\mathcal{B})$ such that

$$\chi_\xi = 1 \quad \text{in } Q_T \ \ \text{and} \ \ \chi_\xi=\xi \ \ \text{in} \ ((0,T)\times\mathcal{B})\setminus Q_T \ \ \text{for } \xi>0. \tag{4.8}$$

We now set

$$p_{\xi,\delta}(\rho,\theta)=p_M(\rho,\theta)+\frac{a_\xi}{3}\theta^4+\delta\rho^\beta, \quad \beta\geq 4,\ \delta>0, \tag{4.9}$$

$$e_\xi(\rho,\theta)=e_M(\rho,\theta)+\frac{a_\xi\theta^4}{\rho}, \quad s_\xi(\rho,\theta)=s_M(\rho,\theta)+\frac{4a_\xi\theta^3}{3\rho}. \tag{4.10}$$

• Let us now define the modified initial data $\rho_{0,\delta}$, $(\rho\mathbf{u})_{0,\delta}$ and $\theta_{0,\delta}$. We consider $\rho_{0,\delta}$ such that

$$\begin{gathered}\rho_{0,\delta}\geq 0, \quad \rho_{0,\delta}\not\equiv 0 \ \text{ in } \Omega_0, \quad \rho_{0,\delta}=0 \ \text{ in } \mathbb{R}^3\setminus\Omega_0, \quad \int_{\mathcal{B}}\left(\rho_{0,\delta}^{\frac{5}{3}}+\delta\rho_{0,\delta}^{\beta}\right)\leq c,\\ \rho_{0,\delta}\to\rho_0 \ \text{ in } L^{\frac{5}{3}}(\mathcal{B}) \ \text{ as } \delta\to 0, \quad |\{\rho_{0,\delta}<\rho_0\}|\to 0 \ \text{ as } \delta\to 0.\end{gathered} \tag{4.11}$$

In above, the constant $c>0$ is independent of the parameter δ.

Next, the initial data for the momentum part is taken in such a way

$$(\rho\mathbf{u})_{0,\delta}=\begin{cases}(\rho\mathbf{u})_0 & \text{if } \rho_{0,\delta}\geq\rho_0,\\ 0 & \text{else.}\end{cases} \tag{4.12}$$

For the temperature part, we consider $0<\underline{\theta}\leq\theta_{0,\delta}\leq\overline{\theta}$ with $\theta_{0,\delta}\in L^\infty(\mathcal{B})\cap\mathcal{C}^{2+\nu_0}(\mathcal{B})$ for some exponent $\nu_0\in(0,1)$ where $\underline{\theta},\overline{\theta}$ are positive real numbers as introduced in Section 1.1.

Moreover, $\rho_{0,\delta}$ and $\theta_{0,\delta}$ are taken in such a way that

$$\int_{\Omega_0}\rho_{0,\delta}e(\rho_{0,\delta},\theta_{0,\delta})\to\int_{\Omega_0}\rho_0 e(\rho_0,\theta_0), \tag{4.13}$$

$$\rho_{0,\delta}s(\rho_{0,\delta},\theta_{0,\delta})\to\rho_0 s(\rho_0,\theta_0) \ \text{ weakly in } \ L^1(\Omega_0). \tag{4.14}$$

4.2 Penalization in the fixed domain and weak formulations

We begin this subsection by shortly describing the strategy of the proof for Theorem 3.1.

1. In the momentum equation, we add the penalized term

$$\frac{1}{\varepsilon}\int_0^T\int_{\Gamma_t}(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\,\varphi\cdot\mathbf{n} \ \text{ for } \varepsilon>0 \text{ small}, \tag{4.15}$$

which was originally proposed by Stokes and Carey in [25]. In principle, this allows to deal with the slip boundary conditions. Indeed, as $\varepsilon \to 0$, this additional term yields the boundary condition $(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = 0$ on Γ_t, after reaching some uniform estimates w.r.t. ε. Accordingly, the reference domain $(0,T) \times \mathcal{B}$ is separated by an impermeable interface $\cup_{t\in(0,T)}\{t\} \times \Gamma_t$ to a *fluid domain* Q_T and a *solid domain* $((0,T) \times \mathcal{B}) \setminus Q_T$.

As a matter of fact, we need to take care the behaviour of the solution in the solid domain. To do so, we consider the variable coefficients $\mu_\omega, \eta_\omega, \kappa_\nu, a_\xi$ as presented in Section 4.1. Moreover, similar to the existence theory developed in [12], we introduce the artificial pressure $p_{\xi,\delta}$ with an extra term $\delta\rho^\beta$ (see (4.9)), which gives some more (regularity) information about the density.

2. We add a term $\lambda\theta^\alpha$ into the energy balance and $\lambda\theta^{\alpha-1}$ into the entropy balance, where $\lambda > 0$ and α is appearing in (1.22) in the hypothesis of heat conductivity coefficient κ. These terms yield a control over the temperature in the solid domain. More precisely, these extra penalized terms help to get rid of some unusual terms in solid domain while passing to the limit as $\xi, \nu \to 0$.

3. Keeping $\varepsilon, \omega, \nu, \lambda, \xi$ and $\delta > 0$ fixed, we use the existence theory for the compressible N-S-F system with nonhomogeneous boundary data in the fixed reference domain, developed in [2] (the part of Poisson equation with the N-S-F can be easily handled in the fixed domain).

4. Taking the initial density ρ_0 vanishing outside Ω_0 and letting $\varepsilon \to 0$ for fixed $\omega, \nu, \lambda, \xi, \delta > 0$ we obtain a "two-fluid" system where the density vanishes in the solid part $\big((0,T) \times B\big) \setminus Q_T$. Then, in order to get rid of the terms in $\big((0,T) \times B\big) \setminus Q_T$, we tend all other parameters to zero. To this end, it is required to introduce a proper scaling to let the parameters $\omega, \nu, \xi, \lambda$ to zero simultaneously. This has been rigorously prescribed in Section 5.5. Finally, we let $\delta \to 0$ in a standard fashion, as already used in other related works.

Now we are ready to state the weak formulation for the penalized problem. We consider that the extended ρ and $\mathbf{u}$ vanish on the boundary $(0,T) \times \partial\mathcal{B}$, that is

$$\mathbf{u}|_{\partial\mathcal{B}} = 0, \qquad \text{for all } t \in (0,T), \tag{4.16}$$

and θ satisfies

$$\theta|_{\partial\mathcal{B}} = \theta_B, \qquad \text{for all } t \in (0,T), \tag{4.17}$$

where the above θ_B is introduced in (1.15) as an smooth extended version of the nonhomogeneous temperature on the boundary Γ_t, $t \in [0,T]$ given by (1.14).

I. Continuity equation. The weak formulation for the continuity equation reads as

$$-\int_0^T \int_{\mathcal{B}} \rho B(\rho)\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) + \int_0^T\int_{\mathcal{B}} b(\rho)\mathrm{div}_x\mathbf{u}\varphi = \int_{\mathcal{B}} \rho_{0,\delta}B(\rho_{0,\delta})\varphi(0,\cdot), \tag{4.18}$$

for any test function $\varphi \in \mathcal{C}^1_c([0,T) \times \mathcal{B};\mathbb{R})$ and any $b \in L^\infty \cap \mathcal{C}([0,+\infty))$ such that $b(0) = 0$ and $B(\rho) = B(1) + \int_1^\rho \frac{b(z)}{z^2}$.

II. Momentum equation. The momentum equation is represented by the family of integral identities

$$\begin{aligned} -\int_0^T\int_{\mathcal{B}} \big(\rho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p_{\xi,\delta}(\rho,\theta)\mathrm{div}_x\boldsymbol{\varphi}\big) + \int_0^T\int_{\mathcal{B}} \mathbb{S}_\omega : \nabla_x\boldsymbol{\varphi} \\ -\int_0^T\int_{\mathcal{B}} \rho\nabla_x\Psi\cdot\boldsymbol{\varphi} + \frac{1}{\varepsilon}\int_0^\tau\int_{\Gamma_t} (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\ \boldsymbol{\varphi}\cdot\mathbf{n} = \int_{\mathcal{B}} (\rho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot), \end{aligned} \tag{4.19}$$

for any test function $\boldsymbol{\varphi} \in \mathcal{C}^1_c([0,T) \times \mathcal{B};\mathbb{R}^3)$ and

$$\mathbb{S}_\omega(\theta,\nabla_x\mathbf{u}) = \mu_\omega(\theta,t,x)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta_\omega(\theta,t,x)\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{4.20}$$

III. Poisson equation. The weak formulation for the Poisson equation is given by

$$\int_0^T \int_{\mathcal{B}} \nabla_x \Psi \cdot \nabla_x \varphi = \int_0^T \int_{\mathcal{B}} \rho \varphi, \tag{4.21}$$

for any test function $\varphi \in \mathcal{C}^1((0,T) \times \mathcal{B}; \mathbb{R})$, under the assumption that $\rho = 0$ outside $\mathcal{B}$.

IV. Entropy inequality. Next, we write the penalized entropy inequality, given by

$$\begin{aligned} -\int_0^T \int_{\mathcal{B}} \left(\rho s_\xi(\rho,\theta) \left(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi\right) - \frac{\kappa_\nu(\theta,t,x)}{\theta} \nabla_x \theta \cdot \nabla_x \varphi \right) \\ - \int_{\mathcal{B}} \rho_{0,\delta} s(\rho_{0,\delta}, \theta_{0,\delta}) \varphi(0,\cdot) + \int_0^T \int_{\mathcal{B}} \lambda \theta^{\alpha-1} \varphi \\ \geq \int_0^T \int_{\mathcal{B}} \frac{\varphi}{\theta} \left(\mathbb{S}_\omega : \nabla_x \mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta} |\nabla_x \theta|^2 \right), \end{aligned} \tag{4.22}$$

for any test function $\varphi \in \mathcal{C}_c^1([0,T) \times \mathcal{B}; \mathbb{R})$ with $\varphi \geq 0$.

V. Ballistic energy inequality. Before giving the penalization for the ballistic energy (2.18), we need to extend the boundary data θ_B to the whole domain $\mathcal{B}$.

To this end, we fix a test function $\widetilde{\theta}$ which is the unique solution to

$$-\Delta_x \widetilde{\theta}(t,\cdot) = 0 \ \text{ in } \mathcal{B}, \quad \widetilde{\theta}(t,\cdot)|_{\partial \mathcal{B}} = \theta_B(t,\cdot), \quad \forall t \in [0,T], \tag{4.23}$$

where θ_B is given by (4.17). We still denote this function $\widetilde{\theta}$ as θ_B in the following part.

Finally, we write the ballistic energy to the penalized problem as follows:

$$\begin{aligned} &-\int_0^T \partial_t \psi \int_{\mathcal{B}} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho e_\xi(\rho,\theta) - \rho s_\xi(\rho,\theta)\theta_B + \frac{\delta}{\beta-1}\rho^\beta \right) + \int_0^T \psi \int_{\mathcal{B}} \lambda \theta^\alpha \\ &+ \int_0^T \psi \int_{\mathcal{B}} \frac{\theta_B}{\theta} \left(\mathbb{S}_\omega : \nabla_x \mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta} |\nabla_x \theta|^2 \right) + \frac{1}{\varepsilon} \int_0^T \psi \int_{\Gamma_t} |(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|^2 \\ &\leq \psi(0) \int_{\mathcal{B}} \left(\frac{1}{2} \frac{|(\rho \mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta} e_\xi(\rho_{0,\delta}, \theta_{0,\delta}) - \rho_{0,\delta} s_\xi(\rho_{0,\delta}, \theta_{0,\delta}) \theta_B(0,\cdot) + \frac{\delta}{\beta-1} \rho_{0,\delta}^\beta \right) \\ &+ \int_0^T \psi \int_{\mathcal{B}} \lambda \theta^{\alpha-1} \theta_B \\ &- \int_0^T \psi \int_{\mathcal{B}} \left(\rho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \mathbf{V} - \mathbb{S}_\omega : \nabla_x \mathbf{V} + p_{\xi,\delta}(\rho,\theta) \mathrm{div}_x \mathbf{V} \right) + \int_0^T \psi \int_{\mathcal{B}} \partial_t(\rho \mathbf{u}) \cdot \mathbf{V} \\ &- \int_0^T \psi \int_{\mathcal{B}} \rho \nabla_x \Psi \cdot \mathbf{V} - \int_0^T \psi \int_{\mathcal{B}} \left[\rho s_\xi \left(\partial_t \theta_B + \mathbf{u} \cdot \nabla_x \theta_B \right) - \frac{\kappa_\nu(\theta,t,x)}{\theta} \nabla_x \theta \cdot \nabla_x \theta_B \right], \end{aligned} \tag{4.24}$$

for any $\psi \in \mathcal{C}_c^1([0,T))$ with $\psi \geq 0$ and $\partial_t \psi \leq 0$.

Definition 2. *We say that the trio* $(\rho, \mathbf{u}, \theta)$ *is a weak solution to the penalized problem with initial data* (4.11)–(4.14) *if the following items hold:*

- $\rho \in L^\infty(0,T; L^{\frac{5}{3}}(\mathbb{R}^3)) \cap L^\infty(0,T; L^\beta(\mathbb{R}^3))$, $\rho \geq 0$, $\rho \in L^q((0,T) \times \mathcal{B})$ *with some certain* $q > 1$,
- $\mathbf{u}$, $\nabla_x \mathbf{u} \in L^2((0,T) \times \mathcal{B})$, $\rho \mathbf{u} \in L^\infty(0,T; L^1(\mathcal{B}))$,
- $\theta > 0$ *a.e. on* Q_T, $\theta \in L^\infty(0,T; L^4(\mathcal{B}))$, $\theta, \nabla_x \theta, \log \theta, \nabla_x \log \theta \in L^2((0,T) \times \mathcal{B})$,
- *relations* (4.18), (4.19), (4.21), (4.22), (4.24) *are satisfied.*

Theorem 4.1. *Assume that* $\mathbf{V} \in C^1([0,T]; C_c^3(\mathbb{R}^3; \mathbb{R}^3))$, *and the hypotheses in subsections 1.3–4.1 for viscosity, conductive heat coefficients, and equations of states are all satisfied. Moreover, the initial data satisfy* (4.11)–(4.14). *Then there exists a weak solution to the penalized problem on any time interval* $(0,T)$ *in the sense of Definition 2.*

Proof. Here we just give a short explanation on the proof. The existence of weak solution with the non-homogeneous Dirichlet condition for the temperature in the fixed domain is similar to [2]. It is necessary to consider the continuity equation with a viscous term $\Delta_x \rho$, solving the momentum equations via Faedo-Galerkin approximations and the Poisson equation. Instead of pursuing the solution to the entropy equation, we look for the solution to the internal energy equation. Note that (4.24) is adopted by dealing with the particular non-homogeneous boundary conditions. As pointed out in [18, Theorem 3.1], here also we face the following difficulties.

- The penalized terms $\frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\,\boldsymbol{\varphi}\cdot\mathbf{n}$ in (4.19) and $\frac{1}{\varepsilon}\int_0^T \psi \int_{\Gamma_t} |(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2$ in (4.24).
- The jumps in functions $\kappa_\nu(\theta,t,x)$ in (4.5), and $a_\xi(t,x)$ in (4.7).

The strategy to overcome these difficulties has already been discussed in the beginning of the proof of Theorem 3.1 in [18]. In the present work, we employ similar methodology for the proof. We emphasize that the term $\lambda\theta^\alpha$ ($\alpha > 6$) is necessary for our modified internal energy equation to provide uniform bounds of high power of the temperature on $\mathcal{B}$. Moreover, additional difficulty will arise to get a proper bound of the term $\rho s_\xi(\rho,\theta)\mathbf{u}$ appearing in right hand side of the energy inequality (4.24). □

The rest of the paper is devoted to proof the main result of this paper, that is, Theorem 3.1.

4.3 Uniform bounds

This subsection is devoted to establish the uniform bounds for the weak solution which is constructed by virtue of Theorem (4.1).

Let us define the following Helmholtz-type function (see [12, Chapter 2.2.3]):

$$\mathcal{H}_{\theta_B,\xi}(\rho,\theta) := \rho e_\xi(\rho,\theta) - \rho s_\xi(\rho,\theta)\theta_B, \tag{4.25}$$

and in what follows, we denote

$$\mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) = \rho_{0,\delta} e_\xi(\rho_{0,\delta},\theta_{0,\delta}) - \rho_{0,\delta} s_\xi(\rho_{0,\delta},\theta_{0,\delta})\theta_B(0,\cdot). \tag{4.26}$$

We now consider $\psi_\zeta \in \mathcal{C}_c^1([0,T))$ with

$$\psi_\zeta(t) = \begin{cases} 1 & \text{for } t < \tau - \zeta, \\ 0 & \text{for } t \geq \tau, \end{cases} \quad \text{for any given } \tau \in (0,T),\ 0 < \zeta < \tau,$$

and using it as a test function in (4.24) we derive after passing to the limit $\zeta \to 0$,

$$\begin{aligned}
&\int_{\mathcal{B}} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \mathcal{H}_{\theta_B,\xi}(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot) + \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} |(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2 \\
&+ \int_0^\tau \int_{\mathcal{B}} \lambda\theta^\alpha + \int_0^\tau \int_{\mathcal{B}} \frac{\theta_B}{\theta}\left(\mathbb{S}_\omega : \nabla_x\mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right) \\
\leq &\int_{\mathcal{B}} \left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta\right) + \int_0^\tau \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\,\theta_B \\
&+ \int_{\mathcal{B}} (\rho\mathbf{u}\cdot\mathbf{V})(\tau,\cdot) - \int_{\mathcal{B}} (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0,\cdot) \\
&- \int_0^\tau \int_{\mathcal{B}} \Big(\rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\mathbf{V} - \mathbb{S}_\omega : \nabla_x\mathbf{V} + p_{\xi,\delta}(\rho,\theta)\operatorname{div}_x\mathbf{V} + \rho\mathbf{u}\cdot\partial_t\mathbf{V}\Big) \\
&- \int_0^\tau \int_{\mathcal{B}} \rho\nabla_x\Psi\cdot\mathbf{V} - \int_0^\tau \int_{\mathcal{B}} \Big[\rho s_\xi(\rho,\theta)\Big(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B\Big) - \frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\Big],
\end{aligned} \tag{4.27}$$

for almost all $\tau \in (0,T)$.

Let us now find the uniform bounds of the right hand side of the modified energy inequality (4.27). First, we recall that the fluid system satisfies the mass conservation law, that is

$$\int_{\mathcal{B}} \rho(\tau,\cdot) = \int_{\mathcal{B}} \rho_{0,\delta}(\cdot) = \int_{\Omega_0} \rho_0(\cdot) = C(\rho_0) > 0.$$

Keeping that in mind, we proceed to find the estimates.

• **Step 1.** (i) For any $\epsilon > 0$ small, we have

$$\int_{\mathcal{B}} (\rho\mathbf{u}\cdot\mathbf{V})(\tau,\cdot) \leq C(\mathbf{V}) \left| \int_{\mathcal{B}} \sqrt{\rho}\sqrt{\rho}\mathbf{u}(\tau,\cdot) \right| \leq C(\mathbf{V},\rho_0) + \epsilon \int_{\mathcal{B}} \rho|\mathbf{u}|^2. \tag{4.28}$$

(ii) Next, recall how we obtain (2.13), and we reach to the following:

$$\begin{aligned} \left| \int_0^\tau \int_{\mathcal{B}} \rho\nabla_x\Psi\cdot\mathbf{V} \right| &\leq C(\mathbf{V}) \int_0^\tau \int_{\mathcal{B}} \rho^{\frac{5}{3}} + C(\mathbf{V},\rho_0,g) \\ &\leq C(\mathbf{V},p_\infty,g)\Big(\int_0^\tau \int_{\mathcal{B}} \rho e_\xi(\rho,\theta) + 1 \Big). \end{aligned} \tag{4.29}$$

(iii) Without loss of generality we assume $0 < \lambda \leq 1$ from now onwards. Then, by using Hölder's and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_0^\tau \int_{\mathcal{B}} \mathbb{S}_\omega : \nabla_x\mathbf{V} &\leq \frac{1}{2}\int_0^\tau \int_{\mathcal{B}} \frac{\theta_B}{\theta}\mathbb{S}_\omega : \nabla_x\mathbf{u} + C(\mathbf{V},\theta_B) \int_0^\tau \int_{\mathcal{B}} \theta \\ &\leq \frac{1}{2}\int_0^\tau \int_{\mathcal{B}} \frac{\theta_B}{\theta}\mathbb{S}_\omega : \nabla_x\mathbf{u} + \epsilon\int_0^\tau \int_{\mathcal{B}} \lambda\theta^\alpha + \frac{C(\mathbf{V},\theta_B,\epsilon)}{\lambda^{1/(\alpha-1)}}. \end{aligned} \tag{4.30}$$

We also have that

$$\left| \int_0^\tau \int_{\mathcal{B}} \rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\mathbf{V} \right| \leq C(\mathbf{V}) \int_0^\tau \int_{\mathcal{B}} \rho|\mathbf{u}|^2, \tag{4.31}$$

$$\left| \int_0^\tau \int_{\mathcal{B}} \rho\mathbf{u}\cdot\partial_t\mathbf{V} \right| \leq C(\mathbf{V},\rho_0) + C\int_0^\tau \int_{\mathcal{B}} \rho|\mathbf{u}|^2. \tag{4.32}$$

(iv) Next, since $0 < \lambda \leq 1$, it is easy to observe that

$$\int_0^\tau \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\,\theta_B \leq \frac{C(\theta_B)}{\epsilon} + \epsilon\int_0^\tau \int_{\mathcal{B}} \lambda\theta^\alpha. \tag{4.33}$$

(v) The pressure term $p_{\xi,\delta}(\rho,\theta)$ in (4.9) can be estimated as follows. First, we recall the point (1.34) which indeed tells that $P'(Z) > 0$ for all $Z > 0$. Further, we recall the fact (1.35) which gives $\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0$. Therefore, we obtain the following bounds on the molecular pressure p_M,

$$\begin{aligned} \underline{c}\rho\theta &\leq p_M \leq \overline{c}\rho\theta \quad \text{if } \rho < \overline{Z}\theta^{\frac{3}{2}}, \\ \underline{c}\rho^{\frac{5}{3}} &\leq p_M \leq \begin{cases} \overline{c}\theta^{\frac{5}{2}} & \text{if } \rho < \overline{Z}\theta^{\frac{3}{2}}, \\ \overline{c}\rho^{\frac{5}{3}} & \text{if } \rho > \overline{Z}\theta^{\frac{3}{2}}, \end{cases} \end{aligned} \tag{4.34}$$

and p_M is monotone in $\underline{Z}\theta^{\frac{3}{2}} \leq \rho \leq \overline{Z}\theta^{\frac{3}{2}}$.

With the above information, we deduce that

$$\begin{aligned} \left| \int_0^\tau \int_{\mathcal{B}} p_{\xi,\delta}(\rho,\theta)\mathrm{div}_x\mathbf{V} \right| \leq C(\mathbf{V}) \int_0^\tau \int_{\mathcal{B}} \frac{\delta}{\beta-1}\rho^\beta + C(\mathbf{V}) \int_0^\tau \int_{\mathcal{B}} a_\xi\theta^4 + C(\mathbf{V}) \int_0^\tau \int_{\mathcal{B}} \rho^{\frac{5}{3}} \\ + \epsilon\int_0^\tau \int_{\mathcal{B}} \lambda\theta^\alpha + \frac{C(\mathbf{V},\epsilon)}{\lambda^{5/(2\alpha-5)}}. \end{aligned} \tag{4.35}$$

In fact, we have that

$$\rho e_\xi \geq a_\xi\theta^4 + \frac{3}{2}p_\infty\rho^{\frac{5}{3}}, \tag{4.36}$$

which can be shown in the same way as we have obtained (1.33), and therefore,

$$\int_0^\tau \int_{\mathcal{B}} \left(a_\xi \theta^4 + \rho^{\frac{5}{3}}\right) \leq C(p_\infty) \int_0^\tau \int_{\mathcal{B}} \rho e_\xi.$$

Using the above inequality in (4.35) and together with all other estimates above, we have from (4.27) (by fixing $\epsilon > 0$ small enough),

$$\begin{aligned}
&\int_{\mathcal{B}} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \mathcal{H}_{\theta_B,\xi}(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot) + \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} |(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2 \\
&\quad + \int_0^T \int_{\mathcal{B}} \lambda\theta^\alpha + \int_0^\tau \int_{\mathcal{B}} \frac{\theta_B}{\theta}\left(\mathbb{S}_\omega : \nabla_x \mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right) \\
&\leq \int_{\mathcal{B}} \left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0,\cdot)\right) \\
&\quad + C\int_0^\tau \int_{\mathcal{B}} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e_\xi(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right) \\
&\quad + \left|\int_0^\tau \int_{\mathcal{B}} \left[\rho s_\xi(\rho,\theta)\left(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B\right) - \frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\right]\right| + C\Big(1 + \frac{1}{\lambda^{5/(2\alpha-5)}}\Big),
\end{aligned} \tag{4.37}$$

for almost all $\tau \in (0,T)$, where $\mathcal{H}_{\theta_B,\xi}(\rho,\theta)$ and $\mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta})$ are defined by (4.25) and (4.26) respectively and $C > 0$ is some constant that may depend on the quantities $\mathbf{V}$, ρ_0, p_∞, g and θ_B but not on the parameters λ, ω, ξ, ν, ε or δ.

• **Step 2.** (i) To the next, we recall the expression of $\mathbb{S}_\omega$ from (4.20) and using (4.1), (4.2) and (1.21), we obtain

$$\int_0^\tau \int_{\mathcal{B}} \frac{\theta_B}{\theta}\mathbb{S}_\omega : \nabla_x\mathbf{u} \geq c_1(\omega) \inf_{(0,T)\times\mathcal{B}} |\theta_B| \int_0^\tau \int_{\mathcal{B}} \left|\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right|^2, \tag{4.38}$$

for some constant $c_1(\omega) > 0$.

On the other hand, by the Korn-Poincaré inequality (see Lemma A.2) we have

$$\begin{aligned}
\|\mathbf{u}\|^2_{W^{1,2}(\mathcal{B};\mathbb{R}^3)} &\leq C\left\|\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right\|^2_{L^2(\mathcal{B},\mathbb{R}^3)} + C\Big(\int_{\mathcal{B}} \rho|\mathbf{u}|\Big)^2 \\
&\leq C\left\|\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right\|^2_{L^2(\mathcal{B},\mathbb{R}^3)} + C(\rho_0)\int_{\mathcal{B}} \rho|\mathbf{u}|^2.
\end{aligned} \tag{4.39}$$

Therefore,

$$c_1(\omega)\int_0^\tau \|\mathbf{u}\|^2_{W^{1,2}(\mathcal{B};\mathbb{R}^3)} \leq C(\theta_B)\int_0^\tau \int_{\mathcal{B}} \frac{\theta_B}{\theta}\mathbb{S}_\omega : \nabla_x\mathbf{u} + C(\rho_0,\theta_B)\int_0^\tau \int_{\mathcal{B}} \rho|\mathbf{u}|^2. \tag{4.40}$$

Remark 1. *From the definition of μ_ω and η_ω it is clear that the constant $c_1(\omega)$ behaves like "$c\omega$" in $\mathcal{B}\setminus\Omega_t$ for some constant $c > 0$ which is independent in ω.*

(ii) We still need to estimate of the last couple of terms containing θ_B in the right hand side of (4.37). Recall the definition of κ_ν from (4.5), we get

$$\begin{aligned}
\int_0^\tau \int_{\mathcal{B}} \frac{\kappa_\nu(\theta)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B &= \int_0^\tau \int_{\mathcal{B}} \nabla_x K_\nu(\theta)\cdot\nabla_x\theta_B \\
&= -\int_0^\tau \int_{\mathcal{B}} K_\nu(\theta)\cdot\Delta_x\theta_B + \int_0^\tau \int_{\partial\mathcal{B}} K_\nu(\theta_B)\nabla_x\theta_B\cdot\mathbf{n},
\end{aligned} \tag{4.41}$$

where $\frac{\partial}{\partial\theta}K_\nu(\theta) = \frac{\kappa_\nu(\theta)}{\theta}$. Now, thanks to the choice of test function θ_B in (4.23), we have $\Delta_x\theta_B = 0$ and as a consequence, one has

$$\left|\int_0^\tau \int_{\mathcal{B}} \frac{\kappa_\nu(\theta)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\right| \leq C(\theta_B). \tag{4.42}$$

(iii) We now need to find a proper bound of $\rho s_\xi(\rho,\theta)|\mathbf{u}|$. Recall (1.39), there exists some $c>0$ such that

$$s_M(\rho,\theta) = S\left(\frac{\rho}{\theta^{\frac{3}{2}}}\right) \le c \quad \text{when } \frac{\rho}{\theta^{3/2}} > 1, \tag{4.43}$$

and therefore,

$$\rho s_\xi(\rho,\theta) = \rho s_M(\rho,\theta) + \frac{4a_\xi}{3}\theta^3 \le c\rho + \frac{4a_\xi}{3}\theta^3, \qquad \text{for } \frac{\rho}{\theta^{3/2}} > 1. \tag{4.44}$$

On the other hand, when $\frac{\rho}{\theta^{3/2}} \le 1$, we use the strategy developed in [13, Section 4, formula (4.6)] and according to that, one has (using the Gibb's relation (1.5), the hypothesis (1.23)–(1.32) and (1.37))

$$s_M(\rho,\theta) \le C(1+|\log\rho| + [\log\theta]^+).$$

This yields

$$\rho s_\xi(\rho,\theta) = \rho s_M(\rho,\theta) + \frac{4a_\xi}{3}\theta^3 \le C\left(\rho + |\rho\log\rho| + |\rho|[\log\theta]^+\right) + \frac{4a_\xi}{3}\theta^3. \tag{4.45}$$

Now, observe that

$$|\rho\log\rho| \le \begin{cases} C\rho^{\frac{1}{2}}, & \text{when } 0<\rho\le 1, \\ \frac{3}{2}\rho[\log\theta]^+, & \text{when } \rho > 1 \ \ (\text{consequently } \theta > 1 \text{ since } \frac{\rho}{\theta^{3/2}} \le 1), \end{cases} \tag{4.46}$$

where we have used the fact that $|\rho^{\frac{1}{2}}\log\rho|$ is bounded for $0<\rho\le 1$.

Using (4.46) in (4.45), we get

$$\rho s_\xi(\rho,\theta) \le C\Big(\rho + \rho^{\frac{1}{2}} + \rho[\log\theta]^+\Big) + \frac{4a_\xi}{3}\theta^3, \qquad \text{for } \frac{\rho}{\theta^{3/2}} \le 1. \tag{4.47}$$

Thus, the inequalities (4.44) and (4.47) yield

$$\begin{aligned} &\int_0^\tau\int_{\mathcal{B}} \rho s_\xi(\rho,\theta)|\mathbf{u}| \\ \le& C\int_0^\tau\int_{\mathcal{B}}\rho|\mathbf{u}| + C\int_0^\tau\int_{\mathcal{B}}\rho^{\frac{1}{2}}|\mathbf{u}| + C\int_0^\tau\int_{\mathcal{B}}\rho|\mathbf{u}|[\log\theta]^+ + C\int_0^\tau\int_{\mathcal{B}} a_\xi\theta^3|\mathbf{u}| \\ \le& C(\rho_0) + C\int_0^\tau\int_{\mathcal{B}}\rho|\mathbf{u}|^2 + C\int_0^\tau\int_{\mathcal{B}}\rho([\log\theta]^+)^2 + \frac{c_1(\omega)}{2}\int_0^\tau\int_{\mathcal{B}}|\mathbf{u}|^2 + \frac{1}{2c_1(\omega)}\int_0^\tau\int_{\mathcal{B}} a_\xi^2\theta^6, \end{aligned} \tag{4.48}$$

where $c_1(\omega)$ is appearing in (4.40).

Here, we observe that the third term in the last inclusion is arising due to the case when $\frac{\rho}{\theta^{3/2}} \le 1$. Keeping in mind this point, we have

$$\int_0^\tau\int_{\mathcal{B}}\rho([\log\theta]^+)^2 \le \int_0^\tau\int_{\mathcal{B}}\theta^{\frac{3}{2}}([\log\theta]^+)^2 \le \int_0^\tau\int_{\mathcal{B}}\theta^{\frac{5}{2}} \le \epsilon\int_0^\tau\int_{\mathcal{B}}\lambda\theta^\alpha + \frac{C(\epsilon)}{\lambda^{5/(2\alpha-5)}} \tag{4.49}$$

for any chosen $\epsilon>0$ (since $[\log\theta]^+ \le \theta^{1/2}$).

We now need further care to compute the integral containing θ^6 in the r.h.s. of (4.48).

- First observe that

$$\int_{\mathcal{B}}\theta_B\kappa_\nu(\theta)\frac{|\nabla_x\theta|^2}{|\theta|^2} \ge \underline{\kappa}c_2(\nu)\inf_{(0,T)\times\mathcal{B}}|\theta_B|\int_{\mathcal{B}}\left(\theta^{-2}+\theta^{\alpha-2}\right)|\nabla_x\theta|^2, \tag{4.50}$$

  in the l.h.s. of (4.37), for some constant $c_2(\nu)>0$ which behaves like "$c\nu$" in $\mathcal{B}\setminus\Omega_t$ for some constant $c>0$ that is independent in ν and in Ω_t, $c_2(\nu)$ does not depend on ν since $\chi_\nu = 1$ in Ω_t for any $t\in[0,T]$.

– Now, using the Hölder's and Young's inequalities, we have

$$\frac{1}{2c_1(\omega)}\int_{\mathcal{B}} a_\xi^2\theta^6 \le \epsilon c_2(\nu)\int_{\mathcal{B}}\theta^\alpha + C(\epsilon)\int_{\mathcal{B}}\left(\frac{a_\xi^2}{2c_1(\omega)}c_2(\nu)^{-\frac{6}{\alpha}}\right)^{\frac{\alpha}{\alpha-6}}, \quad \text{where } \alpha > 6. \tag{4.51}$$

for any given $\epsilon > 0$.

But $W^{1,2}(\mathcal{B}) \hookrightarrow L^6(\mathcal{B})$; using this and the generalized Poincaré inequality (see Lemma A.1), we deduce that

$$\begin{aligned}\int_{\mathcal{B}}\theta^\alpha \le \|\theta^\alpha\|_{L^3(\mathcal{B})} = \left\|\theta^{\frac{\alpha}{2}}\right\|^2_{L^6(\mathcal{B})} &\le \left\|\theta^{\frac{\alpha}{2}}\right\|^2_{W^{1,2}(\mathcal{B})} \\ &\le C\left\|\nabla_x\theta^{\frac{\alpha}{2}}\right\|^2_{L^2(\mathcal{B})} + C\int_{\mathcal{B}}|\theta| \\ &\le C\int_{\mathcal{B}}\theta^{\alpha-2}|\nabla_x\theta|^2 + C\int_{\mathcal{B}}\lambda\theta^\alpha + \frac{C}{\lambda^{1/(\alpha-1)}}.\end{aligned} \tag{4.52}$$

By using (4.52) and (4.51) in (4.48), and thanks to (4.49), we obtain

$$\begin{aligned}&\int_0^\tau\int_{\mathcal{B}}\rho s_\xi(\rho,\theta)|\mathbf{u}| \\ &\le C(\rho_0) + C\int_0^\tau\int_{\mathcal{B}}\rho|\mathbf{u}|^2 + \frac{c_1(\omega)}{2}\int_0^\tau\int_{\mathcal{B}}|\mathbf{u}|^2 + C\epsilon\, c_2(\nu)\int_0^\tau\int_{\mathcal{B}}\theta^{\alpha-2}|\nabla_x\theta|^2 \\ &\quad + C\epsilon\int_0^\tau\int_{\mathcal{B}}\lambda\theta^\alpha + \frac{C(\epsilon)}{\lambda^{5/(2\alpha-5)}} + \frac{C}{\lambda^{1/(\alpha-1)}} + C(\epsilon)\int_{\mathcal{B}}\left(\frac{a_\xi^{2\alpha}}{c_1(\omega)^\alpha c_2(\nu)^6}\right)^{\frac{1}{\alpha-6}}.\end{aligned} \tag{4.53}$$

Collecting the bounds (4.42), (4.53) along with (4.38)–(4.40), we have from (4.37) (by fixing $\epsilon > 0$ small enough) that

$$\begin{aligned}&\int_{\mathcal{B}}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \mathcal{H}_{\theta_B,\xi}(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot) + \frac{1}{\varepsilon}\int_0^\tau\int_{\Gamma_t}|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2 + \int_0^T\int_{\mathcal{B}}\lambda\theta^\alpha \\ &+ c_1(\omega)\int_0^\tau\|\mathbf{u}\|^2_{W^{1,2}(\mathcal{B};\mathbb{R}^3)} + c_2(\nu)\int_0^\tau\int_{\mathcal{B}}\left(\theta^{-2}+\theta^{\alpha-2}\right)|\nabla_x\theta|^2 \\ \le &\int_{\mathcal{B}}\left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0,\cdot)\right) \\ &+ C\int_0^\tau\int_{\mathcal{B}}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e_\xi(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right) + C\left(1 + \frac{1}{\lambda^{5/(2\alpha-5)}} + \left(\frac{\xi^{2\alpha}}{\omega^\alpha\nu^6}\right)^{\frac{1}{\alpha-6}}\right),\end{aligned} \tag{4.54}$$

for almost all $\tau \in (0,T)$, and the constant $C > 0$ may depend on the quantities $\mathbf{V}$, ρ_0, p_∞, g and θ_B but not on the parameters λ, ω, ξ, ν, ε or δ.

Applying the Grönwall's inequality in (4.54), we deduce that

$$\begin{aligned}&\int_{\mathcal{B}}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \mathcal{H}_{\theta_B,\xi}(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot) + \frac{1}{\varepsilon}\int_0^\tau\int_{\Gamma_t}|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2 + \int_0^\tau\int_{\mathcal{B}}\lambda\theta^\alpha \\ &+ c_1(\omega)\int_0^\tau\|\mathbf{u}\|^2_{W^{1,2}(\mathcal{B};\mathbb{R}^3)} + c_2(\nu)\int_0^\tau\int_{\mathcal{B}}\left(\theta^{-2}+\theta^{\alpha-2}\right)|\nabla_x\theta|^2 \\ \le &C\int_{\mathcal{B}}\left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0,\cdot)\right) \\ &+ C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^{2\alpha}}{\omega^\alpha\nu^\alpha}\right)^{\frac{1}{\alpha-6}}\right),\end{aligned} \tag{4.55}$$

for almost all $\tau \in (0,T)$ and $C > 0$ constant which has been specified in (4.54). In above, we have used the following facts: since $0 < \lambda \le 1$ and $\alpha > 6$, one has

$$\frac{1}{\lambda^{5/(2\alpha-5)}} < \frac{1}{\lambda^{5/7}} \quad \text{as} \quad \frac{5}{(2\alpha-5)} < \frac{5}{7},$$

and, moreover

$$\nu^6 > \nu^\alpha \quad \text{since } 0 < \nu \leq 1.$$

To ensure that the left hand side of (4.55) is positive. we proceed as follows. Setting a constant $\overline{\rho}$ such that $\displaystyle\int_{\mathcal{B}} (\rho - \overline{\rho}) = 0$ for almost all $\tau \in [0, T)$ and we rewrite the estimate (4.55) as the total dissipation inequality:

$$\begin{aligned}
&\int_{\mathcal{B}} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \mathcal{H}_{\theta_B,\xi}(\rho,\theta) - (\rho - \overline{\rho}) \frac{\partial \mathcal{H}_{\theta_B,\xi}(\overline{\rho},\theta_B)}{\partial \rho} - \mathcal{H}_{\theta_B,\xi}(\overline{\rho},\theta_B) + \frac{\delta}{\beta-1}\rho^\beta \right)(\tau,\cdot) \\
&+ \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} |(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|^2 + \int_0^\tau \int_{\mathcal{B}} \lambda \theta^\alpha \\
&+ c_1(\omega) \int_0^\tau \|\mathbf{u}\|^2_{W^{1,2}(\mathcal{B};\mathbb{R}^3)} + c_2(\nu) \int_0^\tau \int_{\mathcal{B}} \left(\theta^{-2} + \theta^{\alpha-2}\right) |\nabla_x \theta|^2 \\
\leq & C \int_{\mathcal{B}} \left(\frac{1}{2} \frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0,\cdot) \right) \\
&- \int_{\mathcal{B}} \left((\rho_{0,\delta} - \overline{\rho}) \frac{\partial \mathcal{H}_{\theta_B,\xi}(\overline{\rho},1)}{\partial \rho} + \mathcal{H}_{\theta_B,\xi}(\overline{\rho},1) \right) + C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right),
\end{aligned} \tag{4.56}$$

for almost all $\tau \in (0, T)$. In (4.56), the left hand side is positive due to the hypothesis of thermodynamic stability (1.26), (1.27).

• **The uniform bounds.** (i) From (4.56), we directly have

$$\int_0^T \int_{\Gamma_t} |(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|^2 \leq \varepsilon\, C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right), \tag{4.57}$$

$$\operatorname*{ess\,sup}_{\tau\in[0,T]} \|\delta\rho^\beta(\tau,\cdot)\|_{L^1(\mathcal{B})} \leq C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right), \tag{4.58}$$

$$\operatorname*{ess\,sup}_{\tau\in[0,T]} \|\sqrt{\rho}\mathbf{u}(\tau,\cdot)\|^2_{L^2(\mathcal{B})} \leq C\left(\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right), \tag{4.59}$$

$$\|\lambda\theta^\alpha\|_{L^1((0,T)\times\mathcal{B})} \leq C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right). \tag{4.60}$$

(ii) We also have

$$c_1(\omega)\|\mathbf{u}\|^2_{L^2(0,T;W^{1,2}(\mathcal{B};\mathbb{R}^3))} \leq C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right). \tag{4.61}$$

(iii) From (4.56), we get

$$c_2(\nu) \int_0^T \int_{\mathcal{B}} \left(\theta^{-2} + \theta^{\alpha-2}\right) |\nabla_x \theta|^2 \leq C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right). \tag{4.62}$$

In other words,

$$c_2(\nu) \int_0^T \int_{\mathcal{B}} \left(\left|\nabla_x \log(\theta)\right|^2 + \left|\nabla_x \theta^{\frac{\alpha}{2}}\right|^2 \right) \leq C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right). \tag{4.63}$$

(iv) Now, since $\mathcal{H}_{\theta_B,\xi}$ is coercive (this can be proved in accordance with [12, Proposition 3.2]) and bounded from below, we get

$$\operatorname*{ess\,sup}_{\tau\in[0,T]} \|\rho e_\xi(\tau,\cdot)\|_{L^1(\mathcal{B})} \leq C\left(1 + \frac{1}{\lambda^{5/7}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right), \tag{4.64}$$

and consequently we have

$$\operatorname*{ess\,sup}_{\tau\in(0,T)} \|a_\xi \theta^4(\tau,\cdot)\|_{L^1(\mathcal{B})} \leq C\Big(1+\frac{1}{\lambda^{5/7}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{\alpha-6}}\Big), \tag{4.65}$$

$$\operatorname*{ess\,sup}_{\tau\in(0,T)} \|\rho(\tau,\cdot)\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\mathcal{B})} \leq C\Big(1+\frac{1}{\lambda^{5/7}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{\alpha-6}}\Big). \tag{4.66}$$

(v) The bound (4.66) gives the uniform bound for the gravitational potential Ψ w.r.t. "ε". In fact, by following the steps as we obtained (2.12), one could get

$$\|\Psi\|^2_{L^2(0,T;W^{1,2}(\mathcal{B}))} \leq \|\rho\|^{\frac{5}{3}}_{L^\infty(0,T;L^{\frac{5}{3}}(\mathcal{B}))} + C(\rho_0,g) \leq C\Big(1+\frac{1}{\lambda^{5/7}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{\alpha-6}}\Big). \tag{4.67}$$

(vi) Then by (4.63), (4.60) and generalized Poincaré inequality from Lemma A.1 (since the condition (1.10) satisfies), we deduce that

$$\|\theta^\gamma\|^2_{L^2(0,T;W^{1,2}(\mathcal{B}))} \leq \widehat{C}_1, \quad \text{for any } 1\leq\gamma\leq\frac{\alpha}{2}, \quad \text{where } \alpha>6, \tag{4.68}$$

where the constant $\widehat{C}_1>0$ may depend on the parameters ξ,ν,ω,λ but not on ε.

The estimate (4.63) also provides us

$$\|\nabla_x \log\theta\|^2_{L^2(0,T;L^2(\mathcal{B}))} \leq \widehat{C}_2, \tag{4.69}$$

for some constant $\widehat{C}_2>0$ that is independent in ε.

(vii) Further, we have

$$\begin{aligned}\int_0^T\int_{\mathcal{B}} \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta| &\leq \frac{1}{2}\int_0^T\int_{\mathcal{B}}\frac{\kappa_\nu(\theta,t,x)}{\theta^2}|\nabla_x\theta|^2 + \int_0^\tau\int_{\mathcal{B}}\kappa_\nu(\theta,t,x)\\ &\leq C(\nu)\int_0^T\int_{\mathcal{B}}\Big(\big|\nabla_x\log(\theta)\big|^2+\big|\nabla_x\theta^{\frac{\alpha}{2}}\big|^2\Big) + C(\nu)\int_0^T\int_{\mathcal{B}}(1+\theta^\alpha)\\ &\leq \widehat{C}_3,\end{aligned} \tag{4.70}$$

where we have used the bounds (4.63) and (4.52). The constant $\widehat{C}_3>0$ does depend on the parameters ξ,ν,ω,λ but not on ε.

(viii) Using the technique based on the *Bogovskii operator*, one can get more information about the modified pressure, namely

$$p_{\xi,\delta}(\rho,\theta) = p_M(\rho,\theta)+\frac{a_\xi}{3}\theta^4+\delta\rho^\beta \quad (\beta\geq 4).$$

The idea is to use the multipliers of the form

$$\psi\,\mathcal{L}\big[\rho-\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}\rho\big], \quad \psi\in\mathcal{D}(0,T), \quad 0\leq\psi\leq 1$$

in the momentum equation(1.2), where $\mathcal{L}$ is defined as follows: the function $\mathbf{v}=\mathcal{L}[f]$ such that it solves the problem

$$\operatorname{div}(\mathbf{v}) = f \quad \text{in } \mathcal{B}, \quad \mathbf{v}|_{\partial\mathcal{B}}=0.$$

In what follows, one can ensure that there exists some $\upsilon>0$ such that

$$\iint_K \Big(p_{\xi,\delta}(p,\theta)\rho^\upsilon+\delta\rho^{\beta+\upsilon}\Big) \leq C(K), \tag{4.71}$$

for any compact set $K\subset(0,T)\times\mathcal{B}$ such that

$$K\cap\Big(\cup_{\tau\in[0,T]}\big(\{\tau\}\times\Gamma_\tau\big)\Big)=\emptyset. \tag{4.72}$$

Moreover, the constant υ can be chosen independently of ε, ω, λ, ξ, ν and δ. For more details, we refer [10, Section 4.2] or [15].

(ix) To find a suitable estimate for the term $\rho s_\xi(\rho,\theta)$, we recall the estimates (4.44) and (4.47), which gives

$$\rho s_\xi(\rho,\theta) \leq \begin{cases} c\rho + \frac{4a_\xi}{3}\theta^3, \quad \text{for } \frac{\rho}{\theta^{3/2}} > 1, \\ C\Big(\rho^{\frac{1}{2}} + \rho + \rho[\log\theta]^+\Big) + \frac{4a_\xi}{3}\theta^3 \leq C\Big(\theta^{\frac{3}{4}} + \theta^{\frac{3}{2}} + \theta^2\Big) + \frac{4a_\xi}{3}\theta^3, \quad \text{for } \frac{\rho}{\theta^{3/2}} \leq 1. \end{cases}$$

Now, using the estimates (4.60), (4.66), one can deduce that

$$\|\rho s_\xi(\rho,\theta)\|_{L^p((0,T)\times\mathcal{B})} \leq \widehat{C}_4, \quad \text{for certain } p \geq 1, \tag{4.73}$$

where the constant $\widehat{C}_4 > 0$ that may depend on the parameters ξ,ν,λ,ω but not on ε.

5 Passing to the limit

In this section, we first perform the limit $\varepsilon \to 0$ and then together we pass to the limit λ, ξ, ν, ω and finally δ to 0.

5.1 Penalization limit: passing with $\varepsilon \to 0$

In this subsection, we fix all the parameters δ, λ, ν, ξ and ω. Then, passing to the limit $\varepsilon \to 0$, we directly obtain

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}\big|_{\Gamma_\tau} = 0, \quad \text{for a.a. } \ \tau \in [0,T], \tag{5.1}$$

so we retrieve the impermeability boundary condition (1.13).

• By (4.65), (4.66) and (4.68), we respectively have (up to a suitable subsequence)

$$\theta_\varepsilon \to \theta \quad \text{weakly}^* \text{ in } \ L^\infty(0,T;L^4(\mathcal{B})) \ \text{ as } \varepsilon \to 0, \tag{5.2}$$

$$\rho_\varepsilon \to \rho \quad \text{weakly}^* \text{ in } \ L^\infty(0,T;L^{\frac{5}{3}}(\mathcal{B})) \ \text{ as } \varepsilon \to 0, \tag{5.3}$$

$$\theta_\varepsilon \to \theta \quad \text{weakly in } \ L^2(0,T;W^{1,2}(\mathcal{B})) \ \text{ as } \varepsilon \to 0, \tag{5.4}$$

Due to (4.60) and (4.65), we also have

$$\theta_\varepsilon^\alpha \to \overline{\theta^\alpha} \quad \text{weakly in } \ L^1((0,T)\times\mathcal{B}) \ \text{ as } \varepsilon \to 0, \tag{5.5}$$

$$\theta_\varepsilon^4 \to \overline{\theta^4} \quad \text{weakly in } \ L^1((0,T)\times\mathcal{B}) \ \text{ as } \varepsilon \to 0. \tag{5.6}$$

Here and in the sequel, the “bar” denotes a weak limit of a composed or nonlinear function.

• Thanks to (4.61) and (4.67), we respectively have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } \ L^2(0,T;W^{1,2}(\mathcal{B},\mathbb{R}^3)) \ \text{ as } \varepsilon \to 0, \tag{5.7}$$

$$\Psi_\varepsilon \to \Psi \quad \text{weakly in } \ L^2(0,T;W^{1,2}(\mathcal{B})) \ \text{ as } \varepsilon \to 0. \tag{5.8}$$

• We have also better convergence result of $\{\rho_\varepsilon\}_\varepsilon$ than (5.3): using the continuity equation (4.18) one indeed get

$$\rho_\varepsilon \to \rho \quad \text{in } \ \mathcal{C}_{\text{weak}}([0,T];L^{\frac{5}{3}}(\mathcal{B})) \ \text{ as } \varepsilon \to 0. \tag{5.9}$$

The above fact, together with (5.7) and the fact $L^{\frac{5}{3}}(\mathcal{B}) \hookrightarrow W^{-1,2}(\mathcal{B})$ is compact, one has

$$\rho_\varepsilon\mathbf{u}_\varepsilon \to \rho\mathbf{u} \quad \text{weakly}^* \text{ in } \ L^\infty(0,T;L^{\frac{5}{4}}(\mathcal{B},\mathbb{R}^3)) \ \text{ as } \varepsilon \to 0 \tag{5.10}$$

• Now, by using (5.6) and (4.71), the asymptotic behavior (4.34) of p_M, and then by utilizing (5.9), (5.4) we have

$$p_{\xi,\delta}(\rho_\varepsilon,\theta_\varepsilon) = p_M(\rho_\varepsilon,\theta_\varepsilon) + \frac{a_\xi}{3}\theta_\varepsilon^4 + \delta\rho_\varepsilon^\beta \to \overline{p_M(\rho,\theta)} + \frac{a_\xi}{3}\overline{\theta^4} + \delta\overline{\rho^\beta} \quad \text{weakly in} \quad L^1(K), \tag{5.11}$$

where K is as given by (4.71)–(4.72).

• Further, since $W_0^{1,2}(\mathcal{B},\mathbb{R}^3) \hookrightarrow L^6(\mathcal{B},\mathbb{R}^3)$ is compact, using (5.7), the convective term satisfies (by following the steps of [19, Section 4] or [12, Chapter 3.6.4])

$$\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\rho\mathbf{u}\otimes\mathbf{u}} \quad \text{weakly in } L^2(0,T;L^{\frac{30}{29}}(\mathcal{B},\mathbb{R}^3)), \tag{5.12}$$

and indeed,

$$\overline{\rho\mathbf{u}\otimes\mathbf{u}} = \rho\mathbf{u}\otimes\mathbf{u} \quad \text{a.a. in } (0,T)\times\mathcal{B}, \tag{5.13}$$

since $L^{\frac{5}{4}}(\mathcal{B}) \hookrightarrow W^{-1,2}(\mathcal{B})$ is compact.

5.2 Pointwise convergence of the temperature and the density

• In order to show a.e. convergence of the temperature, we follow the technique based on the Div–Curl Lemma and Young measures methods (see [12, Section 3.6.2]). Let us set

$$\mathbf{U}_\varepsilon = \left[\rho_\varepsilon s_\xi(\rho_\varepsilon,\theta_\varepsilon),\ \rho_\varepsilon s_\xi(\rho_\varepsilon,\theta_\varepsilon)\mathbf{u}_\varepsilon + \frac{\kappa_\nu(\theta_\varepsilon)\nabla_x\theta_\varepsilon}{\theta_\varepsilon}\right], \tag{5.14}$$

$$\mathbf{W}_\varepsilon = \left[G(\theta_\varepsilon),0,0,0\right], \tag{5.15}$$

where G is bounded and globally Lipschitz function in $[0,\infty)$. Then due to the estimates obtained in previous section, $\mathrm{div}_{t,x}\mathbf{U}_\varepsilon$ is precompact in $W^{-1,s}((0,T)\times\mathcal{B})$ and $\mathrm{Curl}_{t,x}\mathbf{W}_\varepsilon$ is precompact in $W^{-1,s}((0,T)\times\mathcal{B})^{4\times 4}$ with certain $s>1$. Therefore using the Div-Curl lemma for $\mathbf{U}_\varepsilon$ and $\mathbf{W}_\varepsilon$, we may derive that

$$\overline{\rho s_\xi(\rho_\varepsilon,\theta_\varepsilon)G(\theta_\varepsilon)} = \overline{\rho s_\xi(\rho_\varepsilon,\theta_\varepsilon)}\ \overline{G(\theta_\varepsilon)} \tag{5.16}$$

In fact, by applying the theory of parameterized (Young) measures (see [12, Section 3.6.2]), one can show that

$$\overline{\rho s_M(\rho,\theta)G(\theta)} \ge \overline{\rho s_M(\rho,\theta)}\ \overline{G(\theta)}, \qquad \overline{\theta^3 G(\theta)} \ge \overline{\theta^3}\ \overline{G(\theta)}. \tag{5.17}$$

Combining (5.16)–(5.17) and taking $G(\theta)=\theta$, we deduce

$$\overline{\theta^4} = \overline{\theta^3}\,\theta,$$

which yields

$$\theta_\varepsilon \to \theta \quad \text{a.a. in} \quad (0,T)\times\mathcal{B}. \tag{5.18}$$

Moreover, thanks to (4.69) and using the generalized Poincaré inequality in Lemma A.1, one can prove that $\log\theta \in L^2((0,T)\times\mathcal{B})$ which ensures that the limit temperature is positive a.e. on the set $(0,T)\times\mathcal{B}$.

• Next proceeding as [18, Section 4.1.2], one can further obtain

$$\rho_\varepsilon \to \rho \quad \text{a.a. in} \quad (0,T)\times\mathcal{B}. \tag{5.19}$$

• Then, using (5.18), (5.19) and (5.7), we identify the following limits:

$$\begin{cases} \mathbb{S}_\omega(\theta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) \to \mathbb{S}_\omega(\theta,\nabla_x\mathbf{u}) & \text{weakly in} \quad L^1((0,T)\times\mathcal{B}), \\ \rho_\varepsilon s_\xi(\rho_\varepsilon,\theta_\varepsilon) \to \rho s_\xi(\rho,\theta) & \text{weakly in} \quad L^1((0,T)\times\mathcal{B}), \\ \rho_\varepsilon s_\xi(\rho_\varepsilon,\theta_\varepsilon)\mathbf{u}_\varepsilon \to \rho s_\xi(\rho,\theta)\mathbf{u} & \text{weakly in} \quad L^1((0,T)\times\mathcal{B}), \end{cases} \tag{5.20}$$

up to a suitable subsequence.

5.3 The limiting system as $\varepsilon \to 0$

In this subsection, we summarize the limiting behaviors of all the quantities from the previous two subsections, and write weak formulation for the limiting system (as $\varepsilon \to 0$).

• Passing to the limit as $\varepsilon \to 0$, we can obtain the continuity equation exactly as (4.18).

• Next, we proceed to pass to the limit in the momentum equation (4.19). Having in hand the local estimates (and limits) of the pressure term (see (4.71) and (5.11)), we consider the test functions

$$\varphi \in \mathcal{C}_c^1([0,T); W^{1,\infty}(\mathcal{B},\mathbb{R}^3)), \quad \mathrm{Supp}\,[\mathrm{div}_x \varphi(\tau,\cdot)] \cap \Gamma_\tau = \emptyset, \quad \varphi \cdot \mathbf{n}|_{\Gamma_\tau} = 0, \quad \forall \tau \in [0,T]. \tag{5.21}$$

Then in accordance with the limits in the previous two subsections, we have upon $\varepsilon \to 0$,

$$\begin{aligned} -\int_0^T \int_{\mathcal{B}} \left(\rho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p_{\xi,\delta}(\rho,\theta)\mathrm{div}_x\boldsymbol{\varphi}\right) + \int_0^T \int_{\mathcal{B}} \mathbb{S}_\omega : \nabla_x\boldsymbol{\varphi} \\ \int_0^T \int_{\mathcal{B}} \rho\nabla_x\Psi\cdot\nabla_x\varphi = \int_{\mathcal{B}} (\rho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot), \end{aligned} \tag{5.22}$$

for any test function as in (5.21).

• The weak formulations for Ψ has same expressions as (4.21) after passing to the limit as $\varepsilon \to 0$.

• Further, by using (4.70), we have

$$\frac{\kappa_\nu(\theta_\varepsilon)}{\theta_\varepsilon}|\nabla_x\theta_\varepsilon| \to \frac{\kappa_\nu(\theta)}{\theta}|\nabla_x\theta| \quad \text{weakly in} \ \ L^1((0,T)\times\mathcal{B}). \tag{5.23}$$

We also have that the terms $\frac{1}{\theta_\varepsilon}\mathbb{S}_\omega(\theta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathbf{u}_\varepsilon$ and $\frac{\kappa_\nu(\theta_\varepsilon)|\nabla_x\theta_\varepsilon|^2}{\theta_\varepsilon}$ are weakly lower semicontinuous. These, together with (5.6), (5.18) and (5.19), the entropy inequality follows (as $\varepsilon \to 0$)

$$\begin{aligned} -\int_0^T \int_{\mathcal{B}} \left(\rho s_\xi(\rho,\theta)\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) - \frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\varphi\right) \\ - \int_{\mathcal{B}} \rho_{0,\delta}s(\rho_{0,\delta},\theta_{0,\delta})\varphi(0,\cdot) + \int_0^T \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\varphi \\ \geq \int_0^T \int_{\mathcal{B}} \frac{\varphi}{\theta}\left(\mathbb{S}_\omega : \nabla_x\mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right), \end{aligned} \tag{5.24}$$

for any test function $\varphi \in \mathcal{C}_c^1([0,T)\times\mathcal{B};\mathbb{R})$ with $\varphi \geq 0$.

• Let us pass to the limit $\varepsilon \to 0$ in the modified energy inequality (4.24). Thanks to the almost everywhere convergence results (5.18), (5.19), the bound (4.64) and the fact that $\{\rho_\varepsilon e_\xi(\rho_\varepsilon,\theta_\varepsilon)\}_\varepsilon$ is nonnegative, we have, using the Fatou's lemma,

$$\limsup_{\varepsilon\to 0} \int_0^T \int_{\mathcal{B}} \rho_\varepsilon e_\xi(\rho_\varepsilon,\theta_\varepsilon)\partial_t\psi \leq \int_0^T \int_{\mathcal{B}} \rho e_\xi(\rho,\theta)\partial_t\psi$$

by choosing $\psi \in \mathcal{C}_c^1([0,T))$ such that $\partial_t\psi \leq 0$. Similarly, one has

$$\limsup_{\varepsilon\to 0} \int_0^T \int_{\mathcal{B}} \rho_\varepsilon|\mathbf{u}_\varepsilon|^2\partial_t\psi \leq \int_0^T \int_{\mathcal{B}} \rho|\mathbf{u}|^2\partial_t\psi.$$

Using the above information and gathering all the limits (5.18), (5.19), (5.11), (5.23), (5.12), the limiting ballistic energy inequality reads as (passing to the limit as $\varepsilon \to 0$ in (4.24))

$$
\begin{aligned}
&-\int_0^T \partial_t \psi \int_{\mathcal{B}} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \mathcal{H}_{\theta_B,\xi}(\rho,\theta) + \frac{\delta}{\beta-1}\rho^\beta\right) + \int_0^T \psi \int_{\mathcal{B}} \lambda\theta^\alpha \\
&+ \int_0^T \psi \int_{\mathcal{B}} \frac{\theta_B}{\theta}\left(\mathbb{S}_\omega : \nabla_x \mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right) \\
\leq& \psi(0)\int_{\mathcal{B}} \left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \mathcal{H}_{\theta_B,\xi}(\rho_{0,\delta},\theta_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_0\mathbf{V}(0)\right) \\
&+ \int_0^T \psi \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\theta_B - \int_0^T \int_{\mathcal{B}} \rho\mathbf{u}\cdot\partial_t(\psi\mathbf{V}) \\
&- \int_0^T \psi \int_{\mathcal{B}} \Big(\rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\mathbf{V} - \mathbb{S}_\omega : \nabla_x\mathbf{V} + p_{\xi,\delta(\rho,\theta)}\operatorname{div}_x\mathbf{V}\Big) \\
&- \int_0^T \psi \int_{\mathcal{B}} \rho\nabla_x\Psi\cdot\mathbf{V} - \int_0^T \psi \int_{\mathcal{B}} \Big[\rho s_\xi(\rho,\theta)\big(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B\big) - \frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\Big],
\end{aligned}
\tag{5.25}
$$

for all $\psi \in \mathcal{C}_c^1([0,T))$ with $\psi \geq 0$ and $\partial_t\psi \leq 0$.

5.4 Get rid of the density dependent solid part

Let us take care the density-dependent terms in the solid part $((0,T)\times\mathcal{B})\setminus Q_T$. We use [11, Lemma 4.1] (see also [19, Section 4.1.4]) to conclude that the density ρ remains "zero" on the solid part if it was so initially, thanks to the continuity equation and the fact that density is square-integrable. Here, we must mention that the square-integrability of density is identified from the estimate of $\delta\rho^\beta$ given by (4.58).

We also recall that, the parameters ω, ξ and ν are not involved in Ω_t ($t \in [0,T]$) as per our extension strategy given in the beginning of Section 4.

• Since, we have set the initial data $\rho_{0,\delta}$ to be zero in $\mathcal{B}\setminus\Omega_0$, it remains zero in $\mathcal{B}\setminus\Omega_t$ due to [11, Lemma 4.1]. This leads to the following weak formulation for the continuity equation upon passing to the limit as $\varepsilon \to 0$,

$$
-\int_0^T \int_{\Omega_t} \rho B(\rho)\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) + \int_0^T \int_{\Omega_t} b(\rho)\operatorname{div}_x\mathbf{u}\varphi = \int_{\Omega_0} \rho_{0,\delta}B(\rho_{0,\delta})\varphi(0,\cdot),
\tag{5.26}
$$

for any test function $\varphi \in \mathcal{C}_c^1([0,T)\times\mathcal{B};\mathbb{R})$ and any $b \in L^\infty \cap \mathcal{C}([0,+\infty))$ such that $b(0) = 0$ and $B(\rho) = B(1) + \displaystyle\int_1^\rho \frac{b(z)}{z^2}$.

• Using the fact $\rho = 0$ in $\mathcal{B}\setminus\Omega_t$ for any $t\in[0,T]$, the momentum equation now reads

$$
\begin{aligned}
-\int_0^T \int_{\Omega_t} \big(\rho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p_\delta(\rho,\theta)\operatorname{div}_x\boldsymbol{\varphi}\big) + \int_0^T \int_{\Omega_t} \mathbb{S} : \nabla_x\boldsymbol{\varphi} - \int_0^T \int_{\Omega_t} \rho\nabla_x\Psi\cdot\boldsymbol{\varphi} \\
= \int_{\Omega_0} (\rho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot) - \int_0^T \int_{\mathcal{B}\setminus\Omega_t} \mathbb{S}_\omega : \nabla_x\boldsymbol{\varphi} + \int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{a_\xi}{3}\theta^4\operatorname{div}_x\boldsymbol{\varphi},
\end{aligned}
\tag{5.27}
$$

for any test function $\boldsymbol{\varphi}$ satisfying

$$
\boldsymbol{\varphi} \in \mathcal{C}_c^1([0,T)\times\mathcal{B};\mathbb{R}^3) \quad \text{with} \quad \boldsymbol{\varphi}(\tau,\cdot)\cdot\mathbf{n}\big|_{\Gamma_\tau} = 0 \ \text{ for any } \tau\in[0,T].
$$

• Since $\rho = 0$ in $\mathcal{B}\setminus\Omega_t$ for each $t\in[0,T]$, the weak formulation for Ψ from (4.21) simply reduces to

$$
\int_0^T \int_{\Omega_t} \nabla_x\Psi\cdot\nabla_x\varphi = \int_0^T \int_{\Omega_t} \rho\varphi,
\tag{5.28}
$$

for any test function $\varphi \in \mathcal{C}^1((0,T)\times\mathcal{B};\mathbb{R})$.

• Next, we write the limiting entropy inequality (recall the fact that $s_\xi(\rho,\theta) = s_M(\rho,\theta) + \frac{4a_\xi}{3\rho}\theta^3$)

$$
\begin{aligned}
&-\int_0^T\int_{\Omega_t} \rho s(\rho,\theta)\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) - \int_0^T\int_{\mathcal{B}\setminus\Omega_t} \frac{4a_\xi}{3}\theta^3\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right)\\
&\quad + \int_0^T\int_{\Omega_t} \frac{\kappa(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\varphi + \int_0^T\int_{\mathcal{B}\setminus\Omega_t} \frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\varphi\\
&-\int_{\Omega_0}\rho_{0,\delta}s(\rho_{0,\delta},\theta_{0,\delta})\varphi(0,\cdot) - \int_{\mathcal{B}\setminus\Omega_0}\frac{4a_\xi}{3}\theta_{0,\delta}^3\varphi(0,\cdot) + \int_0^T\int_{\mathcal{B}}\lambda\theta^{\alpha-1}\varphi\\
&\qquad \geq \int_0^T\int_{\Omega_t}\frac{\varphi}{\theta}\left(\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right)\\
&\qquad + \int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{\varphi}{\theta}\left(\mathbb{S}_\omega:\nabla_x\mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right),
\end{aligned}
\tag{5.29}
$$

for any test function $\varphi\in\mathcal{C}_c^1([0,T)\times\mathcal{B})$ with $\varphi\geq 0$.

• We shall now look at the ballistic energy inequality (5.25). It has the following form now.

$$
\begin{aligned}
&-\int_0^T\int_{\Omega_t}\partial_t\psi\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\theta) - \rho s(\rho,\theta)\theta_B + \frac{\delta}{\beta-1}\rho^\beta\right) - \int_0^T\int_{\mathcal{B}\setminus\Omega_t} a_\xi\theta^4\partial_t\psi\\
&+\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{4a_\xi}{3}\theta^3\theta_B\partial_t\psi + \int_0^T\psi\int_{\mathcal{B}}\lambda\theta^\alpha + \int_0^T\psi\int_{\Omega_t}\frac{\theta_B}{\theta}\left(\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right)\\
&+\int_0^T\psi\int_{\mathcal{B}\setminus\Omega_t}\frac{\theta_B}{\theta}\left(\mathbb{S}_\omega:\nabla_x\mathbf{u} + \frac{\kappa_\nu(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right)\\
\leq&\,\psi(0)\int_{\Omega_0}\left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta}e(\rho_{0,\delta},\theta_{0,\delta}) - \rho_{0,\delta}s(\rho_{0,\delta},\theta_{0,\delta})\theta_B(0) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0)\right)\\
&+\psi(0)\int_{\mathcal{B}\setminus\Omega_0}a_\xi\left(\theta_{0,\delta}^4 - \frac{4}{3}\theta_{0,\delta}^3\theta_B\right) + \int_0^T\psi\int_{\mathcal{B}}\lambda\theta^{\alpha-1}\,\theta_B - \int_0^T\int_{\Omega_t}\rho\mathbf{u}\cdot\partial_t(\psi\mathbf{V})\\
&-\int_0^T\psi\int_{\Omega_t}\left(\rho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{V} - \mathbb{S}:\nabla_x\mathbf{V} + p_\delta(\rho,\theta)\,\mathrm{div}_x\mathbf{V}\right) - \int_0^T\psi\int_{\Omega_t}\rho\nabla_x\Psi\cdot\mathbf{V}\\
&+\int_0^T\psi\int_{\mathcal{B}\setminus\Omega_t}\mathbb{S}_\omega:\nabla_x\mathbf{V} - \int_0^T\psi\int_{\mathcal{B}\setminus\Omega_t}\frac{a_\xi}{3}\theta^4\mathrm{div}_x\mathbf{V}\\
&-\int_0^T\psi\int_{\Omega_t}\left[\rho s(\rho,\theta)\big(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B\big) - \frac{\kappa(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\right]\\
&-\int_0^T\psi\int_{\mathcal{B}\setminus\Omega_t}\frac{4a_\xi}{3}\theta^3\Big(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B\Big) + \int_0^T\psi\int_{\mathcal{B}\setminus\Omega_t}\frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B
\end{aligned}
\tag{5.30}
$$

for any $\psi\in\mathcal{C}_c^1([0,T))$ with $\psi\geq 0$ and $\partial_t\psi\leq 0$.

5.5 Passing to the limit of other parameters

In this section, we first consider the following scaling for all the parameters ω,ξ,ν,λ. Let

$$
\lambda = \nu^{\frac{1}{3}} = \omega^{\frac{1}{3}} = \xi^{\frac{1}{6}} = h \quad \text{for } h>0. \tag{5.31}
$$

5.5.1 Step 1. Bounds of the integrals in $((0,T)\times\mathcal{B})\setminus Q_T$

In this step, we shall find suitable bounds of the integrals in $((0,T)\times\mathcal{B})\setminus Q_T$.

• We start with the following. Recall the weak formulation (5.27) for the momentum equation and focus on the integrals on $\mathcal{B}\setminus\Omega_t$. First, we compute

$$
\begin{aligned}
\left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{a_\xi}{3}\theta^4\mathrm{div}_x\boldsymbol{\varphi}\right| &\le C\|\mathrm{div}_x\boldsymbol{\varphi}\|_{L^\infty((0,T)\times\mathcal{B})}\frac{\xi}{\lambda^{\frac{4}{\alpha}}}\left(\int_0^T\int_{\mathcal{B}}\lambda\theta^\alpha\right)^{\frac{4}{\alpha}}\\
&\le \frac{C\xi}{\lambda^{\frac{4}{\alpha}}}\left(1+\frac{1}{\lambda^{\frac{5}{7}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{\alpha-6}}\right)^{\frac{4}{\alpha}}\\
&\le \frac{C\xi}{\lambda^{\frac{4}{\alpha}}}\left(1+\frac{1}{\lambda^{\frac{20}{7\alpha}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{4}{(\alpha-6)}}\right)\\
&\le \frac{C\xi}{\lambda^{\frac{2}{3}}}\left(1+\frac{1}{\lambda^{\frac{10}{21}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{4}{(\alpha-6)}}\right)\\
&=: A_1(\xi,\omega,\lambda).
\end{aligned}
\tag{5.32}
$$

thanks to the bound (4.60) and using the fact that

$$
\frac{1}{\lambda^{1/\alpha}}<\frac{1}{\lambda^{1/6}}\quad\text{since }\alpha>6\text{ and }0<\lambda\le 1. \tag{5.33}
$$

Utilizing the scaling (5.31), we conclude

$$
A_1(\xi,\omega,\lambda)=Ch^6\left(\frac{1}{h^{\frac{2}{3}}}+\frac{1}{h^{\frac{8}{7}}}+\Big(\frac{h^{12}}{h^6}\Big)^{\frac{4}{\alpha-6}}\right). \tag{5.34}
$$

• Secondly, we compute the following:

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\mathbb{S}_\omega:\nabla_x\boldsymbol{\varphi}\right|\\
\le&\|\nabla_x\boldsymbol{\varphi}\|_{L^\infty((0,T)\times\mathcal{B})}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\Big(\frac{1}{\sqrt{\theta}}|\mathbb{S}_\omega|\Big)^{\frac{2\alpha}{2\alpha-1}}\right)^{\frac{2\alpha-1}{2\alpha}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\theta^\alpha\right)^{\frac{1}{2\alpha}}\\
\le&\frac{C}{\lambda^{\frac{1}{2\alpha}}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\lambda\theta^\alpha\right)^{\frac{1}{2\alpha}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\Big(\frac{1}{\sqrt{\theta}}\sqrt{|\mathbb{S}_\omega:\nabla_x\mathbf{u}|}\sqrt{|f_\omega|(1+\theta)}\Big)^{\frac{2\alpha}{2\alpha-1}}\right)^{\frac{2\alpha-1}{2\alpha}}\\
\le&\frac{C}{\lambda^{\frac{1}{2\alpha}}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\lambda\theta^\alpha\right)^{\frac{1}{2\alpha}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{1}{\theta}|\mathbb{S}_\omega:\nabla_x\mathbf{u}|\right)^{\frac{1}{2}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\big(f_\omega(1+\theta)\big)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{2\alpha}}\\
\le&\frac{C}{\lambda^{\frac{1}{2\alpha}}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\lambda\theta^\alpha\right)^{\frac{1}{2\alpha}}\left(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{1}{\theta}|\mathbb{S}_\omega:\nabla_x\mathbf{u}|\right)^{\frac{1}{2}}\sqrt{\omega}\left[1+\frac{1}{\lambda^{\frac{1}{2\alpha}}}\Big(\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\lambda\theta^\alpha\Big)^{\frac{1}{2\alpha}}\right]\\
\le&\frac{C\sqrt{\omega}}{\lambda^{\frac{1}{2\alpha}}}\left(1+\frac{1}{\lambda^{\frac{5}{7}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{(\alpha-6)}}\right)^{\frac{1}{2\alpha}}\left(1+\frac{1}{\lambda^{\frac{5}{7}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{(\alpha-6)}}\right)^{\frac{1}{2}}\times\\
&\qquad\left[1+\frac{1}{\lambda^{\frac{1}{2\alpha}}}\left(1+\frac{1}{\lambda^{\frac{5}{7}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{(\alpha-6)}}\right)^{\frac{1}{2\alpha}}\right]\\
\le&\frac{C\sqrt{\omega}}{\lambda^{\frac{1}{2\alpha}}}\left(1+\frac{1}{\lambda^{\frac{5}{14\alpha}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{1}{2(\alpha-6)}}\right)\left(1+\frac{1}{\lambda^{\frac{5}{14}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{2(\alpha-6)}}\right)\\
&\quad+\frac{C\sqrt{\omega}}{\lambda^{\frac{1}{\alpha}}}\left(1+\frac{1}{\lambda^{\frac{5}{7\alpha}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{1}{(\alpha-6)}}\right)\left(1+\frac{1}{\lambda^{\frac{5}{14}}}+\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{2(\alpha-6)}}\right)\\
\le&\frac{C\sqrt{\omega}}{\lambda^{\frac{9}{14}}}+\frac{C\sqrt{\omega}}{\lambda^{\frac{11}{21}}}\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{1}{(\alpha-6)}}+\frac{C\sqrt{\omega}}{\lambda^{\frac{2}{7}}}\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha}{2(\alpha-6)}}+\frac{C\sqrt{\omega}}{\lambda^{\frac{1}{6}}}\Big(\frac{\xi^2}{\omega\nu}\Big)^{\frac{\alpha+2}{2(\alpha-6)}}\quad\text{(using (5.33))}\\
=:&A_2(\xi,\omega,\lambda).
\end{aligned}
\tag{5.35}
$$

Here, we have utilized that

$$
|\mathbb{S}_\omega|\le\sqrt{|\mathbb{S}_\omega:\nabla_x\mathbf{u}|(\mu_\omega(\theta)+\eta_\omega(\theta))}\le C\sqrt{|\mathbb{S}_\omega:\nabla_x\mathbf{u}||f_\omega|(1+\theta)},
$$

thanks to the definitions of $\mu_\omega(\theta), \eta_\omega(\theta)$ given by (4.1)–(4.2) and the hypothesis (1.21). Using the same ingredients and from the expression of $\mathbb{S}_\omega$ in (4.20), we further deduce

$$\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{1}{\theta} |\mathbb{S}_\omega : \nabla_x \mathbf{u}| \leq C |f_\omega| \|\mathbf{u}\|^2_{L^2(0,T;W^{1,2}(\mathcal{B},\mathbb{R}^3))}.$$

By means of the above inequalities followed by the bounds (4.60), (4.61) and the fact $\|f_\omega\|_{L^\infty((0,T)\times\mathcal{B})} \leq c\,\omega$ (see (4.3)), we obtained the required bound in (5.35).

Using the scaling (5.31) in (5.35), we further deduce that

$$A_2(\xi,\omega,\lambda) = \frac{Ch^{3/2}}{h^{\frac{9}{14}}} + \frac{Ch^{3/2}}{h^{\frac{11}{21}}} h^{6/(\alpha-6)} + \frac{Ch^{3/2}}{h^{\frac{2}{7}}} h^{\frac{6\alpha}{2(\alpha-6)}} + \frac{Ch^{3/2}}{h^{1/6}} h^{\frac{6(\alpha+2)}{2(\alpha-6)}}. \tag{5.36}$$

• Next, we look to the entropy inequality (5.29). We observe that

$$\begin{aligned}
&\left| \int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{4a_\xi}{3} \theta^3 (\partial_t \varphi + \mathbf{u}\cdot\nabla_x\varphi) \right| \\
\leq & \frac{C\xi}{\lambda^{\frac{3}{\alpha}}} \left(\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \lambda\theta^\alpha \right)^{\frac{3}{\alpha}} + C\xi \int_0^T \int_{\mathcal{B}\setminus\Omega_t} |\mathbf{u}|^2 + C\xi \int_0^T \int_{\mathcal{B}\setminus\Omega_t} \theta^6 \\
\leq & \frac{C\xi}{\lambda^{\frac{3}{\alpha}}} \left(\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \lambda\theta^\alpha \right)^{\frac{3}{\alpha}} + C\xi \int_0^T \int_{\mathcal{B}\setminus\Omega_t} |\mathbf{u}|^2 + \frac{C\xi}{\lambda^{\frac{6}{\alpha}}} \left(\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \lambda\theta^\alpha \right)^{\frac{6}{\alpha}} \\
\leq & \frac{C\xi}{\lambda^{\frac{3}{\alpha}}} \left(1 + \frac{1}{\lambda^{\frac{15}{7\alpha}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{3}{(\alpha-6)}} \right) + \frac{C\xi}{\omega} \left(1 + \frac{1}{\lambda^{\frac{5}{7}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right) \\
& \qquad + \frac{C\xi}{\lambda^{\frac{6}{\alpha}}} \left(1 + \frac{1}{\lambda^{\frac{30}{7\alpha}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{6}{(\alpha-6)}} \right) \\
\leq & \frac{C\xi}{\lambda^{\frac{1}{2}}} \left(1 + \frac{1}{\lambda^{\frac{5}{14}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{3}{(\alpha-6)}} \right) + \frac{C\xi}{\omega} \left(1 + \frac{1}{\lambda^{\frac{5}{7}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{\alpha-6}} \right) \quad \text{(using (5.33))} \\
& \quad + \frac{C\xi}{\lambda} \left(1 + \frac{1}{\lambda^{\frac{5}{7}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{6}{(\alpha-6)}} \right) \\
=: & A_3(\xi,\omega,\lambda),
\end{aligned} \tag{5.37}$$

thanks to the estimates (4.60), (4.61) and the fact that $\alpha > 6$.

By means of (5.31), we deduce

$$\begin{aligned}
A_3(\xi,\omega,\lambda) = Ch^6 \left(\frac{1}{h^{\frac{1}{2}}} + \frac{1}{h^{\frac{6}{7}}} + \frac{1}{h^{\frac{1}{2}}} h^{\frac{18}{(\alpha-6)}} \right) + Ch^3 \left(1 + \frac{1}{h^{\frac{5}{7}}} + h^{\frac{6\alpha}{(\alpha-6)}} \right) \\
+ Ch^5 \left(1 + \frac{1}{h^{\frac{5}{7}}} + h^{\frac{36}{(\alpha-6)}} \right).
\end{aligned} \tag{5.38}$$

• The next term in the entropy inequality (5.29) in $\mathcal{B}\setminus\Omega_t$ satisfies the following:

$$\begin{aligned}
&\left| \int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{\kappa_\nu(\theta,t,x)}{\theta} \nabla_x\theta \cdot \nabla_x\varphi \right| \\
\leq & \left(\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{\kappa_\nu(\theta,t,x)}{\theta^2} |\nabla_x\theta|^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \kappa_\nu(\theta,t,x) |\nabla_x\varphi|^2 \right)^{\frac{1}{2}} \\
\leq & C \|\nabla_x\varphi\|_{L^\infty((0,T)\times\mathcal{B})} \left(1 + \frac{1}{\lambda^{\frac{5}{14}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{2(\alpha-6)}} \right) \sqrt{\nu} \left(\int_0^T \int_{\mathcal{B}\setminus\Omega_t} (1+\theta^\alpha) \right)^{\frac{1}{2}} \\
\leq & C\sqrt{\nu} \left(1 + \frac{1}{\lambda^{\frac{5}{14}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{2(\alpha-6)}} \right) \left(1 + \frac{1}{\lambda^{\frac{1}{2}}} \left(\int_0^T \int_{\mathcal{B}} \lambda\theta^\alpha \right)^{\frac{1}{2}} \right) \\
\leq & C\sqrt{\nu} \left(1 + \frac{1}{\lambda^{\frac{5}{14}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{2(\alpha-6)}} \right) + \frac{C\sqrt{\nu}}{\sqrt{\lambda}} \left(1 + \frac{1}{\lambda^{\frac{5}{7}}} + \left(\frac{\xi^2}{\omega\nu}\right)^{\frac{\alpha}{(\alpha-6)}} \right) \\
=: & A_4(\nu,\xi,\omega,\lambda).
\end{aligned} \tag{5.39}$$

We further compute (thanks to the choice (5.31))

$$A_4(\nu,\xi,\omega,\lambda) = Ch^{3/2}\left(1+\frac{1}{h^{\frac{5}{14}}}+h^{\frac{6\alpha}{2(\alpha-6)}}\right)+Ch\left(1+\frac{1}{h^{\frac{5}{7}}}+h^{\frac{6\alpha}{(\alpha-6)}}\right). \tag{5.40}$$

• In a similar way, we can bound the following terms appearing in the energy inequality (5.30). Indeed, the estimates (5.32) and (5.37) yield

$$\begin{aligned} &\left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t} a_\xi\Big(\theta^4-\frac{4}{3}\theta^3\theta_B\Big)\partial_t\psi\right| + \left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{4a_\xi}{3}\theta^3\Big(\partial_t\theta_B+\mathbf{u}\cdot\nabla_x\theta_B\Big)\psi\right| \\ &\qquad + \left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{a_\xi}{3}\theta^4\mathrm{div}_x\mathbf{V}\psi\right| \le C\left(A_1(\xi,\omega,\lambda)+A_3(\xi,\omega,\lambda)\right). \end{aligned} \tag{5.41}$$

On the other hand, the information (5.35) and (5.39) give

$$\begin{aligned} &\left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\mathbb{S}_\omega:\nabla_x\mathbf{V}\psi\right| + \left|\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\psi\right| \\ &\qquad \le C\left(A_2(\xi,\omega,\lambda)+A_4(\nu,\xi,\omega,\lambda)\right). \end{aligned} \tag{5.42}$$

5.5.2 Step 2. Finding suitable estimate for the term $\lambda\theta^{\alpha-1}$ in the entropy balance

In this step, we shall find a suitable bound of the term $\displaystyle\int_0^T\int_{\mathcal{B}\setminus\Omega_t}\lambda\theta^{\alpha-1}\varphi$ appearing in the entropy inequality (5.29). To do that, we need a more precise bound of the term $\|\lambda\theta^\alpha\|_{L^1((0,T)\times\mathcal{B})}$.

In fact, we have already computed that the integrals in $\mathcal{B}\setminus\Omega_t$ are "small" w.r.t. all the parameters, and as per construction there is no parameter involved in the integrals in Ω_t. We shall make use of these information to get a sharper estimate of $\|\lambda\theta^\alpha\|_{L^1((0,T)\times\mathcal{B})}$.

Let us first rewrite the energy inequality (5.30) in the following form: (which can be written by using the similar strategy as we have used to obtain (4.27))

$$\begin{aligned} &\int_{\Omega_t}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho e(\rho,\theta)-\rho s(\rho,\theta)\theta_B+\frac{\delta}{\beta-1}\rho^\beta\right)(\tau,\cdot)+\int_0^\tau\int_{\mathcal{B}}\lambda\theta^\alpha \\ &+\int_0^\tau\int_{\Omega_t}\frac{\theta_B}{\theta}\left(\mathbb{S}:\nabla_x\mathbf{u}+\frac{\kappa(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right) \\ \le&\int_{\Omega_0}\left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}}+\rho_{0,\delta}e(\rho_{0,\delta},\theta_{0,\delta})-\rho_{0,\delta}s(\rho_{0,\delta},\theta_{0,\delta})\theta_B(0)+\frac{\delta}{\beta-1}\rho_{0,\delta}^\beta-(\rho\mathbf{u})_{0,\delta}\mathbf{V}(0)\right) \\ &+\int_{\mathcal{B}\setminus\Omega_0}a_\xi\Big(\theta_{0,\delta}^4-\frac{4}{3}\theta_{0,\delta}^3\theta_B\Big)+\int_0^\tau\int_{\mathcal{B}}\lambda\theta^{\alpha-1}\theta_B+\int_{\Omega_t}(\rho\mathbf{u}\cdot\mathbf{V})(\tau,\cdot) \\ &+\left|\int_0^\tau\int_{\Omega_t}\Big(\rho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{V}-\mathbb{S}:\nabla_x\mathbf{V}+p_\delta(\rho,\theta)\,\mathrm{div}_x\mathbf{V}+\rho\mathbf{u}\cdot\partial_t\mathbf{V}\Big)\right| \\ &+\left|\int_0^\tau\int_{\Omega_t}\left[\rho s(\rho,\theta)\Big(\partial_t\theta_B+\mathbf{u}\cdot\nabla_x\theta_B\Big)-\frac{\kappa(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\right]\right| \\ &+\left|\int_0^\tau\int_{\Omega_t}\rho\nabla_x\Psi\cdot\mathbf{V}\right|+\left|\int_0^\tau\int_{\mathcal{B}\setminus\Omega_t}\mathbb{S}_\omega:\nabla_x\mathbf{V}\right|+\left|\int_0^\tau\int_{\mathcal{B}\setminus\Omega_t}\frac{a_\xi}{3}\theta^4\mathrm{div}_x\mathbf{V}\right| \\ &+\left|\int_0^\tau\int_{\mathcal{B}\setminus\Omega_t}\frac{4a_\xi}{3}\theta^3\Big(\partial_t\theta_B+\mathbf{u}\cdot\nabla_x\theta_B\Big)\right|+\left|\int_0^\tau\int_{\mathcal{B}\setminus\Omega_t}\frac{\kappa_\nu(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\right| \end{aligned} \tag{5.43}$$

for almost all $\tau\in(0,T)$.

• Now, the estimations of the terms $\int_{\Omega_t}(\rho\mathbf{u}\cdot\mathbf{V})(\tau,\cdot)$, $\int_0^\tau\int_{\Omega_t}\rho\nabla_x\Psi\cdot\mathbf{V}$, $\int_0^\tau\int_{\Omega_t}\rho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{V}$, $\int_0^\tau\int_{\Omega_t}\rho\mathbf{u}\cdot\partial_t\mathbf{V}$ and $\int_0^\tau\int_{\mathcal{B}}\lambda\theta^{\alpha-1}\theta_B$ can be done in a similar fashion as previous; see (4.28), (4.29), (4.31), (4.32) and (4.33) respectively (since all of those estimates are uniform w.r.t. λ).

• We also recall that all the terms in fluid domain Q_T is independent of the parameters ω,ξ,ν. In what follows, let us estimate

$$\begin{aligned}
\int_0^\tau\int_{\Omega_t}\mathbb{S}:\nabla_x\mathbf{V} &\le \frac{1}{2}\int_0^\tau\int_{\Omega_t}\frac{\theta_B}{\theta}\mathbb{S}:\nabla_x\mathbf{u}+C(\mathbf{V},\theta_B)\int_0^\tau\int_{\Omega_t}\theta\\
&\le \frac{1}{2}\int_0^\tau\int_{\Omega_t}\frac{\theta_B}{\theta}\mathbb{S}:\nabla_x\mathbf{u}+C(\mathbf{V},a,\theta_B)\Big(1+\int_0^\tau\int_{\Omega_t}a\theta^4\Big)\\
&\le \frac{1}{2}\int_0^\tau\int_{\Omega_t}\frac{\theta_B}{\theta}\mathbb{S}:\nabla_x\mathbf{u}+C(\mathbf{V},a,\theta_B)\Big(1+\int_0^\tau\int_{\Omega_t}\rho e(\rho,\theta)\Big),
\end{aligned}\tag{5.44}$$

since $a\theta^4\le\rho e(\rho,\theta)$. Note that, the term $\frac{1}{2}\int_0^\tau\int_{\Omega_t}\frac{\theta_B}{\theta}\mathbb{S}:\nabla_x\mathbf{u}$ can be absorbed by the associated term in the l.h.s. of (5.43).

• In a similar way as we have obtained (4.42), here we get

$$\left|\int_0^\tau\int_{\Omega_t}\frac{\kappa(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B\right|\le C(\theta_B).\tag{5.45}$$

• Let us now recall the pressure term $p_\delta(\rho,\theta)=p_M(\rho,\theta)+\frac{a}{3}\theta^4+\delta\rho^\beta$ and the bound of p_M given by (4.34). To this end, we find

$$\begin{aligned}
\left|\int_0^\tau\int_{\Omega_t}p_{\xi,\delta}(\rho,\theta)\mathrm{div}_x\mathbf{V}\right| &\le C(\mathbf{V})\int_0^\tau\int_{\Omega_t}\Big(\frac{\delta}{\beta-1}\rho^\beta+a\theta^4+\rho^{\frac{5}{3}}+\theta^{\frac{5}{2}}\Big)\\
&\le C(\mathbf{V},p_\infty,a)\int_0^\tau\int_{\Omega_t}\Big(\frac{\delta}{\beta-1}\rho^\beta+\rho e(\rho,\theta)+1\Big),
\end{aligned}\tag{5.46}$$

• Next, using the bounds (4.44)–(4.47), we deduce that

$$\begin{aligned}
&\int_0^\tau\int_{\Omega_t}\rho s(\rho,\theta)|\mathbf{u}|\\
&\le\int_0^\tau\int_{\Omega_t}\rho|\mathbf{u}|+C\int_0^\tau\int_{\Omega_t}\rho|\mathbf{u}|[\log\theta]^++C\int_0^\tau a\theta^3|\mathbf{u}|\\
&\le C(\rho_0)+C\int_0^\tau\int_{\Omega_t}\rho|\mathbf{u}|^2+\frac{C}{\epsilon}\int_0^\tau\int_{\Omega_t}\rho^2([\log\theta]^+)^2+C(a,\epsilon)\int_0^\tau\int_{\Omega_t}a\theta^4+\epsilon\int_0^\tau\int_{\Omega_t}(|\mathbf{u}|^2+|\mathbf{u}|^4)\\
&\le C(\rho_0)+C\int_0^\tau\int_{\Omega_t}\rho|\mathbf{u}|^2+C(a,\epsilon)\int_0^\tau\int_{\Omega_t}a\theta^4+2\epsilon\int_0^\tau\|\mathbf{u}\|^2_{W^{1,2}(\Omega_t)}\\
&\le C(\rho_0)+C\int_0^\tau\int_{\Omega_t}\rho|\mathbf{u}|^2+C(a,\epsilon)\int_0^\tau\int_{\Omega_t}\rho e(\rho,\theta)+2\epsilon\int_0^\tau\|\mathbf{u}\|^2_{W^{1,2}(\Omega_t)}
\end{aligned}\tag{5.47}$$

using the fact that $\rho\le\theta^{3/2}$ and $[\log\theta]^+\le\theta^{1/2}$ in the integral $\int_0^\tau\int_{\Omega_t}\rho^2([\log\theta]^+)^2$. Also, we have used that $a\theta^4\le\rho e(\rho,\theta)$ in the above estimate.

• Now, as we described in (4.38)–(4.40), here we have

$$\int_0^\tau\|\mathbf{u}\|^2_{W^{1,2}(\Omega_t)}\le C(\theta_B)\int_0^\tau\int_{\Omega_t}\frac{\theta_B}{\theta}\mathbb{S}:\nabla_x\mathbf{u}+C(\rho_0,\theta_B)\int_0^\tau\int_{\Omega_t}\rho|\mathbf{u}|^2.\tag{5.48}$$

• One can also deduce that

$$\left|\int_{\mathcal{B}\setminus\Omega_0}a_\xi\Big(\theta^4_{0,\delta}-\frac{4}{3}\theta^3_{0,\delta}\theta_B(0)\Big)\right|\le C(\theta_0,\theta_B)\xi,\tag{5.49}$$

for fixed $0 < \delta < 1$.

• Using all the above estimates and the bounds of the terms in $\mathcal{B} \setminus \Omega_t$ from (5.41)–(5.42), we have from (5.43) (by applying Grönwall's lemma)

$$
\begin{aligned}
&\int_{\Omega_t} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\theta) - \rho s(\rho,\theta)\theta_B + \frac{\delta}{\beta-1}\rho^\beta \right)(\tau,\cdot) + \int_0^\tau \int_{\mathcal{B}} \lambda\theta^\alpha \\
&+ \int_0^\tau \|\mathbf{u}\|^2_{W^{1,2}(\Omega_t)} + \int_0^\tau \int_{\Omega_t} \theta_B \frac{\kappa(\theta,t,x)}{\theta^2}|\nabla_x\theta|^2 \\
\leq & C \int_{\Omega_0} \left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta} e(\rho_{0,\delta},\theta_{0,\delta}) - \rho_{0,\delta} s(\rho_{0,\delta},\theta_{0,\delta})\theta_B(0) + \frac{\delta}{\beta-1}\rho^\beta_{0,\delta} - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0) + 1 \right) \\
&+ C(\theta_B,\theta_0)\xi + C\left(A_1(\xi,\omega,\lambda) + A_2(\xi,\omega,\lambda) + A_3(\xi,\omega,\lambda) + A_4(\nu,\xi,\omega,\lambda) \right),
\end{aligned}
\tag{5.50}
$$

for almost all $\tau \in (0,T)$, where the constant $C > 0$ does not depend on any of the parameters λ,ω,ν,ξ.

Bounds of the terms $\lambda\theta^{\alpha-1}$. Let us recall the entropy balance (5.29). The only term left to estimate is the integral concerning $\lambda\theta^{\alpha-1}$. Indeed, we have

$$
\begin{aligned}
\left| \int_0^T \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\varphi \right| &\leq \lambda^{\frac{1}{\alpha}} \|\varphi\|_{L^\infty((0,T)\times\mathcal{B})} \left(\int_0^T \int_{\mathcal{B}} \lambda\theta^\alpha \right)^{\frac{\alpha-1}{\alpha}} \\
&\leq C\lambda^{\frac{1}{\alpha}} \Big(1 + \xi + A_1(\xi,\omega,\lambda) + A_2(\xi,\omega,\lambda) + A_3(\xi,\omega,\lambda) + A_4(\nu,\xi,\omega,\lambda) \Big)^{\frac{\alpha-1}{\alpha}},
\end{aligned}
\tag{5.51}
$$

where the constant $C > 0$ is independent on the parameters $\lambda,\xi,\omega,\nu,\delta$.

5.5.3 Passing to the limits of ω,ξ,ν,λ

Now, we are in position to pass to the limits of all the parameters ω, ξ, ν, λ together. We keep in mind the scaling introduced in (5.31) w.r.t. h. Then, recall the points (5.32)–(5.34), (5.35)–(5.36), (5.37)–(5.38), (5.39)–(5.40), from which it is not difficult to observe that

$$
\begin{aligned}
A_1(\xi,\omega,\lambda) &= Ch^6 \left(\frac{1}{h^{\frac{2}{3}}} + \frac{1}{h^{\frac{8}{7}}} + h^{\frac{24}{(\alpha-6)}} \right) \to 0 \text{ as } h \to 0, \\
A_2(\xi,\omega,\lambda) &= \frac{Ch^{3/2}}{h^{\frac{9}{14}}} + \frac{Ch^{3/2}}{h^{\frac{11}{21}}} h^{6/(\alpha-6)} + \frac{Ch^{3/2}}{h^{\frac{2}{7}} h^{\frac{6\alpha}{2(\alpha-6)}}} + \frac{Ch^{3/2}}{h^{1/6}} h^{\frac{6(\alpha+2)}{2(\alpha-6)}} \to 0 \quad \text{as } h \to 0, \\
A_3(\xi,\omega,\lambda) &= Ch^6 \left(\frac{1}{h^{\frac{1}{2}}} + \frac{1}{h^{\frac{6}{7}}} + \frac{1}{h^{\frac{1}{2}}} h^{\frac{18}{(\alpha-6)}} \right) + Ch^3 \left(1 + \frac{1}{h^{\frac{5}{7}}} + h^{\frac{6\alpha}{(\alpha-6)}} \right) \\
&\qquad + Ch^5 \left(1 + \frac{1}{h^{\frac{5}{7}}} + h^{\frac{36}{(\alpha-6)}} \right) \to 0 \quad \text{as } h \to 0, \\
A_4(\nu,\xi,\omega,\lambda) &= Ch^{3/2} \left(1 + \frac{1}{h^{\frac{5}{14}}} + h^{\frac{6\alpha}{2(\alpha-6)}} \right) + Ch \left(1 + \frac{1}{h^{\frac{5}{7}}} + h^{\frac{6\alpha}{(\alpha-6)}} \right) \to 0 \quad \text{as } h \to 0.
\end{aligned}
\tag{5.52}
$$

Next, from the estimate (5.41), one has

$$
\begin{aligned}
&\int_0^T \int_{\mathcal{B}\setminus\Omega_t} a_\xi \Big(\theta^4 - \frac{4}{3}\theta^3\theta_B \Big) \partial_t\psi \to 0 \quad \text{as } h \to 0, \\
&\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{4a_\xi}{3} \theta^3 \Big(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B \Big) \psi \to 0 \quad \text{as } h \to 0, \\
&\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{a_\xi}{3} \theta^4 \mathrm{div}_x \mathbf{V} \psi \to 0 \quad \text{as } h \to 0,
\end{aligned}
$$

and analogously, from (5.42) we have

$$\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \mathbb{S}_\omega : \nabla_x \mathbf{V}\psi \to 0 \quad \text{as } h \to 0,$$

$$\int_0^T \int_{\mathcal{B}\setminus\Omega_t} \frac{\kappa_\nu(\theta,t,x)}{\theta} \nabla_x\theta \cdot \nabla_x \theta_B \psi \to 0 \quad \text{as } h \to 0,$$

in the energy inequality (5.30).

Also, from (5.49), one has (in the energy inequality (5.30))

$$\int_{\mathcal{B}\setminus\Omega_0} a_\xi \Big(\theta_{0,\delta}^4 - \frac{4}{3}\theta_{0,\delta}^3 \theta_B(0)\Big) \to 0 \quad \text{as } \xi \to 0$$

We can also show that

$$\int_{\mathcal{B}\setminus\Omega_0} \frac{4a_\xi}{3}\theta_{0,\delta}^3 \varphi(0,\cdot) \to 0 \quad \text{as } \xi \to 0,$$

in the entropy inequality (5.29).

Now, thanks to the estimate (5.51), and the limiting behaviors of A_1, A_2, A_3 and A_4 from (5.52), we have

$$\int_0^T \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\varphi \to 0 \quad \text{as } h \to 0, \tag{5.53}$$

in the entropy inequality (5.29). Similarly, the term $\int_0^T \int_{\mathcal{B}} \lambda\theta^{\alpha-1}\theta_B\psi$ in the energy estimate (5.30) vanishes as $h \to 0$.

5.5.4 Resultant weak formulations.

Taking into account all the above limits of ξ, ω, ν, λ to 0 (equivalently $h \to 0$) from the previous subsection, we now write the weak formulations and the ballistic energy inequality for our system.

• The weak formulation of the continuity equation will be the same as (5.26) after passing to the limits.

• The weak formulation of the momentum equation becomes (from (5.27))

$$\begin{aligned} -\int_0^T \int_{\Omega_t} \big(\rho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p_\delta(\rho,\theta)\mathrm{div}_x\boldsymbol{\varphi}\big) + \int_0^T \int_{\Omega_t} \mathbb{S} : \nabla_x \boldsymbol{\varphi} \\ = \int_0^T \int_{\Omega_t} \rho\nabla_x\Psi\cdot\boldsymbol{\varphi} + \int_{\Omega_0} (\rho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot) \end{aligned} \tag{5.54}$$

for any test function $\boldsymbol{\varphi}$ satisfying

$$\boldsymbol{\varphi} \in \mathcal{C}_c^\infty([0,T]\times\mathcal{B};\mathbb{R}^3) \quad \text{with} \quad \boldsymbol{\varphi}(\tau,\cdot)\cdot\mathbf{n}\big|_{\Gamma_\tau} = 0 \;\text{ for any } \tau \in [0,T].$$

• The weak formulation for the Poisson equation has the same expression as (5.28) after the limiting process. So, we do not write it again.

• The entropy inequality can be written as (from (5.29))

$$\begin{aligned} -\int_0^T \int_{\Omega_t} \rho s(\rho,\theta)\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) + \int_0^T \int_{\Omega_t} \frac{\kappa(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\varphi - \int_{\Omega_0} \rho_{0,\delta}s(\rho_{0,\delta},\theta_{0,\delta})\varphi(0,\cdot) \\ \geq \int_0^T \int_{\Omega_t} \frac{\varphi}{\theta}\left(\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\theta,t,x)}{\theta}|\nabla_x\theta|^2\right), \end{aligned} \tag{5.55}$$

for any test function $\varphi \in \mathcal{C}_c^1([0,T)\times\mathcal{B})$ with $\varphi \geq 0$.

• The ballistic energy inequality (5.30) now reduces to the following:

$$
\begin{aligned}
&-\int_0^T \int_{\Omega_t} \partial_t \psi \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\theta) - \rho s(\rho,\theta)\theta_B + \frac{\delta}{\beta-1}\rho^\beta \right) \\
&+ \int_0^T \int_{\Omega_t} \frac{\theta_B}{\theta} \left(\mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\theta,t,x)}{\theta} |\nabla_x \theta|^2 \right) \psi \\
\leq &\int_{\Omega_0} \psi(0) \left(\frac{1}{2} \frac{|(\rho\mathbf{u})_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta} e(\rho_{0,\delta},\theta_{0,\delta}) - \rho_{0,\delta} s(\rho_{0,\delta},\theta_{0,\delta})\theta_B(0) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta - (\rho\mathbf{u})_{0,\delta}\mathbf{V}(0) \right) \\
&- \int_0^T \int_{\Omega_t} \left(\rho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x \mathbf{V} - \mathbb{S} : \nabla_x \mathbf{V} + p_\delta(\rho,\theta)\,\mathrm{div}_x \mathbf{V} \right) \psi - \int_0^T \int_{\Omega_t} \rho\mathbf{u}\cdot\partial_t(\mathbf{V}\psi) \\
&- \int_0^T \int_{\Omega_t} \left[\rho s(\rho,\theta)\Big(\partial_t\theta_B + \mathbf{u}\cdot\nabla_x\theta_B\Big) - \frac{\kappa(\theta,t,x)}{\theta}\nabla_x\theta\cdot\nabla_x\theta_B \right]\psi - \int_0^T \psi \int_{\Omega_t} \rho\nabla_x\Psi\cdot\mathbf{V},
\end{aligned} \tag{5.56}
$$

for all $\psi \in \mathcal{C}_c^1([0,T))$ with $\psi \geq 0$ and $\partial_t\psi \leq 0$. In the above, we can simply omit the term $\Big(- \int_0^T \int_{\mathcal{B}} \lambda\theta^\alpha\psi \Big)$ from the l.h.s. of the energy inequality since this term is non-positive.

5.6 Conclusion of the proof: vanishing artificial pressure

In the final step, we proceed with $\delta \to 0$ in a similar way as developed in [10]; see also the book [12]. We skip the details in this paper since the arguments are by now well-understood.

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A Appendix

Lemma A.1 (Generalized Poincaré inequality)**.** *Let $1 \leq p \leq \infty$, $0 < \gamma < \infty$, $U_0 > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

Then, there exists a positive constant $C = C(p,\gamma,U_0)$ such that

$$
\|v\|_{W^{1,p}(\Omega)} \leq C \left[\|\nabla_x v\|_{L^p(\Omega,\mathbb{R}^N)} + \Big(\int_U |v|^\gamma \Big)^{\frac{1}{\gamma}} \right] \tag{A.1}
$$

for any measurable $U \subset \Omega$, $|U| \geq U_0$ and any $v \in W^{1,p}(\Omega)$.

A formal proof of the above result is given in [12, Theorem 11.20, Chapter 11.9].

Lemma A.2 (Korn-Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that r is a non-negative function such that*

$$
0 < m_0 \leq \int_\Omega r dx, \quad \int_\Omega r^\gamma \leq K,
$$

for some certain $\gamma > 1$. Then

$$
\|v\|_{W^{1,p}(\Omega,\mathbb{R}^3)} \leq C(p,m_0,K) \left(\left\| \nabla_x v + \nabla_x^t v - \frac{2}{3} div_x v \mathbb{I} \right\|_{L^p(\Omega,\mathbb{R}^3)} + \int_\Omega r|v| \right) \tag{A.2}
$$

for any $v \in W^{1,p}(\Omega,\mathbb{R}^3)$ and $1 < p < \infty$.

We refer [12, Theorem 11.22, Chapter 11.10] for the proof of the above lemma.

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