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A multifluid model with chemically reacting components – construction of weak solutions

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Abstract

We investigate the existence of weak solutions to the multi-component system, consisting of compressible chemically reacting components, coupled with the compressible Stokes equation for the velocity. Specifically, we consider the case of irreversible chemical reaction and assume the nonlinear relation between the pressure and the particular densities. These assumptions cause the additional difficulties in the mathematical analysis, due to the possible presence of vacuum.

It is shown that there exists a global weak solution, satisfying the L^∞ bounds for all the components. Moreover, despite the lack of the regularity on the gradients, we obtain strong compactness of densities in L^p spaces. The applied method captures the properties of the models of high generality, which admit an arbitrary number of components. Furthermore, the framework we develop can handle models that contain both diffusing and non-diffusing elements.

keywords: compressible Stokes system, multi-component flow, weak solutions, irreversible chemical reaction

MSC: 35Q35,76N10, 35D30

1 Description of the multifluid system

In the recent years, a significant progress has been done in the mathematical theory of mixtures, both compressible and incompressible. The physical background for modeling of mixtures can be found in [35], approach closer to modern understanding of continuum mechanics and thermodynamics in [36]. Let us mention also the book [16], where a general model of compressible chemically reacting mixtures under very general conditions is analyzed from the mathematical point of view, however, only in a small neighborhood of a given static solution. We will be more concerned about results which are not restricted to small data. As the results differ with respect to studied complexity, we mention here for the case of incompressible fluids the paper [37] and the book [31].

In the paper we analyze the system of N compressible, reacting fluids with densities

$$\varrho_1, \dots, \varrho_N \colon \Omega \subset \mathbb{R}^d \to \mathbb{R},$$

moving along a common velocity vector $u\colon \Omega \to \mathbb{R}^d$. We assume the slow motion, therefore the momentum equation is given by the Stokes equation. The pressure p depends on the vector of the densities $\vec{\varrho} = (\varrho_1, \dots, \varrho_N)$. The interactions between the particular components are described by the production rates $\omega_1, \dots, \omega_N \colon \mathbb{R}^N_+ \to \mathbb{R}$, $\omega_1, \dots, \omega_N \in C^1(\overline{\mathbb{R}^N_+})$ and dissipation is prescribed by fluxes F_i.

The full system, incorporating all these phenomena reads

$$\begin{aligned} \partial_t \varrho_i + \operatorname{div}(\varrho_i u) - \operatorname{div} F_i = \omega_i(\vec{\varrho}), \quad i = 1, \dots, N, \quad &\text{in } \Omega \times (0,T), \\ -\mu \Delta u - \nabla((\mu + \lambda) \operatorname{div} u) + \nabla p(\vec{\varrho}) = 0, \quad &\text{in } \Omega \times (0,T), \end{aligned} \tag{1.1}$$

where we assume that μ, λ are constants such that $\mu > 0$ and $\lambda + \frac{2}{3}\mu > 0$.

The system is considered with the slip boundary conditions for u and the Neumann type constrain for the fluxes, namely on $\partial\Omega \times (0,T)$

$$\begin{aligned} u \cdot n &= 0 && \text{at } \partial\Omega \times (0,T), \\ n \cdot \mathbb{T}(u,p) \cdot \tau_k + f u \cdot \tau_k &= 0, \quad k = 1, \dots, d-1, && \text{at } \partial\Omega \times (0,T), \\ F_i \cdot n &= 0, \quad i = 1, ..., N, && \text{at } \partial\Omega \times (0,T), \end{aligned} \tag{1.2}$$

where n and τ_k are the normal and tangent vectors respectively, f describes the friction and the stress tensor $\mathbb{T}$ is given by

$$\mathbb{T}(u, p(\vec{\varrho})) = 2\mu\mathbb{D}u + (\lambda \mathrm{div}\, u - p(\vec{\varrho}))\mathbb{I}.$$

The system is finally supplemented by the set of initial data for the species' densities

$$\varrho_{i|_{t=0}} = \varrho_{0,i}.$$

The kernel of the system is the constitutive relation giving us the form of the pressure. We assume the total pressure p is in the form

$$p(\vec{\varrho}) = \sum_{i=1}^{N} p_i(\varrho_i), \quad \text{with} \quad p_i(\varrho_i) = \frac{1}{m_i}\varrho_i^{\gamma_i},$$

where m_i are molar masses of the components and $\gamma_i > 1$. The diffusion fluxes are depending on the pressure. Following [16], we assume that the fluxes have the form

$$F_i = \nabla p_i - \frac{\varrho_i}{\varrho}\nabla p. \tag{1.3}$$

A such form guarantees the fundamental feature of fluxes, namely $\sum_{i=1}^{N} F_i = 0$, and also non-negativity of densities ρ_i. Regarding the reaction terms, we focus on *the irreversible reactions* in the form

$$A_1 + ... + A_K \to C_1 + ... + C_L. \tag{1.4}$$

We impose the conditions on ω_i so that the densities remain non-negative for non-negative initial densities, and that the total mass of the system is preserved. To obtain the first property, we have to assume

$$\omega_i(\vec{\varrho}) \geq 0 \quad \text{for} \quad \varrho_i = 0 \quad \text{and} \quad \varrho_j \geq 0,\ j \neq i. \tag{1.5}$$

Moreover, the latter is satisfied if $\sum_{i=1}^{N} \omega_i(\vec{\varrho}) = 0$. This, together with the fact that $\sum_{i=1}^{N} F_i = 0$, ensures us that the sum of the densities $\varrho = \sum_{i=1}^{N} \varrho_i$ satisfies the continuity equation

$$\partial_t \varrho + \mathrm{div}\,(\varrho u) = 0$$

and as a consequence the total mass $\int_\Omega \varrho \,\mathrm{d}x$ remains constant.

1.1 Main results

To begin with, we state the three main theorems. The first result is the following.

Theorem 1.1. *Let $\vec{\varrho}_0 \in L^\infty(\Omega)$. In addition, let us assume that all $\gamma_i = 2$. Then there exist*

$$\varrho_1, \ldots, \varrho_N \in L^\infty([0,\infty) \times \Omega),$$

$$\bar{F}_1, \ldots, \bar{F}_N \in L^2((0,\infty) \times \Omega), \quad \sum_{i=1}^{N} \bar{F}_i = 0$$

and

$$u \in L^2(0,\infty; H^1), \quad \mathrm{div}\, u \in L^\infty((0,\infty) \times \Omega)$$

solving the system

$$\begin{aligned} \partial_t \varrho_i + \mathrm{div}\,(\varrho_i u) - \mathrm{div}\,\bar{F}_i = \omega_i(\vec{\varrho}), \quad i = 1, \ldots, N, \\ -\mu\Delta u - \nabla((\mu+\lambda)\mathrm{div}\, u) + \nabla p(\vec{\varrho}) = 0 \end{aligned} \tag{1.6}$$

in $(0,\infty) \times \Omega$ in the sense of distributions with the boundary conditions (1.2). Moreover, assuming $\varrho_i \in L^2(0,T;H^1)$, $i = 1, \ldots, N$, it gives us that the fluxes $\bar{F}_i$ satisfy the relation

$$\frac{1}{2}\sum_{i=1}^{N} \frac{1}{\varrho_i}|\bar{F}_i|^2 \leq \sum_{i=1}^{N} \bar{F}_i \cdot \frac{1}{m_i}\nabla \varrho_i \quad a.\ e.\ in \quad (0,\infty) \times \Omega. \tag{1.7}$$

The inequality (1.7) can be also rewritten in the following way: if $F_i = \nabla p_i - \frac{\varrho_i}{\varrho}\nabla p$, then (1.7) is equivalent to

$$\sum_{i=1}^{N} \frac{1}{\varrho_i}\bar{F}_i \cdot (\bar{F}_i - F_i) \leq 0.$$

Remark 1.2. *Let us discuss the above result. A careful analysis notes that the solutions obtained by this way are not satisfactory enough. It opens rather a lot of questions, then it delivers answers. The first question concerns the meaning of obtained fluxes $\bar{F}_i$. The ground regularity of solutions is too weak to determine $\bar{F}_i$ in terms of ρ_i. It is acceptable, but then Theorem 1.1 implies the question:*

Question. *Assume the solution constructed in Theorem 1.1 is smooth. Does $\bar{F}_i$ correspond to F_i given by (1.3)?*

A hint to resolve this problem seems to be in (1.7), but on the other hand it looks that it is not enough. Note that in the literature we find some positive answers to similar questions, but for slightly or significantly different systems. In [41, 42] the author were able to control the fluxes $\bar{F}_i$ in terms of gradients of species, but in that framework the densities have L^2 integrable gradients. It is a consequence of a very special dependence of viscous coefficients on the density. Here, there is no such possibility. In [28], the authors construct the weak solutions to a system of continuity equations involving chemical reactions with a given velocity. In that case, they overcome the lack of regularity for the components by assuming that the total density is in $L^2(0,T;H^1)$, which together with the linear pressure provides the estimates on the gradients of the particular species. However, we are not able to improve the regularity of the total density naturally from the equation.

To underline the nontriviality of Theorem 1.1, we need to refer to the auxillary result of the paper, which obtains the solutions to an approximate system to (1.1). Here the exponents γ_i can already be different from 2:

Theorem 1.3. *Assume that $p_i(\varrho_i) = \frac{1}{m_i}\varrho_i^{\gamma_i}$, where the exponents γ_i satisfy the relation*

$$2\gamma_{\max} < 3\gamma_{\min} - \gamma_S + 1 \tag{1.8}$$

for $\gamma_{\max}$ and $\gamma_{\min}$ being the maximum and minimum of $\{\gamma_1, \ldots, \gamma_N\}$ respectively, and $\gamma_S = \max_{j\in S}\gamma_j$ with S denoting the components on the right hand side of (1.4). Let F_i be given by (1.3). Then, for any $T > 0$, there exists at least one weak solution $(\varrho_1, \ldots, \varrho_N, u)$ to the system

$$\begin{aligned} \partial_t\varrho_i + \operatorname{div}(\varrho_i u) - \operatorname{div} F_i &= \omega_i(\vec{\varrho}) + \varepsilon\Delta\varrho_i, \quad i = 1, \ldots, N, \\ -\mu\Delta u - \nabla((\mu+\lambda)\operatorname{div} u) + \nabla p(\vec{\varrho}) &= 0, \end{aligned} \tag{1.9}$$

in $[0,T]\times\Omega$ with

$$\varrho_{i|_{t=0}} = \varrho_{0,i,\varepsilon} \in C^\infty(\Omega), \quad \varrho_{0,i,\varepsilon} \to \varrho_{0,i} \quad in \quad L^2(\Omega),$$

the Neumann boundary condition $\frac{\mathrm{d}\varrho_i}{\mathrm{d}n} = 0$ on $\partial\Omega$ and the slip boundary conditions (1.2) for u. Moreover, the obtained solution satisfies

$$\varrho_i \geq 0, \quad i = 1, \ldots, N,$$

$$\|\varrho_i\|_{L^\infty((0,T)\times\Omega)}, \left\|\frac{1}{\sqrt{\varrho_i}}F_i\right\|_{L^2((0,T)\times\Omega)}, \sqrt{\varepsilon}\|\varrho_i\|_{L^2(0,T;H^1)} \leq C, \quad i = 1, \ldots, N$$

and

$$\|u\|_{L^2(0,T;H^1)}, \|\nabla u\|_{L^\infty(0,T;BMO)} \leq C,$$

where the constant C does not depend on ε or T.

Note that the above result delivers well defined objects. Gradients of species are in L^2, so F_i are well defined in terms of formula (1.7). The proof of Theorem 1.3 is based on the classical techniques. First, we consider a suitable regularization, which makes our fluxes less degenerate, and densities better integrable. Then starting from the Galerkin methods we prove existence of each step of approximation scheme, obtaining the solutions to (1.9) finally. As Theorem 1.3, one can say, belongs to the classical theory of weak solution to compressible flows, the step to Theorem 1.1 seems to be not. We need to take $\varepsilon \to 0$, and then determine the limits in the equation. What is unexpected, is the strong convergence of all densities. We prove

$$\varrho_i^{(\varepsilon)} \to \rho_i \quad \text{strongly in} \quad L^p(\Omega \times (0, \infty)). \tag{1.10}$$

In order to justify the above limits we need to apply techniques of convergence adopted from [8]. Thanks to the structure of fluxes, limited in this part just to $\gamma_i = 2$, we are able to prove (1.10). Note that other tools from compensated compactness theory do not fit to our problem. This points out the nontriviality of Theorem 1.1. It is also an interesting example of the application of the Bresch & Jabin technique, where all classical compactness methods seem to fail. Another case of the multicomponent setting where this method becomes useful is the result of Dębiec et al. [13], concerning two-species system modelling tumor growth.

The main difficulty in Theorem 1.1 is a possible lack of strict positivity of the particular densities. It turns out that even though we still cannot control the gradients of the densities, assuming that all components are bounded away from zero we can rewrite the fluxes F_i in terms of new variables, which are already bounded in $L^2(0,T;H^1)$. We summarize this result below:

Theorem 1.4. *Let p_i and γ_i be as in Theorem 1.3. Let $(\vec{\varrho}_\varepsilon, u_\varepsilon)$ be a sequence of solutions to (1.9), such that for every $i = 1, \dots, N$*

$$\varrho_{i,\varepsilon} > c \quad \text{in} \quad [0,T] \times \Omega,$$

for some $c > 0$. Then $(\vec{\varrho}_\varepsilon, u_\varepsilon)$ converges to a solution to (1.1), with

$$F_i = \nabla p_i - \frac{\varrho_i}{\varrho} \nabla p.$$

The proof is based on the fact that even though the fluxes F_i form a degenerate system, we can rewrite them as

$$F_i = \sum_{j=1}^{N-1} b_{i,j} \nabla q_j, \tag{1.11}$$

where $\vec{q} = (q_1, \dots, q_N)$ is a certain projection of $\left(\frac{\gamma_1}{\gamma_1 - 1}\frac{p_1}{\varrho_1}, \dots, \frac{\gamma_N}{\gamma_N - 1}\frac{p_N}{\varrho_N}\right)$ on the $N-1$-dimensional space orthogonal to $(1, \dots, 1)$ (in particular $q_N = -\sum_{j=1}^{N-1} q_j$). It is shown that in the case when $\varrho_1, \dots, \varrho_N \neq 0$, the coefficients $b_{i,j}$ from the relation (1.11) form an invertible matrix, which in turn provides an L^2 bound on ∇q_i. Similar approach, with a different definition of the fluxes, was considered for example in [9, 10].

It is worth noting that our method also allows the system to contain both diffusive and non-diffusive components. Assuming N_1 is the number of diffusing elements, the full system reads

$$\begin{aligned} \partial_t \varrho_i + \operatorname{div}(\varrho_i u) - \operatorname{div} F_i = \omega_i(\vec{\varrho}), &\quad i = 1, \dots, N_1, \\ \partial_t \varrho_j + \operatorname{div}(\varrho_j u) = \omega_j(\vec{\varrho}), &\quad j = N_1 + 1, \dots, N, \\ -\mu \Delta u - \nabla((\mu + \lambda) \operatorname{div} u) + \nabla p(\vec{\varrho}) = 0. & \end{aligned} \tag{1.12}$$

Let us now briefly explain the outline of the rest of the paper and discuss the emerging obstacles:

- In Sections 2–5, we consider the case when diffusion occurs for all the components and the most general diffusive – non-diffusive case is shortly commented in Section 6.

- In Section 2, we present the a priori estimates for the solutions. In particular, we show that the densities ϱ_i are in $L^\infty((0,T) \times \Omega)$. The used strategy allows us to derive the required estimates in the case where the exponents γ_i satisfy the relation (1.8).

- Section 3 is devoted to the construction of solutions to equation (1.9) and in consequence ends the proof of Theorem 1.3. The next step towards the proofs of Theorems 1.1 and 1.4 is the limit passage with $\varepsilon \to 0$. This final step is done Sections 4 and 5 under the assumptions of Theorem 1.1 and 1.4 respectively.

- Section 4 contains the end of the proof of Theorem 1.1. Due to the presence of the additional terms in the continuity equations, the standard Lions–Feireisl approach [24, 14] is not well-suited for this framework. Instead, we are able to adapt the method of Bresch and Jabin [7, 8], based on the Kolmogorov compatness criterion, which allows us to show the strong convergence of the densities. However, due to the limitations of this method we need to restrict to the case $\gamma_1 = \ldots, \gamma_N = 2$.
- In Section 5 we perform the final limit passage in the framework of Theorem 1.4, namely assuming that the sequence of approximate solutions is strictly positive. In this case we again do not need any additional assumptions on the exponents γ_i besides (1.8).

1.2 Bibliographical remarks

Let us begin with the seminal work by A. Fick, who summarized on diffusion matter in liquids and formulated in one dimension relation between the molar mass flux J_i^{mol}, the molar concentration of constituent A_i denoted by c_i, $i \in \{1, ..., N\}$, the phenomenological coefficient D_i (the diffusion coefficient). Nowadays, this relation is called Fick's first law and has a form

$$j_i^{\text{mol}} = -D_i \nabla c_i. \tag{1.13}$$

It was realized that such a relation is less realistic and it leads to introduce the general setting of the Theory of irreversible processes. We can refer to De Groot and Mazur 1984, where the deficiencies of Fick's law are described together with a new law - the Fick-Onsager form of diffusion fluxes. The diffusion coefficients D_i are replaced by a matrix phenomenological coefficients L_{ij} called Osanger coefficients for diffusion. Another approach is the Maxwell-Stefan equations - in the so-called diffusive approximation, which ignore the acceleration of the relative motion, see Dreyer, Druet, Gajewski and Guhlke (2020) or Standart, Taylor and Krishna (1975). For more details concerning modelling, we refer to paper by Bothe, Druet [3].

Maxwell - Stefan type models were studied from the engineering point of view but there are only a few results concerning the mathematical theory. Recently such problems attract a lot of attention, see works of [2, 6, 21]. The mass-based Maxwell-Stefan approach to one-phase multicomponent reactive mixture is analyzed [18]. It is shown the local well-posedness in Lp setting and generate a local semiflow on its natural state space. A self-contained introduction to this approach can be find in [20]. Analysis of cross-diffusion systems for fluid mixtures driven by a pressure gradient was analysis for weak and strong solution by Druet, Jungel [12].

Many mathematical results have been proved in direction of multicomponent diffusion systems where the component share the same velocity and the Maxwell-Stefan system is coupled with the Navier-Stokes equations. The model for the compressible chemically reacting mixtures developed by Giovangigli in [16] in the context of the Fick law were considered in [15]; for non-diagonal mobility matrix in [41]. The paper [27] includes the case when the mean viscosity depends on the total density and fulfils the Bresch - Desjardin relation. The steady problems (with non-diagonal mobility matrix) were studied in [40, 17, 32]. More complex situation is considered in [10]. Strong solution of such type of model was investigated by Piasecki, Shibata, Zatorska [33, 34]. Let us also mention the recent work of Druet concerning the singular limit for multicomponent models, [11].

The above results in principle concern the situation, where the chemical reaction is reversible. Concerning the irreversible reaction, from the mathematical point of view not much is known. A fast irreversible reaction of type (2.1) was investigated by Bothe & Pierre in the case of reaction-diffusion system [5], where it appears between species A and B with similar concentrations. Purely numerical and experimental results, involving irreversible reactions, were obtained for example in [4, 19, 22].

2 A priori estimates

Now we will proceed with a priori estimates. We assume for simplicity that $N = 3$ and that we deal with a reaction of the type

$$A + B \to C, \tag{2.1}$$

however from the proof it is clear that the reasoning can be adjusted to a one-way reaction with arbitrary number of components. For the one-way reaction, the production rates have fixed signs. The ones corresponding to reagents are negative, whereas the ones corresponding to products are positive. In the special case (2.1), $\omega_i = -\alpha_i \varrho_1 \varrho_2$ for $i = 1, 2$ and $\omega_3 = (\alpha_1 + \alpha_2)\varrho_1\varrho_2$, where $\alpha_i > 0$ depend on the reaction rate and molecular weights of A and B respectively. The above example can be easily generalized for the case

$$A_1 + ... + A_K \to C_1 + ... + C_L,$$

with suitable definitions of ω_i's.

Throughout this section we will assume that $(\vec{\varrho}, u)$ is a sufficiently smooth solution to (1.1). The main goal is to show that

Lemma 2.1. $(\vec{\varrho}, u)$ *satisfies*

$$\rho_i \geq 0, \qquad \int_\Omega \rho(t)\,\mathrm{d}x = \int_\Omega \varrho_0\,\mathrm{d}x,$$

$$\sum_{i=1}^N \|\varrho_i\|_{L^\infty([0,T]\times\Omega)} + \sum_{i=1}^N \left\| \frac{1}{\sqrt{\varrho_i}} F_i \right\|_{L^2([0,T]\times\Omega)} + \|\mathrm{div}\, u\|_{L^\infty([0,T]\times\Omega)} \leq C,$$

where the constant C depends only on $\|\vec{\varrho}_0\|_{L^\infty(\Omega)}$.

Proof. The proof is splited into four steps.

Step I. First, let us prove the basic properties of the solution: mass conservation and non-negativity of all densities. As the total density satisfies the continuity equation, after integrating over Ω we get $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho\,\mathrm{d}x = 0$, and thus $\displaystyle\int_\Omega \varrho(t,x)\mathrm{d}x = \int_\Omega \varrho_0\,\mathrm{d}x$ for all $t > 0$.

We will now show that $\varrho_i \geq 0$ for all $i = 1, \ldots, N$. For a fixed i, let $\Omega_i^-(t) = \{x \in \Omega : \varrho_i(t,x) < 0\}$. Integrating the equation on ϱ_i over Ω_i^-, we get

$$\int_{\Omega_i^-(t)} \partial_t \varrho_i \mathrm{d}x = \int_{\partial\Omega_i^-(t)} \left(-\varrho_i u + \gamma_i \varrho_i^{\gamma_i - 1} \nabla \varrho_i - \frac{\varrho_i}{\varrho} \nabla p \right) \cdot \mathfrak{n}_i^-(t) \mathrm{d}S + \int_{\Omega_i^-(t)} \omega_i \mathrm{d}x,$$

where $\mathfrak{n}_i^-(t)$ is the outer normal vector to $\Omega_i^-(t)$. To deal with the last term let us observe:

Remark 2.2. *The assumption (1.5) allows us to extend ω_i to a function $\tilde{\omega}_i$ on $\mathbb{R}^N$ in such a way that $\tilde{\omega}_i(\vec{\varrho}) \geq 0$ whenever $\varrho_i < 0$ and $\tilde{\omega}_i$ is locally Lipschitz on $\mathbb{R}^N$. If $\omega_i = 0$ for $\varrho_i = 0$, we simply put $\tilde{\omega}_i = 0$ for $\varrho_i < 0$. If $\omega_i > 0$ for $\varrho_i = 0$ and $\varrho_j \geq 0$, then we first extend continuously the function $\omega_i|_{\varrho_i=0}$ so that $\omega_i|_{\varrho_i=0} \geq 0$ for all $\varrho_j \in \mathbb{R}^N$, $j \neq i$. Then, for $\varrho_i < 0$ we just put $\tilde{\omega}_i(\varrho_1, \ldots, \varrho_i, \ldots, \varrho_N) = \omega_i(\varrho_1, \ldots, 0, \ldots \varrho_N)$. For simplicity we will just denote $\tilde{\omega}_i$ by ω_i.*

Therefore, as by Remark 2.2 $\omega_i \geq 0$ for $\varrho_i \leq 0$ and $\varrho_i|_{\partial\Omega_i^-(t)} = 0$, we get that

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_i^- \mathrm{d}x \geq 0,$$

where $\varrho_i^- \geq 0$ is the negative part of ϱ_i ($\varrho_i = \varrho_i^+ - \varrho_i^-$). In consequence,

$$\int_\Omega \varrho_i^-(t,x)\mathrm{d}x \leq \int_\Omega \varrho_i^-(0,x)\mathrm{d}x = 0,$$

which yields $\varrho_i^-(t,\cdot) \equiv 0$.

Note that we are able to perform the above calculations provided that the set $\Omega_i^-(t)$ is of class C^1. However, from the implicit function theorem and Sard's theorem [38] it follows that we can find $\delta_n > 0$, $\delta_n \to 0$ with $n \to \infty$, such that $\Omega_{i,\delta_n}^-(t) := \{x \in \Omega : \varrho_i(t,x) < \delta_n\}$ have the desired regularity. Then the assertion follows by taking $n \to \infty$.

Step II. In this part we construct the basic energy estimate, in terms of the point-wise bound on the density and the production rates ω_i's. The goal is to show that

$$\int_0^T \int_\Omega \mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 \, \mathrm{d}x\mathrm{d}t + \sup_{t\in[0,T]} \sum_{i=1}^N \frac{1}{\gamma_i - 1}\int_\Omega p_i(\varrho_i)\mathrm{d}x + \sum_{i=1}^N \int_0^T \int_\Omega \frac{1}{\varrho_i}|F_i|^2 \mathrm{d}x\mathrm{d}t$$
$$\leq \sum_{i=1}^N \frac{1}{\gamma_i-1}\int_\Omega p_i(\varrho_{0,i})\mathrm{d}x + C\|\varrho\|^{\gamma_3-1}_{L^\infty([0,T]\times\Omega)}. \tag{2.2}$$

Note that in the general case (1.4), instead of γ_3 we need to put γ_S, which is the maximum of exponents corresponding to $C_1, \dots, C_L$.

Testing the momentum equation by u, we get

$$\mu \int_\Omega |\nabla u|^2 \mathrm{d}x + (\lambda+\mu)\int_\Omega (\operatorname{div} u)^2 \mathrm{d}x + \int_\Omega \nabla p \cdot u = 0.$$

To obtain the desired estimate, we need to deal with the last term. We have

$$\int_\Omega \nabla p \cdot u \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \frac{1}{\gamma_i-1}\frac{1}{m_i}\int_\Omega \varrho_i^{\gamma_i}\,\mathrm{d}x + 2\sum_{i=1}^N \frac{\gamma_i}{\gamma_i-1}\frac{1}{m_i}\int_\Omega \nabla\varrho_i^{\gamma_i-1}\cdot F_i\,\mathrm{d}x - 2\sum_{i=1}^N \frac{\gamma_i}{m_i}\int_\Omega \varrho_i^{\gamma_i-1}\omega_i\,\mathrm{d}x.$$

The choice of the fluxes F_i leads to the following important relation:

$$\sum_{i=1}^N \frac{\gamma_i}{m_i}\nabla\varrho_i^{\gamma_i-1}\cdot F_i = \sum_{i=1}^N \frac{1}{\varrho_i}\left(\frac{1}{m_i}\nabla\varrho_i^{\gamma_i} - \frac{\varrho_i}{\varrho}\nabla p\right)\cdot F_i = \sum_{i=1}^N \frac{1}{\varrho_i}|F_i|^2. \tag{2.3}$$

Moreover, as ω_1 and ω_2 are nonpositive, we have

$$-2\sum_{i=1}^N \frac{\gamma_i}{m_i}\int_\Omega \varrho_i^{\gamma_i-1}\omega_i\,\mathrm{d}x \geq -2\frac{\gamma_3}{m_3}\int_\Omega \varrho_3^{\gamma_3-1}\omega_3\,\mathrm{d}x.$$

In conclusion, our energy estimates read

$$\mu\int_\Omega |\nabla u|^2\,\mathrm{d}x + (\lambda+\mu)\int_\Omega(\operatorname{div} u)^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N\frac{1}{\gamma_i-1}\int_\Omega p_i(\varrho_i)\,\mathrm{d}x$$
$$+\sum_{i=1}^N\int_\Omega\frac{1}{\varrho_i}|F_i|^2\mathrm{d}x - \frac{\gamma_3}{m_3}\int_\Omega \varrho_3\omega_3(\vec{\varrho})\,\mathrm{d}x \leq 0$$

and after integrating over time and estimating the last term by

$$C\|\varrho\|^{\gamma_3-1}_{L^\infty([0,T]\times\Omega)}\int_0^T\int_\Omega \omega_3(\vec{\varrho})\mathrm{d}x\mathrm{d}t,$$

we get the inequality

$$\int_0^T \int_\Omega \mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 \, \mathrm{d}x\mathrm{d}t + \sup_{t\in[0,T]} \sum_{i=1}^N \frac{1}{\gamma_i - 1}\int_\Omega p_i(\varrho_i)\mathrm{d}x + \sum_{i=1}^N \int_0^T \int_\Omega \frac{1}{\varrho_i}|F_i|^2 \mathrm{d}x\mathrm{d}t$$
$$\leq \sum_{i=1}^N \frac{1}{\gamma_i-1}\int_\Omega p_i(\varrho_{0,i})\mathrm{d}x + C\|\varrho\|^{\gamma_3-1}_{L^\infty([0,T]\times\Omega)}\int_0^T\int_\Omega\omega_3(\vec{\varrho})\mathrm{d}x\mathrm{d}t.$$

However, the straightforward integration of the equation on ρ_3 yields

$$\int_0^T\int_\Omega \omega_3(\varrho)\,\mathrm{d}x\mathrm{d}s = \int_\Omega \varrho_3(T,x)\mathrm{d}x - \int_\Omega \varrho_{0,3}(x)\mathrm{d}x \leq C$$

and the inequality (2.2) follows.

Step III. To obtain the estimate for the density, it is required to study the effective viscous flux

$$(2\mu+\lambda)\mathrm{div}\, u = p(\vec{\varrho}) - \frac{1}{|\Omega|}\int_\Omega p(\vec{\varrho})\mathrm{d}x + d, \tag{2.4}$$

where $\Delta d = 0$ in Ω. The formula (2.4) is derived by a projection on the potential part of the momentum equation (1.1).

The structure of the slip conditions allows to split the field into two parts: the divergence-free and potential ones, denoted by $\mathcal{P}u$ and $\mathcal{Q}u$ respectively. Altogether we have $u = \mathcal{P}u + \mathcal{Q}u$, where

$$\begin{array}{llllll} \mathrm{rot}\, \mathcal{P}u = \alpha, & \mathrm{div}\, \mathcal{P}u = 0, & \mathrm{div}\, \mathcal{Q}u = \mathrm{div}\, u, & \mathrm{rot}\, \mathcal{Q}u = 0 & \text{in } \Omega, \\ n\cdot\mathcal{P}u = 0, & & n\cdot \mathcal{Q}u = 0 & & \text{at } \partial\Omega. \end{array} \tag{2.5}$$

The rotation $\alpha = \mathrm{rot}\, u$ can be determined separately from the equation on the vorticity:

$$\begin{array}{ll} -\mu\Delta\alpha = 0 & \text{in } \Omega, \\ \alpha\cdot\tau_1 = (2\chi_1 - f/\mu)u\cdot\tau_1, \quad \alpha\cdot\tau_2 = (f/\mu - 2\chi_2)u\cdot\tau_2 & \text{at } \partial\Omega, \\ \frac{(\alpha\cdot n)}{\partial n} = -(\alpha\cdot\tau_1)_{\tau_1} - (\alpha\cdot\tau_2)_{\tau_2} & \text{at } \partial\Omega. \end{array} \tag{2.6}$$

The form of boundary conditions in (2.6) is a consequence of fine properties of slip boundary conditions, described in more details, e.g. in [25, 29].

The theory of maximal regularity estimate for the elliptic problems yields

$$\|\alpha\|_{L^\infty(0,T;W^1_p)} \le C\|u|_{\partial\Omega}\|_{L^\infty(0,T;W_p^{1-1/p})}, \tag{2.7}$$

see details in [26]. In consequence, as $\|\mathcal{P}u\|_{L^\infty(0,T;W^{2,p})} \le C\|\mathrm{rot} u\|_{L^\infty(0,T;W^{1,p})}$, we conclude that

$$\|\mathcal{P}u\|_{L^\infty(0,T;W^{2,p})} \le C\|u\|_{L^\infty(0,T;W^{1,p})}. \tag{2.8}$$

We will now estimate the harmonic function d from the equation (2.4). We have $\nabla d = \mathcal{P}\Delta\mathcal{P}u$ and $\int d(x,t)\,\mathrm{d}x = 0$. Then from (2.8) we get

$$\|d\|_{L^\infty(0,T;W^{1,p})} \le C\|\mathcal{P}u\|_{L^\infty(0,T;W^{2,p})} \le C\|u\|_{L^\infty(0,T;W^{1,p})}. \tag{2.9}$$

Using (2.2), we are able to estimate different norms of p in terms of $\|\varrho\|_{L^\infty((0,T)\times\Omega)}$. In particular, we have

$$\sup_{t\in[0,T]}\int_\Omega p(\vec{\varrho})\,\mathrm{d}x \le C + C\|\varrho\|^{\gamma_3-1}_{L^\infty([0,T]\times\Omega)}. \tag{2.10}$$

Moreover, the structure of the pressure implies that

$$\frac{1}{Nm_{\max}}\varrho^{\gamma_{\min}} \le p(\vec{\varrho}) \le \varrho^{\gamma_{\max}}\sum_{i=1}^N \frac{1}{m_i}, \tag{2.11}$$

where $\gamma_{\min}$ and $\gamma_{\max}$ are respectively the minimum and maximum of $\{\gamma_1,\ldots,\gamma_N\}$. In consequence, since by the interpolation of L^p spaces

$$\int_\Omega |p(\vec{\varrho})|^p\mathrm{d}x \le \|p(\vec{\varrho})\|^{p-1}_{L^\infty(\Omega)}\int_\Omega p(\vec{\varrho})\mathrm{d}x \le C\|\varrho\|^{\gamma_{\max}(p-1)}_{L^\infty(\Omega)}(1+\|\varrho\|^{\gamma_3-1}_{L^\infty([0,T]\times\Omega)}),$$

we get that

$$\|p(\vec{\varrho})\|_{L^\infty(0,T;L^p)} \le C + C\|\varrho\|^{\tilde{\gamma}}_{L^\infty((0,T)\times\Omega)}$$

for $\tilde{\gamma} = \gamma_{\max} - \frac{\gamma_{\max}-\gamma_3+1}{p}$. Therefore from the elliptic estimates we also have

$$\|u\|_{L^\infty(0,T;W^{1,p})} \le C + C\|\varrho\|^{\tilde{\gamma}}_{L^\infty((0,T)\times\Omega)}$$

and if $p > 3$, then using (2.9) we get

$$\|d\|_{L^\infty((0,T)\times\Omega)} \le C\|d\|_{L^\infty(0,T;W^{1,p})} \le C + C\|\varrho\|^{\tilde{\gamma}}_{L^\infty((0,T)\times\Omega)}. \tag{2.12}$$

Note that if $\gamma_1, \dots, \gamma_N$ satisfy the relation (1.8), then there exists such $p > 3$ that $\tilde{\gamma} < \gamma_{\min}$. Moreover, as $\gamma_{\min} \le \gamma_3 \le \gamma_{\max}$, it also follows that $\gamma_3 - 1 < \gamma_{\min}$.

Step IV. The last step finishes our proof of Lemma 2.1. The key point of the estimate requires a special point-wise conditional estimate controlling the sign of the divergence of "bad" regions.

Lemma 2.3. *If $\varrho(t,x) > \|\varrho\|_{L^\infty((0,T)\times\Omega)} - 1$ and $\|\varrho\|_{L^\infty((0,T)\times\Omega)}$ is sufficiently large, then*

$$\operatorname{div} u(t,x) > 0.$$

Proof. Combining the estimates (2.10), (2.11) and (2.12) with (2.4), we obtain

$$\begin{aligned}
\operatorname{div} u(t,x) &= p(\vec{\varrho}(t,x)) - \frac{1}{|\Omega|}\int_\Omega p(\vec{\varrho})\mathrm{d}x - d \\
&\ge C\varrho(t,x)^{\gamma_{\min}} - C\|\varrho\|^{\gamma_3-1}_{L^\infty((0,T)\times\Omega)} - \|d\|^{\tilde{\gamma}}_{L^\infty((0,T)\times\Omega)} \\
&\ge C\left(\left(\|\varrho\|_{L^\infty((0,T)\times\Omega)} - 1\right)^{\gamma_{\min}} - \|\varrho\|^{\tilde{\gamma}}_{L^\infty((0,T)\times\Omega)} - \|\varrho\|^{\gamma_3-1}_{L^\infty((0,T)\times\Omega)}\right).
\end{aligned}$$

As $\gamma_{\min} > \max(\tilde{\gamma}, \gamma_3 - 1)$, the function $z \mapsto (z-1)^{\gamma_{\min}} - z^{\tilde{\gamma}} - z^{\gamma_3-1}$ is strictly positive for sufficiently large z and thus we arrive to the desired conclusion. ☐

To finish the proof of Lemma 2.1, let us assume that $\|\varrho\|_{L^\infty((0,T)\times\Omega)} > \|\varrho_0\|_{L^\infty(\Omega)} + 1$ (otherwise C from the statement of the Lemma is just equal to $\|\varrho_0\|_{L^\infty(\Omega)} + 1$) and let $k > \|\varrho\|_{L^\infty((0,T)\times\Omega)} - 1$. We test the continuity equation on ϱ by $(\varrho - k)_+$, where f_+ denotes the positive part of f. Then

$$\begin{aligned}
\frac{1}{2}\int_\Omega (\varrho - k)_+^2(t,x)\mathrm{d}x - \frac{1}{2}\int_\Omega (\varrho_0 - k)_+^2\mathrm{d}x &= \int_0^t\int_\Omega \varrho u \cdot \nabla(\varrho - k)_+\mathrm{d}x\mathrm{d}s \\
&= -\int_0^t\int_\Omega \operatorname{div} u\left(\frac{1}{2}(\varrho - k)_+^2 + k(\varrho - k)_+\right)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Therefore using the fact that $\varrho_0 < k$, we obtain

$$\sup_{\tau\in[0,t)}\int_\Omega (\varrho - k)_+^2(x,\tau)\mathrm{d}x \le -\int_0^t\int_\Omega (\varrho - k)_+(\varrho + k)\operatorname{div} u\mathrm{d}x\mathrm{d}s.$$

Now if $\|\varrho\|_{L^\infty((0,T)\times\Omega)}$ is large enough, then from Lemma 2.3 the right hand side is non-positive. In consequence $(\varrho - k)_+ = 0$ a.e. in $(0,T)\times\Omega)$ and thus

$$\varrho < \|\varrho\|_{L^\infty((0,T)\times\Omega)} - 1 \quad \text{a.e. in} \quad [0,T]\times\Omega.$$

This leads to contradiction, and in consequence $\|\varrho\|_{L^\infty((0,T)\times\Omega)}$ is bounded by some constant depending only on $\|\vec{\varrho}_0\|_{L^\infty(\Omega)}$. Thus Lemma 2.1 is proved.

Remark 2.4. *As $\|\varrho\|_{L^\infty((0,T)\times\Omega)} \le C$, by the relation (2.4) $\|\operatorname{div} u\|_{L^\infty((0,T)\times\Omega)} \le C$ as well. Therefore from the elliptic estimates $\|\nabla\mathcal{Q}u\|_{L^\infty(0,T;BMO)} \le C$, and by (2.8)*

$$\|\nabla u\|_{L^\infty(0,T;BMO)} \le C. \tag{2.13}$$

3 Existence of an approximated system

The goal of this section is to prove Theorem 1.3. Similarly as in Section 2, the proof is valid for different γ_i's under the constraint (1.8), however for simplicity we restrict themselves to the case $\gamma_1 = \cdots = \gamma_N = 2$. Note that from the Neumann condition on ϱ_i it also follows that $F_i \cdot n = 0$ for $i = 1, \dots, N$.

Proof of Theorem 1.3: Let us introduce another level of approximation, depending on a small parameter δ. We improve the integrability of the densities, by adding the terms $\delta\tilde{\varrho}^{\beta-2}\varrho_i$ for $\tilde{\varrho} = \sum_{i=1}^N |\varrho_i|$ and β sufficiently large. We also introduce the truncation of ϱ_i, defined as

$$\varrho_i^\delta := \operatorname{sgn}(\varrho_i)\min\left(|\varrho_i|, \frac{1}{\delta}\right).$$

Then we put

$$\omega_i^\delta(\vec{\varrho}) := \omega_i\left(\varrho_1^\delta, \dots, \varrho_N^\delta\right) \quad \text{and} \quad \tilde{\varrho}_i^\delta := \min\left(|\varrho_i|, \frac{1}{\delta}\right) = |\varrho_i^\delta|.$$

There is also the need to approximate the pressure and the diffusion fluxes. We put

$$p_i^\delta(\varrho_i) = \frac{2}{m_i}\int_0^{\varrho_i} \min\left(|w|, \frac{1}{\delta}\right) \mathrm{d}w \tag{3.1}$$

and define F_i^δ as

$$F_i^\delta = \nabla p_i^\delta(\varrho_i) - \frac{\tilde{\varrho}_i^\delta}{\tilde{\varrho}^\delta}\nabla p^\delta, \quad \text{where} \quad \tilde{\varrho}^\delta = \sum_{k=1}^N \left|\varrho_k^\delta\right| \quad \text{and} \quad p^\delta = \sum_{k=1}^N p_k^\delta(\varrho_i). \tag{3.2}$$

Note that F_i^δ satisfy the same necessary properties as F_i, namely $\sum_{i=1}^N F_i^\delta = 0$ and

$$\sum_{i=1}^N \frac{1}{m_i}\nabla\varrho_i \cdot F_i^\delta = \sum_{i=1}^N \frac{|F_i^\delta|^2}{|\varrho_i^\delta|} \geq 0.$$

Finally, the (δ, ε)-approximative system reads

$$\begin{aligned} &\partial_t\varrho_i + \operatorname{div}(\varrho_i u) - \operatorname{div} F_i^\delta + \delta\tilde{\varrho}^{\beta-2}\varrho_i = \omega_i^\delta(\vec{\varrho}) + \varepsilon\Delta\varrho_i, \quad i = 1, \dots, N, \\ &-\mu\Delta u - \nabla((\mu+\lambda)\operatorname{div} u) + \nabla p(\vec{\varrho}) = 0. \end{aligned} \tag{3.3}$$

3.1 The Galerkin approximation

We will obtain the solutions to (3.3) using the Galerkin method.

Lemma 3.1. *The equation (3.3) admits a global weak solution in* $[0,T]\times\Omega$.

Proof. Let $\{w_k\}_{k\in\mathbb{N}} \in C^\infty(\Omega)$ and $\{v_k\}_{k\in\mathbb{N}} \in (C^\infty(\Omega))^d$ be the suitable orthogonal bases of $H^1(\Omega)$ and $(H^1(\Omega))^d$ respectively. Define

$$\varrho_{i,n} = \sum_{k=1}^n a_{i,k}(t)w_k \quad \text{and} \quad u_n = \sum_{k=1}^n b_k(t)v_k,$$

where the coefficients $a_{i,k}$ satisfy the system of $n\times N$ ODEs

$$\begin{aligned} \dot{a}_{i,k} = &\sum_{l,m=1}^n a_{i,l}b_m\int_\Omega w_l v_m \nabla w_k \mathrm{d}x - \int_\Omega F_{i,n}^\delta \cdot \nabla w_k \mathrm{d}x \\ &- \varepsilon a_{i,k}\int_\Omega |\nabla w_k|^2 \mathrm{d}x - \delta\sum_{l=1}^n a_{i,l}\int_\Omega \tilde{\varrho}_n^{\beta-2} w_l w_k \mathrm{d}x + \int_\Omega \omega_i^\delta(\vec{\varrho}_n)w_k \mathrm{d}x \end{aligned} \tag{3.4}$$

and b_k is given by

$$b_k = \int_\Omega p(\vec{\varrho}_n)\operatorname{div} v_k \mathrm{d}x \tag{3.5}$$

for $\vec{\varrho}_n = (\varrho_{1,n}, \dots, \varrho_{N,n})$. Since the functions

$$(z_1, \dots, z_N) \mapsto \frac{z_i z_j}{\sum_{k=1}^N z_k}, \quad i,j = 1, \dots, N$$

are Lipschitz continuous on $\{z \in \mathbb{R}^N : z_k \geq 0 \ \forall_{k=1,\dots,N}\}$, the terms $\int_\Omega F_i^\delta \cdot \nabla w_k \mathrm{d}x$ are locally Lipschitz with respect to $a_{i,k} \in \mathbb{R}^{N\times n}$. Therefore, plugging the relation (3.5) into (3.4), we see that the right hand side of (3.4) is locally Lipschitz and from the Picard-Lindelöf theorem there exists a local in time solution to (3.4), corresponding to the initial condition

$$a_{i,k}(0) = \langle \varrho_{0,i,\varepsilon}, w_k \rangle_{L^2}.$$

To obtain the solution on a whole interval $[0,T]$, we need to find the global estimate. We test the momentum equation by u_n and integrate by parts. Then, we get

$$\begin{aligned}
&\mu \int_0^T \int_\Omega |\nabla u_n|^2 \mathrm{d}x\mathrm{d}t + \sup_{t\in[0,T]} \sum_{i=1}^N \frac{1}{m_i} \int_\Omega \varrho_{i,n}^2 \mathrm{d}x \\
&+ \sum_{i=1}^N \int_0^T \int_\Omega \frac{|F_i^\delta|^2}{\tilde{\varrho}_{i,n}^\delta} \mathrm{d}x\mathrm{d}t + 2\delta \sum_{i=1}^N \frac{1}{m_i} \int_0^T \int_\Omega \tilde{\varrho}_n^{\beta-2} \varrho_{i,n}^2 \mathrm{d}x\mathrm{d}t \\
&+ 2\varepsilon \sum_{i=1}^N \frac{1}{m_i} \int_0^T \int_\Omega |\nabla \varrho_{i,n}|^2 \mathrm{d}x\mathrm{d}t - 2 \sum_{i=1}^N \frac{1}{m_i} \int_0^T \int_\Omega \varrho_{i,n} \omega_i^\delta(\vec{\varrho}_n) \mathrm{d}x\mathrm{d}t \leq \sum_{i=1}^N \frac{1}{m_i} \int_\Omega \varrho_{0,i,\varepsilon}^2 \mathrm{d}x.
\end{aligned}$$

As $\omega_i^\delta \in L^\infty((0,T)\times\Omega)$, from the Cauchy inequality we have

$$\int_0^T \int_\Omega \varrho_{i,n} \omega_i^\delta \mathrm{d}x \leq \eta \|\varrho_{i,n}\|_{L^\infty(0,T;L^2)}^2 + \frac{CT}{\eta} \|\omega_i^\delta\|_{L^\infty((0,T)\times\Omega)}^2.$$

Therefore choosing η sufficiently small, we get the estimates

$$\begin{aligned}
\|\nabla u_n\|_{L^2((0,T)\times\Omega)}^2 + \sum_{i=1}^N \Big(\|\varrho_{i,n}\|_{L^\infty(0,T;L^2)}^2 + \varepsilon \|\nabla \varrho_{i,n}\|_{L^2((0,T)\times\Omega)}^2 + \delta \|\varrho_{i,n}\|_{L^\beta((0,T)\times\Omega)}^\beta \Big) \\
\leq C + C(\delta)T. \quad (3.6)
\end{aligned}$$

In particular, we get the bound on $\sum_{i=1}^N \sum_{k=1}^n a_{i,k}^2(t)$, which provides that we can extend the solution to a whole interval $[0,T]$. The inequality (3.6) also provides the estimates uniform in n, which allow us to extract weakly convergent subsequences (indexed again by n)

$$u_n, \varrho_{1,n}, \dots, \varrho_{N,n} \rightharpoonup u, \varrho_1, \dots, \varrho_N \quad \text{in} \quad L^2(0,T;W^{1,2}).$$

From the estimate on $\|\varrho_{i,n}\|_{L^\beta((0,T)\times\Omega)}$ it follows that $\|p_n\|_{L^{\beta/2}((0,T)\times\Omega)} \leq C$. Therefore, by the momentum equation, we deduce also

$$\|\nabla p_n\|_{L^{\beta/2}(0,T;W^{-1,\beta/2})} + \|u_n\|_{L^{\beta/2}(0,T;W^{1,\beta/2})} \leq C.$$

In particular, if β is large enough, then

$$\|\varrho_{i,n} u_n\|_{L^2((0,T\times\Omega)}, \|F_{i,n}^\delta\|_{L^2((0,T)\times\Omega)} \leq C.$$

In consequence, we obtain the uniform bound on $\|\partial_t \varrho_{i,n}\|_{L^2(0,T;H^{-1})}$ and from the Aubin-Lions Lemma

$$\varrho_{i,n} \to \varrho_i \quad \text{in} \quad L^2((0,T)\times\Omega).$$

This allows us to pass to the limit with $n \to \infty$ in the weak formulation of (3.3) and in consequence obtain a weak solution. □

3.2 Nonnegativity of the components

We will now prove that for $(\varrho_1, \ldots, \varrho_N, u)$ solving (3.3), we have

$$\varrho_i \geq 0 \quad \text{a. e.} \quad \forall_{i=1,\ldots,N}.$$

Note that the low regularity of ϱ_i does not allow us to replicate the argument from Section 2. Instead, we will obtain the result by choosing a suitable test function. Let $\varrho_i = \varrho_i^+ - \varrho_i^-$, where $\varrho_i^+, \varrho_i^- \geq 0$ denote the positive and negative part of ϱ_i. We test the equation on ϱ_i by the function $\psi_i = (l+\varrho_i^-)^\alpha$ for $l, \alpha > 0$ small. Since ϱ_i is in $L^2(0,T;W^{1,2})$, $\psi_i \in L^2(0,T;W^{1,2})$ as well. First, observe that if $\varphi_\eta(t,x)$ is the standard mollifier over time and space, then for $(\varrho_i^-)_\eta = \varrho_i^- * \varphi_\eta$ we have $(l+(\varrho_i^-)_\eta)^\alpha \to (l+\varrho_i^-)^\alpha$ in $L^2(0,T;H^1)$ and $\partial_t(\varrho_i^-)_\eta \rightharpoonup^* \partial_t \varrho_i^-$ in $L^2(0,T;H^{-1})$. Thus

$$\begin{aligned}
\int_0^t \int_\Omega \partial_t \varrho_i^- (l+\varrho_i^-)^\alpha \mathrm{d}x\mathrm{d}s &= \lim_{\eta\to 0} \int_0^t \int_\Omega \partial_t (\varrho_i^-)_\eta (l + (\varrho_i^-)_\eta)^\alpha \mathrm{d}x\mathrm{d}s \\
&= \frac{1}{\alpha+1} \lim_{\eta\to 0} \int_0^t \int_\Omega \partial_t (l + (\varrho_i^-)_\eta)^{\alpha+1} \mathrm{d}x\mathrm{d}s \\
&= \frac{1}{\alpha+1} \lim_{\eta\to 0} \left[\int_\Omega (l + (\varrho_i^-)_\eta(t,x))^{\alpha+1}\mathrm{d}x - \int_\Omega (l + (\varrho_i^-)_\eta(0,x))^{\alpha+1}\mathrm{d}x \right] \\
&= \frac{1}{\alpha+1} \left[\int_\Omega (l + \varrho_i^-(t,x))^{\alpha+1}\mathrm{d}x - \int_\Omega (l + \varrho_i^-(0,x))^{\alpha+1}\mathrm{d}x \right].
\end{aligned}$$

Similarly,

$$\int_0^t \int_\Omega l^\alpha \partial_t \varrho_i^+ \mathrm{d}x\mathrm{d}s = l^\alpha \left(\int_\Omega \varrho_i^+(t,x)\mathrm{d}x - \int_\Omega \varrho_i^+(0,x)\mathrm{d}x\mathrm{d}t \right).$$

Therefore

$$\begin{aligned}
\int_0^t \int_\Omega \partial_t \varrho_i \psi_i \mathrm{d}x\mathrm{d}s =& l^\alpha \int_0^t \int_\Omega \partial_t \varrho_i^+ \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \partial_t \varrho_i^- (l+\varrho_i^-)^\alpha \mathrm{d}x\mathrm{d}s \\
=& l^\alpha \left(\int_\Omega \varrho_-^+(t,x)\mathrm{d}x - \int_\Omega \varrho_i^+(0,x)\mathrm{d}x\mathrm{d}t \right) \\
& - \frac{1}{\alpha+1} \left[\int_\Omega (l+\varrho_i^-(t,x))^{\alpha+1}\mathrm{d}x - \int_\Omega (l+\varrho_i^-(0,x))^{\alpha+1}\mathrm{d}x \right].
\end{aligned}$$

Moreover,

$$\begin{aligned}
-\varepsilon \int_0^t \int_\Omega \Delta \varrho_i \psi_i \mathrm{d}x\mathrm{d}s &= \varepsilon\alpha \int_0^t \int_\Omega (l+\varrho_i^-)^{\alpha-1} (\nabla \varrho_i^+ - \nabla \varrho_i^-) \cdot \nabla \varrho_i^- \mathrm{d}x\mathrm{d}s \\
&= -\varepsilon\alpha \int_0^t \int_\Omega (l+\varrho_i^-)^{\alpha-1} |\nabla \varrho_i^-|^2 \mathrm{d}x\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
-\int_0^t \int_\Omega \operatorname{div} F_i^\delta \psi_i \mathrm{d}x\mathrm{d}s =& \alpha \int_0^t \int_\Omega \left(\frac{1}{m_i} \tilde{\varrho}_i^\delta \nabla(\varrho_i^+ - \varrho_i^-) - \frac{\tilde{\varrho}_i^\delta}{\tilde{\varrho}^\delta} \nabla p^\delta \right) (l+\varrho_i^-)^{\alpha-1} \nabla \varrho_i^- \mathrm{d}x\mathrm{d}s \\
=& -\alpha \int_0^t \int_\Omega \frac{1}{m_i} (l+\varrho_i^-)^{\alpha-1} \tilde{\varrho}_i^\delta |\nabla \varrho_i^-|^2 \mathrm{d}x\mathrm{d}s \\
& -\alpha \int_0^t \int_\Omega (l+\alpha_i^-)^{\alpha-1} \frac{\min(\varrho_i^-, \frac{1}{\delta})}{\tilde{\varrho}^\delta} \nabla p^\delta \cdot \nabla \varrho_i^- \mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Employing the fact that $\varrho_i^-(0,x) = 0$ and combining the above calculations with the remaining terms, we get

$$
\begin{aligned}
&\frac{-1}{\alpha+1}\int_\Omega (l+\varrho_i^-(t,\cdot))^{\alpha+1}\mathrm{d}x+\frac{l^{\alpha+1}|\Omega|}{\alpha+1}\\
&+l^\alpha\int_\Omega \varrho_i^+(t,\cdot)\mathrm{d}x-l^\alpha\int_\Omega \varrho_{0,i,\varepsilon}\mathrm{d}x\\
&-\varepsilon\alpha\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}|\nabla\varrho_i^-|^2\mathrm{d}x\mathrm{d}s\\
&+\delta\left(l^\alpha\int_0^t\int_\Omega \tilde\varrho^{\beta-2}\varrho_i^+\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega \tilde\varrho^{\beta-2}\varrho_i^-(l+\varrho_i^-)^\alpha\mathrm{d}x\mathrm{d}s\right)\\
&-\alpha\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\varrho_i u\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\\
&-\frac{\alpha}{m_i}\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\tilde\varrho_i^\delta|\nabla\varrho_i^-|^2\mathrm{d}x\mathrm{d}s\\
&-\alpha\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\frac{\min(\varrho_i^-,\frac{1}{\delta})}{\tilde\varrho^\delta}\nabla p^\delta\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s=\int_0^t\int_\Omega \omega_i^\delta(\vec\varrho)(l+\varrho_i^-)^\alpha\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Neglecting the nonpositive terms, we get

$$
\begin{aligned}
&\frac{-1}{\alpha+1}\int_\Omega (l+\varrho_i^-(t,\cdot))^{\alpha+1}\mathrm{d}x+\frac{l^{\alpha+1}|\Omega|}{\alpha+1}\\
&+l^\alpha\int_\Omega \varrho_i^+(t,\cdot)\mathrm{d}x-l^\alpha\int_\Omega \varrho_{0,i,\varepsilon}\mathrm{d}x\\
&-\alpha\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\varrho_i^- u\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\\
&+\delta l^\alpha\int_0^t\int_\Omega \tilde\varrho^{\beta-2}\varrho_i^+\mathrm{d}x\mathrm{d}s\\
&-\alpha\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\frac{\min(\varrho_i^-,\frac{1}{\delta})}{\tilde\varrho^\delta}\nabla p^\delta\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\geq\int_0^t\int_\Omega \omega_i^\delta(l-\varrho_i)^\alpha\mathbb{1}_{\varrho_i\leq 0}\,\mathrm{d}x\mathrm{d}s\\
&\qquad+l^\alpha\int_0^t\int_\Omega \omega_i^\delta\mathbb{1}_{\varrho_i\geq 0}\,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

First, we want to pass to the limit with $l\to 0$. For small α we have

$$
(l+\varrho_i^-)^{\alpha-1}\varrho_i^-=\frac{\varrho_i^-}{l+\varrho_i^-}(\varrho_i^-)^\alpha\leq\varrho_i^-
$$

and

$$
(l+\varrho_i^-)^{\alpha-1}\min(\varrho_i^-,\frac{1}{\delta})=\frac{\min(\varrho_i^-,\frac{1}{\delta})}{l+\varrho_i^-}(l+\varrho_i^-)^\alpha\leq\frac{1}{\delta}.
$$

Therefore from the dominated convergence theorem

$$
\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\varrho_i^- u\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\to\int_0^t\int_\Omega (\varrho_i^-)^\alpha u\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s
$$

and

$$
\int_0^t\int_\Omega (l+\varrho_i^-)^{\alpha-1}\frac{\min(\varrho_i^-,\frac{1}{\delta})}{\tilde\varrho^\delta}\nabla p^\delta\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\to\int_0^t\int_\Omega (\varrho_i^-)^{\alpha-1}\frac{\min(\varrho_i^-,\frac{1}{\delta})}{\tilde\varrho^\delta}\nabla p^\delta\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s
$$

as $l\to 0$. After the limit passage, we obtain

$$
\begin{aligned}
&\frac{-1}{\alpha+1}\int_\Omega (\varrho_i^-(t,\cdot))^{\alpha+1}\mathrm{d}x-\alpha\int_0^t\int_\Omega (\varrho_i^-)^\alpha u\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\\
&\quad-\alpha\int_0^t\int_\Omega (\varrho_i^-)^{\alpha-1}\frac{\min(\varrho_i^-,\frac{1}{\delta})}{\tilde\varrho^\delta}\nabla p^\delta\cdot\nabla\varrho_i^-\mathrm{d}x\mathrm{d}s\geq\int_0^t\int_\Omega \omega_i^\delta|\varrho_i|^\alpha\mathbb{1}_{\varrho_i\leq 0}\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Now we want to pass to the limit with $\alpha \to 0$. As the integrals $\int_0^t \int_\Omega (\varrho_i^-)^\alpha u \cdot \nabla \varrho_i^- \mathrm{d}x\mathrm{d}s$ and $\int_0^t \int_\Omega (\varrho_i^-)^{\alpha-1} \frac{\min(\varrho_i^-, \frac{1}{\delta})}{\tilde{\varrho}^\delta} \nabla p^\delta \cdot \nabla \varrho_i^- \mathrm{d}x\mathrm{d}s$ are bounded independently of α, we get

$$\int_\Omega \varrho_i^-(t,\cdot)\mathrm{d}x \le -\int_0^t \int_\Omega \omega_i^\delta \mathbb{1}_{\varrho_i \le 0} \mathrm{d}x\mathrm{d}s.$$

From the assumptions on ω_i and Remark 2.2, for $\varrho_i \le 0$ we have $\omega_i^\delta \ge 0$. Therefore

$$\int_\Omega \varrho_i^-(t,\cdot)\mathrm{d}x \le 0.$$

However, $\varrho_i^- \ge 0$ by definition, and thus $\varrho_i^-(t,\cdot) \equiv 0$ for all $t \in [0,T]$.

3.3 Boundedness of ϱ

It turns out that all components are bounded, and their L^∞ bound does not depend on ε and δ. As we already know that $\varrho_i \ge 0$ for all $i = 1,\dots,N$, it is sufficient to prove the L^∞ estimate for the total density $\varrho = \sum_{i=1}^N \varrho_i$. Since $\varrho = \tilde{\varrho}$, the total density satisfies the equation

$$\partial_t \varrho + \operatorname{div}(\varrho u) - \varepsilon \Delta \varrho + \delta \varrho^{\beta-1} = 0.$$

First, let us show that $\varrho \in L^\infty((0,T)\times\Omega)$. Recall the following result:

Proposition 3.2. *Let $h \in L^p(0,T;L^q)$ for $1 < p,q < \infty$. Then the solution to*

$$\partial_t \varrho - \varepsilon \Delta \varrho = h, \quad \varrho_{|t=0} = \varrho_0$$

satisfies the estimate

$$\varepsilon^{1-1/p}\|\varrho\|_{L^\infty(0,T;W^{2-2/p,q})} + \|\partial_t \varrho\|_{L^p(0,T;L^q)} + \varepsilon\|\varrho\|_{L^p(0,T;W^{2,q})} \le C\left(\varepsilon^{1-1/p}\|\varrho_0\|_{W^{1,q}} + \|h\|_{L^p(0,T;L^q)}\right). \tag{3.7}$$

Moreover, if $h = \operatorname{div} w$, $w \in L^p(0,T;L^q)$ and $p \ge 2$, then

$$\varepsilon^{1-1/p}\|\varrho\|_{L^\infty(0,T;L^q)} + \varepsilon\|\nabla\varrho\|_{L^p(0,T;L^q)} \le C\left(\varepsilon^{1-1/p}\|\varrho_0\|_{L^q} + \|w\|_{L^p(0,T;L^q)}\right). \tag{3.8}$$

Note that if β is large enough, then in particular $\varrho u \in L^p(0,T;L^q)$ for some suitable $p \ge 2$ and $q > 3$. Therefore from Proposition 3.2, we have

$$\varrho \in L^\infty(0,T;L^q).$$

Then, using the property (2.11) from the elliptic estimates we have $p \in L^\infty(0,T;L^{q/2})$ and thus we get that

$$u \in L^\infty(0,T;W^{1,q/2}).$$

In consequence,

$$\varrho u \in L^\infty(0,T;L^r)$$

for some $r > 3$. Therefore applying the De Giorgi technique (see e.g. [39], Chapter 4), we get that

$$\varrho \in L^\infty((0,T)\times\Omega).$$

We will now prove that in fact the L^∞ bound on ϱ depends only on $\|\varrho_0\|_{L^\infty(\Omega)}$, using the reasoning analogous as in Section 2. Similarly as for the a priori estimates, we have

$$-2\sum_{i=1}^N \frac{1}{m_i}\int_\Omega \varrho_i \omega_i^\delta \,\mathrm{d}x \ge -2\frac{1}{m_3}\int_\Omega \varrho_3 \omega_3^\delta \,\mathrm{d}x.$$

Therefore analogously we get the bound

$$
\begin{aligned}
\mu\int_0^T\int_\Omega|\nabla u|^2\mathrm{d}x\mathrm{d}t+&\sup_{t\in[0,T]}\sum_{i=1}^N\frac{1}{m_i}\int_\Omega\varrho_i^2\mathrm{d}x+\sum_{i=1}^N\int_0^T\int_\Omega\frac{|F_i^\delta|^2}{\tilde\varrho_i^\delta}\mathrm{d}x\mathrm{d}t\\
+2\delta\sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega\varrho^{\beta-2}\varrho_i^2\mathrm{d}x\mathrm{d}t&+\varepsilon\sum_{i=1}^N\frac{2}{m_i}\int_0^T\int_\Omega|\nabla\varrho_i|^2\mathrm{d}x\mathrm{d}t\\
&\le\sum_{i=1}^N\frac{1}{m_i}\int_\Omega\varrho_{0,i,\varepsilon}^2\mathrm{d}x+C\|\varrho_3\|_{L^\infty((0,T)\times\Omega)}\int_0^T\int_\Omega\omega_3^\delta\mathrm{d}x.
\end{aligned}\tag{3.9}
$$

As

$$
\int_\Omega\varrho(t,x)\mathrm{d}x+\delta\int_0^t\int_\Omega\varrho^{\beta-1}\mathrm{d}x=\int_\Omega\varrho_{0,\varepsilon}\mathrm{d}x,
$$

we also have

$$
\int_0^T\int_\Omega\omega_3^\delta\mathrm{d}x\mathrm{d}t=\int_\Omega\varrho_i(T,\cdot)\mathrm{d}x-\int_\Omega\varrho_{i,0,\varepsilon}\mathrm{d}x+\delta\int_0^T\int_\Omega\varrho^{\beta-2}\varrho_i\mathrm{d}\mathrm{d}s\le\int_\Omega\varrho_{0,\varepsilon}\mathrm{d}x\le\int_\Omega\varrho_0\mathrm{d}x.\tag{3.10}
$$

Now we repeat Step IV of the proof of Lemma 2.1. If $\|\varrho\|_{L^\infty((0,T)\times\Omega)}\le\|\varrho_0\|_{L^\infty(\Omega)}+1$, then the assertion follows. If $\|\varrho\|_{L^\infty((0,T)\times\Omega)}>\|\varrho_0\|_{L^\infty(\Omega)}+1$, then let $k>\|\varrho\|_{L^\infty((0,T)\times\Omega)}-1$. We test the equation on ϱ by $(\varrho-k)_+$. We have

$$
\begin{aligned}
&\frac12\int_\Omega(\varrho-k)_+^2(t,x)\mathrm{d}x-\frac12\int_\Omega(\varrho_0-k)_+^2\mathrm{d}x\\
+\varepsilon\int_0^t\int_\Omega|\nabla(\varrho-k)_+|^2\mathrm{d}x\mathrm{d}s+\delta\int_0^t\int_\Omega\varrho^{\beta-1}(\varrho-k)_+\mathrm{d}x\mathrm{d}s&=\int_0^t\int_\Omega\varrho u\cdot\nabla(\varrho-k)_+\mathrm{d}x\mathrm{d}s\\
&=-\int_0^t\int_\Omega\mathrm{div}\,u\left(\frac12(\varrho-k)_+^2+k(\varrho-k)_+\right)\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Therefore using the fact that $\varrho_0<k$, we obtain

$$
\int_\Omega(\varrho-k)_+^2\mathrm{d}x\le-\int_0^t\int_\Omega(\varrho-k)_+(\varrho+k)\mathrm{div}\,u\mathrm{d}x\mathrm{d}s.
$$

Combining (3.9) and (3.10), we obtain the estimate

$$
\|p\|_{L^\infty(0,T;L^1)}\le C+C\|\varrho\|_{L^\infty((0,T)\times\Omega)}.
$$

Then we proceed in the same way as in the proof of Lemma 2.3 to get $\mathrm{div}u\mathbb{1}_{\varrho>k}>0$ for k sufficiently large and in consequence

$$
\|\varrho\|_{L^\infty((0,T)\times\Omega)}\le C,
$$

where C is the absolute constant independent from ε and δ, given in terms of the L^∞ norm of ρ_0.

3.4 Limit passage with $\delta\to0$.

Having the uniform L^∞ bounds on ϱ and nonnegativity, we know that for sufficiently small δ

$$
\omega_i^\delta=\omega_i,\quad\varrho_i^\delta=\tilde\varrho_i^\delta=\varrho_i\quad\text{and}\quad F_i^\delta=F_i,\quad i=1,\dots,N.
$$

Moreover, using (3.9) we get the estimates

$$
\|\nabla u\|_{L^2((0,T)\times\Omega)}^2+\sum_{i=1}^N\left(\varepsilon\|\nabla\varrho_i\|_{L^2((0,T)\times\Omega)}^2+\delta\|\varrho_i\|_{L^\beta((0,T)\times\Omega)}^\beta+\left\|\frac{1}{\sqrt{\varrho_i}}F_i\right\|_{L^2((0,T)\times\Omega)}\right)\le C,
$$

where C does not depend on ε and δ. In consequence, we also get

$$\|\partial_t \varrho_i\|_{L^2(0,T;H^{-1})} \leq C.$$

Therefore, if $(\vec{\varrho}_{\varepsilon,\delta}, u_{\varepsilon,\delta})$ is a solution to (3.3), then up to a subsequence

$$\varrho_{1,\varepsilon,\delta}, \ldots, \varrho_{N,\varepsilon,\delta}, u_{\varepsilon,\delta} \rightharpoonup \varrho_{1,\varepsilon}, \ldots, \varrho_{N,\varepsilon}, u_\varepsilon \quad \text{in} \quad L^2(0,T;W^{1,2})$$

and from the Aubin–Lions lemma

$$\varrho_{i,\varepsilon,\delta} \to \varrho_{i,\varepsilon} \quad \text{in} \quad L^2((0,T)\times\Omega).$$

Thus passing to the limit in the weak formulation (note that the terms $\int_0^T \int_\Omega \varrho_{\varepsilon,\delta}^{\beta-2} \varrho_{i,\varepsilon,\delta}\varphi \, \mathrm{d}x\mathrm{d}t$ for $\varphi \in C_0^\infty((0,T)\times\Omega)$ are bounded independently of δ) we obtain the weak solution to the system (1.9).

4 End of the proof of Theorem 1.1. The Bresch & Jabin method

The main goal of this section is to perform the final limit passage with $\varepsilon \to 0$ in context of Theorem 1.1. We will show that if $\vec{\varrho}_k = (\varrho_{1,k}, \varrho_{2,k}, \ldots, \varrho_{N,k})$ and u_k solve the system (1.9) with $\varepsilon = \frac{1}{k}$ and $\gamma_1 = \cdots = \gamma_N = 2$, then for any $i = 1,\ldots,N$ the sequence $(\varrho_{i,k})_{n\in\mathbb{N}}$ is compact in $L^2((0,T)\times\Omega)$. From the calculations in Section 3, we have the following estimates uniform in k:

$$\sum_{i=1}^N \|\varrho_{i,k}\|_{L^\infty((0,T)\times\Omega)} + \frac{1}{\sqrt{k}} \sum_{i=1}^N \|\nabla \varrho_{i,k}\|_{L^2((0,T)\times\Omega)} + \|\operatorname{div} u_k\|_{L^\infty((0,T)\times\Omega)} + \|\nabla u_k\|_{L^\infty(0,T;BMO)} \leq C.$$

Let us now formulate the main result of this section:

Lemma 4.1. *Let $\vec{\varrho}_0 \in L^\infty(\Omega)$ and $(\varrho_{1,k}, \ldots, \varrho_{N,k}, u_k)$ be a sequence of solutions to (1.9) with $\varepsilon = \frac{1}{k}$ and initial conditions $\varrho_{0,i,k} \to \varrho_{0,i}$ in $L^2(\Omega)$ with $k \to \infty$. Then $(\varrho_{i,k})_{k\in\mathbb{N}}$ is compact in $L^2((0,T)\times\Omega)$ for any $i = 1,\ldots,N$.*

Proof. We use the following version of the compactness criterion from [7, 8]:

Proposition 4.2. *Let ϱ_k be a sequence uniformly bounded in $L^p((0,T)\times\Omega)$ for $1 \leq p < \infty$. Assume that $\{K_h\}_{h>0}$ is a family of positive, bounded functions on $\mathbb{R}^3$, satisfying:*

- $\forall_{\eta>0} \quad \sup_{h>0} \int_{\mathbb{R}^3} K_h(x) \mathbb{1}_{\{|x|>\eta\}} \mathrm{d}x < \infty,$
- $\|K_h\|_{L^1} \to \infty$ *as* $h \to 0$.

If $\partial_t \varrho_k$ is uniformly bounded in $L^q(0,T;W^{-1,p})$ for some $q > 1$ and

$$\limsup_k \int_0^T \frac{1}{\|K_h\|_1} \iint_{\Omega\times\Omega} K_h(x-y)|\varrho_k(t,x) - \varrho_k(t,y)|^p \, \mathrm{d}x\mathrm{d}y\mathrm{d}t \to 0 \quad as \quad h \to 0,$$

then $\{\varrho_k\}_{k\in\mathbb{N}}$ is compact in $L^p((0,T)\times\Omega)$. Conversely, if $\{\varrho_k\}_{k\in\mathbb{N}}$ is compact in $L^p((0,T)\times\Omega)$, then the above quantity converges to 0.

Following [7], we choose the functions K_h in the following way:
Let $K_h \colon \mathbb{R}^3 \to \mathbb{R}$ be given by

$$K_h(x) = \frac{1}{(|x|+h)^3} \quad \text{for} \quad |x| \leq \frac{1}{2}$$

and let K_h be independent of h for $|x| > \frac{2}{3}$, zero outside the ball $B(0,\frac{3}{4})$ and such that $K_h \in C^\infty(\mathbb{R}^3)$. Denote also $\overline{K}_h = \frac{K_h}{\|K_h\|_1}$. An important property of K_h is that $\|K_h\|_1 \sim |\log h|$. To use the Proposition 4.2, we will estimate the quantity

$$\mathcal{R}_h(t) = \sum_{i=1}^N \mathcal{R}_h^i(t) = \sum_{i=1}^N \frac{1}{\|K_h\|_1} \frac{1}{m_i} \iint_{\Omega\times\Omega} K_h(x-y)|\varrho_{i,k}(t,x) - \varrho_{i,k}(t,y)|^2 \mathrm{d}x\mathrm{d}y.$$

We need to estimate $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_h(t)$. In the following calculations we drop the index k where it does not raise confusion, and use the notation $f^x := f(t,x)$.

From the equation on ϱ_i, we get

$$\begin{aligned}\partial_t(\varrho_i^x-\varrho_i^y)^2 &= 2(\varrho_i^x-\varrho_i^y)(-\mathrm{div}\,(\varrho_i^x u^x)+\mathrm{div}\,(\varrho_i^y u^y)+\mathrm{div}\,F_i^x-\mathrm{div}\,F_i^y+\omega_i(\vec{\varrho}^x)-\omega_i(\vec{\varrho}^y))\\ &= -\mathrm{div}_x(u^x(\varrho_i^x-\varrho_i^y)^2)-\mathrm{div}_y(u^y(\varrho_i^x-\varrho_i^y)^2)-(\mathrm{div}\,u^x-\mathrm{div}\,u^y)(\varrho_i^x+\varrho_i^y)(\varrho_i^x-\varrho_i^y)\\ &\quad +2(\varrho_i^x-\varrho_i^y)(\mathrm{div}_x F_i^x-\mathrm{div}_y F_i^y)+2(\varrho_i^x-\varrho_i^y)(\omega_i(\vec{\varrho}^x)-\omega_i(\vec{\varrho}^y)).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_h^i(t) =& \frac{2}{\|K_h\|_1}\frac{1}{m_i}\iint_{\Omega\times\Omega}\nabla K_h(x-y)(u^x-u^y)|\varrho_i^x-\varrho_i^y|^2\mathrm{d}x\mathrm{d}y\\ &-\frac{2}{m_i}\iint_{\Omega\times\Omega}\overline{K}_h(x-y)(\mathrm{div}\,u^x-\mathrm{div}\,u^y)(\varrho_i^x-\varrho_i^y)\varrho_i^x\mathrm{d}x\mathrm{d}y\\ &+\frac{2}{m_i}\iint_{\Omega\times\Omega}\overline{K}_h(x-y)(\varrho_i^x-\varrho_i^y)(\mathrm{div}_x F_i^x-\mathrm{div}_y F_i^y)\mathrm{d}x\mathrm{d}y\\ &+\frac{2}{m_i}\iint_{\Omega\times\Omega}\overline{K}_h(x-y)(\varrho_i^x-\varrho_i^y)(\omega_i(\vec{\varrho}^x)-\omega_i(\vec{\varrho}^y))\mathrm{d}x\mathrm{d}y\\ &-\frac{4\varepsilon}{m_i}\iint_{\Omega\times\Omega}\overline{K}_h(x-y)|\nabla\varrho_i^x|^2\mathrm{d}x\mathrm{d}y\\ =&A_1^i+A_2^i+A_3^i+A_4^i+A_5^i.\end{aligned}$$

The last term has a good sign, so we just estimate it by 0. The first term is estimated in the same way as in [7]. By definition of K_h, we have

$$|\nabla K_h(z)|\le\frac{CK_h(z)}{|z|}.$$

Using the inequality

$$|f(x)-f(y)|\le C|x-y|(M|\nabla f|(x)+M|\nabla f|(y)),$$

where M denotes the maximal function, we arrive at

$$A_1^i\le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)(M|\nabla u|^x+M|\nabla u|^y)|\varrho_i^x-\varrho_i^y|^2\mathrm{d}x\mathrm{d}y$$

and in consequence

$$A_1^i\le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)M|\nabla u|^x|\varrho_i^x-\varrho_i^y|^2\mathrm{d}x\mathrm{d}y.$$

We will now estimate A_2^i. From (2.4) and the L^∞ estimates on $\vec{\varrho}$, we get

$$\begin{aligned}|\mathrm{div}\,u^x-\mathrm{div}\,u^y|&\le|d^x-d^y|+|p(\vec{\varrho}^x)-p(\vec{\varrho}^y)|\\ &\le|d^x-d^y|+C\sum_{j=1}^N|\varrho_j^x-\varrho_j^y|.\end{aligned}$$

Therefore, by the Cauchy inequality and the L^∞ bound on ϱ_i,

$$\begin{aligned}A_2^i&\le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)\left(|d^x-d^y|+C\sum_{j=1}^N|\varrho_j^x-\varrho_j^y|\right)|\varrho_i^x-\varrho_i^y|\mathrm{d}x\mathrm{d}y\\ &\le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)\left(\frac{1}{2}|d^x-d^y|^2+\frac{N+1}{2}|\varrho_i^x-\varrho_i^y|^2+\frac{C}{2}\sum_{j=1}^N|\varrho_j^x-\varrho_j^y|^2\right)\mathrm{d}x\mathrm{d}t\\ &\le C\mathcal{R}_h(t)+C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)|d^x-d^y|^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

From (2.9) we know that

$$\|\nabla d\|_{L^\infty(0,T;L^2)} \le C.$$

Denoting by Ed an $H^1(\mathbb{R}^3)$ extension of d, we have

$$\begin{aligned}
\iint_{\Omega\times\Omega} \overline{K}_h(x-y)|d^x - d^y|^2 \mathrm{d}x\mathrm{d}y &\le \iint_{\mathbb{R}^3\times\mathbb{R}^3} \overline{K}_h(z)|Ed(t,y+z) - Ed(t,y)|^2 \mathrm{d}y\mathrm{d}z \\
&\le \int_0^1 \int_{\mathbb{R}^3} \overline{K}_h(z)|z|^2 \int_{\mathbb{R}^3} |\nabla Ed(t,y+sz)|^2 \mathrm{d}y\mathrm{d}z\mathrm{d}s \\
&\le \|\nabla d\|^2_{L^\infty(0,T;L^2)} \frac{1}{|\log h|} \int_{\mathbb{R}^3} K_h(z)|z|^2 \mathrm{d}z.
\end{aligned}$$

In conclusion, since $\sup_h \int_{\mathbb{R}^3} K_h(z)|z|^2 \mathrm{d}z \le C$, we get

$$A_2^i \le C\mathcal{R}_h(t) + \frac{C}{|\log h|}.$$

For the term A_4^i, as ω_i is locally Lipschitz, we get

$$\begin{aligned}
A_4^i &\le \frac{1}{m_i} \iint_{\Omega\times\Omega} \overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)^2 \mathrm{d}x\mathrm{d}y + \iint_{\Omega\times\Omega} \overline{K}_h(x-y)(\omega_i(\vec{\varrho}^{\,x}) - \omega_i(\vec{\varrho}^{\,y}))^2 \mathrm{d}x\mathrm{d}y \\
&\le \mathcal{R}_h^i(t) + C\sum_{j=1}^N \iint_{\Omega\times\Omega} \overline{K}_h(x-y)(\varrho_j^x - \varrho_j^y)^2 \mathrm{d}x\mathrm{d}y
\end{aligned}$$

and in consequence

$$\sum_{i=1}^N A_4^i \le C\mathcal{R}_h(t).$$

Finally, we estimate A_3^i, the key point of the proof of Theorem 1.1. Here we find the place where we are obliged to assume all $\gamma_i = 2$. We have

$$\begin{aligned}
&\frac{1}{m_i} \iint_{\Omega\times\Omega} \overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)(\mathrm{div}_x F_i^x - \mathrm{div}_y F_i^y)\mathrm{d}x\mathrm{d}y \\
&\quad = -\frac{1}{m_i} \iint_{\Omega\times\Omega} F_i^x \cdot \nabla_x \left(\overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)\right) \mathrm{d}x\mathrm{d}y + \frac{1}{m_i} \iint_{\Omega\times\Omega} F_i^y \cdot \nabla_y \left(\overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)\right) \mathrm{d}x\mathrm{d}y \\
&\quad = -\frac{1}{m_i} \iint_{\Omega\times\Omega} F_i^x \cdot \nabla\overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)\mathrm{d}x\mathrm{d}y - \frac{1}{m_i} \iint_{\Omega\times\Omega} \overline{K}_h(x-y) F_i^x \cdot \nabla\varrho_i^x \mathrm{d}x\mathrm{d}y \\
&\quad\quad - \frac{1}{m_i} \iint_{\Omega\times\Omega} F_i^y \cdot \nabla\overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)\mathrm{d}x\mathrm{d}y - \frac{1}{m_i} \iint_{\Omega\times\Omega} \overline{K}_h(x-y) F_i^y \cdot \nabla\varrho_i^y \mathrm{d}x\mathrm{d}y \\
&\quad = -2 \iint_{\Omega\times\Omega} \overline{K}_h(x-y) F_i^x \cdot \frac{1}{m_i}\nabla\varrho_i^x \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Using the relation

$$\sum_{i=1}^N F_i \cdot \frac{1}{m_i}\nabla\varrho_i = \sum_{i=1}^N \frac{1}{\varrho_i}|F_i|^2 \ge 0$$

we arrive at

$$\sum_{i=1}^N A_3^i \le 0.$$

Combining all estimates, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_h(t) \le C \sum_{i=1}^N & \iint_{\Omega\times\Omega} \overline{K}_h(x-y) M|\nabla u^x|(\varrho_i^x - \varrho_i^y)^2 \mathrm{d}x\mathrm{d}y \\
&+ C\mathcal{R}_h(t) + \frac{C}{|\log h|}.
\end{aligned}$$

For the first term, as the maximal function is bounded in BMO (cf. Theorem 4.2 in [1]), we use the following logarithmic inequality from [30]:

$$\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x \leq C\|f\|_{BMO}\|g\|_{L^1}\big(|\log\|g\|_{L^1} + \log(1+\|g\|_{L^\infty})\big)$$

applied to $M|\nabla u^x|$ and $\displaystyle\int_\Omega \overline{K}_h(x-y)(\varrho_i^x - \varrho_i^y)^2\mathrm{d}y$ extended by 0 outside Ω. We get

$$\iint_{\Omega\times\Omega} \overline{K}_h(x-y)M|\nabla u^x|(\varrho_i^x-\varrho_i^y)^2\mathrm{d}x\mathrm{d}y \\ \leq C\|\nabla u\|_{BMO}\mathcal{R}_h^i(t)\left(|\log\mathcal{R}_h^i(t)| + \log(1+C\|\varrho_i\|_{L^\infty}^2)\right).$$

In conclusion, using the uniform bounds on $\varrho_{i,k}$ and ∇u_k and integrating over time, we obtain the following inequality

$$\mathcal{R}_h(t) - \mathcal{R}_h(0) \leq C\sum_{i=1}^N \int_0^t \mathcal{R}_h^i(\tau)(1+|\log\mathcal{R}_h^i(\tau)|)\mathrm{d}\tau + CT|\log h|^{-1}.$$

Using properties of the function $|\log x|$ and the fact that $x(|\log x|+1)$ is concave for small x, we finally arrive at

$$\mathcal{R}_h(t) - \mathcal{R}_h(0) \leq \ C\int_0^t \mathcal{R}_h(\tau)(1+|\log\mathcal{R}_h(\tau)|)\mathrm{d}\tau + C|\log h|^{-1}.$$

Then

$$\limsup_{k\to\infty}\sup_t R_h(t) \to 0 \quad \text{as} \quad h\to 0,$$

by the standard comparison criterion and the following Proposition (with $\varepsilon = |\log h|^{-1}$):

Proposition 4.3. *Let*

$$0 \leq x_\varepsilon(t) \leq x_\varepsilon(0) + \varepsilon + \int_0^t x_\varepsilon(|\ln x_\varepsilon|+1)\mathrm{d}\tau$$

with $x_\varepsilon(0)\to 0$ *as* $\varepsilon\to 0$. *Then* $\sup\limits_{t\in[0,T]} x_\varepsilon(t) \to 0$ *as* $\varepsilon \to 0$.

Proof. Denote $y_\varepsilon(t) = x_\varepsilon(0)+\varepsilon+\displaystyle\int_0^t x_\varepsilon(|\ln x_\varepsilon|+1)\mathrm{d}\tau$. As the function $x(|\ln x|+1)$ is increasing, we have

$$\dot{y}_\varepsilon = x_\varepsilon(|\ln x_\varepsilon|+1) \leq y_\varepsilon(|\ln y_\varepsilon|+1).$$

Therefore from the standard comparison criterion for ODEs we have

$$x_\varepsilon(t) \leq y_\varepsilon(t) \leq z_\varepsilon(t),$$

where z_ε solves

$$\dot{z}_\varepsilon = z_\varepsilon(|\ln z_\varepsilon|+1), \quad z_\varepsilon(0) = x_\varepsilon(0)+\varepsilon.$$

However, it is now easy to see that $\sup\limits_{t\in[0,T]} z_\varepsilon(t) \to 0$ as $\varepsilon\to 0$. Indeed, as $z_\varepsilon(t) \geq z_\varepsilon(0)$, we get $|\ln z_\varepsilon(t)| \leq |\ln z_\varepsilon(0)|$. Then from Gronwall's lemma

$$z_\varepsilon(t) \leq z_\varepsilon(0)e^{t+t|\ln z_\varepsilon(0)|} = e^t(z_\varepsilon(0))^{1-t}.$$

Therefore $\sup_{t\in[0,t_1]} z_\varepsilon(t)\to 0$ for some $t_1<1$. Note that in particular $z_\varepsilon(t_1)\to 0$, hence we are able to repeat these estimates on the consecutive intervals, and in consequence obtain the convergence on any finite interval $[0,T]$. ☐

To finish the proof of Lemma 4.2, it is enough to show that the sequence $(\partial_t\varrho_{i,k})_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;W^{-1,p})$. It is however an immediate consequence of the uniform bounds on $\vec{\varrho}_k$ and u_k. ☐

Using Lemma 4.1 and the estimates from Theorem 1.3, we are able to extract a sequence converging to a solution to (1.6). If $(\varrho_{1,\varepsilon},\dots,\varrho_{N,\varepsilon},u_\varepsilon)$ is a solution to (1.9), then from the uniform estimates we have up to a subsequence

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(0,T;H^1)$$

and

$$F_{i,\varepsilon} \rightharpoonup \bar{F}_i \quad \text{in} \quad L^2((0,T)\times\Omega)$$

for some functions $\bar{F}_i \in L^2((0,T)\times\Omega)$. Moreover, for any $\varphi \in C_0^\infty((0,T)\times\Omega)$,

$$\varepsilon \int_0^T \int_\Omega \nabla\varrho_i \cdot \nabla\varphi \,\mathrm{d}x \to 0.$$

By Lemma 4.1, we also have strong convergence

$$\varrho_{i,\varepsilon} \to \varrho_i \quad \text{in} \quad L^2((0,T)\times\Omega),$$

which allows us to pass to the limit in the weak formulation.

In the end, let us show that if the solution to (1.6) has higher regularity, then it satisfies the relation (1.7). At the level of approximated equation (1.9), testing the continuity by $\frac{1}{m_i}\varrho_{i,\varepsilon}\varphi$ for some smooth, positive $\varphi \in C_0^\infty((0,T)\times\Omega)$ yields

$$\begin{aligned}
-\frac{1}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}^2\varphi_t \,\mathrm{d}x\mathrm{d}t &+ \frac{1}{2m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}^2 \mathrm{div}\,(\varphi u_\varepsilon)\,\mathrm{d}x\mathrm{d}t - \frac{1}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}^2 u_\varepsilon\cdot\nabla\varphi \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T\int_\Omega \varphi F_{i,\varepsilon}\cdot\frac{1}{m_i}\nabla\varrho_{i,\varepsilon}\,\mathrm{d}x\mathrm{d}t + \frac{1}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}F_{i,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \\
&+ \frac{\varepsilon}{m_i}\int_0^T\int_\Omega |\nabla\varrho_{i,\varepsilon}|^2\varphi\,\mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}\nabla\varrho_{i,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t = \frac{1}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}\omega_i(\vec{\varrho}_\varepsilon)\varphi\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Summing the above equality over i and using (2.3), we get

$$\begin{aligned}
-\int_0^T\int_\Omega p_\varepsilon\varphi_t\,\mathrm{d}x\mathrm{d}t &+ \frac{1}{2}\int_0^T\int_\Omega p_\varepsilon \mathrm{div}\,(\varphi u_\varepsilon)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega p_\varepsilon u_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \\
&+ \sum_{i=1}^N\int_0^T\int_\Omega \frac{1}{\varrho_{i,\varepsilon}}|F_{i,\varepsilon}|^2\mathrm{d}x\mathrm{d}t + \sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}F_{i,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \\
&+ \varepsilon\sum_{i=1}^N\int_0^T\int_\Omega \varrho_{i,\varepsilon}\nabla\varrho_{i,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \le \sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega \varrho_{i,\varepsilon}\omega_i(\vec{\varrho}_\varepsilon)\varphi\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.1}$$

Using the strong convergence of $\varrho_{i,\varepsilon}$, weak convergences of $F_{i,\varepsilon}, u_\varepsilon$ and weakly lower semicontinuity of convex functions, after passing to the limit in (4.1) we get

$$\begin{aligned}
-\int_0^T\int_\Omega p\varphi_t\,\mathrm{d}x\mathrm{d}t &+ \frac{1}{2}\int_0^T\int_\Omega p\,\mathrm{div}\,(\varphi u)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega pu\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \\
&+ \sum_{i=1}^N\int_0^T\int_\Omega \frac{1}{\varrho_i}|\bar{F}_i|^2\varphi\,\mathrm{d}x\mathrm{d}t + \sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega \varrho_i\bar{F}_i\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \le \sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega \varrho_i\omega_i\varphi\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.2}$$

Now let us test the limit equation by $\varrho_i\varphi$, provided that $\nabla\varrho_i \in L^2((0,T)\times\Omega)$. Performing the analogous calculations, we arrive at the equality

$$\begin{aligned}
-\int_0^T\int_\Omega p\varphi_t\,\mathrm{d}x\mathrm{d}t &+ \frac{1}{2}\int_0^T\int_\Omega p\,\mathrm{div}\,(\varphi u)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega pu\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \\
&+ \sum_{i=1}^N\int_0^T\int_\Omega \varphi\bar{F}_i\cdot\frac{1}{m_i}\nabla\varrho_i\,\mathrm{d}x\mathrm{d}t + \sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega \varrho_i\bar{F}_i\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t = \sum_{i=1}^N\frac{1}{m_i}\int_0^T\int_\Omega \varrho_i\omega_i\varphi\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.3}$$

After substracting (4.3) from (4.2), we get

$$\int_0^T \int_\Omega \sum_{i=1}^N \left(\frac{1}{2}\frac{1}{\varrho_i} |\bar{F}_i|^2 - \bar{F}_i \cdot \frac{1}{m_i} \nabla \varrho_i \,\mathrm{d}x\mathrm{d}t \right) \varphi \,\mathrm{d}x\mathrm{d}t \leq 0$$

for any positive $\varphi \in C_0^\infty((0,T)\times\Omega)$ and in consequence we obtain the inequality (1.7) a. e. in $(0,T)\times\Omega$.

5 End of the proof of Theorem 1.4

In this section we perform the limit passage $\varepsilon \to 0$ with the additional assumption that all densities are bounded away from zero. First, note that we can rewrite F_i in a following way:

$$F_i = \sum_{j=1}^N C_{i,j} \nabla \frac{1}{m_j} \varrho_j^{\gamma_j} = \sum_{j=1}^N \tilde{C}_{i,j} \frac{\gamma_j}{\gamma_j - 1} \frac{1}{m_j} \nabla \varrho_j^{\gamma_j - 1},$$

where

$$\tilde{C}_{i,j} = \begin{cases} \dfrac{\varrho_i}{\varrho} \displaystyle\sum_{k\neq i} \varrho_k, & i = j, \\ -\dfrac{\varrho_i \varrho_j}{\varrho}, & i \neq j \end{cases}$$

is symmetric.

Using the above relation we can write $F_1, \ldots, F_{N-1}$ as a combination of $\nabla q_1, \ldots, \nabla q_{N-1}$ for

$$q_i = \frac{\gamma_i}{\gamma_i - 1} \frac{1}{m_i} \varrho_i^{\gamma_i - 1} - \frac{\gamma_{i+1}}{\gamma_{i+1} - 1} \frac{1}{m_{i+1}} \varrho_{i+1}^{\gamma_{i+1} - 1},$$

namely

$$F_i = \sum_{j=1}^{N-1} b_{i,j} \nabla q_j$$

with

$$b_{i,j} = \begin{cases} -\dfrac{\varrho_i}{\varrho} \displaystyle\sum_{k=1}^{j} \varrho_k, & j < i, \\ \dfrac{\varrho_i}{\varrho} \displaystyle\sum_{k=j+1}^{N} \varrho_k, & j \geq i \end{cases}$$

By performing elementary operations on the matrix $B = (b_{i,j})_{i,j=1,\ldots,N-1}$, it is easy to see that $\det B = \frac{1}{\varrho} \varrho_1 \ldots \varrho_N$. Indeed, adding the verses $2, \ldots, N-1$ to the first one and dividing it by $\frac{\varrho_N}{\varrho}$, we get the matrix

$$\begin{bmatrix} \varrho_1 & \varrho_1 + \varrho_2 & \cdots & \cdots & \sum_{k=1}^{N-1} \varrho_k \\ -\frac{\varrho_1 \varrho_2}{\varrho} & \frac{\varrho_2}{\varrho} \sum_{k=3}^{N} \varrho_k & \cdots & \cdots & \frac{\varrho_2}{\varrho} \varrho_N \\ -\frac{\varrho_1 \varrho_3}{\varrho} & -\frac{\varrho_3}{\varrho} \sum_{k=1}^{2} \varrho_k & \cdots & \cdots & \frac{\varrho_3}{\varrho} \varrho_N \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ -\frac{\varrho_1 \varrho_{N-1}}{\varrho} & -\frac{\varrho_{N-1}}{\varrho} \sum_{k=1}^{2} \varrho_k & \cdots & \cdots & \frac{\varrho_{N-1}}{\varrho} \varrho_N \end{bmatrix}$$

Now to the i-th verse we add the first one multiplied by $\frac{\varrho_i}{\varrho}$. After that operation, we obtain the triangular matrix with $\varrho_1, \varrho_2, \ldots, \varrho_{N-1}$ on the diagonal. In conclusion

$$\det B = \frac{\varrho_N}{\varrho} \cdot \varrho_1 \ldots \varrho_{N-1}$$

as we claimed.

In consequence, the matrix B is invertible whenever $\varrho_1 \dots \varrho_N \neq 0$ and if $\varrho_1 \dots \varrho_N \geq c$ for some $c > 0$, then

$$\sum_{i=1}^{N-1} |\nabla q_i|^2 \leq C \sum_{i=1}^{N-1} |F_i|^2$$

for some constant C depending on c and B^{-1}. In particular, from the L^2 bound on F_i we get

$$\sum_{i=1}^{N-1} \|\nabla q_i\|_{L^2((0,T)\times\Omega)} \leq C.$$

Now, let us show the Lipschitz estimate on $\varrho_1, \dots, \varrho_N$ with respect to $q_1, \dots, q_{N-1}, \varrho$. Let

$$G(z_1, \dots, z_N) = \begin{bmatrix} \frac{\gamma_1}{\gamma_1 - 1}\frac{1}{m_1} z_1^{\gamma_1 - 1} - \frac{\gamma_2}{\gamma_2 - 1}\frac{1}{m_2} z_2^{\gamma_2 - 1} \\ \frac{\gamma_2}{\gamma_2 - 1}\frac{1}{m_2} z_2^{\gamma_2 - 1} - \frac{\gamma_3}{\gamma_3 - 1}\frac{1}{m_3} z_3^{\gamma_3 - 1} \\ \vdots \\ \frac{\gamma_{N-1}}{\gamma_{N-1} - 1}\frac{1}{m_{N-1}} z_{N-1}^{\gamma_{N-1} - 1} - \frac{\gamma_N}{\gamma_N - 1}\frac{1}{m_N} z_N^{\gamma_N - 1} \\ z_1 + \dots + z_N \end{bmatrix}.$$

Then

$$(\vec{q}, \varrho) = G(\vec{\varrho}).$$

We will now prove that G is invertible. First, let us show two auxillary results:

Proposition 5.1. $\det DG(\vec{z}) \neq 0$ *for any* $\vec{z} \in \mathcal{U} := \{\vec{z} \in \mathbb{R}^N : z_1, \dots, z_N > 0\}$.

Proof. We have

$$DG(\vec{z}) = \begin{bmatrix} a_1 & -a_2 & 0 & \dots & 0 \\ 0 & a_2 & -a_3 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & a_{N-1} & -a_N \\ 1 & \dots & \dots & \dots & 1 \end{bmatrix}$$

for $a_i = \frac{\gamma_i}{m_i} z_i^{\gamma_i - 2}$. By elementary operations, we can transform this matrix into

$$\begin{bmatrix} a_1 & -a_2 & 0 & \dots & 0 \\ 0 & a_2 & -a_3 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & a_{N-1} & -a_N \\ 0 & \dots & \dots & 0 & 1 + a_N \sum_{j=1}^{N-1} \frac{1}{a_j} \end{bmatrix}$$

Therefore

$$\det DG(\vec{z}) = \sum_{i=1}^{N} \prod_{j \neq i} a_j > 0$$

for $z_1, \dots, z_N > 0$ $\square$

In the next Proposition we analyze the codomain of G. For simplicity we assume that $\frac{\gamma_i}{\gamma_i - 1}\frac{1}{m_i} = 1$, otherwise we can additionally rescale the variables to get the same result. For clarity, below we present the precise formulation for only 3 components, however we can proceed analogously for arbitrary N.

Proposition 5.2 (For $N=3$). *$G(\mathcal{U}) \subseteq \mathcal{V} := \{(s_1,s_2,s_3) : s_3 > g(s_1,s_2)\}$, with*

$$g(s_1,s_2) = \begin{cases} (-s_1)^{1/\alpha_2} + (-s_1-s_2)^{1/\alpha_3}, & (s_1,s_2) \in I_1 \\ s_1^{1/\alpha_1} + (-s_2)^{1/\alpha_3}, & (s_1,s_2) \in I_2 \\ (s_1+s_2)^{1/\alpha_1} + s_2^{1/\alpha_2}, & (s_1,s_2) \in I_3 \end{cases}$$

for $\alpha_i = \gamma_i - 1$ and

$$\begin{aligned} I_1 &= \{s_1 < 0,\ s_1+s_2 < 0\}, \\ I_2 &= \{s_1 > 0,\ s_2 < 0\}, \\ I_3 &= \{s_2 > 0,\ s_1+s_2 > 0\} \end{aligned}$$

Proof. It is enough to examine the behavior of G at $\partial\mathcal{U}$. We have

$$G(0,z_2,z_3) = \begin{bmatrix} -z_2^{\alpha_2}, \\ z_2^{\alpha_2} - z_3^{\alpha_3}, \\ z_2 + z_3 \end{bmatrix}.$$

Therefore using the parametrisation $s_1 = -z_2^{\alpha_2}$, $s_2 = z_2^{\alpha_2} - z_3^{\alpha_3}$ we see that

$$z_2 + z_3 = |s_1|^{1/\alpha_2} + (|s_1| - s_2)^{1/\alpha_3}.$$

Moreover, since $z_2, z_3 \geq 0$, we have $(s_1,s_2) \in I_1$. Proceeding analogously for the remaining components of $\partial\mathcal{U}$, we get that $G(\partial\mathcal{U})$ is a graph of a function g. □

Note that in the case of arbitrary N analogous calculations show that the function g has a form

$$g(s_1,\ldots,s_{N-1}) = \sum_{j=1}^{i-2} \left(\sum_{l=j}^{i-1} s_l \right)^{1/\alpha_j} + s_{i-1}^{1/\alpha_{i-1}} + (-s_i)^{1/\alpha_i} + \sum_{j=i+2}^{N} \left(-\sum_{l=i}^{j-1} s_l \right)^{1/\alpha_j}$$

for $(s_1,\ldots,s_{N-1}) \in I_i$, where

$$I_i = \left\{ \sum_{l=j}^{i-1} s_l \geq 0,\ j = 1,\ldots,i-1, \quad \text{and} \quad \sum_{l=i}^{k} s_l \leq 0,\ k = i,\ldots,N-1 \right\}.$$

The above propositions allows us to conclude invertibility of G. By Proposition 5.1 G is a local diffeomorphism. Then, since $\mathcal{V}$ is simply connected and $G\colon \mathcal{U} \to \mathcal{V}$ is proper, by the Hadamard's theorem (see e.g. Theorem 6.2.8 in [23]), G is invertible.

Since G is a diffeomorphism, we know that

$$\vec{\varrho} = G^{-1}(\vec{q}, \varrho)$$

is locally Lipschitz. Therefore in particular

$$|p(\vec{\varrho}(t,x)) - p(\vec{\varrho}(t,y))| \leq C|\varrho(t,x) - \varrho(t,y)| + C \sum_{j=1}^{N-1} |q_j(t,x) - q_j(t,y)|. \tag{5.1}$$

We will use the above information to first show the compactness of the total density again via Bresch & Jabin method (note that in this case $(\partial_t \varrho_{i,\varepsilon})$ is also bounded in $L^2(0,T;W^{-1,p})$ for each i). Let K_h be as in Section 4 and define

$$\mathcal{R}_h(t) = \iint_{\Omega\times\Omega} \overline{K}_h(x-y)|\varrho(t,x) - \varrho(t,y)|^2 \mathrm{d}x\mathrm{d}y.$$

By analogous calculations as in the previous section, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_h(t) =& \frac{2}{\|K_h\|_1}\iint_{\Omega\times\Omega} \nabla K_h(x-y)(u^x-u^y)|\varrho^x-\varrho^y|^2\mathrm{d}x\mathrm{d}y\\
&-2\iint_{\Omega\times\Omega}\overline{K}_h(x-y)(\operatorname{div}u^x-\operatorname{div}u^y)(\varrho^x-\varrho^y)\varrho^x\mathrm{d}x\mathrm{d}y\\
&-\frac{4\varepsilon}{m_i}\iint_{\Omega\times\Omega}\overline{K}_h(x-y)|\nabla\varrho_i^x|^2\mathrm{d}x\mathrm{d}y\\
=&A_1+A_2+A_3.
\end{aligned}
$$

Then similarly as before

$$
A_1 \le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)M|\nabla u|^x|\varrho^x-\varrho^y|^2\mathrm{d}x\mathrm{d}y
$$

and

$$
A_2 \le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)(|d^x-d^y|+|p^x-p^y|)|\varrho^x-\varrho^y|\mathrm{d}x\mathrm{d}y.
$$

Using (5.1) and Young inequality, we can further estimate A_2 by

$$
A_2 \le C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)\left(|d^x-d^y|^2+\sum_{j=1}^{N-1}|q_j^x-q_j^y|^2\right)\mathrm{d}x\mathrm{d}y + C\mathcal{R}_h(t).
$$

We have

$$
\iint_{\Omega\times\Omega}\overline{K}_h(x-y)|d^x-d^y|^2\mathrm{d}x\mathrm{d}y \le \frac{\|\nabla d\|_{L^2}^2}{|\log h|}\int_{\mathbb{R}^3}K_h(z)|z|^2\mathrm{d}z
$$

and similarly

$$
\iint_{\Omega\times\Omega}\overline{K}_h(x-y)|q_j^x-q_j^y|^2\mathrm{d}x\mathrm{d}y \le \frac{\|\nabla q_j\|_{L^2}^2}{|\log h|}\int_{\mathbb{R}^3}K_h(z)|z|^2\mathrm{d}z.
$$

Then, since the integral $\int_{\mathbb{R}^3}K_h(z)|z|^2\mathrm{d}z$ is bounded independently of h, we get

$$
A_2 \le C\mathcal{R}_h(t) + \frac{C}{|\log h|}\left(\|\nabla d\|_{L^2(\Omega)}^2+\sum_{j=1}^{N-1}\|\nabla q_j\|_{L^2(\Omega)}^2\right).
$$

In conclusion,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_h(t) \le& C\iint_{\Omega\times\Omega}\overline{K}_h(x-y)M|\nabla u^x|(\varrho^x-\varrho^y)^2\mathrm{d}x\mathrm{d}y + C\mathcal{R}_h(t)\\
&+\frac{C}{|\log h|}\left(\|\nabla d\|_{L^2(\Omega)}^2+\sum_{j=1}^{N-1}\|\nabla q_j\|_{L^2(\Omega)}^2\right)
\end{aligned}
$$

and after applying the logarithmic inequality from [30] and integrating over time, we end up with

$$
\begin{aligned}
\mathcal{R}_h(t)-\mathcal{R}_h(0) &\le C\int_0^t\mathcal{R}_h(\tau)(1+|\log\mathcal{R}_h(\tau)|)\mathrm{d}\tau + \frac{C}{|\log h|}\left(\|\nabla d\|_{L^2(0,T;L^2)}^2+\sum_{j=1}^{N-1}\|\nabla q_j\|_{L^2(0,T;L^2)}^2\right)\\
&\le C\int_0^t\mathcal{R}_h(\tau)(1+|\log\mathcal{R}_h(\tau)|)\mathrm{d}\tau + \frac{C}{|\log h|}.
\end{aligned}
$$

Then, proceeding in the same way as in Section 4, we arrive at

$$
\limsup_{k\to\infty}\sup_t R_h(t) \to 0 \quad \text{as} \quad h\to 0.
$$

In conclusion, the sequence (ϱ_ε) is compact in $L^2((0,T)\times\Omega)$. Now, from the higher space regularity of q_j, we can extract compactness of all particular densities as well. For each i we simply put

$$\begin{aligned}\int_0^T \iint_{\Omega\times\Omega} &\overline{K}_h(x-y)|\varrho_{i,\varepsilon}(t,x)-\varrho_{i,\varepsilon}(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\leq C\int_0^T \iint_{\Omega\times\Omega} \overline{K}_h(x-y)|\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad + C\sum_{j=1}^{N-1}\int_0^T \iint_{\Omega\times\Omega} \overline{K}_h(x-y)|q_{j,\varepsilon}(t,x)-q_{j,\varepsilon}(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\leq C\int_0^T \iint_{\Omega\times\Omega} \overline{K}_h(x-y)|\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad + \frac{C}{|\log h|}\sum_{j=1}^{N-1}\|\nabla q_{j,\varepsilon}\|^2_{L^2((0,T)\times\Omega)}\end{aligned}$$

and then the compactness of $(\varrho_{i,\varepsilon})$ follows immediately from Proposition 4.2.

6 The general diffusing/non-diffusing case

In the last section, we briefly present the necessary modifications for the system (1.12). In that case, we need to divide the total density and the pressure into two parts, where the first part depends only on the diffusive components. In other words, $p = p^{(1)} + p^{(2)}$ and $\varrho = \varrho^{(1)} + \varrho^{(2)}$, where

$$p^{(1)} = \sum_{i=1}^{N_1} p_i(\varrho_i), \quad \varrho^{(1)} = \sum_{i=1}^{N_1} \varrho_i$$

and N_1 is the number of diffusive components. With such decomposition of p and ϱ, we define the fluxes only in terms of the first N_1 components, namely

$$F_i = \nabla p_i - \frac{\varrho_i}{\varrho^{(1)}}\nabla p^{(1)}, \quad i=1,\dots,N_1.$$

Note that with the above definition we preserve the important properties of F_i's. In consequence, we are able to repeat the arguments from Sections 2 and 3 and derive the existence of solutions to the approximate system

$$\begin{aligned}\partial_t\varrho_i + \operatorname{div}(\varrho_i u) - \operatorname{div} F_i &= \omega_i(\vec{\varrho}) + \varepsilon\Delta\varrho_i, \quad i=1,\dots,N_1,\\
\partial_t\varrho_j + \operatorname{div}(\varrho_j u) &= \omega_j(\vec{\varrho}) + \varepsilon\Delta\varrho_j, \quad j=N_1+1,\dots,N,\\
-\mu\Delta u - \nabla((\mu+\lambda)\operatorname{div} u) + \nabla p(\vec{\varrho}) &= 0\end{aligned} \tag{6.1}$$

in the same way as before. To apply the reasoning from Section 4, we can perform the Bresch & Jabin argument together for all the components. Defining

$$\mathcal{R}_h(t) = \sum_{i=1}^N \mathcal{R}_h^i(t) = \sum_{i=1}^N \frac{1}{\|K_h\|_1}\frac{1}{m_i}\iint_{\Omega\times\Omega} K_h(x-y)|\varrho_{i,k}(t,x)-\varrho_{i,k}(t,y)|^2\mathrm{d}x\mathrm{d}y$$

and computing $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_h$, it is clear that for the non-diffusing components the equivalents of the terms A_1, A_2, A_4 and A_5 can be dealt with in the same manner, whereas A_3 is just equal to 0. In consequence, the computations from Section 4 give the strong convergence of densities for both diffusing and non-diffusing components.

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